Optimal dynamic asset allocation for pension funds
in the presence of a minimum guarantee

Marina Di Giacinto
Università degli studi di Cassino*

Fausto Gozzi
Libera Università internazionale degli studi sociali “Guido Carli” - Roma

Abstract

In this paper we propose a continuous time stochastic model of optimal allocation for a
defined contribution pension fund with a minimum guarantee. Traditionally, portfolio selection
models are interested in maximizing the total expected discounted utility from consumption
and from final wealth, whereas our target is to maximize the total expected discounted utility
from current wealth. In our model the dynamics of wealth takes directly into account the flows
of contributions and benefits, so that in general the portfolio is not self-financing and the level
of wealth is constrained to stay above a “solvability level”. The fund manager can invest in a
riskless asset and in a risky asset but borrowing and short selling are prohibited.

Pension funds are exposed to financial and demographic risks, but in our model we consider
these risks stochastically independent and we only analyze the financial aspects. Although this
is the common assumption, it is questionable and we leave the inclusion of demographic risk to
future research.

Applying dynamic programming techniques, we discuss the existence, uniqueness and regu-
larity of the value function which solves the related Hamilton-Jacobi-Bellman equation. Lastly
we obtain the existence and uniqueness of the allocation strategy in a feedback form.

Keywords: defined contribution pension fund, minimum guarantee, stochastic control problem,
Hamilton-Jacobi-Bellman equation, viscosity solution.

J.E.L. classification: C61, G11, G23.

A.M.S. Subject Classification: 91B28, 93E20, 49L25.

1 Introduction

This paper proposes to determine an optimal allocation of resources pertaining to a defined con-
tribution pension fund with a minimum guarantee by dynamic programming techniques and by
applying as control variable the proportion of wealth to invest in a risky asset.

The first analysis on optimal selection portfolio in a continuous time model and using the
dynamic programming approach was faced by R. C. Merton. Maximizing expected utility from

*Corresponding author. Mailing address: Dipartimento Economia e territorio, Via Mazzaroppi, s.n.c., 03043
Cassino (FR), Italy. E-mail: digiacinto@unicas.it.
consumption and from final wealth, he proved in [Merton, 1969] that explicit solutions exist if the individual utility function belongs to the CRRA (constant relative risk aversion) family, and in [Merton, 1971] if it belongs to the HARA (hyperbolic absolute risk aversion) family.

Traditionally, most of pension plans created in the past were based on defined benefits, but the situation has changed in recent years and actually we are going towards a rapid development of defined contributions pension plans. For this reason only recent literature is oriented to models concerning defined contribution pension schemes.

Stochastic optimization approaches to a defined contribution but with the constraint that the terminal wealth must not be inferior to a minimum guarantee have been recently introduced in [Boulier et al., 2001] and [Deelstra et al., 2003]. In these models the traditional Merton approach maximizing the total expected discounted utility from final wealth is applied. They assume that the terminal date corresponds to the retirements of a representative worker, so that they only consider the accumulation phases.

This modelization does not take into account the dynamical evolution of contributions and benefits which are related with new workers that adhere to the pension fund and those that have accrued the right of the pension. In our model the target is to maximize the total expected discounted utility from current wealth and the dynamics of wealth takes directly into account the flows of contributions and benefits over an infinite time horizon.

Pension funds are exposed to financial and demographic risks. In this paper we analyze the financial aspects: it is the first step of a work project and we leave the inclusion of demographic risk to future research.

The stochastic control problem, arising from the simplest case of a market with a riskless asset and a single risky asset, can be solved by the dynamic programming approach and by deriving an Hamilton-Jacobi-Bellman (HJB) equation for the state constraint problem (for a general reference on stochastic control see [Fleming & Soner, 1993] and [Yong & Zhou, 1999]). We discuss and prove the unique existence of a continuous viscosity solution of this equation and its regularity. Finally we show the existence and uniqueness of the optimal feedback map.

2 The model

Over an infinite continuous-time model we consider a security market competitive\(^1\), frictionless\(^2\), consistent\(^3\), default free and continuously open. The investor faces the following trading constraints: bankruptcy never occurs and borrowing and short positions are not allowed for all available securities.

The hypothesis that investors are price takers is usual in literature regarding financial management models of pension funds and it is realistic if the single financial agent does not invest a big amount of money. As a matter of fact, the volume of securities exchanged by pension funds is such that they could affect the price of securities (i.e. investors are price makers).

Pension funds are subject to financial and demographic risks. We hypothesize these two kinds of risks stochastically independent to develop their respective analysis separately. Therefore, to concentrate the analysis on the financial items, we can suppose the population of fund members

\(^1\) Investors' behavior are optimizing: they optimize their utility function on the whole time horizon and believe their actions cannot affect the probability distribution of returns on the available securities (i.e. investors are price takers).

\(^2\) All securities are perfectly divisible and there are no transaction costs or taxes.

\(^3\) There is no opportunity to gain without assuming risk with probability not null, i.e. the market is arbitrage free.
is a stationary open collectivity: there can be new entrees, nevertheless there will be no changes during the time in the numerousness as well as in the distribution per age class.

Let us consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t^B\}_{t \geq 0}\), where \(t \geq 0\) is the time variable. The space \(\Omega\) is the set of all possible states of nature, the set \(\mathcal{F}\) is the associated \(\sigma\)-algebra and \(\mathbb{P}\) is the probability measure observed, defined on the measurable space \((\Omega, \mathcal{F})\). The filtration \(\{\mathcal{F}_t^B\}_{t \geq 0}\) describing the information structure, is generated by the trajectories of a one-dimensional standard Brownian motion \(B(t), t \geq 0\), defined on the same probability space and completed with the addition of a null measure set of \(\mathcal{F}\). Moreover we hypothesize that \(\mathcal{F}_{t}^{B_{\infty}} = \mathcal{F}\).

The security market is composed of two kinds of assets: a riskless asset and a risky asset.

**Hypothesis 2.1** The price of the riskless asset, denoted by \(S^0(t), t \geq 0\), evolves according to the equation

\[
\frac{dS^0(t)}{S^0(t)} = rdt, \quad S^0(0) = 1,
\]

where \(r \geq 0\) is the instantaneous spot rate of return.

**Hypothesis 2.2** The price of risky asset \(S^1(t), t \geq 0\), follows an Itô process and satisfies the equation

\[
\frac{dS^1(t)}{S^1(t)} = \mu dt + \sigma dB(t),
\]

where \(\mu \geq 0\) is the instantaneous rate of expected return and \(\sigma > 0\) is the instantaneous rate of volatility.

The coefficient \(\mu\) verifies the relation \(\mu = r + \sigma \lambda\), where \(\lambda\) is the instantaneous risk premium of the market, i.e. the price that the market assigns to the randomness expressed by the standard Brownian motion \(B\). If we accept the further hypothesis that the investor is risk adverse, as we do, this premium assumes strictly positive values.

We expect to decide the optimal portfolio strategy of a defined contribution pension fund with a minimum guarantee between the two alternative investments maximizing his utility. Then the decisional variable is represented by the proportion of wealth that the manager can invest respectively into the two assets offered by the market.

Let us indicate by \(X(t), t \geq 0\), the \(\{\mathcal{F}_t^B\}_{t \geq 0}\)-adapted process that describes the amount of the pension fund wealth at any time and by \(\theta(t) > 0, t \geq 0\), the \(\{\mathcal{F}_t^B\}_{t \geq 0}\)-adapted process that represents the proportion of fund wealth to invest into the risky asset. Then the dynamics of wealth is expressed in the following state equation

\[
\begin{cases}
    dX(t) = \frac{\theta(t)X(t)}{S^1(t)} dS^1(t) + \frac{[1 - \theta(t)]X(t)}{S^0(t)} dS^0(t) + c(t)dt - p(t)dt, & t \geq 0 \\
    X(0) = x_0 > 0
\end{cases}
\]

where \(\frac{\theta(t)X(t)}{S^1(t)}\) and \(\frac{[1 - \theta(t)]X(t)}{S^0(t)}\) are the quantity in portfolio of risky and riskless asset, respectively; while the non-negative integrable function \(c(t), t \geq 0\), indicates the contribution flow and the non-negative function \(p(t), t \geq 0\), represents the benefit flow.

---

4 We mean a class of people that have the same characteristics (same age, same professional qualification, same skill, and so on).
If we want the pension fund to be able to refund the accrued capitals to the fund members at any time, including those who do not have the right to retirement yet, we can suppose that wealth $X$ is not inferior to a given amount we define solvability level. Then this amount concerns the contributions of active workers, revalued to the rate of return of minimum guarantee. For this reason let us include the following

**Hypothesis 2.3** The process $X$ describing the fund wealth is subject to the following constraint

$$X(t) \geq l(t) \quad \text{P-a.s.,} \quad \forall t \geq 0,$$

where the strictly positive function $l(t), t \geq 0$, represents the solvability level.

This hypothesis is very interesting because it avoids improper behavior of the fund manager. If this assumption is not held, he could keep the fund wealth at a negative level for long periods and at a positive level only for fixed balanced dates.

The state equation (1) can be rewritten in the following way

$$\begin{cases} dX(t) = \left\{ \theta(t) \sigma_{\lambda} + r \right\} X(t) + (c(t) - p(t)) dt + \theta(t) \sigma X(t) dB(t), & t \geq 0 \\ X(0) = x_0 \geq l(0) > 0 \end{cases}$$

such that $X(t) \geq l(t) \quad \text{P-a.s.,} \quad \forall t \geq 0$.

In the population stationarity hypothesis, contributions could be considered exogenous. Instead, benefits can be hypothesized exogenous only if depending on the guarantee, while it is endogenous if we suppose it is a function, direct or indirect, of the wealth level reached by the pension fund.

**Hypothesis 2.4** The payment of aggregate contributions occurs at any time according to the following relation

$$c(t) := \alpha w, \quad 0 < \alpha < 1, \quad \forall t \geq 0,$$

where the parameter $\alpha$ represents the average contribution rate and $w \geq 0$ the wage bill earned by the fund members.

**Hypothesis 2.5** The aggregate benefits are paid at any time according to the following relation

$$p(t) := \begin{cases} 0 & \text{if } 0 \leq t < T \\ g(t) + s(t, X) & \text{if } t \geq T \end{cases}$$

where $g$ is the aggregate flow of money at any time that the pension fund pays to its fund members in retirement as a minimum guarantee and $s$, that we define surplus, is a function depending on the time and on the fund return in the time interval $T, T > 0$, during which, on average, the fund members adhere to the pension fund. We suppose that $T$ is an exogenous fixed value.

By the given hypothesis, the investment risk is charged to the fund members for the amount related to the surplus, where it weighs on the pension fund for the share related to the minimum guarantee. Then the guarantee assumes a particular importance for the fund risk management.

**Hypothesis 2.6** The minimum guarantee is defined in the following way

$$g(t) := \eta \int_{t-T}^{t} c(u)e^{\delta(t-u)} du, \quad 0 \leq \eta < 1, \quad t \geq T,$$

where $\eta$ is the exit rate from the fund members that have accrued the right of the pension, and $\delta \geq 0$ is the guaranteed rate of return. We note the rate $\eta$ is constant for the given demographic hypothesis.
Hypothesis 2.4 implies that the minimum guarantee paid from the pension fund at any time is the following quantity

\[ g(t) = \eta \alpha w \frac{e^{\delta T} - 1}{\delta}, \quad t \geq T. \]

If we hypothesize in (3) that the minimum guarantee \( g \) is identically null, the model is adapted to describe a simple defined contribution pension fund and the investment risk is totally charged to fund members.

Vice versa, the fund manager that assures a minimum pension could decide, as it often happens, not to pay any amount in addition to the guarantee. Then this situation is described hypothesizing that in (3) the surplus is identically null.

We solve our stochastic control problem in this last simpler case, introducing the following

**Hypothesis 2.7** The payment of any surplus is not provided.

Hence benefits, that is described in (3), is simply expressed in the following way

\[
p(t) := \begin{cases} 
0 & \text{if } 0 \leq t < T \\
\eta \alpha w \frac{e^{\delta T} - 1}{\delta} & \text{if } t \geq T.
\end{cases}
\]

We aim to substitute the above hypothesis with the most natural

**Hypothesis 2.8** The surplus is described by the following function

\[ s(t, X) := \xi \eta \left[ X(t) - e^{\delta T} X(t - T) \right], \quad t \geq T, \]

where \( \xi \) is the retrocession rate.

In this case the state equation (2) becomes a delay differential equation and the related problem becomes more complex. We propose to study and solve the problem associated to the backward state equation in a future work.

**Remark 2.9** The solvability level introduced in hypothesis 2.3 is represented by the following relation

\[
l(t) = \begin{cases} 
l_0 + \zeta (1 - \eta) \int_0^t c(u) e^{\delta(t-u)} du & \text{if } 0 \leq t < T \\
l_0 + \zeta (1 - \eta) \int_{t-T}^t c(u) e^{\delta(t-u)} du & \text{if } t \geq T,
\end{cases}
\]

where \( l_0 > 0 \) and \( 0 \leq \zeta \leq 1 \).

Straightly from (4), we get

\[
l(t) = \begin{cases} 
l_0 + \zeta (1 - \eta) \alpha w \frac{e^{\delta t} - 1}{\delta} & \text{if } 0 \leq t < T \\
l_0 + \zeta (1 - \eta) \alpha w \frac{e^{\delta T} - 1}{\delta} & \text{if } t \geq T.
\end{cases}
\]
Fixing the observation initial time at $t = t_0$, $t_0 \geq T$, the analysis is made on a running pension fund and the dynamics of wealth, in accordance with (2) and with the hypotheses just enunciated, evolves according to the state equation

$$
\begin{align*}
    dX(t) &= \left\{ [\theta(t)\sigma + r]X(t) + \alpha w - \eta \alpha w \frac{e^{\delta T} - 1}{\delta} \right\} dt + \theta(t)\sigma X(t)dB(t), \ t \geq t_0 \\
    X(t_0) &= x \geq l_{t_0}
\end{align*}
$$

(5)

such that $X(t) \geq l(t)$ $\mathbb{P}$-a.s., for any $t \geq t_0$, and where $l(t_0) = l_{t_0}$.

The case of $0 \leq t < t_0$, $t_0 < T$, can be easily studied and we aim to do so.

**Remark 2.10** The state equation, for any $\{\mathcal{F}^B_t\}_{t \geq t_0}$-adapted process $\theta$, has a strong unique solution on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^B_t\}_{t \geq t_0}, \mathbb{P})$ (see, e.g., Theorem 2.5 in [Karatzas & Shreve, 1991] p. 287).

The total expected discounted utility coming from wealth is given by

$$
J(x; \theta) := \mathbb{E} \left[ \int_{t_0}^{+\infty} e^{-\rho t} U [X(t; x, \theta_{t_0})] \, dt \right],
$$

(6)

where

- $X(t; x, \theta_{t_0})$ is the random amount of the fund wealth that at time $t = t_0$ assumes the value $x$ and has control $\theta_{t_0}$;
- the discount rate $\rho \geq 0$, is a psychological parameter. Higher is $\rho$, more is the weight the fund manager assigns to an amount of wealth higher now rather than in the future;
- $U : [l_{t_0}, +\infty) \longrightarrow [0, +\infty)$ is the fund manager’s utility function, which is assumed to be integrable, strictly increasing, strictly concave and belonging to class $C^2((l_{t_0}, +\infty) ; [0, +\infty))$. Moreover $\lim_{y \to l_{t_0}^+} DU(y) = +\infty$ and $\lim_{y \to +\infty} DU(y) = 0$.

Let us indicate by $\Theta_{ad}(x)$ the set of admissible controls for the trajectories that at initial time $t = t_0$ have initial point $x$ and control $\theta(t_0) = \theta_{t_0}$, i.e.

$$
\Theta_{ad}(x) := \left\{ \theta : [t_0, +\infty) \times \Omega \longrightarrow [0, 1] \right. \text{ adapted to } \{\mathcal{F}^B_t\}_{t \geq t_0} \text{ s.t. } X(t; x, \theta_{t_0}) \in [l_{t_0}, +\infty), \ t \geq t_0 \}
$$

To solve our stochastic control problem we have to find the solution of the following family of stochastic control problems

$$
\sup_{\theta(\cdot) \in \Theta_{ad}(x)} J(x; \theta) = \sup_{\theta(\cdot) \in \Theta_{ad}(x)} \mathbb{E} \left[ \int_{t_0}^{+\infty} e^{-\rho t} U [X(t; x, \theta_{t_0})] \, dt \right] \ \text{s.t.} \ X(t) \geq l(t) \ \mathbb{P} \text{-a.s., } \forall t \geq t_0. \ (7)
$$

**Remark 2.11** The set of admissible strategies is constrained by the borrowing and the short selling prohibition and by the minimum solvability level.

The value function associated of the control problem (7) is given by

$$
V(x) := \sup_{\theta(\cdot) \in \Theta_{ad}(x)} \mathbb{E} \left[ \int_{t_0}^{+\infty} e^{-\rho t} U [X(t; x, \theta_{t_0})] \, dt \right], \quad x \in [l_{t_0}, +\infty). \ (8)
$$
and the HJB equation associated is the following

$$\rho v(x) - H(x, Dv(x), D^2v(x)) = 0, \quad \forall x \in [l_0, +\infty),$$

(9)

where $\rho$ is a discount rate, already defined, and

$$H(x, Dv(x), D^2v(x)) := \sup_{\theta \in [0,1]} \left\{ U(x) + \left[ (\theta \sigma \lambda + r)x + \alpha w - \eta \alpha w \frac{e^{\delta T} - 1}{\delta} \right] Dv(x) + \frac{1}{2} \theta^2 \sigma^2 x^2 D^2v(x) \right\} \quad x \in [l_0, +\infty),$$

is the generalized hamiltonian function.

### 3 Technical results

We can show some basic properties of the value function $V$ defined in (8). In particular we can prove the following statements:

**Proposition 3.1** The value function $V$ is concave.

**Proposition 3.2** The value function $V$ is strictly increasing.

**Proposition 3.3** The value function $V$ is uniformly continuous on the interval $[l_0, +\infty)$ and Lipschitz continuous in $[a, +\infty)$, for any $a > l_0$.

Moreover, using also results of [Ishii & Loreti, 2002] and [Zariphopoulou, 1994], we can show that the value function is the unique viscosity solution of the HJB equation and that it is also a smooth solution. More precisely we can prove the following

**Theorem 3.4** The value function $V$ defined in (8) is a constrained viscosity solution of the HJB equation (9) on the interval $[l_0, +\infty)$.

and we can derive the comparison theorem, which assures the uniqueness of a viscosity solution. In 1990 H. Ishii and P.L. Lions (see [Ishii & Lions, 1990]) investigated comparison results for a large class of bordering conditions related to the degenerate elliptic non linear second order partial differential equations. In particular, using a modification of these arguments introduced by H. Ishii and P. Loreti in [Ishii & Loreti, 2002], we prove the following

**Theorem 3.5** The value function $V$ defined in (8) is the unique constrained viscosity solution of the HJB equation (9) associated to the control problem (7).

Lastly we can prove the smoothness of $V$, which is assured by

**Theorem 3.6** The value function $V$ defined in (8) is in the class of concave functions the unique solution belonging to class $C([l_0, +\infty); \mathbb{R}) \cap C^2((l_0, +\infty); \mathbb{R})$ of the HJB equation (9) associated to the control problem (7).

which allows us to determine explicitly the optimal control policies.
4 The optimal feedback map

At this point from the HJB equation, we derive the optimal control strategies in feedback form depending on the optimal wealth process, and we study its properties.

We can prove the following important results

**Theorem 4.1** The optimal control policy can be expressed, for any $t \geq t_0$, in the feedback form

$$
\theta(X(t)) = G(X(t), DV(X(t)), D^2V(X(t))).
$$

Moreover, we have

$$
G(X(t), DV(X(t)), D^2V(X(t))) = \begin{cases} 
\min \left\{ 1, \frac{\lambda}{\sigma X(t)} \frac{D^2V(X(t))}{DV(X(t))} \right\} & \text{if } D^2V \neq 0 \\
1 & \text{if } D^2V = 0.
\end{cases}
$$

(10)

is the unique optimal feedback map.

**Remark 4.2** Assuming the risk premium $\lambda$, the volatility $\sigma$ and the wealth process $X(t)$, for any $t$, strictly positive values by hypothesis, and having proved $DV > 0$ and $D^2V \leq 0$, we can also indicate the feedback map (10) in the following way

$$
G(X(t), DV(X(t)), D^2V(X(t))) = \begin{cases} 
\min \left\{ 1, \frac{\lambda}{\sigma X(t)} \frac{DV(X(t))}{D^2V(X(t))} \right\} & \text{if } D^2V \neq 0 \\
1 & \text{if } D^2V = 0.
\end{cases}
$$

It is interesting to note the optimal investment policy in risky asset is explicitly linked to the risk premium $\lambda$, since the feedback map goes to augment when the risk premium increases. This means when the investment risk is remunerated with a high premium the fund manager is stimulated to invest a greater proportion of wealth in a risky asset, while when this premium comes close to zero, the allocation of wealth is oriented to a riskless asset.

On the contrary the relation between the optimal control and the level of wealth is not clear. As a matter of fact, on one hand the feedback map tends to decrease when wealth grows for two reasons: because of its inverse relation with wealth and because of its direct relation with the first derivative of the value function, which decreases when wealth increases (the second derivative of the value function is not positive); on the other hand there exists a direct relation with the second derivative of the value function whose behavior we do not know.

The feedback map can be interpreted as the elasticity of the first derivative of the value function $E_{DV}$, defined from the following formula

$$
E_{DV}(X(t)) := \frac{D^2V(X(t))}{DV(t)} X(t).
$$

Considering the previous formula, the feedback map (10) can be expressed in the following way

$$
G(X(t), DV(X(t)), D^2V(X(t))) = \begin{cases} 
\min \left\{ 1, \frac{\lambda}{\sigma} \frac{1}{E_{DV}(X(t))} \right\} & \text{if } D^2V \neq 0 \\
1 & \text{if } D^2V = 0.
\end{cases}
$$

(11)

The ratio $\frac{\lambda}{\sigma}$, with which the feedback map has a straight proportional link, can be interpreted as the remuneration assigned to every unit of risk and therefore, obviously, the more this remuneration is high the more the fund manager is encouraged to invest in a risky asset.
High values of elasticity express appreciable surplus values. The optimal allocation policy expressed by the feedback map (11) suggests that if the elasticity increases, the proportion to invest in a risky asset decreases; vice versa if the elasticity decreases, the proportion to invest in a riskless asset increases.

An optimal portfolio strategy that invests a greater share in a risky asset when the level of wealth is low and tends to the investment of a riskless asset if the level increases, can be justified by the following reasons:

(a) the investor tries to reach a high value of wealth as quickly as possible without caring about the risk this strategy involves;

(b) the investor is interested in protecting a wealth which has reached a conspicuous amount.

References


