Option Pricing for Pure Jump Processes with Markov Switching Compensators.

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Abstract

This paper proposes a model for asset prices which is the exponential of a pure jump process with an \( N \)-state Markov switching compensator. We argue that such a process has a good chance of capturing all the empirical stylized regularities of stock price dynamics and we provide a closed form representation of its characteristic function. We also provide a parsimonious representation of the (not necessarily unique) risk neutral density and show how to price and hedge a large class of options on assets whose prices follow this process.

1 Introduction

Since the work of Mandelbrot and Taylor (1967) and Clark (1973) it has been recognized that the dynamics of asset returns cannot be adequately described by geometric Brownian motion with constant drift and volatility. Certain empirical features such as fat tails and the leverage effect\(^1\) imply that for any model of asset returns to be considered adequate, it must not only have a location and scale (volatility) parameter, but must also be able to capture tail decays and asymmetric tail patterns. Other empirical regularities such as volatility clustering and aggregational Gaussianity,\(^2\) imply that a suitable model for asset returns must also be able to capture time varying volatility and non-flat term structures of moments. Furthermore, these moment term structures, in particular the volatility term structure and the kurtosis term structure, display stylized patterns that imply a model which must allow for time inhomogeneity.

On the other hand, due to the economic argument of market efficiency, it is reasonable to model asset returns as discounted martingales with independent increments. Thus, a popular approach is to use drift-adjusted Lévy processes which are a fairly large class of continuous time processes with stationary independent increments, a class that includes

\(^1\)The unconditional distribution of asset returns show heavier tails than that of the normal distribution, implying greater probability of extreme events. Also the different behaviour of the two tails leads to a gain/loss asymmetry.

\(^2\)Volatility clustering is the observed tendency of high volatility periods to be followed by more high volatility periods, while aggregational Gaussianity is the observation that at short time scales, the distribution of returns is highly non-normal with fat tails, while at longer time scales the distribution tends to look more and more normal.
Brownian motion, jump diffusions and the Variance Gamma process. However, as Konikov and Madan (2001) have shown, for time-homogeneous Lévy processes, the theoretical behaviour of the term structure of the moments does not match empirical observations. For example, the variance theoretically increases with a factor $t$ (the length of the holding period), skewness decreases with a factor $t$, while kurtosis decreases with a factor $\sqrt{t}$. Empirically, however, these moments do not show patterns of growth or decay that are even close to these factors. Given all this, in order to allow for time-inhomogeneity, there has developed an interest in modelling asset returns using switching processes. For example, Buffington and Elliott (2001) look at a two-state switching process where the underlying processes are geometric Brownian motions. Konikov and Madan (2001) consider a two-state Markov chain where the underlying state processes are Variance Gamma processes as that of Madan, Carr, and Chang (1998). The empirical results of Konikov and Madan suggest that more than two states should be considered, but the mathematical approach they use cannot accommodate more than two states.

The main contribution of this paper is the extension of the work of Konikov and Madan to more than two states, an extension which is supported by the data and is thus more than just a theoretical exercise. Therefore, in this paper, we generalize their model and introduce an $N$-state Markov switching model where the underlying process is a pure jump Lévy process. Although the model we use can be extended to a general Lévy process, recent studies indicate that asset returns can be modelled as pure jump processes. Geman, Madan and Yor (1998) provide a theoretical argument that asset prices follow time changed, or subordinated, Brownian motion with local uncertainty, resulting in a pure jump process. Carr, Geman, Madan and Yor (2001) demonstrate empirically that a diffusion component is unnecessary if the pure jump process allows for infinite activity around the origin. Ait-Sahalia (2002) uses a condition on the transition function to test whether stock prices are diffusion processes. This condition, which he labels a “diffusion criterion”, essentially says that if the difference between two sample paths of a diffusion process starts out positive and ends up positive, it must always be positive in the interim. Using data from European options on the S&P 500 index, he finds that the underlying process cannot be a diffusion. We, therefore, feel justified in ignoring the diffusion component of the Lévy process and focusing only on the pure jump component. We note that for a pure jump process, the statistical parameters of the process and the risk neutral parameters will differ. Therefore, we provide a representation of an equivalent martingale measure which allows us to link the statistical and risk neutral parameters in terms of a few additional transformation parameters. Obviously, our equivalent martingale representation is not unique but it is parsimonious and represents a first order approximation for any such measure. The number of additional transformation parameters is equal to the number of switching states and they are required to remain stable across options of differing strikes and differing maturities if the representation is to be considered adequate.

Section 2 below of this paper describes the pure jump process that serves as the building block for our model. Section 3 introduces our stock price process which is a pure jump exponential. Of interest here is the compensator for the jump process which captures its statistical properties; we leave the actual specification of the compensator to section 5. As the probability density function will, in general, not have a closed form expression, in section 4 we instead determine the characteristic function. In section 4, and all subsequent sections, we focus on the Variance Gamma process developed by Madan and Senata (1987). In section 5 we specify the compensator as a Markov switching process and provide a closed form representation for the characteristic function of a jump exponential price process with such a compensator. In section 6 we derive the empirical martingale representation of this process. Since the characteristic function contains the same information as the distribution function, in section 7 we give two examples of methods for obtaining empirical estimates of
the parameters of the model using the characteristic function, and in section 8 we provide
a valuation representation for a large class of derivatives on the stock, again using the
characteristic function. Section 9 obtains Delta and Risk Minimizing hedge ratios and we
conclude with Section 10.

2 Pure Jump Processes

2.1 Jump Process Dynamics

Consider a real valued pure jump process \( X = \{ X_t, t \geq 0 \} \) defined on a probability space
\((\Omega, \mathcal{F}, \mathbb{P})\). Clearly

\[
X_t = X_0 + \sum_{0 < s \leq t} (X_s - X_{s-})
\]

So if \( \gamma \) is the random measure which selects the random jump times and random jump
sizes \( x = X_s - X_{s-} \), we can write (see Jacod and Shiryaev (1987))

\[
X_t = X_0 + \int_0^t \int \gamma(dx, ds).
\]

(1)

Under the historical probability \( \mathbb{P} \), the statistical properties of \( X \) are determined by
its compensator or dual predictable projection. We suppose this measure has the form
\( \bar{k}(x) dx ds \), where \( \bar{k}(x) \geq 0 \). Then under \( \mathbb{P}_M = \mathbb{P} \)

\[
M_t = X_t - \int_0^t \int \bar{k}(x) dx ds = X_0 + \int_0^t \int \gamma(dx, ds) - \bar{k}(x) dx ds
\]

is a local martingale with respect to the filtration generated by \( X \). We note that compensator
is unique and must be predictable with respect to the filtration generated by \( X \). As mentioned in the introduction, we could easily extend the process \( X \) to include a
diffusion component. However, as demonstrated in Carr, Geman, Madan, and Yor, this is
unnecessary if we allow for infinite jump activity around the origin. This means that we
have infinitely many very small jumps which model the behavior of a diffusion component.
We allow for this pattern of jump activity by specifying that \( \int \bar{k}(x) dx = \infty \). We can
also allow for decreasing activity as we move away from the origin which means that as
the jump sizes become larger, their occurrence becomes rarer.

2.2 The Girsanov theorem for Jump Processes

The Girsanov theorem for jump processes is now described. Suppose \( h : \mathbb{R} \rightarrow [0, \infty) \) is
such that the process \( \Lambda \) is a martingale, where

\[
\Lambda_t = 1 + \int_0^t \Lambda_{s-} \left( \int_{-\infty}^{+\infty} (h(x) - 1) \gamma(dx, ds) - \bar{k}(x) dx ds \right)
\]

(3)

\[
= \exp \left\{ -\int_0^t \int_{-\infty}^{+\infty} (h(x) - 1) \bar{k}(x) dx ds + \int_0^t \int_{-\infty}^{+\infty} \log(h(x)) \gamma(dx, ds) \right\}.
\]

Write \( \{ \mathcal{F}_t \} \) for the right continuous complete filtration generated by \( X \) and define a new
probability measure \( \mathbb{P} \) by setting

\[
\frac{d\mathbb{P}}{d\mathbb{P}_{\mathcal{F}_t}} = \Lambda_t
\]
Then, for $M = \{M_t, t \geq 0\}$, we obtain from Elliott (1982) Chapter 13, that

$$M_t - \int_0^t \int_{-\infty}^{+\infty} x (h(x) - 1) \tilde{k}(x) \, dx \, ds$$

is a $P$-martingale, where

$$M_t - \int_0^t \int_{-\infty}^{+\infty} x (h(x) - 1) \tilde{k}(x) \, dx \, ds \quad (4)$$

$$= X_t - \int_0^t \int_{-\infty}^{+\infty} x \tilde{k}(x) \, dx \, ds - \int_0^t \int_{-\infty}^{+\infty} x (h(x) - 1) \tilde{k}(x) \, dx \, ds$$

$$= X_t - \int_0^t \int_{-\infty}^{+\infty} x h(x) \tilde{k}(x) \, dx \, ds$$

$$= X_0 + \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds) - \int_0^t \int_{-\infty}^{+\infty} x h(x) \tilde{k}(x) \, dx \, ds.$$

More generally, under $P$, the compensator of $\gamma(dx, ds)$ is $h(x) \tilde{k}(x) \, dx \, ds$.

3 Jump Process Exponentials and the Stock Price Process

Our model for a price process $S$ will be the exponential of a pure jump process $X$. Let $X_0 = 0$. Then, if $X = \{X_t, t \geq 0\}$ where

$$X_t = \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds)$$

we take $S = \{S_t, t \geq 0\}$ to be of the form

$$S_t = S_0 \exp(X_t).$$

We also suppose the bank pays, and charges, a risk free interest at the constant continuously compounded rate $r$, so 1 dollar at time 0 becomes $e^{rt}$ dollars at time $t$.

The differentiation rule we shall use has the form

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-}) \, dX_s + \sum_{0 < u \leq t} [F(X_s) - F(X_{s-}) - F'(X_{s-}) \, \Delta X_s],$$

$$= F(X_0) + \int_0^t \int_{-\infty}^{+\infty} (F(X_{s-} + x) - F(X_{s-})) \gamma(dx, ds).$$

Applied to $S_t = S_0 \exp(X_t) = F(X_t)$, this gives

$$S_t = S_0 + \int_0^t S_{s-} \left( \int_{-\infty}^{+\infty} (e^x - 1) \gamma(dx, ds) \right). \quad (6)$$

The appearance of the factor $(e^x - 1)$ motivates us to consider the process $Z_t$ defined as

$$Z_t = S_0 \exp\left(X_t - \int_0^t \int_{-\infty}^{+\infty} (e^x - 1) h(x) \tilde{k}(x) \, dx \, ds\right)$$

$$= S_0 \exp\left(\int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds) - \int_0^t \int_{-\infty}^{+\infty} (e^x - 1) h(x) \tilde{k}(x) \, dx \, ds\right).$$
Using the differentiation rule, we obtain

\[ Z_t = S_0 + \int_0^t Z_{s-} \left( \int_{-\infty}^{+\infty} (e^x - 1) \right) \gamma(dx, ds) - \int_0^t Z_{s-} \left( \int_{-\infty}^{+\infty} (e^x - 1) h(x) \tilde{k}(x) dx \right) ds \]

Consequently, under some integrability conditions, Z is a martingale under the measure P.

**Lemma 1** The discounted price process is a martingale under P if the function h(·) satisfies

\[ r = \int_{-\infty}^{\infty} (e^x - 1) h(x) \tilde{k}(x) dx. \]

**Proof:** Taking the price process \( S_t = \exp(X_t) \) where \( X_t = X_0 + \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds) \), the discounted price process is

\[ e^{-rt} S_t = \exp(X_t - rt) \]

We have noted above that

\[ \exp \left( X_t - \int_0^t \int_{-\infty}^{\infty} (e^x - 1) h(x) \tilde{k}(x) dx ds \right) \]

is a P-martingale, so the result is immediate.

We shall refer to P as the risk neutral measure and \( Z_t \) as the discounted stock price process under the risk neutral measure. The function h(·) thus relates the historical process \( S_t \) to the risk neutral process \( Z_t \).

### 4 Characteristic Functions

In general, it is not possible to obtain a closed form expression for the density function of \( X \). However, we show in this section that we can still derive an essentially closed form for its characteristic function.

Suppose, as in Section 2, \( X \) is a pure jump process: \( X_t = X_0 + \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds) \), and that \( \gamma(dx, ds) \) has a compensator \( \tilde{k}(x) dxds \) under \( P \).

Applying the differentiation rule to \( e^{iuX_t} \) gives

\[ e^{iuX_t} = e^{iuX_0} + \int_0^t \int_{-\infty}^{+\infty} e^{iuX_s-} (e^{iuX} - 1) \gamma(dx, ds) = e^{iuX_0} + \int_0^t \int_{-\infty}^{+\infty} e^{iuX_s-} (e^{iuX} - 1) (\gamma(dx, ds) - \tilde{k}(x) dx, ds) + \int_0^t e^{iuX_s} \left( \int_{-\infty}^{+\infty} (e^{iuX} - 1) \tilde{k}(x) dx \right) ds \]
Under integrability conditions, the first integral above is a martingale. Taking expected values gives the characteristic function

\[ \Phi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] \]

\[ = e^{iuX_0} + \int_0^t \mathbb{E}\left[e^{iuX_t}\right] \left( \int_{-\infty}^\infty (e^{iuX} - 1) \tilde{k}(x) \, dx \right) \, ds \]

so that

\[ \Phi_{X_t}(u) = e^{iuX_0} \cdot \exp \left[ t \left( \int_{-\infty}^\infty (e^{iuX} - 1) \tilde{k}(x) \, dx \right) \right] \] (8)

The unit characteristic function is the value when \( t = 1 \) and \( X_0 = 0 \).

Examples include the normal inverse Gaussian process of Barndorff-Nielsen (1998) for which \( \Phi_{X_1}(u) \) is expressed in terms of parameters \( \theta, \delta \) and \( \phi \) as

\[ \Phi_{X_1}(u) = e^{iuX_0} \cdot \exp \left\{ -\delta \left[ \sqrt{\theta^2 + \phi^2 - (\theta + iu)^2} - \phi \right] \right\}, \]

and the variance gamma (VG) process for which \( \Phi_{X_t}(u) \) is in terms of parameters \( \theta, \sigma \) and \( \nu \) as

\[ \Phi_{X_t}(u) = E\left[e^{iuX_t}\right] = \left( 1 - i\theta\nu u + \frac{\sigma^2\nu}{2} u^2 \right)^{-t/\nu}. \]

For the variance gamma process

\[ \tilde{k}(x) = \frac{1}{\nu|x|} \exp \left( -\left( \xi x + \frac{\sqrt{2}}{\lambda\sqrt{\nu}|x|} \right) \right) \]

where

\[ \xi = -\frac{\theta}{\sigma^2}, \]

\[ \lambda = \frac{\sigma}{\sqrt{1 + \left( \frac{\sigma}{\theta} \right)^2} \frac{\xi}{\nu}} \]

In this paper we shall employ the variance gamma process.

4.1 An Economic Interpretation: Subordinated Brownian Motion

An alternate construction of the VG process is to consider the process

\[ Z = \{ Z(t), t \geq 0 \}, \text{ where } Z(t) = \theta t + \sigma B_t \]

Here \( B = \{ B_t, t \geq 0 \} \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Suppose \( L(t, \nu) \) is a Gamma process with unit mean rate and variance \( \nu \). Consider the time changed, or subordinated, version of \( Z \) given by \( X = \{ X_t, t \geq 0 \} \) where

\[ X_t = Z(L(t, \nu)). \]

Then \( X \) is a VG process with the characteristic function given above:

\[ \Phi_{X_t}(u) = E\left[e^{iuX_t}\right] = \left( 1 - i\theta\nu u + \frac{\sigma^2\nu}{2} u^2 \right)^{-t/\nu}. \] (9)
This provides a nice economic interpretation for the modeling of asset prices as VG processes. Since asset prices do not necessarily respond to the passage of calendar time, but rather to the arrival of relevant new information, if there is no new information relevant to the value of the asset we may expect that there will be no change in the price of the asset. If information rapidly arrives in the markets, then price changes occur much faster as expectations are updated more rapidly. Since, by definition, the arrival of new information itself must be a randomly occurring process, intuitively it makes sense to adjust the time scale from calendar time to some random information arrival time. This results in a time changed, or subordinated, stochastic process.\footnote{Further economic rationale for such time changes can be found in the papers of Geman, Madan, and Yor (1998) or Ghysels, Gourieroux and Jasiak (1995). Thus, as expressed above, the VG process has an expected information arrival time of one, while the parameters $\theta$, and $\nu$ control for the (conditional) skewness and excess kurtosis.}

### 5 Switching Pure Jump Lévy Process

Recall that empirical studies suggest that the dynamics of stock prices are not time-homogenous. To capture this, we allow for the compensator measure - and hence the statistical properties - to switch between a finite set of related measures. Suppose $U = \{U_t, t \geq 0\}$ is a Markov chain, independent of the pure jump process, and with a state space

$$\{e_1, e_2, \ldots e_N\}$$

where $e_i = (0, \ldots, 1, 0, \ldots) \in \mathbb{R}^N$.

Suppose the generator, or Q-matrix of $U$ is $\Pi = \{\Pi_{ij}\}, 1 \leq i, j \leq N$. Then, from Elliott, Aggoun and Moore (1994), we have that

$$U_t = U_0 + \int_0^t \Pi U_s ds + M_t$$

where $M = \{M_t, t \geq 0\}$ is an $\mathbb{R}^N$-valued martingale with respect to the filtration generated by $U$.

Suppose for each $j, 1 \leq j \leq N$, we have a compensator measure $k_j(x) dx dt$ where

$$k_j(x) = \frac{1}{\nu_j |x|} \exp \left( - \frac{\sqrt{2}}{\lambda_j \nu_j} |x| \right).$$

Here, each $k_j(x)$ is associated with a process $\theta_j t + \sigma_j B_t$ time changed by a Gamma process $L(t, \nu_j)$,

$$\xi_j = -\frac{\theta_j}{\sigma_j^2} \quad \text{and} \quad \lambda_j = \frac{\sigma_j}{\sqrt{1 + \left( \frac{\theta_j}{\sigma_j} \right)^2 \frac{\nu_j}{2}}}$$

That is, each $k_j(x), 1 \leq j \leq N$, is the Lévy kernel of a VG process. Recall that the jump process $X$ has the representation, with $X_0 = 0$,

$$X_t = \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds)$$

\footnote{The procedure is sometimes called time deformation.}
Suppose that, under the historical measure $\tilde{P}$, the compensator of $\gamma$ is $\nu$, where

$$\nu(dx, ds) = \sum_{j=1}^{N} \langle U_{s-}, e_j \rangle k_j(x) \, dx ds$$  \hspace{1cm} (11)$$

Note that, unlike Konikov and Madan (2001), we consider only one jump process $X$; it is the compensators which switch according to the state of the Markov chain $U$. For our jump process $X$ with a compensator as given in equation (11) above, we have

$$e^{iuX_t} = 1 + \int_0^t \int_{-\infty}^{+\infty} e^{iuX_s} - (e^{iuX} - 1) \gamma(dx, ds)$$

$$= 1 + \int_0^t e^{iuX_s} - \left( \int_{-\infty}^{+\infty} (e^{iuX} - 1) \left( \gamma(dx, ds) - \sum_{j=1}^{N} \langle U_{s-}, e_j \rangle k_j(x) \, dx ds \right) \right)$$

$$+ \int_0^t e^{iuX_s} - \sum_{j=1}^{N} \langle U_{s-}, e_j \rangle \int_{-\infty}^{+\infty} k_j(x) \, dx ds$$  \hspace{1cm} (12)$$

Write $\{\mathcal{F}_t^U\}$ for the filtration generated by $U$ and

$$\lambda(u, U, s) = \mathbb{E} \left[ e^{iuX_s} \mid \mathcal{F}_t^U \right].$$

Now $U$ is independent of $X$, so conditioning on $\mathcal{F}_t^U$ and noting that the first integral in equation (12) is a martingale, we have that

$$\lambda(u, U, t) = 1 + \int_0^t \lambda(u, U, s) \int_{-\infty}^{+\infty} (e^{iuX} - 1) \left( \sum_{j=1}^{N} \langle U_{s-}, e_j \rangle k_j(x) \, dx ds \right).$$

Therefore

$$\lambda(u, U, t) = \exp \left( \sum_{j=1}^{N} \int_0^t \langle U_{s-}, e_j \rangle \int_{-\infty}^{+\infty} (e^{iuX} - 1) k_j(x) \, dx ds \right).$$

Write $J_t^j = \int_0^t \langle U_s, e_j \rangle \, ds$ for the amount of time the process $U$ has spent in state $j$ up to time $t$. That is, $J_t^j$ is the occupation time spent in state $j$ up to time time $t$. Also, write

$$\phi_j(u) = \int_{-\infty}^{+\infty} (e^{iuX} - 1) k_j(x) \, dx = \log \Phi_{X_t}(u) = -\frac{1}{v_j} \log \left( 1 + i \theta_j v_j u + \frac{\sigma_j^2 v_j}{2} u^2 \right)$$  \hspace{1cm} (13)$$

for the unit log-characteristic function of the jump process with compensator measure $k_j(x) \, dx ds$. Then,

$$\lambda(u, U, t) = \exp \left( J_t^1 \phi_1(u) + J_t^2 \phi_2(u) + \ldots + J_t^N \phi_N(u) \right).$$

The characteristic function of the pure jump process $X$ is, therefore,

$$\Phi_{X_t}(u) = \mathbb{E} \left[ e^{iuX_t} \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{iuX_t} \mid \mathcal{F}_t^U \right] \right]$$

$$= \mathbb{E} \left[ \lambda(u, U, t) \right] = \Phi_{J_t}(\phi_1(u), \phi_2(u), \ldots, \phi_N(u))$$  \hspace{1cm} (14)$$
where
\[
\Phi_{J(t)}(\lambda) = \Phi_{J(t)}(\lambda^1, \lambda^2, ..., \lambda^N) = E \left[ \exp \left( \lambda^1 J^1_t + \lambda^2 J^2_t + ... + \lambda^N J^N_t \right) \right]
\] (15)
is the Laplace transform of the vector of occupation times \(J_t = (J^1_t, J^2_t, ..., J^N_t)\).

It remains to obtain a closed form expression for \(\Phi_{J(t)}(\lambda)\). The proposition below accomplishes this.

**Proposition 2** For the \(N\)-state Markov switching model, the Laplace transform of the occupation times \(J_t\) is given by:
\[
\Phi_{J(t)}(\lambda) = E \left[ \exp (\lambda, J_t) \right] = \exp \{ (\Pi + \text{diag}(\lambda)) t \} \cdot E[U_0],
\]
where \(\mathbf{1} \in \mathbb{R}^N\) is a vector of ones and \(\Pi\) is the \(Q\)-matrix of the Markov chain \(U\).

**Proof:** For a vector of transform variables \(\lambda = (\lambda^1, \lambda^2, ..., \lambda^N) \in \mathbb{R}^N\), the Laplace transform of \(J\) is:
\[
E \left[ \exp (\lambda, J_t) \right] = E \left[ \exp \left( \lambda^1 J^1_t + \lambda^2 J^2_t + ... + \lambda^N J^N_t \right) \right] = E \left[ \exp(\lambda^1 J^1_t) \exp(\lambda^2 J^2_t) ... \exp(\lambda^N J^N_t) \right].
\]
Define \(Y_t\) as the random vector process
\[
Y_t = \exp((\lambda, J_t)) \cdot U_t = \exp \left( \int_0^t \langle \lambda, U_s \rangle \, ds \right) \cdot U_t.
\]
Then \(Y_t \in \mathbb{R}^N\) and
\[
dY_t = \langle \lambda, U_t \rangle \, dY_t + \exp \left( \int_0^t \langle \lambda, U_s \rangle \, ds \right) \, dU_t.
\]
Recall that \(U_t = U_0 + \int_0^t \Pi U_s \, ds + M_t\) and so we can substitute to get
\[
dY_t = \langle \lambda, U_t \rangle \, Y_t \, dt + \Pi Y_t \, dt + \exp \left( \int_0^t \langle \lambda, U_s \rangle \, ds \right) \, dM_t.
\]
Now, \(\langle \lambda, U_t \rangle \, Y_t = \text{diag}(\lambda) Y_t\), so
\[
Y_t = U_0 + \int_0^t (\Pi + \text{diag}(\lambda)) \, Y_s \, ds + \int_0^t \exp \left( \int_0^s \langle \lambda, U_v \rangle \, dv \right) \, dM_s.
\]
The last integral in this expression is a martingale, and taking the expectation, we have
\[
E[Y_t] = E[U_0] + \int_0^t (\Pi + \text{diag}(\lambda)) \, E[Y_s] \, ds
\]
Solving gives
\[
E[Y_t] = \exp \{(\Pi + \text{diag}(\lambda)) \, t\} \, E[U_0].
\]
Let a vector of ones in $\mathbb{R}^N$ be denoted as $\mathbb{1}$. We observe that $E \{\exp (\langle \lambda, J_t \rangle)\} = E \{\exp (\langle \lambda, J_t \rangle) U_t, \mathbb{1}\}$ and denoting $\Phi_{J(t)}(\lambda) = E \{\exp (\langle \lambda, J_t \rangle)\}$ we obtain that

$$\Phi_{J(t)}(\lambda) = \langle E [Y_t], \mathbb{1} \rangle$$

$$= \langle E [\exp (\langle \lambda, J_t \rangle) U_t], \mathbb{1} \rangle$$

$$= \langle \exp \{(\Pi + \text{diag}(\lambda)) t\} E [U_0], \mathbb{1}\rangle$$

$$= \langle E [U_0], \exp \{(\Pi + \text{diag}(\lambda)) t\} \mathbb{1}\rangle$$

which is the desired result and, in principle, is a closed form solution.

Proposition 2 holds for any finite number of states and, although we have focused on the VG process, proposition 2 also applies as a closed form solution to any process has a closed form representation for $k_j(x)$, the compensators associated with each state.

In fact, from state to state, these compensators need not even have the same functional form. Thus, it can be argued that by altering the $k_j(x)$’s and the number of states, the pure jump switching compensator process can be made to accommodate all empirically observed regularities.

6 Measure Change for the Switching Process

We suppose that the jump process $X_t = \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds)$ has, under the historical probability $\bar{P}$, the compensator $\nu$ where

$$\nu(dx, ds) = \sum_{j=1}^N \langle U_{s-}, e_j \rangle k_j(x) dx ds.$$

Consider the exponential process $\Lambda$ where, for positive functions $h_j$

$$\Lambda_t = 1 + \int_0^t \Lambda_{s-} \left( \sum_{j=1}^N \int_{-\infty}^{+\infty} \langle U_{s-}, e_j \rangle (h_j(x) - 1) (\gamma(dx, ds) k_j(x) dx ds) \right)$$

$$= \exp \left( - \int_0^t \sum_{j=1}^N \langle U_{s-}, e_j \rangle \int_{-\infty}^{+\infty} (h_j(x) - 1) k_j(x) dx ds \right.$$  

$$\left. + \int_0^t \sum_{j=1}^N \langle U_{s-}, e_j \rangle \int_{-\infty}^{+\infty} \log h_j(x) \gamma(dx, ds) \right).$$

Suppose, $\Lambda$ is a martingale. A new probability $P$ can be defined by setting

$$\frac{dP}{d\bar{P}} \bigg|_{\mathcal{A}_t} = \Lambda_t.$$  

Then, Girsanov’s theorem states that under $P$ the process $M$ is a martingale, where

$$M_t = X_t - \sum_{j=1}^N \int_0^t \langle U_{s-}, e_j \rangle \left( \int_{-\infty}^{+\infty} x h_j(x) k_j(x) dx \right) ds.$$  

We take the model for the asset price to be

$$S_t = S_0 \exp X_t = S_0 \exp \left( \int_0^t \int_{-\infty}^{+\infty} x \gamma(dx, ds) \right).$$
Consequently, from Lemma 1, the discounted asset price \( e^{-rt} S_t = S_0 \exp (X_t - rt) \) is a martingale under the measure \( P \) if

\[
rt = \sum_{j=1}^{N} \int_{0}^{\infty} \int_{-\infty}^{+\infty} \langle U_{s-}, e_j \rangle (e^x - 1) h_j (x) \overline{k}_j (x) \, dx \, ds.
\]

This is certainly the case if

\[
\int_{-\infty}^{\infty} (e^x - 1) h_j (x) \overline{k}_j (x) \, dx = r \quad \text{for each } j, 1 \leq j \leq N.
\]

(16)

Note there are many functions \( h_j (\cdot) \) which satisfy these conditions. In the proposition below, we consider a particular parsimonious functional form.

**Proposition 3** Suppose \( h_j (x) = e^{\alpha_j x} \) where \( \alpha_j \in \mathbb{R}^+ \), \( \phi_j (u) \) is the unit log-characteristic function given in (13), and the risk free rate is given by \( r \). If for each \( j, 1 \leq j \leq N, \alpha_j \) satisfies:

\[
r = \phi_j (-\alpha_j i) - \phi_j (-\alpha_j + 1)i
\]

then the discounted price \( e^{-rt} S_t \) is a martingale under the measure \( P \).

**Proof:** Substituting for \( h_j (x) = e^{\alpha_j x} \) gives

\[
(e^x - 1) h_j (x) = (e^x - 1) e^{\alpha_j x} = (e^{(\alpha_j + 1)x} - 1) - (e^{\alpha_j x} - 1).
\]

Now, note that

\[
\int_{\mathbb{R}} (e^{\alpha_j x} - 1) \overline{k}_j (dx) dx = \phi_j (-\alpha_j i)
\]

and

\[
\int_{\mathbb{R}} (e^{(\alpha_j + 1)x} - 1) \overline{k}_j (dx) dx = \phi_j (-\alpha_j + 1)i.
\]

Substituting into equation 14 gives the desired result.

We can interpret \( \alpha_j \) as the market price of the jump risk in state \( j \) and the probability measure \( P \) as the risk neutral measure that we will use for derivative pricing. We note that in defining \( h_j (x) \) as \( e^{\alpha_j x} \) we not only obtain a parsimonious representation of the risk neutral density, but we also have a representation that is a first order approximation of any other risk neutral density. With this in mind, proposition 3 then implies that under \( P \), the compensator for the jump process \( X_t \) is \( \sum_{j=1}^{N} (U_{s-}, e_j) k_j (x) \, dx \, ds \), where \( k_j (x) = e^{\alpha_j x} \overline{k}_j \).

Denote the risk neutral unit log-characteristic function of the Lévy process in state \( j \) as \( \phi_j^P (u) \). Then, we have that

\[
\phi_j^P (u) = \int_{-\infty}^{+\infty} (e^{iu} - 1) k_j (x) \, dx
\]

= \[
\int_{-\infty}^{+\infty} (e^{iu} - 1) e^{\alpha_j x} \overline{k}_j (x) \, dx
\]

= \[
\int_{-\infty}^{+\infty} (e^{(u-i\alpha_j)x} - 1) \overline{k}_j (x) \, dx - \int_{-\infty}^{+\infty} (e^{\alpha_j x} - 1) \overline{k}_j (x) \, dx
\]

= \[
\phi_j (u - i\alpha_j) - \phi_j (-i\alpha_j)
\]

11
and \( \Phi_X(t) = \Phi_{J(t)}(\phi_1^P(u), \phi_2^P(u), ..., \phi_N^P(u)) \)

The characteristic function is a powerful tool and, as we subsequently show, can be used both for estimation of the parameters of the stock price process and for the pricing of options.

7 Parameter Estimation

In this section we present two possible procedures by which the parameters of the stock price process can be estimated using the \( \Phi \) characteristic function \( \Phi_X(z) \) given in equation (13) of Section 5. These methods assume that stock prices are in a stationary state and thus avoid the use of filtering. As mentioned previously, the parameters obtained this way will be the statistical parameters which, in general, differ from the risk neutral parameters. Here, we do not attempt to determine which method is superior or how many states are sufficient, and we defer a comprehensive econometric analysis to a later study.

7.1 Maximum Likelihood Estimation

The characteristic function \( \Phi_X(z) \) can be inverted using the inverse Fourier transform to obtain the density of the returns. Assuming that the individual state processes are VG processes, a maximum-likelihood estimation (MLE) of the parameters of the stock price can then be carried out. In principle, this requires that we invert the characteristic function at every return point for each pass of the MLE optimization routine. As this is computationally expensive, we instead suggest the implementation of a Fast Fourier Transform (FFT) on binned sampled returns. In the appendix we describe this FFT and its application.

7.2 Generalized Method of Moments

An alternative estimation method which exploits the characteristic function directly is a generalized method of moments (GMM) estimation using the empirical characteristic function (ECF). Applications of this method can be found in Tran (1998), and Knight and Yu (2002).

To implement this GMM method, recall that the characteristic function, which depends on a set of parameters \( \psi \), is defined as

\[
\Phi_X(u, \psi) = \mathbb{E}[e^{iuX_t}] .
\]

The sample counterpart to this is the ECF which is defined as

\[
\hat{\Phi}_N(u) = \frac{1}{N} \sum_{j=1}^{N} e^{iuX_j}.
\]

Construct a grid \( u^q = (u_1, u_2, ..., u_q) \) of transform variables, which forms the basis for the moment conditions, and the let the \( k^{th} \) moment be

\[
h_k(u_k, X_j, \psi) = e^{iu_kX_j} - \Phi_X(u_k, \psi)
\]

so that \( \mathbb{E}[h(u_k, X_k, \psi)] = 0 \). The corresponding sample moment is

\[
h_N(u_k, \psi) = \hat{\Phi}_N(u_k) - \Phi_X(u_k, \psi)
\]
which is stacked to form a vector of sample moments $h_N(u^q, \psi)$. The GMM estimator for the parameter vector $\psi$ is found by minimizing the quadratic function $h_N(u^q, \psi)^T W h_N(u^q, \psi)$ where $W$ is an optimal weighting matrix.

7.3 Remark

As mentioned above, in order for the time series observations to represent draws from the same distribution, we assume that the stock prices are in a stationary state. This imposes a certain restriction on the initial probabilities and they can no longer be freely estimated. Recall that we denote the initial probability of being in state $i$ as $p^0_i = \Pr(U_0 = e_i)$ and $p_0 = (p^0_1, p^0_2, ..., p^0_K)$. Also recall that the transition rate matrix $\Pi$ solves

$$\frac{dp_i}{dt} = \Pi p_i$$

For the process to be stationary

$$\frac{dp_i}{dt} = 0.$$  

This is the case if the initial probability distribution $p_0$ satisfies

$$\Pi p_0 = 0.$$  

As an example, Table 1 gives some results from the MLE parameter estimation for S&P500 index using daily data from January 1995 to December 2000. As we can see, when the 2-state model is estimated we obtain parameters that allow us to interpret the two states as a high volatility state and a low volatility state and we typically get a stationary probability of 98% to 99% of being in the high volatility state. Moving to three states shows that the high volatility state can itself be decomposed into two states.

8 Option Pricing

Define the log stock price as $s_T = \ln(S_T)$ and suppose the strike price of an option is $K$. The valuation of options using characteristic functions generally proceeds by writing the option’s value in Fourier transform space. This technique has developed along two lines depending on which variable is transformed into Fourier space, the (log) stock price $s_T$ or the strike price $K$. That is, if we denote the value of an option that primarily depends on $s_T$ and $K$ as $V(s_T, K)$, then the Fourier Transform may be

$$\int_{-\infty}^{+\infty} e^{-isz} V(s, .) ds$$

or

$$\int_{-\infty}^{+\infty} e^{-izK} V(., K) dK.$$  

Lewis (2002) and Raible (2000) utilize the first transformation while Carr and Madan (1999) and Konikov and Madan (2000) utilize the second transformation. In this section we shall utilize the first transformation to provide a general representation for the values of a large class of options. Members of this class include the standard call and put options and are characterized by the Fourier integrability of their payoff functions. The approach used in this section is similar to that of Lewis (2002). We first extend the domain of the Fourier Transform to the complex numbers.
Definition: The generalized or complex Fourier transform of a function \( g(x) \) is denoted as \( \hat{g}(z) \) and is given by
\[
\hat{g}(z) = \int_{-\infty}^{+\infty} e^{izx} g(x) dx
\]
where \( z \) belongs to some subset of the complex numbers for which the integral converges.

8.1 Correlation Theorem

The correlation function between two suitably integrable functions \( g(.) \) and \( h(.) \) is
\[
c(z) = \int_{-\infty}^{+\infty} g(z + x)h(x) dx.
\]

When the functions \( g(.) \) and \( h(.) \) are the same function, then \( c(z) \) is termed the auto-correlation function. Note the close resemblance between the correlation and the more widely known convolution of the functions \( g(.) \) and \( h(.) \). In fact a theorem similar to the theorem of Fourier transforms for convolutions exists for correlations.

Theorem: The Fourier transform of a correlation integral is equal to the product of the Fourier transform of the first function and the complex conjugate of the Fourier transform of the second function.

Proof: For \( u \in C \), the generalized Fourier transform of \( c(z) \) is
\[
\hat{c}(u) = \int_{-\infty}^{+\infty} c(z) e^{iuz} dz
\]
\[
= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} g(z + x)h(x) dx \right) e^{iuz} dz
\]
\[
= \int_{-\infty}^{+\infty} h(x) \left( \int_{-\infty}^{+\infty} e^{iuz} g(z + x) dz \right) dx
\]
\[
= \int_{-\infty}^{+\infty} e^{-iux} h(x) \left( \int_{-\infty}^{+\infty} e^{iuy} g(y) dy \right) dx
\]
\[
= \hat{g}(u) \int_{-\infty}^{+\infty} e^{-iux} h(x) dx
\]

It is well known that \( \int_{-\infty}^{+\infty} e^{-iux} h(x) dx \) is the complex conjugate of \( \int_{-\infty}^{+\infty} e^{iux} h(x) dx \), and so the result is obtained.

Denote the payoff function of an option, written in terms of the log stock price, as \( w(s_T) \). It is well known that under an (not necessarily unique) equivalent martingale measure \( P \) the value of this option is given by
\[
V(s_0) = E^P[e^{-rT} w(s_T)].
\]
Recall that in our model, under the risk neutral probability $P$, $s_T = \ln S_t = \ln(S_0) + X_T$ where the pure jump process $X_T$ has a compensator $\sum_{j=1}^{N} \langle U_{j-}, e_j \rangle k_j(x) \, dx \, ds$. Here $k_j(x) = e^{\alpha_j x} \bar{k}_j$. Denoting $Y = s_0 = \ln(S_0)$ and substituting for $s_T$ gives

$$V(Y) = \int_{-\infty}^{+\infty} e^{-rT} w(Y + X) f^P(X) \, dX$$

where $f^P(X)$ is the risk neutral density of $X$.

That is, the value of the option is the correlation between the option’s payoff and the risk-neutral density of the Markov switching pure jump process discounted by the risk free rate.

We now appreciate the importance of the risk-neutral characteristic function for derivative valuation. As we have seen in Section 3, the density of $X_t$ will, in general, not have a closed form; however, its characteristic function does. Since the characteristic function is the Fourier transform of the density, we use the relationship between a correlation function and its Fourier transform to obtain an expression for the option’s value.

The generalized (complex) Fourier transform of the option’s payoff $w(x)$ is denoted as $\hat{w}(z)$ and is given by

$$\hat{w}(z) = \int_{-\infty}^{+\infty} e^{ixz} w(x) \, dx$$

where $z$ belongs to a subset of the complex numbers for which the integral converges. In fact we make the following:

**Assumption:** There exists a strip in the complex plane, parallel to the real axis such that $\hat{w}(z)$ exists if $\text{Im}(z)$ is restricted to this strip.

Another way of stating the above assumption is to say that we restrict our discussion to options which have payoff functions that are *Fourier integrable in a strip*. For example, for a standard European call option with a strike price $K$, the payoff function is $w(x) = (e^x - K)^+$ and its Fourier transform is $\hat{w}(z) = -K^{i(z+1)/(z^2 - iz)}$ for $\text{Im}(z) > 0$. The reason for using the generalized transform is that, when real-valued transform variables are used, option payoffs will typically not have Fourier transforms. In Table 2 we give expressions for the Fourier transforms of European Call and Put Options, as well as for capped options and binary options.

Proposition 1 in Section 4 derived a closed form expression for the characteristic function of $X_T$ which we denoted as $\phi_{X_T}(z)$. In order to apply the correlation theorem, we extend the domain of the characteristic function by use of a complex-valued transform variable. We denote the conjugate of the resulting generalized characteristic function as $\phi_{X_T}^*(z)$. That is, $\phi_{X_T}^*(z) = \text{Re}(\phi_{X_T}(z)) - i \text{Im}(\phi_{X_T}(z))$. It is straightforward to show that Proposition 2 holds for the generalized transform variable and we can therefore use it to establish the following option pricing representation:
Proposition 4 Suppose the payoff of an option satisfies Assumption 1 for some strip $S = \{z \in C : a \leq \text{Im}(z) \leq b\}$. Then the value of the option is given by

$$V(S_0) = e^{-rT} \frac{1}{2\pi} \int_{i \text{Im}(z) - \infty}^{i \text{Im}(z) + \infty} e^{-iz\ln(S_0)} \frac{1}{S_t} \frac{\partial V}{\partial Y_t} \left( i \phi_{X_T}^*(z) \right) dz$$

where $\text{Im}(z) \in (a, b)$

Implementation of this option pricing formula will require application of the Fast Fourier Transform to the product $\hat{w}(z)\phi_{X_T}^*(z)$. In addition to pricing, Proposition 4 can be used to calibrate the risk neutral process from option prices and generate implied market prices of $\alpha_j$, the jump risks.

9 Hedge Ratios

Aside from the valuation of options, application of the model developed in this paper will require the determination of the appropriate hedging strategy. In this section we provide representations for the Delta hedge ratio and the Risk Minimizing hedge ratio in terms of the characteristic function of the (log) stock price and the Fourier transform of the option payoff. Of course, since we are in an incomplete market setting, the correct hedge ratio to use is the Risk Minimizing ratio, but we compare the two.

Delta Hedge Ratio

The delta hedge ratio at time $t$, $DHR_t$, is given as

$$DHR_t = \frac{\partial V}{\partial S_t} = \frac{1}{S_t} \frac{\partial V}{\partial Y_t} = \frac{1}{S_t} \frac{\partial V'}{\partial Y_t} = \frac{1}{S_t} E^P[e^{-r(T-t)}w'(Y_t + X_T)|\mathcal{F}_t]$$

where we recall from the previous section that $Y_t = \ln(S_t) \equiv s_t$. We again assume the existence of a strip in the complex plane, parallel to the real axis such that the Fourier transform of $w'(s_t)$, denoted as $\hat{w}'(z)$ exists if $\text{Im}(z)$ is restricted to this strip. Then, noting that $\hat{w}'(z) = iz\hat{w}(z)$, we have

$$DHR_t = \frac{e^{-r(T-t)}}{2\pi} \int_{i \text{Im}(z) - \infty}^{i \text{Im}(z) + \infty} iz e^{-iz\ln(S_t)} \frac{1}{S_t} \frac{\partial V}{\partial Y_t} \left( i \phi_{X_T}^*(z) \right) dz$$

Risk Minimizing Hedge Ratio

To determine the Risk Minimizing hedge ratio, $RMHR$, we consider a hedge portfolio $\Psi = \{\psi_t\}_{t \in [0, T]}$ where $\psi_t$ is predictable w.r.t $\mathfrak{F}$ and represents the amount of stock held in the hedge portfolio at time $t$. If we denote the value of the hedge portfolio as the $P$ martingale $F_t(S_t) = E^P[e^{-rT}w(S_T)|\mathfrak{F}_t]$, then we can define the cumulative cost process as

$$C_t(\Psi) = F_t - \int_0^t \psi_u dS_u$$

That is, $C_t(\Psi)$ captures all the cash infusions need to maintain the hedging strategy. If $C_t(\Phi)$ were constant for all $0 \leq t \leq T$ and equal to the initial cost of setting up the strategy $C_0(\Psi)$, then the hedge would be self-financing and riskless, and for $F_T(S_T) = V_T(S_T)$, the value of the option would be equal to $C_0(\Psi)$. When a riskless hedge cannot be found,
we focus on hedging strategies that are risk minimizing in the sense of risk defined as the expected quadratic hedging error.

**Lemma 5** The hedge portfolio $F_t(S_t)$ has the dynamics

$$F_t(S_t) = F_0(S_0) + \int_0^t \int_{-\infty}^{\infty} (F_t(e^x S_{s-}) - F_t(S_{s-})) \left( \gamma(dx, ds) - \nu^P(x)dxds \right)$$

where the compensator measure $\nu^P(.)$ is under the risk neutral probability $P$ and is given by

$$\nu(dx, ds) = \sum_{j=1}^N \langle U_{s-}, e_j \rangle e^{\alpha_j T_j} (x) dxds.$$

**Proof:** See Colwell and Elliott (6).

Following Schweizer (15) and Colwell and Elliott (6) we employ the risk minimizing condition which states that if $\Psi$ is a risk minimizing hedge, then it must be that the cost process $C_t(\Psi)$ is a $P$ martingale orthogonal to the martingale component of $S_t$, which we denote as $M_t$. This requirement means that the product $C_t(\Psi) \cdot M_t$ is a $P$ martingale. We impose a mean-self-financing condition on the hedge portfolio which requires that $C_t(\Psi)$ be a $P$ martingale and with these two requirements, the following proposition determines the $RMHR$.

**Proposition 6** The Risk Minimizing hedge ratio at time $t$ is

$$RMHR_t = \psi_t = \frac{E \left[ \int_{\mathbb{R}} (e^x - 1) (F_t(e^x S_t) - F_t(S_t)) v(x)dx \right]}{S_t E \left[ \int_{\mathbb{R}} (e^x - 1)^2 v(x)dx \right]}$$

where

$$E \left[ \int_{\mathbb{R}} (e^x - 1)^2 v(x)dx \right] = \sum_{j=1}^N (\phi_j(-2i) - 2\phi_j(-i)) p_i^j$$

$$E \left[ \int_{\mathbb{R}} (e^x - 1) F_t(S_{t-}) v(x)dx \right] = V_t(S_t) \sum_{j=1}^N \phi_j(-i)p_i^j.$$

and

$$p_i^j = \Pr(U_t = e_j)$$

The evaluation of the remaining term $E \left[ \int_{\mathbb{R}} (e^x - 1) F_t(e^x S_{t-}) v(x)dx \right]_{\mathcal{F}_t}$ will depend on the option’s payoff function.

**Proof:** From the dynamics of $F_t(S_t)$ given in Lemma 5 we can establish the dynamics of the cost process $C_t(\Psi)$. Apply the product rule to $C_t(\Psi) \cdot M_t$ and impose the mean-self-financing condition. Then, taking expectation over the Markov chain $U$, and setting the drift term equal to zero provides the result.
10 Conclusion

In this paper we propose a model for asset prices that follows a pure jump process whose statistical behavior is allowed to switch between \( N \) states. This is accomplished by employing a Markov switching compensator. The process not only theoretically accommodates observed empirical moments and term structures of moments, but is also allows for diffusion-like behavior by allowing infinite activity around the origin. We derive a closed form expression for the characteristic function, provide an equivalent martingale representation, and show how to price and optimally hedge European options on a stock that follows this process. We also give examples on how to estimate parameters using the characteristic function.
Appendix A - Implementing the Fast Fourier Transform

Recall that the characteristic function is given by

$$\phi_{X(t)}(u) = \exp((\Pi - \lambda) t) \cdot E[U_0, \mathbf{1}].$$

We can recover the density for the distribution by inverting this characteristic function. That is, the density is given by

$$f(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuX} \phi_{X(t)}(u) ds.$$ 

We approximate this continuous Fourier Transform by the discrete Fourier Transform

$$f(X_k) \approx \frac{1}{2\pi} \sum_{j=1}^{N-1} e^{-iu_j X_k} \phi_{X(t)}(u_j) \Delta,$$

where $\Delta$ is the partition size in the domain of the characteristic function. We also divide the domain of the density into $N$ partitions each of size $\lambda$. Write

$$u_j = -a + \Delta(j - 1)$$

$$X_k = -b + \lambda(k - 1),$$

where

$$a = \frac{\Delta N}{2}$$

$$b = \frac{\lambda N}{2}.$$

Substituting gives

$$f(X_k) \approx \frac{1}{2\pi} \sum_{j=1}^{N-1} \exp\{-i(-a + \Delta(j - 1))(-b + \lambda(k - 1))\} \phi_{X(t)}(u_j) \Delta$$

$$= \frac{1}{2\pi} e^{ia\lambda(k-1)} \sum_{j=1}^{N-1} e^{-i\Delta\lambda(j-1)(k-1)} e^{iu_j b} \phi_{X(t)}(u_j) \Delta.$$ 

Normally, this would entail performing $N$ inversions at $N$ different density points which may be computationally expensive. However, if we denote $x(j)$ as

$$x(j) = e^{iu_j b} \phi_{X(t)}(u_j) \Delta$$

and restrict $\Delta, \lambda$ and $N$ to satisfy

$$\Delta \lambda = \frac{2\pi}{N},$$

then we get

$$f(X_k) \approx e^{ia\lambda(k-1)} \left[ \frac{1}{2\pi} \sum_{j=1}^{N-1} e^{-i\Delta \lambda(j-1)(k-1)} x(j) \right].$$

The Fast Fourier Transform is an algorithm that allows us to evaluate the expression in square brackets at each of the $N$ density points in a single step.
Table 1 - SP500
Parameter estimates for the 2-State and 3-State VG switching compensator model using daily data from January 1995 to December 2000. $\theta_i$, $\sigma_i$, and $\nu_i$ represent the drift, volatility, and gamma parameter for state $i$, while the $\lambda_{ij}$'s are the transition rates from state $i$ to $j$. Standard errors are given in parentheses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>2-State</th>
<th>3-State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>0.00489 (8.00e-8)</td>
<td>0.00372 (1.20e-7)</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>-0.01757 (1.30e-7)</td>
<td>-0.01638 (9.1e-7)</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>-</td>
<td>0.00534 (4.97e-6)</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.00825 (4.00e-8)</td>
<td>0.00636 (5.70e-7)</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>8.01e-5 (1.13e-6)</td>
<td>5.04e-5 (9.10e-6)</td>
</tr>
<tr>
<td>$\sigma_3$</td>
<td>-</td>
<td>0.01851 (2.93e-5)</td>
</tr>
<tr>
<td>$\nu_1$</td>
<td>0.24760 (8.10e-4)</td>
<td>0.00346 (3.82e-4)</td>
</tr>
<tr>
<td>$\nu_2$</td>
<td>0.13502 (2.05e-4)</td>
<td>9.93e-5 (1.33e-4)</td>
</tr>
<tr>
<td>$\nu_3$</td>
<td>-</td>
<td>0.00535 (3.72e-4)</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
<td>0.00104 (2.92e-4)</td>
<td>7.57e-5 (4.20e-4)</td>
</tr>
<tr>
<td>$\lambda_{21}$</td>
<td>0.01552 (1.12e-4)</td>
<td>0.35708 (5.73e-1)</td>
</tr>
<tr>
<td>$\lambda_{13}$</td>
<td>-</td>
<td>1.75e-4 (1.49e-3)</td>
</tr>
<tr>
<td>$\lambda_{31}$</td>
<td>-</td>
<td>1.36611 (1.7690)</td>
</tr>
<tr>
<td>$\lambda_{23}$</td>
<td>-</td>
<td>1.28913 (5.8457)</td>
</tr>
<tr>
<td>$\lambda_{32}$</td>
<td>-</td>
<td>0.23097 (6.10e-2)</td>
</tr>
</tbody>
</table>

Historical Mean: -0.000729
Historical Std. Dev.: 0.000116
Figure 1- SP500 (3 State Model)

Empirical and VG distributions of daily returns on the S&P 500 (January 1995 - December 2000). The solid line represents the empirical distribution. The dashed line (indistinguishable from the solid) represents the theoretical distribution using estimated parameters for the 3-state VG switching compensator model in Table 1.

Table 2

<table>
<thead>
<tr>
<th>Option</th>
<th>Payoff Function</th>
<th>Fourier Transform</th>
<th>Reg. in the Strip:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Option</td>
<td>((e^{rT} - K)^+)</td>
<td>(-\frac{K^{iz+1}}{2^z - iz})</td>
<td>(\text{Im}(z) &gt; 1)</td>
</tr>
<tr>
<td>Put Option</td>
<td>((K - e^{rT})^+)</td>
<td>(-\frac{K^{iz+1}}{2^z - iz})</td>
<td>(\text{Im}(z) &lt; 0)</td>
</tr>
<tr>
<td>Capped Option</td>
<td>(\max((e^{rT} - K)^+, L))</td>
<td>(-\frac{K^{iz+1}}{2^z - iz} + \frac{L^{iz}<em>{\left(L - K\right)} + K(L^{iz}</em>{\left(L - K\right)}}{2^z - iz})</td>
<td>Entire z-plane</td>
</tr>
<tr>
<td>Digital Option</td>
<td>(e^{rT}1{s_T &gt; \ln(K)})</td>
<td>(-\frac{K^{iz+1}}{iz + 1})</td>
<td>(\text{Im}(z) &gt; 1)</td>
</tr>
</tbody>
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References


