Pricing Exchange Options with Discontinuous Stock Prices

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Abstract

An exchange option allows the holder of the option to exchange one stock for another stock at maturity. Margrabe provides a pricing formula where the distributions of both stock prices are log-normal with correlated components. Merton’s formula prices a European call option on a single stock where the stock price process contains a compound Poisson jump component, in addition to a continuous log-normally distributed component. We extend Margrabe’s and Merton’s results to the case of exchange options where both stock price processes also contain compound Poisson jump components.

Key words: Exchange option, Equivalent martingale measure, Cumulant transform, Compound Poisson processes, Radon-Nikodym derivative.

JEL Classification: C00, G12, G13

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1 Introduction

The seminal Black-Scholes model (Black and Scholes 1973) provides a solution to the then long-standing call option pricing problem and is related to the valuation of a firm’s equity in terms of its total value and debt. The Black-Scholes model assumes a log-normally distributed stock price process and a riskless bond growing at a fixed riskless interest rate. A frictionless and continuous market is also assumed in the model. Other extensions to the basic Black-Scholes model include extensions to include for example, option pricing in stochastic interest rates and stochastic volatility environments. Extensive reviews of further extensions and applications of the Black-Scholes model can be found in, for example, Smith (1976) and Merton (1998). An early extension by Merton (1976) introduces the jump-diffusion process into the stock price model which accounts for unsystematic risk and the leptokurtic distribution of the stock returns. Margrabe (1978) later priced an exchange option between two correlated stocks that each satisfy the assumptions in the Black-Scholes model. We extend the results of Merton (1976) and Margrabe (1978) to the case of exchange options where both stock price processes are each composed of a continuous component and a jump component which is a geometric compound Poisson jump process. Two broad cases are considered – one case with independent jump components and another case with correlated jump components.
2 Risk-Neutral Valuation of Claims

In a complete market, the argument for an arbitrage free price based on martingale theory and the Fundamental Theorem of Asset Pricing (Harrison and Krebs 1979; Harrison and Pliska 1981, 1983; Schachermayer 1992; Delbaen and Schachermayer 1994) suggests that the fair price of any simple European option must be the conditional expectation of the discounted payoff at maturity, where the expectation is taken under some appropriate risk-neutral measure.

Let $C(T, S(T))$ be the value of a claim (e.g., a European option) that can only be exercised at time $T$, where $S(T)$ is the value of the underlying asset (e.g., a stock) at maturity time $T$. We assume a filtered probability measure space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ where the filtration $\{\mathcal{F}_t\}$ is the natural filtration generated by the stock price process $\{S(t), t \geq 0\}$. The market probability measure is denoted by $P$. The existence of a risk-free money market account process $\{M(t), t \geq 0\}$ is assumed and

$$M(t) = e^{rt},$$

where $r$ is the risk-free interest rate. The discounted stock price process $\left\{ \frac{S(t)}{M(t)} \right\}$ is a martingale with respect to the risk-neutral measure $\hat{P}$, that is,

$$S(t) = M(t)E_{\hat{P}} \left[ \frac{S(T)}{M(T)} \bigg| \mathcal{F}_t \right].$$

(2)
In a complete market, the arbitrage free value of the option at any time \( t \) is

\[
C(t, S(t)) = M(t)E_P \left[ \frac{C(T, S(T))}{M(T)} \middle| \mathcal{F}_t \right].
\]

In other words, the discounted option price process \( \{ C(t) \} \) is a \( \tilde{P} \)-martingale. Indeed, the discounted price process of all risky assets are \( \tilde{P} \)-martingales in a complete market situation.

The stock price process can generally be expressed as

\[
S(t) = S(0)e^{X(t)},
\]

where \( \{ X(t) \} \) is a Lévy process (Gerber and Shiu 1994), in which case the filtration \( \{ \mathcal{F}_t \} \) is the same as the natural filtration generated by the Lévy process \( \{ X(t) \} \). The returns process \( \{ X(t) \} \) is actually the accumulated returns (also called continuously-compounded returns or log-returns). For example, the Black-Scholes model (Black and Scholes 1973) uses a scaled Brownian motion with drift for the returns process. Cox, Ross and Rubinstein (1979) use a binary random walk process. Other examples using other types of Lévy processes under which the market is still complete can be found in Gerber and Shiu (1994) and Cheang (2003).

In the case of stock price returns modeled by Lévy processes of the exponential type, the Radon-Nikodým derivative based on the cumulant transform of the distribution of the stock price returns can be used to generate a parametric family of similarly distributed stochastic processes (Gerber and
The search for a risk-neutral measure is then equivalent to a search for a particular value of the parameter that yields the risk-neutral measure. Closed-form expressions for the option prices, if available, can be obtained by deriving expressions for the cumulative probabilities that depend on the value of the parameter determining the risk-neutral measure. Gerber and Shiu (1994) introduced an identity involving the parameter, which can be solved in order to obtain the correct value of the parameter that induces the required risk-neutral measure. The identity has a unique solution under the assumptions of complete market conditions, and the identity is equivalent to the martingale condition (2).

Merton (1976) introduced the jump-diffusion model for option pricing in which the returns process \( \{ X(t) \} \) is modeled by two components. The first component is a scaled Brownian motion with drift, as in the Black-Scholes model (Black and Scholes 1973). The second component is a compound Poisson process with normally distributed jumps. In other words,

\[
X(t) = X_1(t) + X_2(t),
\]

or equivalently

\[
S(t) = e^{X_1(t)+X_2(t)}.
\]

Merton (1976) set the continuous part to be

\[
X_1(t) = \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W(t),
\]
where \( \{W(t)\} \) is standard Brownian motion in the original measure \( P \). The jump part (after compensation) to be

\[
X_2(t) = -\lambda kt + \sum_{n=1}^{N(t)} J_n,
\]

where \( \{N(t)\} \) is a pure Poisson counting process with rate \( \lambda \), and \( k = E_P[e^{J_n} - 1] \). The two components are assumed to be independent of each other and the jump components \( J_n \) are assumed to be identically and independently normally distributed, and are also independent of the counting process and independent of the Brownian motion component in the continuous part.

Merton’s (1976) model is inherently incomplete because we have one asset and more than one random factor driving the dynamics of the returns process. Hence there may exist many measures under which the discounted stock price process \( \left\{ \frac{S(t)}{M(t)} \right\} \) is a martingale. Appealing to financial risk theory, Merton (1976) identified the Brownian motion component (7) as the component contributing to systematic risk. The jump component (8) is due to unsystematic risk. As such, the correct risk-neutral measure to be selected must be one that preserves the distribution of the jump component under the change of measure. In an extension by Cheang (2003) of the Gerber and Shiu (1994) method to price European options in Merton’s (1976) model, the identity involving the cumulant transforms of the two processes \( \{X_1(t)\} \) and \( \{X_2(t)\} \) has two parameters – one associated with each component. By
setting the parameter associated with the jump component to be zero, this is equivalent to the selection of a risk-neutral measure that makes the discounted stock price process a martingale and preserves the distribution of the jump component under the change of measure. As such, there is still a risk-neutral measure $\tilde{P}$, one that does not change the distribution of the jump component, so that (2) and (3) are satisfied and any European option under Merton’s (1976) model can be priced by (3).

3 Margrabe’s Exchange Option Model

In Margrabe’s (1978) exchange option model, the final value of the option at maturity is

$$C(T, S_1(T), S_2(T)) = (S_1(T) - S_2(T))^+. \tag{9}$$

Each of the two stocks are assumed to be log-normally distributed as in the Black-Scholes model (Black and Scholes 1973), with price dynamics given by

$$dS_1(t) = S_1(t)[\mu_1dt + \sigma_1dW_1(t)], \tag{10}$$

and

$$dS_2(t) = S_2(t)[\mu_2dt + \sigma_2dW_2(t)], \tag{11}$$

where $(W_1(t), W_2(t))$ are components of correlated standard Brownian motion with instantaneous correlation $dW_1(t)dW_2(t) = \rho dt$. Margrabe (1978) showed that one could hedge the value of the option at any time if the same
frictionless and continuous market conditions of the Black-Scholes model are satisfied. The hedge is obtained by going long on the first stock and short on the second stock. If the existence of a money market account $M(t) = e^{rt}$, where $r$ is the risk-free interest rate, is assumed, one could price the value of the exchange option

\[
C(t, S_1(t), S_2(t)) = M(t)E_P \left[ \frac{C(T, S_1(T), S_2(T))}{M(T)} \right| \mathcal{F}_t ]
\]

Indeed (12) is equivalent to (3) with the terminal value of the option replaced by the terminal value of the exchange option. In (12), the money market account process $\{M(t)\}$ is used as the numéraire process. The bivariate version of the cumulant transform identity used by Gerber and Shiu (1994) to price the exchange option in Margrabe’s (1978) model is equivalent to the use of the money market account process $\{M(t)\}$ as the numéraire process.

However, a more convenient method to price the exchange option is to use the second stock price process $\{S_2(t)\}$ as the numéraire process as in Margrabe (1978). Although the money market account process $\{M(t)\}$ is not used explicitly, it is required in order to obtain an equivalent martingale measure $Q$ such that the first stock price process, the option price process, as well as the money market account, are all $Q$-martingales after discounting by the second stock price, that is, $\left\{ \frac{S_1(t)}{S_2(t)} \right\}$, $\left\{ \frac{C(t, S_1(t), S_2(t))}{S_2(t)} \right\}$ and $\left\{ \frac{M(t)}{S_2(t)} \right\}$ are all $Q$-martingales. Hence the option price is

\[
C(t, S_1(t), S_2(t)) = S_2(t)E_Q \left[ \frac{C(T, S_1(T), S_2(T))}{S_2(T)} \right| \mathcal{F}_t ]
\]
\[ S_2(t)E_Q \left[ \left( \frac{S_1(t)}{S_2(t)} - 1 \right)^+ \right| \mathcal{F}_t \]

and the calculation of the option price by (13) or (12) yields the same answer.

The process \( \{ \frac{S_1(t)}{S_2(t)} \} \) is a \( Q \)-martingale where

\[ d \left( \frac{S_1(t)}{S_2(t)} \right) = \left( \frac{S_1(t)}{S_2(t)} \right) \left[ \sigma_1 d\tilde{W}_1(t) - \sigma_2 d\tilde{W}_2(t) \right], \quad (14) \]

where \( \tilde{W}_1(t) \) and \( \tilde{W}_2(t) \) are correlated standard Brownian motion components in the \( Q \)-measure, and

\[ d\tilde{W}_1(t) = \left[ \frac{\mu_1 - r}{\sigma_1} - \rho \sigma_2 \right] dt + dW_1(t) \quad (15) \]

and

\[ d\tilde{W}_2(t) = \left[ \frac{\mu_2 - r}{\sigma_2} - \sigma_2 \right] dt + dW_2(t). \quad (16) \]

The term \( E_Q \left[ \left( \frac{S_1(t)}{S_2(t)} - 1 \right)^+ \right| \mathcal{F}_t \] in (13) is analogous to pricing a call option on a log-normally distributed stock with strike price one. Thus the exchange option price at time \( t \) when \( S_1(t) = s_1 \) and \( S_2(t) = s_2 \) is

\[ C(t, s_1, s_2) = s_1 \Phi(d) - s_2 \Phi(d - \sigma \sqrt{T - t}), \quad (17) \]

where

\[ d = \frac{\ln(s_1/s_2) + \frac{\sigma^2}{2}(T - t)}{\sigma \sqrt{T - t}}, \]

and

\[ \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}. \]
4 Extended Merton’s Model

The call option price of an extended Merton’s (1976) stock price model containing two independent jump components is needed to help us price exchange options between two stocks with discontinuous stock prices. In this section, we set out to derive a pricing formula for the extended Merton’s option pricing model. Following Merton, we assume that both jump components are due to unsystematic risks, so that there are no change in the jump parameters corresponding to the change in measure. The call option price is stated in the following theorem.

Theorem 1. Let stock price be

\[ S(t) = S(0)e^{X_1(t) + X_2(t) + X_3(t)}, \]

where the continuous part of the returns process is

\[ X_3(t) = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t), \]

and the two discontinuous parts of the returns process are

\[ X_1(t) = -\lambda_1 k_1 t + \sum_{i=0}^{N_1(t)} J_{1,i}, \]

and

\[ X_2(t) = -\lambda_2 k_2 t + \sum_{j=0}^{N_2(t)} J_{2,j}, \]

where \( J_{1,n} \) are independently and identically distributed as \( N(\alpha_1, \delta_1^2) \), and \( k_1 = E_P[e^{h} - 1] \), and \( J_{2,n} \) are independently and identically distributed as
\( N(\alpha_2, \delta_2^2) \), and \( k_2 = E_P[e^{J_2} - 1] \). The Lévy processes \( X_1(t), X_2(t) \) and \( X_3(t) \) are all independent and the jump arrivals and the jump sizes are also independent of each other. Then the call option price \( C(0, s) \) at time zero when the stock price \( S(0) = s \) and the strike price is \( K \), is

\[
C(0, s) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e^{-|\lambda_1(1+k_1) + \lambda_2(1+k_2)|T}[\lambda_1(1 + k_1)T]^n[\lambda_2(1 + k_2)T]^m}{n!m!} \times \left[s \Phi(d_{1,n,m}) - Ke^{-r_n,mT} \Phi(d_{2,n,m})\right],
\]

where

\[
d_{1,n,m} = \frac{\ln s + \left(r_{n,m} + \frac{\sigma_{n,m}^2}{2}\right)T}{\sqrt{\sigma_{n,m}^2 T}},
\]

and

\[
d_{2,n,m} = \frac{\ln s + \left(r_{n,m} - \frac{\sigma_{n,m}^2}{2}\right)T}{\sqrt{\sigma_{n,m}^2 T}};
\]

where

\[
r_{n,m} = r - \lambda_1 k_1 - \lambda_2 k_2 + \frac{n\alpha_1}{T} + \frac{m\alpha_2}{T} + \frac{n\delta_1^2}{2T} + \frac{m\delta_2^2}{2T},
\]

and

\[
\sigma_{n,m}^2 = \sigma^2 + \frac{n\delta_1^2}{T} + \frac{m\delta_2^2}{T}.
\]

**Remark.** The call option price \( C(t, s) \) at time \( t \) when the stock price \( S(t) = s \) is obtained by setting the time \( T \) to be in the forward time to maturity \( T - t \) in the call option pricing formula (22) and in the expressions for \( d_{1,n,m} \) and \( d_{2,n,m} \).
By application of Itô’s formula to semi-martingales, we see that $S(t)$ satisfies the stochastic differential form

\begin{equation}
    dS(t) = (\mu - \lambda_1 k_1 - \lambda_2 k_2)S(t-)dt + \sigma dW(t)
    + [e^{J_1} - 1]S(t-)dN_1(t) + [e^{J_2} - 1]S(t-)dN_2(t).
\end{equation}

and

\begin{equation}
    dS(t) = (r - \lambda_1 k_1 - \lambda_2 k_2)S(t-)dt + \sigma d\tilde{W}(t)
    + [e^{J_1} - 1]S(t-)dN_1(t) + [e^{J_2} - 1]S(t-)dN_2(t),
\end{equation}

where $\{\tilde{W}(t)\}$ is standard Brownian motion in the risk-neutral measure $Q$ which makes the discounted process $\{S(t)M(t)\}$ a $Q$-martingale and preserves the distribution of the jump components under the change of measure.

**Outline of Proof.** The proof of Theorem 1 is similar to the derivation of Merton’s (1976) original option pricing formula using the cumulant transform method (Cheang 2003).

Following Merton’s argument that jumps are due to unsystematic risk, the distribution of the jump sizes and the arrival rates do not change under the change of measure. From an extension of the Gerber and Shiu (1994) method (Cheang 2003), we solve an identity involving the cumulant transform of $X_1(t) + X_2(t) + X_3(t)$, that is,

\begin{equation}
    \kappa(\Theta^* + \ell) = \kappa(\Theta^*) + r,
\end{equation}

12
where

\[ \kappa(\Theta) = \ln E_P[e^{\theta_1 X_1(1) + \theta_2 X_2(1) + \theta_3 X_3(1)}] \tag{26} \]

and \( \ell \) is a vector of ones. Solving (25) is equivalent to solving

\[ \kappa_1(\theta^*_1 + 1) + \kappa_2(\theta^*_2 + 1) + \kappa_3(\theta^*_3 + 1) = \kappa_1(\theta^*_1) + \kappa_2(\theta^*_2) + \kappa_3(\theta^*_3) + r. \]

The call option price at time zero is

\[ C(0, s) = E_P[(S(T) - K)^+] = E_{P_{\Theta^*}}[(S(T) - K)^+] = E_P\left[ \frac{dP_{\Theta^*}}{dP}(S(T) - K)^+ \right], \tag{27} \]

where

\[ \frac{dP_{\Theta^*}}{dP} = e^{\theta^*_1 X_1(T) + \theta^*_2 X_2(T) + \theta^*_3 X_3(T) - T \kappa(\Theta^*)}, \tag{28} \]

and \( \Theta^* \) satisfies (25). The correct solution of \( \Theta^* \) in (25) yields the Radon-Nikodým derivative \( \frac{dP_{\Theta^*}}{dP} \) which induces the change of measure from \( P \) to the risk-neutral measure \( \tilde{P} \). In order to preserve the distributions of the jump components under the change of measure, we set \( \theta^*_1 = \theta^*_2 = 0 \) and obtain \( \theta^*_3 = \frac{r - \mu}{\sigma^2} \), which induces a change of drift from \( \mu \) to \( r \) in \( X_3(t) \), yielding

\[ X_3(t) = \left( r - \frac{\sigma^2}{2} \right) t + \sigma \tilde{W}(t) \]

in a risk-neutral measure \( P_{\Theta^*} = \tilde{P} \). Thus the call option price is obtained by evaluating the tail probabilities in the expression

\[ C(0, s) = s P_{\Theta^* + \ell} \left[ X_1(T) + X_2(T) + X_3(T) > \ln \frac{K}{s} \right], \tag{29} \]
$-K e^{-rt} P_{\Theta} \left[ X_1(T) + X_2(T) + X_3(T) > \ln \frac{K_s}{s} \right]. \square$

5 Exchange Options with Jumps

In this section, we derive the exchange option price between two stocks that have discontinuous stock prices. In the first scenario, the stocks have correlated jump components. Let the first stock price process $\{S_1(t)\}$ satisfy

$\begin{align*}
\frac{dS_1(t)}{S_1(t)} &= (\mu_1 - \lambda_1 k_1) S_1(t-) dt + \sigma_1 S_1(t-) dW_1(t) \\
&\quad + [e^{J_1} - 1] S_1(t-) dN_1(t),
\end{align*}$

where the marginal distribution of $J_{1,n}$ are independently and identically distributed as $N(\alpha_1, \delta_1^2)$ and $k_1 = E_P[e^{J_1} - 1]$. Let the second stock price process $\{S_2(t)\}$ satisfy

$\begin{align*}
\frac{dS_2(t)}{S_2(t)} &= (\mu_2 - \lambda_1 k_2 + \lambda_2 k_2 - \lambda_3 k_1) S_2(t-) dt + \sigma_1 S_2(t-) dW_2(t) \\
&\quad + [e^{J_3} - 1] S_2(t-) dN_1(t) + [e^{J_2} - 1] S_2(t-) dN_2(t),
\end{align*}$

where the marginal distribution of $J_{3,n}$ are independently and identically distributed as $N(\alpha_2, \delta_2^2)$ and $k_2 = E_P[e^{J_2} - 1]$; and both $J_{1,n}$ and $J_{3,n}$ are correlated with correlation coefficient $\rho_j$; and $J_{2,m}$ are independently and identically distributed as $N(\alpha_2, \delta_2^2)$, and $k_2 = E_P[e^{J_2} - 1]$. An application for such a scenario could be for example, a stock $S_1(t)$ and an index $S_2(t)$, which contains the first stock. Jumps from the first stock
contribute to jumps in the index. However, jumps in the index may also be due to jumps from other stocks that are part of the index. Hence the index \( S_2(t) \) also contains another jump process \( \sum_{m=1}^{N_2(t)} J_{2,m} \) in (31) independent of the jump processes driven by \( N_1(t) \) in (30) and (31). The option in this scenario is actually an outperformance option of one unit of the stock against one unit of the index.

**Theorem 2.** Let the stock price processes \( \{S_1(t)\} \) and \( \{S_2(t)\} \) satisfy (30) and (31) respectively where the marginal distribution of \( J_{1,n} \) are independently and identically distributed as \( N(\alpha_1, \delta_1^2) \) and \( k_1 = E_P[e^{J_1} - 1] \); the marginal distribution of \( J_{3,n} \) are independently and identically distributed as \( N(\alpha_3, \delta_3^2) \) and \( k_3 = E_P[e^{J_3} - 1] \); and both \( J_{1,n} \) and \( J_{3,n} \) are correlated with correlation coefficient \( \rho_j \); and \( J_{2,m} \) are independently and identically distributed as \( N(\alpha_2, \delta_2^2) \), and \( k_2 = E_P[e^{J_2} - 1] \); with \( dW_1(t).dW_2(t) = \rho dt \). The exchange option price \( C(0, s_1, s_2) \) at time zero, with \( S_1(0) = s_1 \) and \( S_2(0) = s_2 \), when the terminal value is \( (S_1(T) - S_2(T))^+ \) is

\[
C(0, s_1, s_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-\left[\lambda_1(1+k_1) + \lambda_2(1+\tilde{k}_2)\right]T} \left[\frac{\lambda_1(1+k_4)T^n\lambda_2(1+\tilde{k}_2)T^m}{n!m!}\right] \\
\times \left[ s_1 \Phi(d_{3,n,m}) - s_2 e^{-r_{n,m}T}\Phi(d_{4,n,m}) \right],
\]  

(32)

where

\[
d_{3,n,m} = \frac{\ln \frac{s_1}{s_2} + \left(r_{n,m} + \frac{\sigma_{n,m}^2}{2}\right)T}{\sqrt{\sigma_{n,m}^2 T}},
\]
and

\[ d_{4,n,m} = \frac{\ln s_1 + \left( r_{n,m} - \frac{\sigma_{n,m}^2}{2} \right) T}{\sqrt{\sigma_{n,m}^2 T}}; \]

where

\[ r_{n,m} = -\lambda_1 k_4 - \lambda_2 \tilde{k}_2 + \frac{n\alpha_4}{T} - \frac{m\alpha_2}{T} + \frac{n\delta^2_4}{2T} + \frac{m\delta^2_2}{2T}, \]

and

\[ \sigma_{n,m}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 + \frac{n\delta_4^2}{T} + \frac{m\delta_2^2}{T} \]

and \( \tilde{k}_2 = \mathbb{E}_P[e^{-J_2} - 1] \); where

\[ \alpha_4 = \alpha_1 - \alpha_3 \]

and

\[ \delta_4 = \sqrt{\delta_1^2 + \delta_3^2 - 2\rho\delta_1\delta_3}. \]

**Proof.** Using (13), the price of the option at time \( t \) is

\[ C(t, S_1(t), S_2(t)) = S_2(t)\mathbb{E}_Q \left[ \frac{(S_1(T) - S_2(T))^+}{S_2(T)} \bigg| \mathcal{F}_t \right] \]

\[ = S_2(t)\mathbb{E}_Q \left[ \left( \frac{S_1(T)}{S_2(T)} - 1 \right)^+ \bigg| \mathcal{F}_t \right], \]

for an equivalent martingale measure under which both discounted processes \( \{M(t)/S_2(t)\} \) and \( \{S_1(t)/S_2(t)\} \) are (local) \( Q \)-martingales. In this case, they are also \( Q \)-martingales even after the jump components are averaged out. Thus

\[ d \left( \frac{M(t)}{S_2(t)} \right) = \left( \frac{M(t)}{S_2(t-)} \right) [(r - \mu_2 + \lambda_2 k_2 + \lambda_3 k_3 + \sigma_2^2)dt - \sigma_2 dW_2(t)] \]
\[(34) \quad + \left( \frac{M(t)}{S_2(t^-)} \right) \left( [e^{-J_2} - 1]dN_2(t) + [e^{-J_3} - 1]dN_1(t) \right) \]

and
\[
(35) \quad d \left( \frac{S_1(t)}{S_2(t)} \right) = \left( \frac{S_1(t^-)}{S_2(t^-)} \right) \left( \mu_1 - \mu_2 + \sigma_2^2 - \rho \sigma_1 \sigma_2 - \lambda_1 (k_1 - k_3) + \lambda_2 k_2 \right) \, dt \\
+ \left( \frac{S_1(t^-)}{S_2(t^-)} \right) \left( \sigma_1 dW_1(t) - \sigma_2 dW_2(t) \right) \\
+ \left( \frac{S_1(t^-)}{S_2(t^-)} \right) \left( [e^{J_4} - 1]dN_1(t) + [e^{-J_2} - 1]dN_2(t) \right).
\]

The jump size $J_4$ is normally distributed as $N(\alpha_4, \delta_4^2)$ where $\alpha_4 = \alpha_1 - \alpha_3$ and $\delta_4 = \sqrt{\delta_1^2 + \delta_3^2 - 2\rho \delta_1 \delta_3}$. Following Merton’s (1976) argument that jumps in stock prices are due to unsystematic risks, the distribution of the jumps components should not change in the risk-neutral measure (where $\{M(t)\}$ is the numéraire process) and by extension, also should not change in the the equivalent martingale measure $Q$ where $\{S_2(t)\}$ is the numéraire process.

By setting
\[
\theta_1 = \frac{\mu_1 - r - \lambda_1 (k_1 - k_4 + \tilde{k}_3)}{\sigma_1} - \rho \sigma_2
\]
and
\[
\theta_2 = \frac{\mu_2 - r - \lambda_1 (k_3 + \tilde{k}_3) - \lambda_2 (k_2 + \tilde{k}_2)}{\sigma_2} - \sigma_2,
\]
where $\tilde{k}_2 = E_P[e^{-J_2} - 1]$ and $\tilde{k}_3 = E_P[e^{-J_3} - 1]$, and setting
\[
(36) \quad d\tilde{W}_i(t) = \theta_i \, dt + dW_i(t), \text{ for } i = 1, 2,
\]
the stochastic differentials (34) and (35) become respectively
\[
d \left( \frac{M(t)}{S_2(t)} \right) = \left( \frac{M(t)}{S_2(t^-)} \right) \left( [e^{-\lambda_2 \tilde{k}_2} - \lambda_1 \tilde{k}_3 + \sigma_2^2] \, dt - \sigma_2 d\tilde{W}_2(t) \right)
\]
The correlated Brownian motion components \( \{ \tilde{W}_1(t) \} \) and \( \{ \tilde{W}_2(t) \} \) are each correlated standard Brownian motion components with \( d\tilde{W}_1(t)d\tilde{W}_2(t) = \rho dt \).

The stochastic differential (38) shows that in the equivalent martingale measure \( Q \), the process \( \{ \frac{S_1(t)}{S_2(t)} \} \) behaves like a stock price process with two independent jump components (see (24)), and the term \( E_Q \left[ \left( \frac{S_1(T)}{S_2(T)} \right)^+ \bigg| \mathcal{F}_t \right] \) is analogous to pricing a call option on such a stock with strike price one, with risk-free rate \( r = 0 \) and combined volatility of the continuous part \( \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \). Application of Theorem 1 completes the proof. \( \square \)

We consider another scenario which is a special case of Theorem 2. In this case the two stocks are only correlated through the Brownian motion components in the respective continuous parts. The jumps are due to unsystematic risk and are assumed to be uncorrelated. The following Corollary to Theorem 2 prices an exchange option in this scenario.
Corollary 1. Let the stock price processes \( \{S_1(t)\} \) and \( \{S_2(t)\} \) satisfy

\[
\begin{align*}
\text{(39)} & \quad dS_1(t) = (\mu_1 \lambda k_1)S_1(t-)dt + \sigma_1S_1(t-)dW_1(t) \\
& \quad + [e^{J_1(t)} - 1]S_1(t-)dN_1(t),
\end{align*}
\]

where the marginal distribution of \( J_{1,n} \) are independently and identically distributed as \( N(\alpha_1, \delta_2^2) \) and \( k_1 = E[e^{J_1} - 1] \); and

\[
\begin{align*}
\text{(40)} & \quad dS_2(t) = (\mu_2 \lambda k_2)S_2(t-)dt + \sigma_2S_2(t-)dW_2(t) \\
& \quad + [e^{J_2(t)} - 1]S_2(t-)dN_2(t),
\end{align*}
\]

where the marginal distribution of \( J_{2,n} \) are independently and identically distributed as \( N(\alpha_2, \delta_2^2) \) and \( k_2 = E[e^{J_2} - 1] \), respectively. The jump arrivals and the jump sizes in (39) and (40) are assumed to be independent and the instantaneous correlation between the correlated standard Brownian motion components is \( dW_1(t).dW_2(t) = \rho dt \). Then the options price \( C(0, s_1, s_2) \) at time \( t = 0 \), when \( S_1(0) = s_1 \) and \( S_2(0) = s_2 \), is

\[
\begin{align*}
C(0, s_1, s_2) = & \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-[\lambda_1(1+k_1)+\lambda_2(1+\tilde{k}_2)]T}[\lambda_1(1+k_1)T]^n[\lambda_2(1+\tilde{k}_2)T]^m \\
& \quad \times \frac{n!m!}{\sqrt{\sigma_1^2 T} \sqrt{\sigma_2^2 T}} \\
& \quad \times \left[ s_1 \Phi(d_{5,n,m}) - s_2 e^{-r_{n,m}T} \Phi(d_{6,n,m}) \right],
\end{align*}
\]

where

\[
\begin{align*}
\text{d}_{5,n,m} = & \quad \frac{\ln \frac{s_1}{s_2} + \left( r_{n,m} + \frac{\sigma_{n,m}^2}{2} \right) T}{\sqrt{\sigma_{n,m}^2 T}}, \\
\text{d}_{6,n,m} = & \quad \frac{\ln \frac{s_1}{s_2} + \left( r_{n,m} - \frac{\sigma_{n,m}^2}{2} \right) T}{\sqrt{\sigma_{n,m}^2 T}};
\end{align*}
\]
where
\[ r_{n,m} = -\lambda_1 k_1 - \lambda_2 \tilde{k}_2 + \frac{n\alpha_1}{T} - \frac{m\alpha_2}{T} + \frac{n\delta_1^2}{2T} + \frac{m\delta_2^2}{2T}, \]
and
\[ \sigma_{n,m}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 + \frac{n\delta_1^2}{T} + \frac{m\delta_2^2}{T}, \]
with \( \tilde{k}_2 = E_P[e^{-J_2} - 1]. \)

The proof of Corollary 1 is immediate from Theorem 2. In both scenarios, the option price at time \( t \) when \( S_1(t) = s_1 \) and \( S_2(t) = s_2 \) can be obtain by setting \( T \) to be the forward time \( T - t \) in the prices (32) and (41) respectively. Both pricing formulae (32) and (41) are also applicable to American style exchange option under each respective scenario since such an American style option is optimally exercised at maturity. Similar arguments made in Margrabe (1978) for the extension to American style exchange options are valid here.

6 Conclusion

We price exchange options between two stocks with discontinuous prices. The marginal distributions of both stocks are mixtures of log-normal distribution since the returns jump sizes are assumed to be normally distributed. In the two broad cases considered, one case can be applied to an outperformance option of one stock against some portfolio index that contains the stock. The
other case assumes that both stocks have uncorrelated unsystematic risks. Both cases can be considered extensions of Merton’s (1976) jump-diffusion model for option pricing and Margrabe’s (1978) model for exchange options.

References


