Optimal Hedge Ratio Estimation in the Presence of Conditional Moments: A GARCH-X Approach

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\textbf{Abstract}

The paper investigates the problems involved in effectively modelling the (time-varying) basis risk and the subsequent calculation of effective hedge ratios. The specific purpose of this research involves the identification of conditional variance models that accommodate the time series properties commonly encountered in many asset price return series, by targeting changes in basis volatility over time. The research analyses the problem of incorporating conditional basis variance into the time series model by extending popular specifications to include long-run (cointegration) information.

The analysis investigates whether the omission of cointegration information in the underlying econometric time series model leads to inappropriate modelling of long-run and short-run time series behaviour. Cointegration information, through the squared spread between (price level) series, may provide potential predictive power in modelling the volatility of asset returns, which is not captured effectively by the GARCH (1,1) model. Consequences of omitting dynamic adjustments include inadequate modelling of the time series behaviour, sub-optimal decision making and the possible progressive degeneration of such modelling and decision making over time.

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Dynamic hedges are compared to constant procedures to determine the conditions under which allowance for conditional variance significantly increases hedging effectiveness. Hedging effectiveness is subsequently defined as the percentage reduction in the variance of the portfolio achieved by implementing a hedged rather than an unhedged position. Where dynamic variance-covariance matrices are effectively modelled, hedge ratios may be constructed that subsequently minimise basis risk.

Another major objective of the research involves the derivation of the conditions where periodic re-balancing of the optimal hedge ratio leads to increased hedging effectiveness. The analysis determines criteria that must be triggered in order for an alternative hedging strategy to become more optimal than the strategy currently in effect.

The most important contribution of the analysis is to ensure that in time series modelling of financial series, the basic model adopted is capable of accounting for any significant short-run and long-run characteristics found to be typical of these series. Concentration is focussed on the calculation of optimal hedge ratios in order to simplify the analysis, by using a very common example of the conditional variance situation. However, the fundamental contribution involves attempting to ensure the basic econometric specification is appropriate in modelling typical time series behaviour so it does not need further adjustment.

**Keywords:** Cointegration, Conditional Variance, GARCH, Hedging
INTRODUCTION

Cointegration provides a precise, effective test for a fundamental property of much of finance theory - equilibrium - by enabling an investigation into the long-run as well as short-run behaviour in price series relationships, such as between spot and futures prices. Without the added information that cointegration provides, little may be concluded about the relationship between financial time series other than to note there is significant change in the difference between spot and futures prices in the short-run. However, with cointegration, the error correction model may be applied to obtain information about the correction path implied for short-run changes in basis risk.

Once it can be established that two or more series are cointegrated, their dynamic structure can be exploited for further investigation. Engle and Granger (1987) show that cointegration implies an error correction representation of the component series. The following task involves analysis into the short-run situation over time and investigates the size of short-run conditional covariance changes between two series in cases where the relationship between spot and futures prices is dynamic in nature.

This paper explains the steps in relation to forming the GARCH-X model as well as applying this model to both simulated and empirical data. Section ?? discusses the modelling of the conditional variance via the method of cointegration in conjunction with the GARCH specification (the GARCH-X model), and the application of this model in a hedging framework to two examples, one empirical and one simulated. The purpose of Section ?? is to examine whether there is evidence of the effectiveness of such a model when compared to other specifications, such as the GARCH model. Section ?? compares various hedging strategies and determines the conditions under which a certain hedging strategy
is more effective than another. Section ?? discusses the effectiveness of the
GARCH-X model both in a modelling as well as a hedging framework. Section ??
provides a distillation on the applicability of the analysis to future research and,
in particular, to cross-hedging situations.

APPLICATIONS OF GARCH-X PROCEDURE

Lee (1994) examined the potential relationship between disequilibrium and
uncertainty in the cointegrated system. Fama and French (1987) and Viswanath
(1993) show that the difference between spot and futures prices (that is, the
basis) is a significant information variable. Lee (1994) states that since the error
correction term may influence the conditional mean, it may also influence the
conditional variance and “if disequilibrium (measured by the error correction
term) is responsible for uncertainty (measured by the conditional variance) the
conditional heteroscedasticity may be modelled with a function of several lagged
error correction terms” to see whether some variables for the conditional means
affect conditional variances. Lee (1994) considered a system of error correction
models for the conditional mean and an extended bivariate GARCH model with
the error correction term for the conditional variance. The model seems
appropriate for testing for causality in variance through the error correction term.
The purpose of this section is to discuss the logic behind the inclusion of
cointegration information (via the error correction term) in a GARCH
specification. Consider a spot security and a futures contract traded on the basis
of the spot security. Let $S_t$ and $F_t$ denote the natural logarithms of spot and
futures prices respectively at time $t$. Since the futures contract is priced off the
spot security, the error correction term is given by

$$z_t = S_t - \beta F_t$$
The term $z_t$ imposes the long-run cointegration relationship between spot and futures prices and measures how the dependent variable adjusts to the previous period’s deviation from long-run equilibrium. The constant $\beta$ is known as the cointegration parameter that links spot and futures prices (logarithms) such that the error correction term is stationary. At any given time, $z_t$ is expected to differ from its long-run equilibrium level, known as a “disequilibrium” state. The expectation of $z_t$ gives the long-run equilibrium relationship between $S_t$ and $F_t$ and short-term periods of disequilibrium occur as the observed value of $z$ varies around its expected value. Therefore, cointegration information relating to the series ($S_t$ and $F_t$) may indeed be a significant variable in modelling the conditional variances and covariances of financial asset returns.

The BEKK formulation of the GARCH-X model is given by

$$H_t = C' C + A'e_{t-1}e_{t-1}'A + B'H_{t-1}B + D'Dz_{t-1}^2,$$

(1)

where $D$ is a $1 \times 2$ matrix of coefficients and $z_t$ is the error correction term equalling $S_t - \beta F_t$ (where $S_t$ and $F_t$ - the natural logarithms of the spot and futures prices respectively - are cointegrated with cointegration parameter $\beta$).

When long-run equilibrium exists, the GARCH-X model abstracts information from the error correction term ($S_t - \beta F_t$), so the size of the deviations (or the variance of the deviations) from the long-run equilibrium level may provide added information about the relationship between spot and futures rates. A direct implication is the further the deviations from long-run equilibrium, the more volatile the asset prices and the more difficult they are to forecast. Such added information may be exploited to obtain more precise time-varying confidence intervals for point forecasts of asset returns.

Two examples are provided to test the applicability of the GARCH-X method in
practical situations. The purpose of these examples is to outline the methodology involved in applying the GARCH-X technique and showing such a model may lead to increased hedging effectiveness. In particular, the first example tests whether the GARCH-X model increases hedging effectiveness above and beyond that obtained via the implementation of constant hedge ratios and the hedge ratios obtained via the GARCH method. The following four tasks are involved in such a process:

1. testing for skewness and leptokurtosis in spot and futures returns and any possible autocorrelation in these returns, determining the significance of any deviation from normality.

2. testing whether cointegration exists between the two price series. If there is no such cointegration there is no logic for implementation of a model that accounts for cointegration and, as a consequence, it appears illogical to hedge using this particular futures instrument, as the two series do not appear to track each other.

3. choosing the specification of the conditional mean and conditional second moments based on information obtained from the previous two tasks.

4. constructing the GARCH-X hedge ratios and applying these hedge ratios to spot and futures returns in order to obtain values for the effectiveness of various hedging techniques.

The GARCH-X specification is used to model the conditional variances and conditional covariance for both a simulated and an empirical example. Example 1 uses actual futures prices for both heating oil and crude oil, where it is expected the spread between the two rates is constrained (cointegrated) in the long-run, yet volatile enough for time-varying procedures to be applicable in the short-run.
In this empirical example, the cointegration vector is calculated for the within-sample period and is not updated throughout the out-of-sample period. Example 2 involves data simulated from a GARCH-X model using the S-Plus statistical package. The benefits of including a simulated example is to ensure the conditional variance specification is correct and to reduce any potential errors that may stem from the inaccurate estimates of the coefficients in the conditional variance model.

EFFECTIVENESS OF ALTERNATIVE HEDGING STRATEGIES

Letting $\Delta S_t$ and $\Delta F_t$ denote the changes in spot and futures prices respectively between time $t - 1$ and $t$, the profit equation at time $t$ is given by

$$\Pi_t = \Delta S_t - h_t \Delta F_t,$$

where $\Pi_t$ denotes the profit at time $t$ in holding one unit of the spot asset and, for each unit held, adopting a short position in $h_t$ ($h_t$ is a measurable function of the variables in $F_{t-1}$) units in the futures market, under the assumption the futures prices are martingales. The conditional variance of any such hedge at time $t$ is equal to

$$E_t(\Pi_t^2) = H_{11,t} - 2h_t H_{12,t} + h_t^2 H_{22,t},$$

where $h_t$ is the hedge ratio at time $t$. In this analysis, $h_t$ may either be equal to the minimum-variance hedge ratio, $h_{MV,t}$, the naive hedge ratio, 1, or the dynamic hedge ratio, $h_t$.

The minimum-variance hedging technique provides a better hedge than the naive procedure when

$$(h_{MV,t} + 1)H_{22,t} - 2H_{12,t} > 0.$$
That is, when the quantity \( (h_{MV,t} + 1)H_{22,t} > 2H_{12,t} \), the minimum-variance hedge is preferred over the naive procedure (in terms of greater risk-reduction). Similarly, the dynamic hedge produces a more effective hedge than the naive technique when

\[ -(H_{12,t} - H_{22,t})^2 < 0 \]

The dynamic hedge produces a better hedge than the naive hedge as the conditional covariance between the spot and futures returns deviates from the conditional variance of the futures returns. The naive hedge produces an adequate hedge when the quantities \( H_{12,t} \) and \( H_{22,t} \) are approximately equal (that is, near-perfect correlation between spot and futures returns). Finally, the dynamic technique provides a better hedge than the minimum-variance technique when

\[ -(\frac{H_{12,t}}{H_{22,t}} - h_{MV,t})^2 < 0 \]

In Example 2, the true form of the variance function is known. Therefore, the conditional variance of each of the hedging methods (naive, minimum-variance and GARCH-X) may be calculated and compared to determine the effectiveness of each method. If the error correction term is influential in modelling the conditional covariance and conditional variances, a GARCH model without cointegration will result in model mis-specification.

APPLICATIONS TO FINANCIAL DATA
Example 1
The purpose of this example is to examine whether the GARCH-X model is effective in empirical situations. The GARCH-X specification will be tested using financial data. This example focusses on empirical data, obtained from the Turtle – Trader website (www.turtletrader.com). The data analysed involve the closing prices for heating oil futures and crude oil futures contracts, traded on the

The hedger is assumed to have a long position in the heating oil futures contract and wishes to hedge the exposure to this contract by shorting the crude oil futures contract. The analysis is equally applicable if the hedger has a short position in the crude oil futures contract and wishes to hedge the exposure to this contract by shorting the heating oil futures contract. In the analysis, the natural logarithms of the heating oil futures prices are denoted by $Heat$ while the natural logarithms of the crude oil futures prices are denoted by $Crude$. The hedger would like to cover the exposure to $Heat$ using $Crude$. The time series plots of the prices (natural logarithms) of the heating oil and crude oil futures contracts, as well as the time series plots of the returns for both the heating oil and crude oil futures contracts, are shown in Figures ??-??.

Figures ?? and ?? verify that both return series are time-varying. Furthermore, $\Delta Heat$ and $\Delta Crude$ are found to be non-normal. Skewness and kurtosis for both returns are significant at the 1% level. The returns are skewed left and heavy-tailed. The Jarque-Bera test reveals significant non-normality in both returns. The Ljung-Box and Lagrange-Multiplier tests show the returns are autocorrelated. The tests are all significant at the 1% level of significance, with the exception of the Ljung-Box test for $\Delta Heat$, which is significant at the 5% level. Both returns series therefore are characterised by ARCH effects. The
summary statistics are included in Table 2.

The mean equation is expressed as

$$\Delta y_t = \mu + \epsilon_t,$$

$$y_t = \begin{pmatrix} Heat_t \\ Crude_t \end{pmatrix} = \begin{pmatrix} S_t \\ F_t \end{pmatrix},$$

$$\epsilon_t = \begin{pmatrix} \epsilon_{Heat,t} \\ \epsilon_{Crude,t} \end{pmatrix},$$

where $S_t$ and $F_t$ - the natural logarithms of the heating and crude oil prices respectively - are cointegrated with cointegration parameter $\beta$ and $\epsilon_t|\mathcal{F}_{t-1} \sim N(0, H_t)$. The conditional structure of the covariance matrix $H_t$ is based on the BEKK specification of Baba, Engle, Kraft and Kroner, as proposed by Engle and Kroner (1995). The BEKK specification of the GARCH model is represented by

$$H_t = C'C + A'\epsilon_{t-1}e'_{t-1}A + B'H_{t-1}B,$$  \hfill (2)

where

$$H = \begin{pmatrix} H_{11} & H_{21} \\ H_{21} & H_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

$$A = \begin{pmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{21} \\ B_{21} & B_{22} \end{pmatrix},$$

The conditional variance GARCH-X specification is an extension of the GARCH formulation and is given by Equation (2). Via implementation of the PCFIML econometric package, the cointegration parameter, $\beta$, that links the two series is found to be equal to 0.88466 (and is significant at the 1% level of significance).

Therefore, $\text{ln}(Heat) - 0.88466 \times \text{ln}(Crude)$ provides a stationary series, where ln denotes the natural logarithm. The time series graph of the linear combination of the cointegration vector is shown in Figure 2.
The diagonal representations of the GARCH and GARCH-X models are both considered as they are time-varying models and one may like to investigate whether the GARCH-X model has potential to produce a better hedge and/or provide more information in regards to the dynamic properties of the heating oil and crude oil returns when compared to its GARCH counterpart (Kavussanos and Nomikos 2000a,b). The estimates of the GARCH and GARCH-X coefficients within-sample, calculated via the econometric package RATS using the BFGS algorithm, are included in Table ??.

The parameters $D_{11}$ and $D_{12}$ - describing the influence of the error correction term on the conditional variance - are both significant (at the 1% level of significance) when analysing the whole sample, indicating these terms have potential predictive power in modelling the conditional variance-covariance matrix of the returns. Therefore, last period's equilibrium error has significant impact on the adjustment process of the subsequent returns. Within-sample, only the parameter $D_{12}$ is significant (at the 1% level of significance). The log-likelihood function values are higher for the GARCH-X model than the GARCH model (both within-sample and analysing the total sample). This indicates that the GARCH-X model is a more adequate model in analysing the conditional variance-covariance matrix of the returns.

Implementing a GARCH model without the restriction that the matrices A and B are diagonal achieves a similar result to the diagonal GARCH-X model, that is, both parameters $D_{11}$ and $D_{12}$ are significant for the whole sample and the parameter $D_{12}$ is significant in-sample (all at the 1% level of significance). Therefore the (square of the) error correction term remains a significant explanatory variable for the conditional variance-covariance matrix even after allowing all components of the matrices A and B to vary through time.
In spite of the superiority of the GARCH-X model in modelling the variance-covariance matrix of the returns in this example, the hedge ratios obtained by the GARCH-X formulation are similar to those obtained by the GARCH specification. Therefore, the benefits of incorporating cointegration information into the formation and adjustment of hedge ratios may, in some instances, be minimal.

APPLICATIONS TO SIMULATED DATA

Example 2

In this simulated example, two time series, $\Delta S_t$ and $\Delta F_t$, are generated from a GARCH-X model. The GARCH-X methodology is then applied to these return series to determine whether the application of this model results in increased hedging effectiveness when compared to static hedging procedures. The first step is to construct the model. The objective is to generate two series that are cointegrated, with both return series exhibiting GARCH effects. The series $z_t$ and $S_t$ are initially formed, the latter series generated from the GARCH-X model. The series $F_t$ is then calculated as a linear combination of $z_t$ and $S_t$. The first differences of $S_t$ (that is, the spot return series) are denoted by $e_{1,t}$. The series $e_{1,t}$ is required to be a random walk and independent of $z_t$, satisfying the condition that

$$E_{t-1}(e_{1,t}^2) = c + a_1 e_{t-1}^2 + b_1 H_{11,t-1} + d_1 z_{t-1}^2,$$

where $E_{t-1}$ denotes the conditional expectation given information available up to and including time $t - 1$. $H_{11,t}$ denotes the conditional variance of spot returns at time $t$ and $c$, $a_1$, $b_1$ and $d_1$ are (constant) coefficients. Given $e_{1,t}$ and $z_t$, assume spot and futures prices, $S_t$ and $F_t$, are generated from the following equations:

$$S_t = S_{t-1} + e_{1,t}, \quad \text{and} \quad F_t = \sum_{i=1}^{n} a_i S_{t-i} + b_i H_{11,t-1} + d_i z_{t-1},$$
\[
    z_t = S_t - \beta F_t,
\]

where \( z_t = \delta_t + \alpha \delta_{t-1} \) is a stationary series (when \( |\alpha| < 1 \)) and \( \delta_t \) is a sequence of normally distributed mutually independent random variables. The next step is to obtain expectations of the conditional variances and conditional covariance. Therefore,

\[
    \Delta S_t = e_{1,t}, \quad \Delta F_t = \frac{1}{\beta} (z_{t-1} + e_{1,t} - z_t) = e_{2,t}, \quad \text{and}
\]

\[
    H_t = \begin{pmatrix}
        H_{11,t} & H_{12,t} \\
        H_{21,t} & H_{22,t}
    \end{pmatrix}
    = E_{t-1} \begin{pmatrix}
        (e_{1,t})^2 & (e_{1,t} e_{2,t}) \\
        (e_{2,t})^2 & (e_{1,t} e_{2,t})
    \end{pmatrix}
    = \begin{pmatrix}
        E_{t-1}(e_{1,t}^2) & E_{t-1}(e_{1,t} e_{2,t}) \\
        E_{t-1}(e_{1,t} e_{2,t}) & E_{t-1}(e_{2,t}^2)
    \end{pmatrix},
\]

where

\[
    E_{t-1}(e_{1,t}^2) = H_{11,t} \\
    E_{t-1}(e_{1,t} e_{2,t}) = \frac{H_{11,t}}{\beta} \\
    E_{t-1}(e_{2,t}^2) = \frac{1}{\beta^2} \left[ \left( 1 - \alpha \right) \delta_{t-1} + \alpha \delta_{t-2} \right]^2 + H_{11,t} + \sigma_\delta^2
\]

The above equations for the conditional covariance and conditional variances were derived as follows:

\[
    E_{t-1}(e_{1,t}^2) = H_{11,t-1} \\
    E_{t-1}(e_{1,t} e_{2,t}) = E_{t-1}(e_{1,t} z_{t-1} + e_{1,t} - z_t) = \frac{1}{\beta} E_{t-1}[e_{1,t} z_{t-1} + e_{1,t} - z_t] \\
    = \frac{1}{\beta} E_{t-1}[e_{1,t} z_{t-1} + e_{1,t}^2 - e_{1,t} z_t] \\
    = \frac{1}{\beta} E_{t-1}[e_{1,t} (\delta_{t-1} + \alpha \delta_{t-2})] + \frac{1}{\beta} E_{t-1}(e_{1,t}^2) - \frac{1}{\beta} E_{t-1}[e_{1,t} (\delta_t + \alpha \delta_{t-1})] \\
    = \frac{1}{\beta} E_{t-1}(e_{1,t}^2)
\]
From the above construction, $\Delta S_t$ and $\Delta F_t$ follow a GARCH(1,1)-X model. The series $S_t$ and $F_t$ are both $I(1)$ and cointegrated (under the assumption that $e_{1,t}, e_{2,t}$ and $z_t$ are all stationary). The conditional second moments of $e_{1,t}$ and $e_{2,t}$ and the conditional covariance of $e_{1,t}e_{2,t}$ are all dependent upon the specification of the conditional second moment, $H_{11,t}$. The next task in examining the applicability of the GARCH-X model involves the simulation of data from the bivariate GARCH-X model outlined above. The sample size is 1210 with the first 10 observations subsequently discarded. The last 200 observations make up the out-of-sample (forecasting) period. The cointegration parameter ($\beta$) that links the two series is equal to 1, that is, the relationship $S_t - F_t = z_t$ forms a stationary series. After generating the $I(1)$ series $S_t, F_t$ is formed by ensuring $z_t$ is a stationary series and implementing the relationship $F_t = S_t - z_t$ to generate the sequence of futures prices. In this example,
\( z_t = \delta_t + 0.6\delta_{t-1} \), where \( \delta_t \) is a sequence of independently and identically distributed random variables from the normal distribution with mean 0 and variance 0.5. The variance function of \( \Delta S_t \) is of the form \( H_t = 0.2e_t^2 + 0.8H_{t-1} + 0.1z_{t-1}^2 \). The time series plots of \( S_t \) and \( F_t \) are shown in Figures ?? and ?? respectively while the plots of the returns, \( \Delta S_t \) and \( \Delta F_t \), appear in Figures ?? and ?? respectively. The two series, \( S_t \) and \( F_t \), are verified to be cointegrated with \( \beta = 1 \).

Figures ?? and ?? verify that both return series are time-varying and conditional variance specifications may be applied to model the conditional volatility. The normality assumption is tested by examining the measures of skewness and kurtosis. There exists significant skewness (at the 1% level of significance) and kurtosis (at the 1% level of significance) in both \( \Delta S_t \) and \( \Delta F_t \). Both return series are also not normally distributed (at the 1% level of significance) and exhibit significant autocorrelation (once again at the 1% level of significance). The absolute values of \( \Delta S_t \), \( \Delta F_t \), \( \Delta S_t^2 \) and \( \Delta F_t^2 \) exhibit similar autocorrelation (at the 1% level of significance). The sign for the skewness of both returns series is negative, indicating the distribution of each return series is skewed left. Likewise, both \( \Delta S_t \) and \( \Delta F_t \) exhibit significant kurtosis, indicating the distributions of both spot and futures returns are heavy-tailed. The Jarque-Bera test for normality also reveals significant non-normality in \( \Delta S_t \) and \( \Delta F_t \) (at the 1% level of significance).

The Ljung-Box and Lagrange-Multiplier tests reveal \( \Delta S_t \) and \( \Delta F_t \) are both autocorrelated (at the 1% level of significance) and therefore significantly impacted by ARCH effects. The summary statistics are included in Table ??.

The final task in this example involves the application of various hedging techniques to the returns series. The purpose of these applications is to enable a comparison of the hedge ratios generated by these constant ratio techniques with that of a GARCH-X model in order to compare relative hedging effectiveness.
The performances of various hedging methods are compared by constructing daily returns as implied by the computed hedge ratios and the variance of the returns of the constructed portfolios are calculated over the entire sample period. When the hedge ratios are unstable, allowance for such stochastic movements significantly increases hedging effectiveness by reducing the volatility of the hedged portfolio. The focus turns to the effectiveness of each type of hedge. In Example 2, the true type of variance function is known as well as the resulting coefficients in this function. The time series plot of the GARCH-X hedge ratios for the out-of-sample period is shown in Figure ??.

The GARCH-X hedge ratios vary from 0.4934 to 0.9708 in the out-of-sample period. The mean value of these hedge ratios is 0.8134. The minimum-variance hedge ratio, on the other hand, generates a ratio of 1.0009 within-sample and this ratio is held constant throughout the out-of-sample period. In this example, the minimum-variance method overestimates the correlation between spot and futures returns in most periods. The plots of the differences between the conditional variances for all combinations of hedging rules appear in Figures ??-??. In this example, the hedging effectiveness value for the GARCH-X method is 0.8122. This value compares favourably to the hedging effectiveness values obtained via both the naive and minimum-variance hedges, which are 0.7596 and 0.7591 respectively.

The main conclusion reached from Examples 1 and 2 is that the GARCH-X model may be utilised in practical situations to provide greater knowledge of how the individual components in the variance-covariance matrix behave over time. However, this may not necessarily translate into increased hedging effectiveness. The GARCH-X model may be more effective in inefficient markets that have large volatilities or markets with no futures contracts, thus invoking the need for
cross-hedging.

EFFECTIVENESS OF GARCH-X MODEL

The technique of cointegration when applied to common trends in multivariate time series allows the retention and modelling of both long-run and short-run dynamics in a system. This represents a major potential improvement over GARCH models in situations where the series may change in the short-run, even though there exists a long-run equilibrium relationship between spot and futures prices. These short-run fluctuations in the relationship between the two (cointegrated) series do not persist indefinitely. The relationship between the two prices eventually reverts back to a long-run equilibrium level.

The studies of Fama and French (1987) and Viswanath (1993) show the basis (or more generally, the error correction term) can influence the conditional mean. Lee (1994) hypothesised that since the error correction term influences the conditional mean, it may also impact on the conditional variance. The GARCH-X model provides a measure for investigating the adjustments in the short-run shocks or deviation from the long-run equilibrium level. The measure used is the square of the long-run cointegration relationship. The square is then taken as a predictor variable of the conditional variance-covariance matrix of the returns.

This paper discussed the modelling of the conditional variance via the GARCH-X model of Lee (1994). Lee (1994) suggested that since conditional means are usually specified as a function of conditional second moments, an examination of the converse specification may distinguish between the notions of disequilibrium and uncertainty in a system of cointegrated variables. The empirical example in this paper indicated that whilst the GARCH-X model explains the conditional variances of, and the conditional covariance between, spot and futures returns
better than the GARCH model, there is no significant difference in the hedge ratios using the GARCH and GARCH-X models.

This examination is deliberately selective as it is not possible to deal extensively with all the issues involved in the application of GARCH-X models in a hedging framework. There may well be a possible link between the length of the out-of-sample time period and the effectiveness of the GARCH-X model (with respect to its GARCH counterpart). A longer out-of-sample time period may show short-run deviations between the two prices are corrected in later time periods. The out-of-sample period may also be too short to be influenced by the cointegration parameter. However, such issues must be left for later research.

The major purpose of this paper is to lay down the methodology for deriving a GARCH-X model and briefly test its performance in relation to other models.

CONCLUSION

Theoretically, cointegration can provide substantial improvement when modelling the variance-covariance matrix of two or more returns series. The incorporation of cointegration relationships into the conditional variance model provides added information that enables the inclusion of long-run behaviour between two or more cointegrated series. The addition also allows analysis of the dynamics of short-run “price shocks” to the system through an error correction model. Both are significant contributions to the modelling of time series behaviour through GARCH models.

Several factors may mitigate against any great improvement in a hedging framework:

1. the estimates of variance and volatility may be accurate via the implementation of static techniques. There would therefore be little room for
improvement and often this would typify an efficient market where the potential for a better hedge is low.

2. even where there is potential for improvement, application of the GARCH model (as in Example 2) provides this improvement, so the GARCH-X model may not significantly outperform the GARCH model.

Cointegration brings added information about long-run (and short-run) correlations between asset prices. It also permits analysis of the dynamics of deviations from this long-run equilibrium level. By using cointegration, investors may obtain added information in forming and/or progressively re-adjusting hedges. This re-adjustment may help in maintaining or improving the hedging effectiveness since new information impacts on asset prices.

In modelling both conditional variances and conditional covariances through the GARCH-X model, hedge ratios may be determined using an alternative specification to the GARCH model in explaining the correlation between returns. As a result, substantial basis risk may have important implications in cross-hedging situations. Cointegration may help create more effective hedges than GARCH models because the impact of cointegration on the adjustment process between spot and futures returns allows the variance of the hedged portfolio to be minimised, creating a more effective hedge. The simulated example emphasised the significant improvement the GARCH-X model provides in a hedging framework. Applying a GARCH model in such instances would result in a mis-specification of the variance-covariance matrix of returns.

It appears as though some unmodelled conditional heteroscedasticity in the GARCH(1,1) model may be explained by a function of the spread. This has been shown to be true for an empirical example involving commodity futures data. The
magnitude of deviations from the cointegration level may provide added
information about the relationship between two rates. However, this does not
necessarily translate to a more effective hedge. The GARCH-X model appears to
explain the relationship between disequilibrium and conditional volatility better
than the GARCH(1,1) specification and should be utilised as it is highly probable
that such a model will not provide a less inferior hedging strategy when compared
to the GARCH model alone.
References


Table 1: Summary statistics for the heating oil futures returns and crude oil futures returns in Example 1.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\Delta Heat_t$</th>
<th>$\Delta Crude_t$</th>
<th>5% Critical Values</th>
<th>1% Critical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>1252</td>
<td>1252</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Skewness</td>
<td>-1.044&lt;sup&gt;a&lt;/sup&gt;</td>
<td>-0.378&lt;sup&gt;a&lt;/sup&gt;</td>
<td>(-0.152,0.152)</td>
<td>(-0.200,0.200)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>6.449&lt;sup&gt;a&lt;/sup&gt;</td>
<td>5.200&lt;sup&gt;a&lt;/sup&gt;</td>
<td>(2.696,3.304)</td>
<td>(2.601,3.399)</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>2396.841&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1440.258&lt;sup&gt;a&lt;/sup&gt;</td>
<td>5.991</td>
<td>9.210</td>
</tr>
<tr>
<td>Ljung-Box $Q(24)$</td>
<td>41.962&lt;sup&gt;b&lt;/sup&gt;</td>
<td>46.483&lt;sup&gt;a&lt;/sup&gt;</td>
<td>36.415</td>
<td>42.980</td>
</tr>
<tr>
<td>Ljung-Box $</td>
<td>Q(24)</td>
<td>$</td>
<td>342.664&lt;sup&gt;a&lt;/sup&gt;</td>
<td>188.460&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>Ljung-Box $Q^2(24)$</td>
<td>170.831&lt;sup&gt;a&lt;/sup&gt;</td>
<td>158.520&lt;sup&gt;a&lt;/sup&gt;</td>
<td>36.415</td>
<td>42.980</td>
</tr>
<tr>
<td>Lagrange-Multiplier</td>
<td>112.791&lt;sup&gt;a&lt;/sup&gt;</td>
<td>105.909&lt;sup&gt;a&lt;/sup&gt;</td>
<td>36.415</td>
<td>42.980</td>
</tr>
</tbody>
</table>

Notes:
1. <sup>a</sup> denotes significance at the 1% level of significance.
2. <sup>b</sup> denotes significance at the 5% level of significance.
Table 2: Parameter estimates resulting from the implementation of the bivariate GARCH and bivariate GARCH-X models respectively for Example 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Within-Sample</th>
<th>Within-Sample</th>
<th>Total-Sample</th>
<th>Total-Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GARCH</td>
<td>GARCH-X</td>
<td>GARCH</td>
<td>GARCH-X</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>-0.0003( ^a )</td>
<td>-0.0004( ^a )</td>
<td>-0.0003( ^a )</td>
<td>-0.0004( ^a )</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>-0.0001( ^a )</td>
<td>-0.0002( ^a )</td>
<td>-0.0001( ^a )</td>
<td>-0.0002( ^a )</td>
</tr>
<tr>
<td>( C_{11} )</td>
<td>0.0025( ^a )</td>
<td>0.0021( ^a )</td>
<td>0.0025( ^a )</td>
<td>0.0024( ^a )</td>
</tr>
<tr>
<td>( C_{12} )</td>
<td>0.0023( ^a )</td>
<td>0.0012</td>
<td>0.0024( ^a )</td>
<td>0.0018( ^a )</td>
</tr>
<tr>
<td>( C_{22} )</td>
<td>0.0017( ^a )</td>
<td>0.0000</td>
<td>0.0017( ^a )</td>
<td>0.0000</td>
</tr>
<tr>
<td>( A_{11} )</td>
<td>0.2764( ^a )</td>
<td>0.2735( ^a )</td>
<td>0.2778( ^a )</td>
<td>0.2760( ^a )</td>
</tr>
<tr>
<td>( A_{22} )</td>
<td>0.2306( ^a )</td>
<td>0.2251( ^a )</td>
<td>0.2691( ^a )</td>
<td>0.2695( ^a )</td>
</tr>
<tr>
<td>( B_{11} )</td>
<td>0.9543( ^a )</td>
<td>0.9550( ^a )</td>
<td>0.9543( ^a )</td>
<td>0.9546( ^a )</td>
</tr>
<tr>
<td>( B_{22} )</td>
<td>0.9629( ^a )</td>
<td>0.9628( ^a )</td>
<td>0.9554( ^a )</td>
<td>0.9536( ^a )</td>
</tr>
<tr>
<td>( D_{11} )</td>
<td>-</td>
<td>-0.0009</td>
<td>-</td>
<td>-0.0006( ^a )</td>
</tr>
<tr>
<td>( D_{12} )</td>
<td>-</td>
<td>-0.0020( ^a )</td>
<td>-</td>
<td>-0.0019( ^a )</td>
</tr>
</tbody>
</table>

Log-likelihood

<table>
<thead>
<tr>
<th></th>
<th>Function Value</th>
</tr>
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<tbody>
<tr>
<td>Within-Sample</td>
<td>9380.77</td>
</tr>
<tr>
<td>Within-Sample</td>
<td>9383.70</td>
</tr>
<tr>
<td>Total-Sample</td>
<td>9833.70</td>
</tr>
<tr>
<td>Total-Sample</td>
<td>9835.95</td>
</tr>
</tbody>
</table>

Notes:
1. \( ^a \) denotes significance at the 1% level of significance.
2. \( ^b \) denotes significance at the 5% level of significance.

Table 3: Summary statistics for the spot and futures returns for Example 2.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>( \Delta S_t )</th>
<th>5% Critical Values</th>
<th>( \Delta F_t )</th>
<th>1% Critical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>1000</td>
<td>1000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.802( ^a )</td>
<td>-0.834( ^a )</td>
<td>(-0.152,0.152)</td>
<td>(-0.200,0.200)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>14.606( ^a )</td>
<td>14.193( ^a )</td>
<td>(2.696,3.304)</td>
<td>(2.601,3.399)</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>8996.124( ^a )</td>
<td>8508.730( ^a )</td>
<td>5.991</td>
<td>9.210</td>
</tr>
<tr>
<td>Ljung-Box ( Q(24) )</td>
<td>329.428( ^a )</td>
<td>317.495( ^a )</td>
<td>36.415</td>
<td>42.980</td>
</tr>
<tr>
<td>Ljung-Box (</td>
<td>9759.903( ^a )</td>
<td>9629.152( ^a )</td>
<td>36.415</td>
<td>42.980</td>
</tr>
<tr>
<td>Ljung-Box ( Q^2(24) )</td>
<td>4592.962( ^a )</td>
<td>4606.894( ^a )</td>
<td>36.415</td>
<td>42.980</td>
</tr>
<tr>
<td>Lagrange-Multiplier</td>
<td>503.901( ^a )</td>
<td>504.084( ^a )</td>
<td>36.415</td>
<td>42.980</td>
</tr>
</tbody>
</table>

Notes:
1. \( ^a \) denotes significance at the 1% level of significance.
2. \( ^b \) denotes significance at the 5% level of significance.
Figure 1: Time series plot of the heating oil futures prices for Example 1.

Figure 2: Time series plot of the crude oil futures prices for Example 1.

Figure 3: Time series plot of the heating oil futures returns for Example 1.
Figure 4: Time series plot of the crude oil futures returns for Example 1.

Figure 5: Time series plot of the linear combination of the Johansen cointegration vector for Example 1.

Figure 6: Time series plot of the spot prices for Example 2.
Figure 7: Time series plot of the futures prices for Example 2.

Figure 8: Time series plot of the spot returns for Example 2.

Figure 9: Time series plot of the futures returns for Example 2.
Figure 10: Time series plot of the hedge ratios obtained via implementation of the bivariate GARCH-X model for Example 2.

Figure 11: Conditional variance of the time-varying (GARCH-X) hedge minus the conditional variance of the minimum-variance hedge for Example 2.

Figure 12: Conditional variance of the time-varying (GARCH-X) hedge minus the conditional variance of the naive hedge for Example 2.
Figure 13: Conditional variance of the minimum-variance hedge minus the conditional variance of the naive hedge for Example 2.

Figure 14: Time series plot of the linear combination of the Johansen cointegration vector for Example 1.