A Markov Model for Valuing Asset Prices in a Dynamic Bargaining Market *

Masaaki Kijima a† and Yoshihiko Uchida b

a Graduate School of Economics, Kyoto University
b Institute for Monetary and Economic Studies, Bank of Japan

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Abstract. This paper proposes a Markov chain model for studying the impact on asset prices of illiquidity associated with search and bargaining in an economy. The economy consists of finitely many agents who can trade only when they find each other, and any trade between agents changes the population of the agent types which affects the asset price in future. Assuming that the equilibrium utility as well as the trade price is proportional to the asset dividend, we obtain the asset prices in steady state. Through extensive numerical experiments, we observe that the equilibrium prices exhibit the cut-off phenomenon (i.e. crash) as the fraction of pessimistic agents gets large. Models with a marketmaker as well as irrational agents are also considered.

Keywords: Markov chain, limiting distribution, Walrasian equilibrium, marketmaker, inventory, irrational agents

*One of the authors (M.K.) acknowledges the financial support by Daiwa Securities Group Inc.
†Send all correspondence to Masaaki Kijima, Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan.
TEL:+81-75-753-3456, FAX:+81-75-753-3492, Email: kijima@econ.kyoto-u.ac.jp
1 Introduction

Through the experiences of the Asian crisis in 1997 and the Russian crisis followed by the LTCM bankruptcy in 1998, the importance of liquidity risk\(^1\) has been widely recognized in financial markets. Recently, Duffie, Garleanu and Pedersen (hereafter DGP) (2004) built a dynamic asset-pricing model with a *continuum* of agents on the interval \([0, 1]\) to study the impact on asset prices of illiquidity associated with search and bargaining in an economy in which the agents can trade only when they find each other. This paper studies the same subject by constructing a finite (and hence *discrete*) Markov chain model to describe the mechanism of trades more explicitly.\(^2\)

The discrete nature in our model enables us to study the *joint* distribution of the types of agents in equilibrium, while DGP (2004) considers only the *mean* fractions of the types of agents. Hence, the present model will provide more information about the structure of equilibrium prices. For example, using our model, we can analyze the impact on asset prices of the number of market participants. Also, our model converges to the DGP model by the law of large numbers.

More specifically, an agent who wants to sell an asset must search for a buyer and, when they meet each other, their bilateral relationship is strategic. At any time and state, the seller’s strategy is either to sell or to do nothing (the buyer’s strategy is either to buy or to do nothing), and their decisions are merely based on the price of the asset. The trade mechanism is discrete by its very nature, and we are interested in the joint distribution of the four types of agents in equilibrium, i.e., an agent with an asset who wants to sell it (*ho* type), an agent with an asset who does not want to sell (*lo* type), an agent without assets who wants to buy (*ln* type), and an agent without assets who does not want to buy (*hn* type). The asset price is affected by the population of the agent types\(^3\) and set through a bargaining process. Assuming that the equilibrium utility as well as the trade price is proportional to the asset dividend, we obtain the asset prices in steady state. Through extensive numerical experiments, we observe that the equilibrium prices exhibit the cutoff phenomenon (i.e. crash) as the fraction of pessimistic agents gets large.

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\(^1\)We define liquidity risk as the risk that one cannot liquidate his/her position at a desired time by the market price at that time.

\(^2\)See DGP (2004) for the relevant literature to the current work.

\(^3\)Because the number of agents is finite, any trade between agents changes the population of the agent types which affects the asset price in future. This is the significant difference of our finite model from the infinite counterpart.
Of interest are the asymptotic results in our economy, i.e. the limiting cases when the number of agents, $M$ say, and investors’ search intensities, $\lambda$ say, get large. It is shown that, when $M \to \infty$, our model converges to the DGP model under regularity conditions. On the other hand, for a fixed $M$, our model exhibits significantly different properties from the DGP model as $\lambda \to \infty$. In particular, our equilibrium price does not converge to the Walrasian price obtained in DGP (2004).\footnote{In the DGP model with a continuum of agents, the Walrasian price is the value of holding the asset forever for a hypothetical agent who always has a low (high, respectively) discount rate.} In our model, the Walrasian equilibrium is attained if the majority of agents in our discrete market becomes either optimistic or pessimistic about the economy.

Of importance in the real market is the existence of irrational investors. For example, a fraction of $lo$ type agents may want to sell the asset irrationally in order to get instant cash. It seems interesting to study the impact on asset prices of such irrational traders when they present in the market. We will show through numerical experiments that the impact of irrational traders is surprisingly strong.

Our discrete model also allows a marketmaker to have an inventory that is controlled through the marketmaker’s bargaining power. This seems important, because many studies such as BIS (1999a, 1999b) pointed out that market stresses may be amplified by a fall in market liquidity due to the malfunctioning of the marketmaker’s mechanism to supply liquidity to the market. The Markov chain model provides a tool to study the relationship between the market stability and market making. We will show how the equilibrium prices depend on the marketmaker’s inventory level and search intensity through numerical examples.

This paper is organized as follows. In the next section, we develop a Markov chain model without marketmakers as the basic market model. Using the theory of Markov chains, we can evaluate the equilibrium distribution of agent types numerically. Assuming that the equilibrium utility as well as the trade price is proportional to the asset dividend, equilibrium asset prices are obtained by solving simultaneous equations numerically. The agent-type distribution becomes binomial as investors’ search intensities get large. Section 3 considers a market with irrational traders, while Section 4 introduces a marketmaker with some inventory level. Through numerical examples, we investigate the impact on asset prices of the irrationality as well as the marketmaker’s inventory level. Section 5 concludes this paper. Proofs are given in Appendix.

Throughout this paper, we fix a probability space $\left( \Omega, F, P \right)$ and a filtration $\left\{ F_t; t \geq 0 \right\}$ of sub-$\sigma$-algebras satisfying the usual conditions that represents the resolution over
time of information commonly available to investors. We note that the information does not include the masses of different types of agents. In other words, each agent knows his/her type, but does not know the exact number of agents of the same type in the market. Investors are assumed to calculate the joint distribution of the types of agents in equilibrium.\(^5\)

## 2 The Model without Marketmakers

We consider a continuous-time economy in which there are \(M\) agents and \(S\) units of a single asset. Each agent can hold at most one unit of the asset and cannot shortsell, and they can trade only when they meet each other. It is assumed that \(S < M\).

Each agent is risk-neutral and infinitely lived, with a time-preference type, high and low. High-type agents are pessimistic about the economy with a high discount rate \(r_h\) and want to sell the asset, while low-type agents are optimistic about the economy with a low discount rate \(r_l\) and want to buy the asset. We assume that high-type agents lose a fraction \(\delta \geq 0\) of any asset cash flows, while \(r_l \leq r_h\), with at least one being strict, setting up strict gains from trade. For simplicity of presentation, we set \(\delta = 0\) so that \(r_l < r_h\).

The switching intensity of low to high of an agent is \(\lambda_u\); the switching intensity of high to low is \(\lambda_d\). When \(\lambda_u > \lambda_d\), there are more pessimistic agents about the economy in the market. Hence, the case \(\lambda_u > \lambda_d\) (\(\lambda_u < \lambda_d\), respectively) is related to the situation that the market is in financial distress (boom).

Next, note that trades are between high-type asset owners and low-type non-owners.\(^6\) We assume that they meet each other at the events of a Poisson process with intensity \(\lambda\) and the contract processes of agents are pair-wise independent. Then, the market is considered to be of high liquidity as the search intensity \(\lambda\) gets large.

### 2.1 The Markov Chain Model

We recognize \(\mathcal{T} = \{ho, hn, lo, ln\}\) as the full set of agent types, where the letters \(h\) and \(l\) designate the agent’s time-preference state while \(o\) and \(n\) indicate whether the agent owns the asset or not, respectively.

Let \(m_\sigma(t), \sigma \in \mathcal{T}\), be the number of agents with type \(\sigma\) at time \(t\). By definition, we

\(^5\)This paper does not consider the case of asymmetric information.

\(^6\)As is shown below, it is optimal for them to trade the asset, when they meet each other, at any time and state.
have
\[ m_{ho}(t) + m_{hn}(t) + m_{lo}(t) + m_{ln}(t) = M \]  
(1)
as well as
\[ m_{ho}(t) + m_{lo}(t) = S \]  
(2)
at any time \( t \). Thus, once we recognize the pair \((m_{ho}(t), m_{ln}(t))\), we can recover the complete distribution of the four types of agents at any time \( t \).

Let \( \{Y_t; t \geq 0\} \) be a two-dimensional Markov chain in continuous time that represents the pair \((m_{ho}(t), m_{ln}(t))\) with state space
\[ S = \{(i, j) : i = 0, 1, \ldots, S, \ j = 0, 1, \ldots, M - S\}. \]
The transition intensity from \((i, j)\) to \((i - 1, j)\) is given by \(i \lambda_d\), because one of \(i\) agents with type \( ho \) changes his/her time preference from high to low. The transition intensity from \((i, j)\) to \((i, j - 1)\) is \(j \lambda_u\), because one of \(j\) agents with type \( ln \) changes his/her time preference from low to high. Similarly, the transition intensities from \((i, j)\) to \((i + 1, j)\) and from \((i, j)\) to \((i, j + 1)\) are obtained as \((S - i) \lambda_u\) and \((M - S - j) \lambda_d\), respectively.

In order to specify the transition intensities for trades of the asset, we recall that trades are between high-type asset owners and low-type non-owners. Since they meet each other at the events of a Poisson process with intensity \( \lambda \) and the contract processes of agents are pair-wise independent, the total intensity that a contract is made is given by \( \lambda \) times the number of such Poisson processes. The number of possible processes is equal to \( i \) times \( j \), when \( m_{ho}(t) = i \) and \( m_{ln}(t) = j \), because an agent with type \( ho \) is chosen at random from the population with \( i \) agents and, at the same time, an agent with type \( ln \) is chosen at random from \( j \) agents. It follows that the transition intensity from \((i, j)\) to \((i - 1, j - 1)\) is given by \( ij \lambda \).\(^7\) Other transitions do not take place, and the transition intensities for exiting the state space \( S \) are zero. The transition diagram of the Markov chain \( \{Y_t; t \geq 0\} \) is depicted in Figure 1.\(^8\) It is readily seen that the state space is irreducible (see, e.g., Seneta, 1981).

\(^7\)Our random matching formulation slightly differs from DGP (2004) in which they assume that \( 2ij \) when \( m_{ho}(t) = i \) and \( m_{ln}(t) = j \). The difference is due to the definition of the Poisson events. If needed, we replace the matching intensity \( \lambda \) by \( \lambda/2 \).

\(^8\)Note that, in order for \( \{Y_t; t \geq 0\} \) to be Markovian, our implicit assumption is that the random switches in time-preference types as well as the random matchings are mutually independent.
The Markov chain \( \{Y_t; t \geq 0\} \) is specified completely, once we state the initial condition. Let 
\[
\pi_{ij}(t) = P^0\{m_{ho}(t) = i, m_{ln}(t) = j\}, \quad (i, j) \in S,
\]
and define 
\[
\pi_{ij} = \lim_{t \to \infty} \pi_{ij}(t), \quad (i, j) \in S,
\]
if they exist, where \( P^0 \) denotes the probability measure given the initial condition. The bivariate distribution \( \pi = (\pi_{ij}; (i, j) \in S) \), if it exists and satisfies the condition 
\[
\pi_{ij} \geq 0, \quad \sum_{(i,j) \in S} \pi_{ij} = 1,
\]
is called the \textit{limiting} distribution of the Markov chain \( \{Y_t; t \geq 0\} \).

In order to derive an equilibrium distribution of agent types, we rely on the well-known result in the theory of Markov chains. That is, for any finite, time-homogeneous Markov chain in continuous time, there exists a limiting distribution such that \( \pi_{ij} > 0 \) for all \((i, j) \in S\) and they are independent of the initial condition. Also, the limiting distribution is the unique equilibrium (stationary) distribution. See, e.g., Kijima (1997) for details. We thus have the following.

**Proposition 1** There exists a unique equilibrium distribution \( \pi = (\pi_{ij}; (i, j) \in S) \) for the agent types such that

\[
\widetilde{P}\{m_{ho}(t) = i, m_{ln}(t) = j, m_{hn}(t) = M - S - j, m_{lo}(t) = S - i\} = \pi_{ij} > 0
\]
for any \( t \), where \( \widetilde{P} \) denotes the probability measure with the initial distribution \( \pi \).

Consulting Figure 1, the \textit{full balance equations} for the equilibrium probabilities \( \pi_{ij} \) are obtained as follows: For state \((i, j) \in S\), we have

\[
(i\lambda_d + j\lambda_u + (S - i)\lambda_u + (M - S - j)\lambda_d + ij\lambda)\pi_{ij} = (i + 1)\lambda_d\pi_{i+1,j} + (j + 1)\lambda_u\pi_{i,j+1} + (S - i + 1)\lambda_u\pi_{i-1,j} + (M - S - j + 1)\lambda_d\pi_{i,j-1} + (i + 1)(j + 1)\lambda\pi_{i+1,j+1},
\]
with understanding that \( \pi_{ij} = 0 \) if \((i, j) \notin S\). The left-hand side in (3) is the probability flux going out from state \((i, j)\), while the right-hand side is the probability flux coming into state \((i, j)\); in equilibrium, they must be equal. It is well known that there is a unique

\[\footnote{We will not specify the initial condition, because the initial condition is irrelevant to the equilibrium distribution in our setting.}\]
solution to the simultaneous equations given in (3). Thus, the equilibrium distribution \( \pi \) is obtained by solving the simultaneous equations.\(^{10}\)

### 2.2 Dynamic Bargaining Equilibrium

We derive a dynamic bargaining equilibrium in two steps. That is, we first obtain an equilibrium distribution for the agent types and, then, compute agents’ value functions and transaction prices (taking as given the equilibrium distribution).

The first step has been done. To compute the second step, we assume that a single non-storable consumption good is used as the numeraire and the asset pays strictly positive dividends continuously. The dividend process \( \{X_t\} \) is assumed to be adapted to the filtration and have a constant, conditional expected dividend growth rate, \( c > 0 \) say. It follows that

\[
\tilde{E}_t[X_{t+s}] = X_t e^{cs}, \quad s > 0,
\]

for any \( t \geq 0 \), where \( \tilde{E}_t \) denotes the conditional expectation operator under the stationary probability measure \( \tilde{P} \) given \( \mathcal{F}_t \), the information available at time \( t \).

Let \( V(X_t, \sigma_t) \) be the equilibrium utility at time \( t \) for remaining lifetime consumption of a particular agent with current type \( \sigma_t \) and current dividend rate \( X_t \). Also, let \( P(X_t) \) be the trade price at time \( t \) of the asset.

In order to calculate \( V \) and \( P \), we introduce the following stopping times. Let \( \tau_r \) denote the next time at which that agent’s time-preference type changes, let \( \tau_m \) denote the next time at which a counterpart with gain from trade is met, and let \( \tau = \min\{\tau_r, \tau_m\} \).

Let \( 1_A \) be the indicator function, meaning that \( 1_A = 1 \) if \( A \) is true and \( 1_A = 0 \) otherwise. Then, for an agent with \( \text{ho} \) type, we obtain

\[
V(X_t, \text{ho}) = \tilde{E}_t\left[ \int_t^\tau e^{-r_h(u-t)}X_u du + e^{-r_h(\tau_r-\tau)}V(X_{\tau_r}, \text{lo})1_{\{\tau_r<\tau_m\}} + e^{-r_h(\tau_m-\tau)}(V(X_{\tau_m}, \text{hn}) + P(X_{\tau_m}))1_{\{\tau_r>\tau_m\}} \right].
\]

Here, the first term in the right-hand side is the dividend that the agent receives until time \( \tau \), because the agent possesses the asset as a high-type agent. The second term

\(^{10}\)A simple way to solve the simultaneous equations is the following. For the equilibrium distribution \( \pi = (\pi_{ij}; (i, j) \in S) \), we renumber the states in a lexicographical manner. The resulting state space is one-dimensional with \((S + 1)(M - S + 1)\) states, which we denote by \( \hat{S} \). Let \( A \) be the infinitesimal generator of a Markov chain defined on the state space \( \hat{S} \). It is well known that the stationary distribution \( \hat{\pi} = (\hat{\pi}_n) \) on \( \hat{S} \) is the unique solution of the simultaneous equations \( \hat{\pi}A = 0 \) and \( \sum_{n \in \hat{S}} \hat{\pi}_n = 1 \), where \( 0 \) denotes the row vector with all the components being zero. The univariate distribution \( \hat{\pi} = (\hat{\pi}_n) \) is then converted back to the desired bivariate distribution \( \pi = (\pi_{ij}; (i, j) \in S) \) in an obvious manner.
represents the utility value when the agent changes the time-preference type at time $\tau_r$ before meeting a counterpart to trade. The last term represents the utility value when the agent finds a counterpart to trade at time $\tau_m$ before changing the time-preference type. The selling price $P(X_{\tau_m})$ is added to this case as a profit. Similarly, we obtain

$$V(X_t, hn) = \tilde{E}_t \left[ e^{-\rho_h(\tau_r-t)}V(X_{\tau_r}, ln) \right],$$

$$V(X_t, lo) = \tilde{E}_t \left[ \int_{\tau_r}^{\tau_m} e^{-\rho_l(u-t)} X_u du + e^{-\rho_l(\tau_r-t)}V(X_{\tau_r}, ho) \right],$$

and

$$V(X_t, ln) = \tilde{E}_t \left[ e^{-\rho_l(\tau_r-t)}V(X_{\tau_r}, hn) 1_{\{\tau_r < \tau_m\}} \right. + \left. e^{-\rho_l(\tau_m-t)} (V(X_{\tau_m}, lo) - P(X_{\tau_m})) 1_{\{\tau_r > \tau_m\}} \right].$$

Next, for the equilibrium price $P$, suppose that

$$P(X_t) = \Delta V_l(X_t)(1-q) + \Delta V_h(X_t)q, \quad 0 < q < 1,$$

where $\Delta V_l(X_t) \equiv V(X_t, lo) - V(X_t, ln)$ denotes the reservation value for a low-type non-owner to buy the asset, while $\Delta V_h(X_t) \equiv V(X_t, ho) - V(X_t, hn)$ denotes the reservation value for a high-type owner to sell the asset. The gain from trade between these agents is $\Delta V_l(X_t) - \Delta V_h(X_t)$; hence, the trade price $P$ defined in (8) indicates that the seller gets a fixed fraction $q$ of the gain in equilibrium.\(^\text{11}\)

Suppose that there exists an equilibrium in which the value functions and prices are proportional to $X$, i.e., $V(X_t, \sigma) = v_{\sigma}X_t$, $\sigma \in T$, and $P(X_t) = pX_t$ for some unknown coefficients $v_{\sigma}$ and $p$. Then, from (4)–(8), we have the following. The proof is given in Appendix A.

**Proposition 2** Suppose that $0 < q < 1$. Then, the value and price coefficients uniquely solve the following simultaneous equations:

$$v_{ho} = \sum_{j=0}^{M-S} \pi_j^{ho} \frac{\lambda_d v_{lo} + \lambda v_{ho} + 1}{r_h + \lambda_d + \lambda_j - c},$$

$$v_{hn} = \frac{\lambda_d v_{ln}}{r_h + \lambda_d - c},$$

$$v_{lo} = \frac{\lambda v_{ho} + 1}{r_l + \lambda - c},$$

$$v_{ln} = \sum_{i=0}^{S} \pi_i^{ln} \frac{\lambda v_{hn} + \lambda_i (v_{lo} - p)}{r_l + \lambda + \lambda_i - c}.\quad (12)$$

\(^{11}\)This means that the seller’s bargaining power is $q$. See Osbone and Rubinstein (1990) for further discussions about bargaining power.
and

\[ p = (v_{ho} - v_{hn})(1 - q) + (v_{lo} - v_{ln})q, \]

(13)

where

\[ \pi^h_{ij} = \frac{\sum_{i=1}^{S} \pi_{ij}}{\sum_{i=1}^{S} \sum_{j=0}^{M-S} \pi_{ij}}, \quad j = 0, 1, \ldots, M - S, \]

\[ \pi^l_{ij} = \frac{\sum_{j=1}^{M-S} \pi_{ij}}{\sum_{i=0}^{S} \sum_{j=1}^{M-S} \pi_{ij}}, \quad i = 0, 1, \ldots, S, \]

denotes the conditional probability given \( m_{ho} \geq 1 \) and where

denotes the conditional probability given \( m_{ln} \geq 1 \).

An agent with current type \( ho \) wants to sell the asset, when meeting an agent who wants to buy it in equilibrium, if and only if \( v_{ho} < p + v_{hn} \). This condition also guarantees that an agent with current type \( hn \) does not want to buy the asset. Similarly, an agent with current type \( ln \) wants to buy the asset, when meeting an agent who wants to sell it in equilibrium, if and only if \( v_{ln} + p < v_{lo} \), and this condition guarantees that an agent with current type \( lo \) does not want to sell the asset. Hence, trades occur only between an agent with type \( ho \) and an agent with type \( ln \) if and only if

\[ \Delta v_h < p < \Delta v_l, \]

(14)

where \( \Delta v_h = v_{ho} - v_{hn} \) and \( \Delta v_l = v_{lo} - v_{ln} \). Note that trades are not postponed, if any, because the discount rates are positive. Hence, the proposed trading strategies make sense if and only if the condition (14) holds. In this case, when two agents with gain from trade meet at any time, both prefer to immediately trade at the agreed price \( p \) which is set through the bargaining processes.

### 2.3 Asymptotic Results

In order to discuss the relationship between our results and the results in DGP (2004), we shall denote \( s \equiv S/M, \quad 0 < s < 1 \), and fix it throughout this subsection. Note that, in our model, there are two asymptotic situations of interest, namely \( M \to \infty \) and \( \lambda \to \infty \). That is, we can investigate the impact on asset prices of not only the matching intensity \( \lambda \) but also the number \( M \) of the agents in the asymptotic case. This is one of the advantages of considering finite models. In this subsection, we consider the case that \( M \to \infty \), while the other case \( \lambda \to \infty \) is treated in the next subsection.

Let \( m_\sigma = m_\sigma(\infty), \quad \sigma \in T, \) and define

\[ \mu_\sigma = \frac{1}{M} E[m_\sigma], \quad \sigma \in T, \]
Then, for sufficiently large $M$ and $S$, we obtain the same results as DGP (2004) with $t \to \infty$, i.e.

\[ 0 = -\lambda_d \mu_{ho} - \lambda' \mu_{ho} \mu_{ln} + \lambda_u \mu_{lo} \quad (15) \]

and

\[ 0 = -\lambda_u \mu_{ln} - \lambda' \mu_{ho} \mu_{ln} + \lambda_d \mu_{hn}, \quad (16) \]

where $\lambda' = \lambda M$. Here, it should be noted that we make the matching intensity proportional to the number of agents in the economy. In other words, in order to keep the $\lambda'$ to be constant, we must let the matching intensity $\lambda$ tend to zero as $M \to \infty$.

To prove Equations (15) and (16), let $A_i = \{(k, l) \in S : k < i\}$. The balance equation between the sets $A_i$ and $A_{ci}$ is given by

\[ \sum_j (i \lambda_d + ij) \pi_{ij} = \sum_j (S - i + 1) \lambda_u \pi_{i-1,j}. \quad (17) \]

Note that the left-hand side is the probability flux from $A_{ci}$ to $A_i$, while the right-hand side is the probability flux from $A_i$ to $A_{ci}$; in equilibrium, they must be equal. Now, summing (17) over $i$ yields

\[ \lambda_d E[m_{ho}] + \lambda E[m_{ho} m_{ln}] = \lambda_u E[m_{lo}] \quad (18) \]

for sufficiently large $M$ and $S$. Then, since $m_{\sigma}/M$ converges to $\mu_{\sigma}$ as $M \to \infty$ by the law of large numbers and since $\lambda' = \lambda M$, we obtain (15) from (18). Equation (16) can be obtained similarly. Note from (1) and (2) that

\[ \mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) = 1 \]

and

\[ \mu_{ho}(t) + \mu_{lo}(t) = s \]

at any time $t$. Therefore, we recover the equilibrium masses of the different agent types obtained in DGP (2004).

### 2.4 Towards the Walrasian Equilibrium

A Walrasian equilibrium is characterized by a single price process at which agents may buy and sell instantly, such that supply equals demand at each state and time in a perfectly competitive market. In a Walrasian allocation, either all the assets are held by agents with a low discount rate, i.e. $m_{ho} = 0$, or all the agents with a low discount rate hold the assets, i.e. $m_{ln} = 0$. In our finite Markov model, these cases occur only if $\lambda \to \infty$. 
Then, any state with $\lambda$ as an outflow intensity becomes an *instantaneous state*,\(^{12}\) so that the resulting Markov chain, when $\lambda \to \infty$, has the state space

$$S_W = \{(i, 0) \cup (0, j) : i = 0, 1, \ldots, S, \ j = 0, 1, \ldots, M - S\}.$$

Note that, as in Figure 2, the state space can be depicted to be one-dimensional. Hence, the resulting Markov chain is thought of as a finite birth–death process and the limiting distribution exists, which can be obtained with ease. See Appendix B for the proof of the next result.

(Figure 2 is placed here.)

**Proposition 3** Suppose that $0 < q < 1$, and let $\lambda \to \infty$. Then, there exists a unique equilibrium for the agent types, which is associated with a binomial distribution with parameters $M$ and $1 - \xi$, where $\xi = \lambda_d / (\lambda_u + \lambda_d)$.

The trade price when $\lambda \to \infty$ is obtained as follows. Suppose that, in equilibrium, $m_{ho} = i > 0$, i.e. the number of type $ho$ agents is equal to $i$. Then, by definition, we have $m_{ln} = 0$ and $m_{hn} = M - S > 0$. A trade occurs only when an agent with type $hn$ changes the time preference from high to low, because an agent with type $ln$ can find the agent immediately to trade (recall that $\lambda = \infty$). Similarly, when $m_{ln} = j > 0$, we have $m_{ho} = 0$ and $m_{lo} = S > 0$. A trade occurs only when an agent with type $lo$ changes the time preference from low to high. The proof of the next result is given in Appendix C.

**Proposition 4** Suppose that $0 < q < 1$, and let $\lambda \to \infty$. Then, the trade price is uniquely determined as

$$p = \frac{r_l - c + \lambda_d + q(r_h - r_l + (M - S)\lambda_d) + \lambda_u + (1 - q)S\lambda_u}{(r_l - c)(r_h - c + (1 + Mq - S)\lambda_d) + (1 + (1 - q)S)(r_h - c)\lambda_u}.$$  \(19\)

We note that the bargaining power $q$ is present in Proposition 4 despite that $\lambda \to \infty$. This does not contradict the result in Rubinstein (1982), because the discount factors $r_l$ and $r_h$ in our model do not represent the time preference in the bargaining process.

Now, in the same spirit as the previous subsection, let $s \equiv S/M$, $0 < s < 1$, and consider the case that $M \to \infty$. Recall that, under the conditions of Proposition 3, the agent-type distribution follows a binomial distribution with parameters $M$ and $1 - \xi$, denoted by $B(M, 1 - \xi)$, in equilibrium. Let $Z$ be a random variable distributed by $B(M, 1 - \xi)$. Then, the mean of $Z$ is equal to $M(1 - \xi)$ while its variance is given by $M\xi(1 - \xi)$.

\(^{12}\)See Kijima (1997) for the definition of instantaneous states.
Let $\zeta = Z - M + S$. By the law of large numbers, it is readily seen that $\zeta / M$ converges to $(1 - \xi) - (1 - s) = s - \xi$ almost surely as $M \to \infty$. Hence, if $\xi > s$, then the agent-type equilibrium converges to $E[m_{lu}] = \xi - s$ almost surely as $M \to \infty$, while it converges to $E[m_{ho}] = s - \xi$ almost surely in the case that $\xi < s$.

On the other hand, when $M \to \infty$, the equilibrium price given in Proposition 4 does not converge to the one obtained in DGP (2004). In fact, when $M \to \infty$ in (19), we obtain

$$p^* = \frac{q(1 - s)\lambda_d + (1 - q)s\lambda_u}{(r_t - c)q(1 - s)\lambda_d + (r_h - c)(1 - q)s\lambda_u}. \quad (20)$$

Recall that, in DGP (2004), the Walrasian price coefficient is given by $p^*_l \equiv (r_t - c)^{-1}$ if $\xi > s$ and $p^*_h \equiv (r_h - c)^{-1}$ otherwise.\(^\text{13}\) Note that, in our model, this happens only when $\lambda_u \to 0$ or $\lambda_d \to 0$, respectively.

### 2.5 Numerical Examples

In this subsection, we provide an illustrative example, where the base case parameters are listed in Table 1. For these parameters, the low discount rate is $r_t = 5\%$, while the high discount rate in ‘financial distress’ is $r_h = 15\%$. Since the dividend growth rate is $c = 3\%$, we obtain $p^*_l = 50$ and $p^*_h = 8.33$.

<table>
<thead>
<tr>
<th>$r_l$</th>
<th>$r_h$</th>
<th>$c$</th>
<th>$\lambda$</th>
<th>$M$</th>
<th>$S$</th>
<th>$q$</th>
<th>$\lambda_u + \lambda_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>15%</td>
<td>3%</td>
<td>10</td>
<td>32</td>
<td>10</td>
<td>0.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 1: Base case parameters

Before proceeding, we note that, when $\lambda_u \to 0$ ($\lambda_d \to 0$, respectively), the equilibrium price coefficient $p$ converges to the limit $p^*_l$ ($p^*_h$), as explicitly observed in the figures below, even in the finite model. This is so, because the price $p^*_l$ ($p^*_h$) is the value of holding the asset forever for a hypothetical agent who always has a low (high, respectively) discount rate.\(^\text{14}\)

Figure 3 shows how the (bivariate) equilibrium distribution $\pi = (\pi_{ij}; (i, j) \in S)$ changes with respect to the intensity $\lambda_u$ (recall that $\lambda_d = 1 - \lambda_u$). Figure (a) depicts

\(^{13}\)Their model can be thought of as the limiting case of our model by taking $M \to \infty$ with $\lambda = \lambda'/M$ first and then $\lambda' \to \infty$. However, if we take $\lambda \to \infty$ first, the resulting model (i.e. the birth-death model) can no longer recover the DGP model. Note that interchange of the order of iterated limits is in general not permissible.

\(^{14}\)Mathematically, we have $\pi_{0,M-S} \to 1$ ($\pi_{S0} \to 1$) as $\lambda_u \to 0$ ($\lambda_d \to 0$), and a simple argument leads to the claim.
the equilibrium distribution for the case that $\lambda_u = 0.1$ and $\lambda_d = 0.9$, i.e., 1 year out of 10 years is in financial distress on average, (b) for the case that $\lambda_u = \lambda_d = 0.5$, i.e., 5 years out of 10 years are in financial distress, and (c) for the case that $\lambda_u = 0.9$ and $\lambda_d = 0.1$, i.e., 9 years out of 10 years are in financial distress. It is explicitly observed that, in the ‘normal case’ (a), most of the assets are held by agents with low discount rate. As $\lambda_u$ gets large, however, the distribution shifts so that agents with high discount rate turn to possess the assets as depicted in Case (c). This case is economically inefficient, because agents with high discount rate want to sell the assets.

(Figure 3 is placed here.)

Next, we calculate the equilibrium price $P(X) = pX$, where we set $X = 1$ for simplicity, based on Proposition 2. Figure 4 shows how the equilibrium price changes with respect to the intensities $\lambda$ and $\lambda_u$. As the matching intensity $\lambda$ gets small, the slope becomes moderate, as expected. Note that the case $\lambda = 10$ means that agents contact other agents at an expected rate of less than once per month. It is interesting to note that, even for the fairly small $\lambda$, the price keeps very high (near $p^*$) until some point of $\lambda_u$ and then drops suddenly to the low Walrasian price $p_h^*$. This seems to explain why the market often crashes when the prospects for the market become negative.

(Figure 4 is placed here.)

For sufficiently large $\lambda$, the price converges to the equilibrium price given in (19), while our model converges to the DGP model as $M \to \infty$. Hence, it is expected that the equilibrium price (19) converges to the Walrasian price obtained in DGP (2004) as $M \to \infty$. However, this is not the case as depicted in Figure 5. Recall that the Walrasian price in DGP (2004) is such that $p^* = 50$ for $\lambda_u < 22/32$ and $p^* = 8.33$ for $\lambda_u > 22/32$, while our price strictly decreases with respect to $\lambda_u$. As explained earlier, this discrepancy occurs because the order of iterated limits is not interchangeable. It is interesting to note that the equilibrium price with $\lambda = \infty$ is very robust with respect to $M$ (see Figure 5), while the trade price with finite $M$ is sensitive to $\lambda$ (see Figure 4).

(Figure 5 is placed here.)

Finally, Figure 6 shows how the equilibrium price changes with respect to the switching intensity $\lambda_u$ for various values of $S$, the number of assets in the market. Since the number of agents is the same ($M = 32$), the bigger the number $S$, the more likely the agent
with high discount rate ought to possess the undesirable asset. It follows that the price drops more rapidly as \( \lambda_u \) gets large for a larger value of \( S \). This phenomenon is explicitly observed in Figure 6.

(Figure 6 is placed here.)

### 3 The Model with Irrational Investors

In this section, we consider a market in which there are some irrational investors. Namely, suppose that a fraction of \( lo \) type agents want to sell the asset irrationally in order to get instant cash. \(^{15}\)

Let \( \alpha \) denote the fraction of the irrational traders in \( lo \) type agents. When an \( lo \) type agent irrationally wants to sell the asset, he/she must search for a buyer and the buyer must be an \( ln \) type agent. That is, only when an irrational \( lo \) type agent and a rational \( ln \) type agent meet each other, the irrational trade occurs. This means that the irrational trade does not change the state of the bivariate Markov chain \( \{Y_t; t \geq 0\} \), and hence the equilibrium distribution of the agent types remains the same.

What is changed by the irrational traders is the equilibrium utility of \( lo \) type agents. Namely, we obtain

\[
V(X_t, lo) = \tilde{E}_t \left[ \int_t^\tau e^{-r_l(u-t)}X_u du + e^{-r_l(\tau_r-t)}V(X_{\tau_r}, ho)1_{\{\tau_r < \tau_m\}} + e^{-r_l(\tau_m-t)}(V(X_{\tau_m}, ln) + P(X_{\tau_m}))1_{\{\tau_r > \tau_m\}} \right],
\]

where \( \tau_m \) denotes the next time at which a counterpart to trade is met. Here, we assume that \( lo \) traders sell to \( ln \) traders at the same price as \( ho \) traders. This makes sense, because \( ln \) traders cannot distinguish the type of sellers. The equilibrium utilities of the other types of agents are unchanged. The next result can be obtained by the exact same arguments as Proposition 2.

**Proposition 5** Suppose that \( 0 < q < 1 \), and let \( \alpha \) be the fraction of the irrational traders in \( lo \) type agents. Then, the value and price coefficients in the equilibrium uniquely solve

\(^{15}\)Other types of irrationality is not considered in this paper. A market with other types of irrational investors has been studied in Uchida (2003). For example, our model can be used to analyze an economy with irrational investors who buy the asset irrationally. The existence of such investors may produce a ‘bubble’ in the market.
the following simultaneous equations:

\[
\begin{align*}
    v_{ho} &= \frac{\sum_{j=0}^{M-S} \pi^h_{ij} \lambda_d v_{lo} + \lambda j (p + v_{hn}) + 1}{r_h + \lambda_d + \lambda j - c}, \\
    v_{hn} &= \frac{\lambda_d v_{ln}}{r_h + \lambda_d - c}, \\
    v_{lo} &= \frac{\sum_{j=0}^{M-S} \pi^h_{ij} \lambda_u v_{ho} + \alpha \lambda j (p + v_{ln}) + 1}{r_l + \lambda_u + \alpha \lambda j - c}, \\
    v_{ln} &= \frac{\sum_{i=0}^{S} \pi^l_{ij} \lambda_u v_{hn} + \lambda i (v_{lo} - p) + \alpha \lambda (S - i) (v_{lo} - p)}{r_l + \lambda_u + \lambda i + \alpha \lambda (S - i) - c},
\end{align*}
\]

and

\[
p = (v_{ho} - v_{hn})(1 - q) + (v_{lo} - v_{ln})q,
\]

where

\[
\pi^h_{ij} = \frac{\sum_{i=1}^{S} \pi_{ij}}{\sum_{i=1}^{S} \sum_{j=0}^{M-S} \pi_{ij}}, \quad j = 0, 1, \ldots, M - S,
\]

and

\[
\pi^l_{ij} = \frac{\sum_{j=1}^{M-S} \pi_{ij}}{\sum_{i=0}^{S} \sum_{j=1}^{M-S} \pi_{ij}}, \quad i = 0, 1, \ldots, S.
\]

As before, we calculate the equilibrium price \( P(X) = pX \), where we set \( X = 1 \) for simplicity, based on Proposition 5. Figure 7 shows how the equilibrium price changes with respect to the factor \( \alpha \) of irrationality. Here, we depict the relative difference of prices, \( (p_{\alpha=0} - p_{\alpha})/p_{\alpha=0} \), for various \( \lambda_u \). The presence of irrational traders decreases the trade price significantly even when the fraction \( \alpha \) is small. In particular, the impact of \( \alpha \) on the asset prices is surprisingly strong for a medium \( \lambda_u \), i.e. when the prospects for the market are uncertain.

(Figure 7 is placed here.)

4 The Model with a Marketmaker

This section introduces a (monopolistic) marketmaker with the maximum inventory capacity \( I \) and investigates the impact of the inventory level on asset prices.\(^\text{16}\) The marketmaker is assumed to be risk-neutral and infinitely lived. However, we assume that the marketmaker’s time preference, \( r_m \), is constant and somewhere between \( r_l \) and \( r_h \), i.e. \( r_l < r_m < r_h \), and the marketmaker behaves as if he/she were an agent in the

\(^{16}\text{See, e.g., Garman (1976) and O’Hara (1995) for other inventory models.}\)
market. More specifically, a high-type asset owner (type ho investor) can sell the asset to either a low-type asset non-owner (type ln investor) or the marketmaker, if the capacity is available, while a type ln investor can buy an asset from either a type ho investor or the marketmaker, if any. Note that the trading mechanism is the same as the previous section.

4.1 The Three-Dimensional Markov Chain

In order to construct a Markov chain model, we assume that each inventory space behaves as if it were an individual marketmaker with inventory capacity of unity. Namely, the \( i \)th inventory space is either occupied or empty, and it can sell (buy, respectively) the asset only if it is occupied (empty).

Denote the type of each inventory space by \( mo \) or \( mn \) if it is occupied or empty, respectively. Then, since each inventory space is seen as an agent, the full set of agent types is in turn given by \( T = \{ ho, hn, lo, ln, mo, mn \} \), where the letter \( m \) designates the marketmaker’s time preference.

As in the previous section, let \( m_\sigma(t), \sigma \in T \), be the number of agents with type \( \sigma \) at time \( t \). By definition, we have

\[
m_{ho}(t) + m_{hn}(t) + m_{lo}(t) + m_{ln}(t) = M, \quad m_{mo}(t) + m_{mn}(t) = I \tag{22}
\]

and

\[
m_{ho}(t) + m_{lo}(t) + m_{mo}(t) = S \tag{23}
\]
at any time \( t \). Thus, once we recognize the triple \((m_{ho}(t), m_{ln}(t), m_{mo}(t))\), we can recover the complete distribution of the agent types at any time \( t \).

Let \( \{Y_t; t \geq 0\} \) be a three-dimensional Markov chain in continuous time, adapted to the filtration, that represents the triple \((m_{ho}(t), m_{ln}(t), m_{mo}(t))\) with state space

\[
S = \{(i, j, k) : k = 0, 1, \ldots, I, \ i = 0, 1, \ldots, S - k, \ j = 0, 1, \ldots, M - S + k\}.
\]

The transition intensities of this Markov chain are given as follows: The intensity from \((i, j, k)\) to

- \((i - 1, j, k)\) is \(i\lambda_d\),
- \((i, j - 1, k)\) is \(j\lambda_u\),
- \((i + 1, j, k)\) is \((S - i - k)\lambda_u\),
• \((i, j + 1, k)\) is \((M - S + k - j)\lambda_d\),
• \((i - 1, j - 1, k)\) is \(ij\lambda\),
• \((i - 1, j, k + 1)\) is \(i(I - k)\rho_{bid}\), and
• \((i, j - 1, k - 1)\) is \(jk\rho_{ask}\).

Here, \(\rho_{bid}\) (\(\rho_{ask}\), respectively) denotes the matching intensity for an agent with type \(ho\) (\(ln\)) to meet an inventory space with type \(mn\) (\(mo\)). Recall that trades are between \(ho\) and \(ln\) type agents and the marketmaker, if possible. Assuming that they meet each other at the events of a Poisson process and the contract processes are mutually independent, the above transition intensities can be obtained.

The Markov chain \(\{Y_t; t \geq 0\}\) is now specified completely, once we state the initial condition; however, the equilibrium distribution of the agent types is independent of the initial condition. Let us denote the equilibrium distribution by \(\pi = (\pi_{ijk}; (i, j, k) \in S)\). The equilibrium distribution exists, since \(\{Y_t; t \geq 0\}\) is a finite, time-homogeneous Markov chain in continuous time. The full balance equations for the equilibrium probabilities \(\pi_{ijk}\) are similar to those in (3) and omitted.

### 4.2 Dynamic Bargaining Equilibrium

Following the arguments given in the previous section, we can derive a dynamic bargaining equilibrium. That is, as before, let \(V(X_t, \sigma_t)\) be the equilibrium utility at time \(t\) for remaining lifetime consumption of a particular agent (or the marketmaker) with current type \(\sigma_t\) and current dividend rate \(X_t\). Also, let \(P(X_t)\) be the trade price between agents in the market at time \(t\) of the asset, let \(A(X_t)\) be the ask price between an agent and the marketmaker at time \(t\), and let \(B(X_t)\) be the bid price between an agent and the marketmaker at time \(t\).

As in (8), suppose that

\[
A(X_t) = \Delta V_m(X_t)(1 - z_{ask}) + \Delta V_l(X_t)z_{ask}, \quad 0 < z_{ask} < 1, \tag{24}
\]

where \(\Delta V_m(X_t) \equiv V(X_t, mo) - V(X_t, mn)\) denotes the reservation value for the marketmaker and \(z_{ask}\) represents the marketmaker’s bargaining power for ask trades. Also, suppose that

\[
B(X_t) = \Delta V_m(X_t)(1 - z_{bid}) + \Delta V_h(X_t)z_{bid}, \quad 0 < z_{bid} < 1, \tag{25}
\]

where \(z_{bid}\) represents the marketmaker’s bargaining power for bid trades.
Supposing \( V(X_t, \sigma) = v_\sigma X_t, \sigma \in \mathcal{T}, P(X_t) = pX_t, A(X_t) = aX_t \) and \( B(X_t) = bX_t \) for some unknown coefficients \( v_\sigma, p, a \) and \( b \), we then have the following. Recall that, in order for trades to occur only between \( \text{ho} \) type and \( \text{ln} \) type agents, we need condition (14). Similarly, we need \( \Delta v_l > a > b > \Delta v_h \) in the current model with a marketmaker. The proof of the next result is similar to Proposition 2 and omitted.

**Proposition 6** Suppose that condition (26) holds. Then, the value and price coefficients uniquely solve the following simultaneous equations:

\[
\begin{align*}
v_{\text{ho}} &= \sum_{k=0}^{I} \sum_{j=0}^{M-S+k} \pi_{jk}^{\text{ho}} \lambda_d v_{\text{lo}} + \lambda_j (p + v_{\text{hn}}) + \rho_{\text{bid}} (I - k) (b + v_{\text{hn}}) + 1 \over r_h + \lambda_d + \lambda_j + \rho_{\text{bid}} (I - k) - c, \\
v_{\text{hn}} &= \lambda_d v_{\text{ln}} \over r_h + \lambda_d - c, \\
v_{\text{lo}} &= \lambda_u v_{\text{ho}} + 1 \over r_l + \lambda_u - c, \\
v_{\text{ln}} &= \sum_{k=0}^{I} \sum_{i=0}^{S-k} \pi_{ik}^l \lambda_u v_{\text{hn}} + \lambda_l (v_{\text{lo}} - p) + \rho_{\text{ask}} k (v_{\text{lo}} - a) \over r_l + \lambda_u + \lambda_l + \rho_{\text{ask}} k - c, \\
v_{\text{mo}} &= \sum_{k=1}^{I} \sum_{j=0}^{M-S+k} \pi_{jk}^m \rho_{\text{ask}} j (a + v_{\text{mn}}) + 1 \over r_m + \rho_{\text{ask}} j - c, \\
v_{\text{mn}} &= \sum_{k=0}^{I-1} \sum_{i=0}^{S-k} \pi_{ik}^m \rho_{\text{bid}} i (v_{\text{mo}} - b) \over r_m + \rho_{\text{bid}} i - c,
\end{align*}
\]

and

\[
\begin{align*}
p &= (v_{\text{ho}} - v_{\text{hn}}) (1 - q) + (v_{\text{lo}} - v_{\text{ln}}) q, \\
a &= (v_{\text{mo}} - v_{\text{mn}}) (1 - z_{\text{ask}}) + (v_{\text{lo}} - v_{\text{ln}}) z_{\text{ask}}, \\
b &= (v_{\text{ho}} - v_{\text{hn}}) z_{\text{bid}} + (v_{\text{mo}} - v_{\text{mn}}) (1 - z_{\text{bid}}),
\end{align*}
\]

where each \( \pi^\sigma, \sigma \in \mathcal{T} \), denote the conditional probabilities such that

\[
\begin{align*}
\pi_{jk}^{\text{ho}} &= \frac{\sum_{i=1}^{S-k} \pi_{ij}^{\text{ho}}}{\sum_{k=0}^{I} \sum_{i=1}^{S-k} \sum_{j=0}^{M-S+k} \pi_{ijk}}, \\
\pi_{ik}^{\text{ln}} &= \frac{\sum_{j=1}^{M-S+k} \pi_{ijk}}{\sum_{k=0}^{I} \sum_{i=0}^{S-k} \sum_{j=1}^{M-S+k} \pi_{ijk}}, \\
\pi_{jk}^{\text{mo}} &= \frac{\sum_{i=0}^{S-k} \pi_{ijk}}{\sum_{k=1}^{I} \sum_{i=0}^{S-k} \sum_{j=1}^{M-S+k} \pi_{ijk}},
\end{align*}
\]
\[ \pi_{ik}^{mn} = \frac{\sum_{j=0}^{M-S+k} \pi_{ijk}}{\sum_{k=0}^{I-1} \sum_{i=0}^{S-k} \sum_{j=0}^{M-S+k} \pi_{ijk}}. \]

Note that, in Proposition 6, we need to assume condition (26). If it is violated, the unique solution obtained by solving the simultaneous equations is not the trade price and no economical equilibrium exists.

4.3 Numerical Example, Continued

In this numerical example, we illustrate the impact of the capacity of the marketmaker’s inventory on the asset prices for the base case parameters listed in Table 1. For other parameters, we set \( z_{ask} = z_{bid} = 0.5 \) and \( \rho = \rho_{ask} = \rho_{bid} \).

We calculate the equilibrium price \( P(X) = pX \), where we set \( X = 1 \) for simplicity, based on Proposition 6. Figure 8 shows how the equilibrium price changes with respect to the switching intensity \( \lambda_u \) (recall that \( \lambda_d = 1 - \lambda_u \)) for various inventory capacities \( I \) when \( \rho = 100 \), while Figure 9 shows the effect of the switching intensity for various \( I \) when \( \rho = 1 \).\(^{17} \) When the matching intensity \( \rho \) is large, the effect by increasing the inventory capacity is significant. That is, until a larger value of \( \lambda_u \), the price keeps the high price \( p^*_l = 50 \). On the other hand, when \( \rho \) is small, the effect by increasing the inventory capacity is limited. Note that the marketmaker’s discount rate \( r_m \) is constant and close to \( r_l \) in this particular example. Hence, as \( \lambda_u \) gets large, i.e. the prospects for the market become negative, the marketmaker can buy the assets more frequently and, as a result, it plays the role of protecting a market crash. The effect, however, is limited and deceases as investors’ and marketmaker’s search intensities become small.

(Figures 8 and 9 are placed here.)

5 Conclusion

In this paper, we propose a Markov chain model for studying the impact on asset prices of illiquidity associated with search and bargaining in an economy. The economy consists of finitely many investors (rational or irrational) and a marketmaker who can trade only when they find each other.

\(^{17}\)The case that \( \rho = 1 \) (\( \rho = 100 \), respectively) implies that the marketmaker contacts agents once a year (twice per week) on average.
Our model has several desirable features by its discrete nature. First, the discrete Markov chain describes the mechanism of trades more explicitly. Second, the joint distribution about the types of agents in equilibrium is calculated, and we can obtain more information about the structure of equilibrium prices. Third, two types of asymptotics can be considered. Namely, when the number of agents gets large, our model converges to the DGP model under regularity conditions. On the other hand, for any fixed number of agents, our model exhibits significantly different properties from the DGP model as investors’ search intensities get large.

The Walrasian price in DGP (2004) is the value of holding the asset forever for a hypothetical agent who always has a low (high, respectively) discount rate. Hence, the equilibrium is attained if the majority of agents in our finite market becomes optimistic (pessimistic) about the economy. Let $p_l^*$ ($p_h^*$) be the price in the optimistic (pessimistic) market. Of course, $p_l^* > p_h^*$. Through extensive numerical experiments, we observe that prices exhibits the cutoff phenomenon (i.e. crash) from $p_l^*$ to $p_h^*$ as the fraction of pessimistic agents increases.

Even when irrational investors are present in our market, the equilibrium distribution of the agent types is unchanged. What is changed is the equilibrium utility of agents, and the calculated trade price is affected considerably by the presence of irrational investors. More precisely, the presence of irrational traders (who sell the asset irrationally) decreases the trade price. Through numerical experiments, we observe that the impact of irrational investors on the asset prices is strong especially when the prospects for the market are uncertain.

Our model also allows a marketmaker to have an inventory that is controlled through the marketmaker’s bargaining power. This seems important, because market stresses may be amplified by a fall in market liquidity due to the malfunctioning of the marketmaker’s mechanism to supply liquidity to the market. The Markov chain model provides a tool to study the relationship between market stability and market making.

\section*{A Proof of Proposition 2}

First, we shall prove that there is a unique solution in the simultaneous equations (9)–(13) that satisfy (14). To this end, let

\[ K_1 = \sum_{j=0}^{M-S} \pi_j^{ho} \frac{\lambda_d}{r_h + \lambda_d + \lambda_j - c}, \quad K_2 = \sum_{j=0}^{M-S} \pi_j^{ho} \frac{\lambda_j}{r_h + \lambda_d + \lambda_j - c} \]
and
\[ K_3 = \sum_{j=0}^{M-S} \pi_j h_0 \frac{1}{r_h + \lambda_d + \lambda_j - c}, \]
so that \( v_{ho} = K_1 v_{lo} + K_2 (p + v_{hn}) + K_3 \). Note that \( K_i > 0 \), \( i = 1, 2, 3 \), \( K_1 + K_2 < 1 \) and
\[ \frac{K_3}{1 - K_1 - K_2} = \frac{1}{r_h - c}. \] (A.1)

Also, let
\[ \alpha = \frac{\lambda_d}{r_h + \lambda_d - c}, \quad \beta_1 = \frac{\lambda_u}{r_l + \lambda_u - c}, \quad \beta_2 = \frac{1}{r_l + \lambda_u - c}, \]
so that \( v_{hn} = \alpha v_{ln} \) and \( v_{lo} = \beta_1 v_{ho} + \beta_2 \). Here \( \alpha, \beta_1, \beta_2 > 0 \) and \( \alpha, \beta_1 < 1 \). Finally, let
\[ L_1 = \sum_{i=0}^{S} \pi_i l_n \frac{\lambda_u}{r_l + \lambda_u + \lambda_i - c}, \quad L_2 = \sum_{i=0}^{S} \pi_i l_n \frac{\lambda_i}{r_l + \lambda_u + \lambda_i - c}, \]
so that \( v_{ln} = L_1 v_{hn} + L_2 (v_{lo} - p) \). Note that \( L_1, L_2 > 0 \) and \( L_1 + L_2 < 1 \).

Now, given \( p \), we have
\[ \begin{pmatrix} v_{ho} \\ v_{hn} \\ v_{lo} \\ v_{ln} \end{pmatrix} = \begin{pmatrix} 0 & K_2 & K_1 & 0 \\ 0 & 0 & 0 & \alpha \\ \beta_1 & 0 & 0 & 0 \\ 0 & L_1 & L_2 & 0 \end{pmatrix} \begin{pmatrix} v_{ho} \\ v_{hn} \\ v_{lo} \\ v_{ln} \end{pmatrix} + \begin{pmatrix} K_3 + K_2 p \\ 0 \\ \beta_2 \\ -L_2 p \end{pmatrix}. \]

Let us denote the matrix by \( \Gamma \). Since the matrix \( \Gamma \) is strictly substochastic, its Perron–Frobenius eigenvalue is strictly less than unity (see, e.g., Seneta, 1981). It follows that the matrix \( (I - \Gamma) \) is non-singular, and the linear equation can be solved uniquely.

After some tedious algebra, we obtain
\[ \Delta v_h = \frac{K_3(1 - \alpha L_1 - \alpha L_2 \beta_1) + (K_1(1 - \alpha L_1) - \alpha(1 - K_2)L_2)\beta_2}{1 - (K_1 + \alpha K_2 L_2)\beta_1 - \alpha L_1(1 - K_1 \beta_1)} \]
\[ + \frac{\alpha L_2(1 - K_1 \beta_1) + K_2(1 - \alpha L_1 - \alpha L_2(1 + \beta_1))}{1 - (K_1 + \alpha K_2 L_2)\beta_1 + \alpha L_1(1 - K_1 \beta_1)} p \]
and
\[ \Delta v_l = \frac{(1 - \alpha L_1 + L_2)(K_3 \beta_1 + \beta_2)}{1 - (K_1 + \alpha K_2 L_2)\beta_1 - \alpha L_1(1 - K_1 \beta_1)} \]
\[ + \frac{L_2 - (K_1 L_2 - K_2(1 - \alpha L_1 - (1 + \alpha)L_2))\beta_1}{1 - (K_1 + \alpha K_2 L_2)\beta_1 - \alpha L_1(1 - K_1 \beta_1)} p. \]

Since \( p = \Delta v_h (1 - q) + \Delta v_l q \), we finally have
\[ \Delta v_l - \Delta v_h = \frac{(1 - \alpha L_1 - L_2)((1 - K_1 - K_2)\beta_2 - K_3(1 - \beta_1))}{f(q)}, \] (A.2)
where
\[
f(q) = (1 - \alpha L_1 - (q + \alpha - q\alpha)L_2)(1 - K_1\beta_1) - K_2(1 - q - (1 - q)\alpha(L_1 + L_2) + q(1 - \alpha L_1 - L_2)\beta_1).
\]
Recall that \(\alpha, L_1 + L_2 < 1\) and \(\beta_2/(1 - \beta_1) = (r_l - c)^{-1}\). Hence, the numerator of (A.2) is positive due to (A.1). Also, since \(\beta_1, K_1 + K_2 < 1\), we have
\[
f(0) = (1 - \alpha(L_1 + L_2))(1 - \beta_1 K_1 - K_2) > 0
\]
and
\[
f(1) = (1 - \alpha L_1 - L_2)(1 - \beta_1(K_1 + K_2)) > 0.
\]
Note that \(f(q)\) is linear in \(q\). It follows that the denominator of (A.2) is positive, whence \(\Delta v_l > \Delta v_h\). Since \(0 < q < 1\) and \(p = \Delta v_h(1 - q) + \Delta v_lq\), we obtain (14), as desired.

It remains to show that Equations (9)–(13) hold. In this appendix, however, we shall only prove that (4) implies Equation (9). Other equations can be obtained similarly.

Consider a particular agent of high-type owner at time \(t\). Then, given \(m_{ho} = i \geq 1\) and \(m_{ln} = j\), the agent can find a counterpart to trade with intensity \(\lambda_j\). Hence, in this situation, the stopping time \(\tau_m\) is exponentially distributed with rate \(\lambda_j\). Also, because \(\tau_r\) and \(\tau_m\) are independent and \(\tau_r\) is exponentially distributed with rate \(\lambda_d\), we conclude that \(\tau = \min\{\tau_r, \tau_m\}\) is also exponentially distributed with rate \(\lambda_d + \lambda j\).

Now, by the law of total probability and the independence assumption, we obtain
\[
\tilde{E}_t\left[\int_t^\tau e^{-r_h(u-t)}X_u\,du\right]
= \tilde{E}_t\left[\int_t^\tau e^{-r_h(u-t)}X_u\,du\bigg|m_{ln}\right]
= X_t\tilde{E}_t\left[\int_t^\tau\int_t^\tau e^{(c-r_h)(u-t)}\,du\lambda_d + \lambda m_{ln}\right]e^{-(\lambda_d + \lambda m_{ln})(x-t)}\,dx
= X_t\tilde{E}_t\left[\int_t^\infty e^{-(\lambda_d + \lambda m_{ln} + r_h-c)(u-t)}\,du\right]
= X_t\sum_{j=0}^{M-S} \pi^{ho}_j \frac{1}{\lambda_d + \lambda m_{ln} + r_h - c},
\]
where \(\pi^{ho}_j\) denotes the conditional probability given \(m_{ho} \geq 1\).\(^{18}\) Similarly, we obtain
\[
\tilde{E}_t\left[e^{-r_h(\tau_r-t)}V(X_{\tau_r}, ho)1_{\{\tau_r < \tau_m\}}\right]
\]
\(^{18}\)We need the conditional probability, because there is at least one ho agent to calculate \(V(X_t, ho)\).
\[ \tilde{E}_t \left[ \int_t^\infty \int_t^u e^{-r_h(x-t)} V(x_t, t_0) \lambda_d e^{-\lambda_d(x-t)} dx \lambda_m u e^{-\lambda_m(u-t)} du \right] \\
= X_t v_{lo} \lambda_d \tilde{E}_t \left[ \int_t^\infty e^{-(\lambda_d + \lambda_m u + r_h - c)(x-t)} dx \right] \\
= X_t \sum_{j=0}^{M-S} \pi_j^{ho} \frac{v_{lo} \lambda_d}{\lambda_d + \lambda_j + r_h - c}. \]

Finally,
\[
\tilde{E}_t \left[ e^{-r_h(\tau_m - t)} (V(x_{\tau_m}, h n) + P(x_{\tau_m})) 1_{\{\tau > \tau_m\}} \right] \\
= X_t \tilde{E}_t \left[ \int_t^\infty \int_t^u e^{-r_h(x-t)} (V(x_t, h n) + P(x_t)) \lambda_m u e^{-\lambda_m(u-t)} du \right] \\
= X_t (v_{hn} + p) \tilde{E}_t \left[ \lambda_m u \int_t^\infty e^{-(\lambda_d + \lambda_m u + r_h - c)(x-t)} dx \right] \\
= X_t \sum_{j=0}^{M-S} \pi_j^{ho} (v_{hn} + p) \lambda_j \frac{\lambda_d + \lambda_j + r_h - c}{\lambda_d + \lambda_j + r_h - c}. \]

Note that the ho agent can start bargaining only if his/her counterparty is of type ln, and hence this term vanishes for the case that \( m_{ln} = 0 \). In summary, we have obtained
\[ v_{ho} = \sum_{j=0}^{M-S} \pi_j^{ho} \frac{1 + v_{lo} \lambda_d + (v_{hn} + p) \lambda_j}{\lambda_d + \lambda_j + r_h - c}, \]
whence Equation (4) is proved.

### B  Proof of Proposition 3

Since the resulting Markov chain is a birth–death process, it is convenient to consider detailed balance equations\(^{19}\) rather than full balance equations (3). More specifically, we obtain
\[ (S + j) \lambda_u \pi_{0j} = (M - S - j + 1) \lambda_d \pi_{0,j-1} \quad (B.1) \]
and
\[ (M - S + i) \lambda_d \pi_{i0} = (S - i + 1) \lambda_u \pi_{i-1,0} \quad (B.2) \]
for all \((i, j) \in S_W\), with understanding that \( \pi_{ij} = 0 \) if \((i, j) \not\in S_W\). Let \( \pi_{0,M-S} = K \). Then, from (B.1), we obtain
\[ \pi_{0,M-S-j} = M C_j \left( \frac{\lambda_u}{\lambda_d} \right)^j K. \]

\(^{19}\)See, e.g., Kijima (1997) for details of balance equations.
Similarly, from (B.2), we obtain

\[ \pi_i^0 = M C_{M-S+i} \left( \frac{\lambda_u}{\lambda_d} \right)^{M-S+i} K. \]

Since \( \sum_{(i,j) \in S_w} \pi_{ij} = 1 \), it follows that \( K = (1 + \lambda_u / \lambda_d)^{-M} \), whence

\[ \pi_{0,M-S-j} = M C_j (1 - \xi)^j \xi^{M-j}, \quad j = 0, 1, \ldots, M - S, \quad (B.3) \]

and

\[ \pi_i^0 = M C_{M-S+i} (1 - \xi)^{M-S+i} \xi^{S-i}, \quad i = 0, 1, \ldots, S, \quad (B.4) \]

where \( \xi = \lambda_d / (\lambda_u + \lambda_d) \). Note that the equilibrium distribution is the binomial distribution with parameters \((M, 1 - \xi)\).

C Proof of Proposition 4

By the exactly same arguments as in the proof of Proposition 2, it is shown that the value and price coefficients uniquely solve the following simultaneous equations:

\[ v_{ho} = \frac{\lambda_d v_{lo} + (M - S)\lambda_d(p + v_{hn}) + 1}{r_h + \lambda_d + (M - S)\lambda_d - c}, \]

\[ v_{hn} = \frac{\lambda_d v_{ln}}{r_h + \lambda_d - c}, \]

\[ v_{lo} = \frac{\lambda_u v_{ho} + 1}{r_l + \lambda_u - c}, \]

\[ v_{ln} = \frac{\lambda_u v_{hn} + S\lambda_u(v_{lo} - p)}{r_l + \lambda_u + S\lambda_u - c}, \]

and

\[ p = (v_{ho} - v_{hn})(1 - q) + (v_{lo} - v_{ln})q. \]

The result is obtained by solving the above equations with respect to \( p \).

References


Figures

Figure 1: Transition diagram of the two-dimensional Markov chain \( \{Y_t; t \geq 0\} \)

Figure 2: Transition diagram of the birth-death process
(a) $\lambda_u = 0.1$, $\lambda_d = 0.9$  
(b) $\lambda_u = \lambda_d = 0.5$  
(c) $\lambda_u = 0.9$, $\lambda_d = 0.1$

Figure 3: Bivariate equilibrium distributions for type $ln$ and $ho$ agents
Figure 4: Dependence of the price $p$ on the switching intensity $\lambda_u$ for various matching intensity $\lambda$.

Figure 5: Convergence of the price (19) to the Walrasian price (20).
Figure 6: Dependence of the price $p$ on the switching intensity $\lambda_u$ for various values of $S$, the number of assets in the market.

Figure 7: Dependence of the price $p$ on the switching intensity $\lambda_u$ for various irrationality factor $\alpha$, where difference means the relative difference of prices $(p_{\alpha=0} - p_\alpha)/p_{\alpha=0}$.
Figure 8: Dependence of the price $p$ on the switching intensity $\lambda_u$ for various inventory capacity $I$ ($\rho = 100$)

Figure 9: Dependence of the price $p$ on the switching intensity $\lambda_u$ for various inventory capacity $I$ ($\rho = 1$)