A Study on Portfolio Value-at-Risk

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ABSTRACT:

Value-at-Risk (VaR) has become one of the most popular risk measures since it was recommended and adopted by the Bank of International Settlements and USA regulatory agencies in 1988. The straightforward interpretation of VaR makes this risk measure an intuitive criterion for asset management decisions. The VaR concept has also been extended to the portfolio Value-at-Risk (PVaR) measure used for managing risks and returns under a multiple-asset portfolio. Although VaR and PVaR are widely used in practice, recent criticisms have focused on the financial risks firms face if the VaR or PVaR estimates are based on poor information. One potentially important source of estimation error is in the assumptions regarding the probability model of asset returns.

In this paper, we propose an integrated method to compute PVaR from a weaker set of assumptions that may be more robust and representative of the returns distributions encountered in practice. The proposed approach combines the generalized Pareto (GP) distribution function and empirical density function (EDF) to model the return
distributions for each asset in the portfolio. We then use a copula function to form the joint distribution of the correlated returns for the set of assets in the portfolio, and we use Monte Carlo sampling methods to form PVaR estimates. We use backtesting and stress testing to compare the validity and performance of the proposed method relative to other popular methods, including the variance-covariance approach, Monte Carlo simulation, historical simulation, and the Jorion approach (Jorion 2001). We also derive some fundamental properties and the associated implications of PVaR, including the selection of the appropriate confidence level, the evaluation of PVaR as a coherent risk measure, and applications to optimal hedging strategies.

Keywords: Value-at-Risk, Portfolio Value-at-Risk, kernel density function, Monte Carlo simulation, variance-covariance approach, historical simulation, extreme value theory, generalized Pareto distribution, empirical density function, copula, stress testing, backtesting, coherent risk measure, optimal hedge ratio, hedging strategies

**IMPORTANCE OF THE RESEARCH:**

VaR is straightforward to estimate and interpret as a measure of risk exposure, and these advantages often appeal to asset managers (Culp, Mensink, and Neves 1998). However, most of the current research on Value-at-Risk (VaR) estimation focuses on the one-dimension (univariate) case. One of the first attempts to compute PVaR from a model of the joint returns distribution was reported by Frauendorfer, Moix, and Schmid (1995), but applications of this method are limited because the PVaR model cannot be stated in closed form and can only be approximated with complex computational algorithms. Alternatively, Wang and Wu (2001) use linear combinations of returns models based on extreme value theory to approximate the tail areas of heavy-tailed distributions, but this approach may be undesirable because it
only focuses on the lower-tail. The alternative approaches that are currently popular include the variance-covariance (VC) method, Monte Carlo simulation, delta-Normal simulation, and historical simulation (HS) (Dowd 1998).

The model assumptions underlying these alternative methods are quite distinct, and the relative accuracy of a particular method depends on the suitability of the model to the actual set of asset returns. For example, some of these methods rely heavily on particular assumptions about the probability models of asset returns, and the resulting PVaR estimators may incur significant estimation bias (Tsay 2002). In particular, normal or lognormal return assumptions provide estimators that are convenient to use, but these estimators may suffer from substantial estimation error in the presence of significant leptokurtosis (Chow and Kritzman 2001), which is commonly exhibited in financial return series. Other methods use a nonparametric approach to avoid the specification error associated with particular probability models, but PVaR estimation bias may yet arise due to modeling errors in the return distribution tails, the areas between the tails, and the correlation structure of the joint returns. For example, kernel density or other smoothed nonparametric estimators of the returns distribution (Danielsson 2000) are known to have properties that depend critically upon the bandwidth selection problem (Wand and Jones 1995). Some PVaR estimation methods are based solely on models of the return distribution tails. However, it may be important to model the entire return distribution, especially if the center part of the return distribution exhibits skewness or other non-normal or non-lognormal properties. Finally, the inclusion of derivatives in the portfolio may introduce significant nonlinear correlation among the return series, which cannot be captured the traditional dependence measures such as the Pearson product-moment correlation coefficient (Embrechts 2000). In particular, these biases may be especially important because
they become larger in the more extreme quantiles of the estimated return distribution, and these estimated quantiles are the foundation for PVaR estimates. Consequently, it is very difficult to find a general parametric joint probability model that may avoid each of these potential pitfalls.

If these potential estimation errors and biases lead to inaccurate PVaR estimates, firms may maintain insufficient risk capital reserves and may have inadequate cushion to absorb large financial shocks. Conversely, inaccurate PVaR estimates may also lead to redundant amount of risk capital maintained, which will reduce capital management efficiency. For example, major financial institutions crashed not long after the breakout of recent financial crises (e.g., East Asian financial crisis of 1997), and some of these failures have been associated with substantial PVaR estimation error. Thus, the presence of PVaR estimation biases may have very important practical implications, and estimation methods that may mitigate or avoid these biases may be potentially generate considerable value as risk management tools.

One promising modeling approach that may help to mitigate PVaR estimation biases is extreme value theory (EVT), which has been developed to model the occurrence of extreme events. Applications of EVT to research problems in financial modeling are well established (Gencay and Selcuk 2004; Rachev 2003). Among the class of tail probability models developed under extreme value theory, the generalized Pareto (GP) distribution is popularly acknowledged for its parsimony and ease of use. Further, the GP model has outperformed alternative models in capturing the tails of the asset return distributions, which are the essential objects for PVaR estimation (Gencay, Selcuk, and Ulugulyacib 2003; Dupuis 1999). Although there exist multivariate EVT models that may be used for PVaR estimation, the currently developed models
may exhibit undesirable properties and may not avoid substantial PVAR estimation bias. First, the marginal distribution properties of the univariate return series are not retained under the multivariate EVT models. Second, the class of multivariate distributions does not reduce to a finite-dimensional parametric family, so the model specification is potentially arbitrary. Third, the multivariate EVT models may not be rich enough to encompass all forms of tail behaviors. Finally, multivariate EVT models are designed for the distribution tails, and estimation efficiency is potentially reduced if we ignore the center sections of the return distributions. Therefore, we propose an estimation strategy for modeling the portfolio returns that decomposes this modeling task into two parts: modeling the univariate return of each asset and then modeling the joint distribution structure conditional on the marginal distributions and the observed correlation structure among the asset returns.

For the univariate return distribution, we fit separate GP models to both the lower and upper distribution tails. Given that the observed set of asset returns is typically dense in the region between the tails, we use the empirical distribution function (EDF) to form a consistent, nonparametric estimator of the center section of the return distribution. By combining the discrete EDF and the continuous parametric GP models of the tails, we form a mixed (discrete-continuous) estimator of the univariate return distributions for each asset in the portfolio. Given the known properties of the GP and EDF models, we can show that the univariate return distribution estimators are consistent.

To form the joint distribution of the asset returns, we must model the correlation or dependence structure among the assets. Usually there is significant nonlinearity in the correlation structure of the portfolio return if a derivative is involved, and the
traditional correlation measures may fail to capture this dependence. We use copula functions to capture the arbitrary correlation structure to construct the joint probability models. Based on Sklar’s theorem, copula functions can dynamically capture the nonlinear correlation structure point by point, which is not possible under the traditional correlation measures (Frees and Valdez 1998; Nelson 1999; Roncalli 2001; Mendes and Souza 2004). Further, copula functions may represent conditional correlation, which is an important feature of financial return series. In the presence of nonlinearities and conditional dependence, it is well known that some of the current popular PVaR estimation methods (e.g., variance-covariance, Monte Carlo approaches) may produce considerable biases. Consequently, the copula approach may capture these features of the data and outperform the alternative methods for estimating PVaR.

MODEL ESTIMATION DETAILS:

Given observed peak-over-threshold (POT) observations for the asset returns, we estimate the generalized Pareto (GP) model following the development presented by Davidson and Smith (1990). Under the $\gamma$-parameterization of the GP tail model, the upper tail of the return distribution is represented by the complement of the GP cumulative distribution function (CDF):

$$
G_{\gamma, \omega, \psi}(x) = 1 - \left(1 - \gamma \left(\frac{x - \psi}{\omega}\right)\right)^{-\gamma}
$$

(1)

where $\psi$, $\omega$, and $\gamma$ are the location, scale and shape parameters, respectively. Accordingly, the lower tail model for outcomes below a particular threshold ($T_{lo}$) is represented by $1 - G_{\gamma, \omega, \psi}(-x)$. In either case, the parameters of the GP model may
be estimated by maximum likelihood and other estimation methods (Reiss 2001), and
the tail estimator is consistent under a general set of regularity conditions.

For the sample space between the lower and upper thresholds, we use the EDF estimator of the CDF based on the observed data. Equal mass or weight (i.e. the inverse of the total number of observations) is assigned to each data point between the threshold values, and we then add the GP lower and upper tail weights to account for the mass assigned to the lower tails. The mixed discrete-continuous CDF for the outcomes in the center of the return distribution takes the form

\[ F(x) = G(T_{lo}) + \frac{1}{n} \sum_{i=n_0}^{n_1} I(x_i \leq x) \quad \text{for } x \geq T_{lo} \]

Here, G denotes the lower tail CDF of GP model evaluated at the lower threshold, \( T_{lo} \).

Also, I(x) is an indicator function used to add empirical mass for observations between the thresholds (numbered \( n_0 \) to \( n_1 \)). Finally, the upper tail of the distribution is constructed by adding the GP model of outcomes above the upper threshold. The resulting mixed discrete-continuous CDF is based on the large number of sample outcomes between the thresholds and is smooth in the upper and lower tails.

A copula function is used to approximate the nonlinear correlation structure and derive the joint distribution of asset returns (Nelson 1999). In general, copula
models generate joint distributions from the marginal distributions and impose the
desired degree of correlation or dependence among the component random variables
(e.g., asset returns). We use the copula function to combine the estimated mixed
(discrete-continuous) marginal distribution for each asset to form a joint probability
model of the returns for each asset in the portfolio. Copula models of financial price
and returns series are being used in an increasing number of research projects,
including recent papers by Embrechts et al. (2001) and by Liu and Miller (2001).

Although there are a large number of parametric copula families, we use the
multivariate normal copula. For example, e.g. Duan (2002) recently employed the
multivariate normal copula to conduct research on option pricing. The general
functional form of the multivariate normal copula model is

\[
C(x_1, \ldots, x_k) = \Phi_k \left( \Phi^{-1}(G_i(x_i)), \ldots, \Phi^{-1}(G_k(x_k)); \Sigma \right)
\]  

(3)

where \(G_i\) is the marginal CDF for the ith return series, \(\Phi\) is the univariate standard
normal CDF, and \(\Phi_k\) is the k-variate normal CDF with null mean and covariance \(\Sigma\).
The k-variate normal CDF \(\Phi_k\) is used to impose dependence on the collection of
marginal distributions for each asset, and \(\Sigma\) represents the required covariation among
the transformed returns, \(\Phi^{-1}(G_i(x_i))\). The marginal distributions of \(C\) are simply
the set of \(G_i\), and the copula provides a convenient way to model the dependence
among two or more return series and yet retain the heavy tails and other characteristics of the marginal distributions.

Given the estimated joint distribution of asset returns, we simulate the returns for a particular portfolio (e.g., a short hedge position) and use the simulated quantiles to estimate PVaR. As such, we can compute estimates of PVaR from a return distribution that avoids the pitfalls of the other methods (e.g. failure to fit leptokurtosis or skewness in the center of the distribution). In addition, our proposed method attains some of the desirable properties of multivariate probability models (Joe 1997). First, our method may be directly interpreted as a joint probability representation (and not just the lower tails). Second, the joint distribution is derived from the marginal distributions, which have a closed form representation. Third, the copula can allow a flexible and wide range of dependence among the asset returns. Fourth, the cumulative density function does not have a closed-formed representation but is computationally feasible.

By simulating the portfolio return quantiles, we may introduce some sampling error in the PVaR estimates. However, this error is relatively easy to control, and we may be able to generate more accurate measures of the tail probabilities than under parametric methods by avoiding the model specification error (Jackson et al., 1997). As such,
Monte Carlo methods are now considered valuable research tools in applied finance. Although this approach is computationally more intensive than local valuation, simulation can help overcome many of the weakness evidenced by local valuation methods (Penza and Bansal, 2001). Simulation methods can also improve estimation performance, especially when derivatives are involved (Jorion, 2001; Smithson and Pearson, 2000)

Therefore, the PVAR is computed by Monte Carlo simulation of portfolio returns generated from the fitted copula model of the joint distribution. We refer to the proposed method as the CE (copula-EVT) method. For comparison purposes, we consider three alternatives to PVAR computed from the copula model of joint returns. First, the VC model and HS are two of the oldest and most widely used methods (CAPITAL MARKET RISK Advisors 2000). To conduct HS, we first derive the required moments of the historical data and approximate the PVAR by brute-force Monte Carlo simulation. The variance-covariance method also relies on brute-force Monte Carlo simulation to enhance accuracy. Both the HS and VC methods reflect an implicit normality assumption. However, unless the number of Monte Carlo replications is adequately large, and these methods cannot model the extremes outcomes very well (Dowd 1998). The third alternative estimation method we consider is the approximation method described by Jorion (2001).

Regarding the selection of a demonstration dataset, we note that both equity and derivatives markets in developing countries are known to exhibit high volatility and more extreme comovements. In particular, these properties are prominent in the Taiwan stock market, and this emerging market is an excellent case for study. The sample market dataset focuses on the underlying weighted stock index in Taiwan and
the two corresponding futures contracts issued in Taiwan and Singapore markets, respectively. The dataset includes daily observations from January, 1998, to December, 2003. Although the futures contracts on the Taiwan stock index were actually launched in 1997, we exclude this early portion of the sample period (1997-1998) to avoid observations that are not representative of current market performance. We account for the implicit weekend effects or holidays effects by equally allocating returns across the non-traded periods.

EXPECTED RESULTS:
First, we validate the fitted models for the candidate PVaR estimation methods by backtesting and stress testing (Longin 2000; Lauridsen 2000). The procedure outlined in Christoffersen (2003) is implemented, and it may be used to determine if the proposed approach (CE) is superior to the other popular alternatives. Second, we compare the PVaR estimates from the different estimations approaches at various quantile levels in order to determine the potential estimation bias and the appropriate confidence level for estimating VaR. This step is important because P VaR can be validated if it is qualified as a coherent risk measure (Delbaen 2000). Third, we compose a synthetic portfolio designed to compare the performance of the alternative methods for optimal hedging purposes. The derivative components in the portfolio are assigned varying weights and these different compositions imply different hedging strategies. We use the performance results to derive the practical implications of using the competing P VaR methods for risk management of specific hedge portfolios.

REFERENCES


