Bargaining Under Institutional Challenges

Leyla D. Karakas*

January 29, 2015

Abstract

Standard legislative bargaining models assume that an agreed-upon allocation is final, whereas in practice, there exist mechanisms for challenging passed legislation when there is lack of sufficient consensus. Such mechanisms include popular vote requirements following insufficient majorities in the legislation. Motivated by such episodes, this paper analyzes a legislative bargaining game whose outcome can be challenged through a referendum. I study the effects of this institution on the bills passed in the legislature and analyze the incentives they provide for reaching grand bargains. The proposer party’s trade-off between choosing a costly partner or a threatening opponent for the referendum summarizes the bargaining problem. The results indicate that it is possible to observe surplus coalitions formed in equilibrium even though smaller coalitions are sufficient for the passage of a bill and that measures of post-bargaining power do not necessarily translate into higher equilibrium payoffs. Moreover, disparities in post-bargaining power such as campaigning resources incentivize challenge procedures to the disadvantage of the proposer party. These results carry policy implications for various forms of post-bargaining power, such as caps on campaign contributions.

Keywords: Legislative bargaining, Referendum campaigns, Campaign finance.

JEL Classification: C78, D72.

*Department of Economics, Maxwell School of Citizenship and Public Affairs, Syracuse University, Syracuse, NY 13244. Email: lkarakas@maxwell.syr.edu.
1 Introduction

Most existing legislative bargaining models assume that the agreed-upon allocation is final, whereas in practice, there exist mechanisms for challenging passed legislation when there is lack of sufficient consensus. Specifically, such mechanisms include popular vote requirements following insufficient majorities in the legislature. In most parliamentary systems, a bill that fails to win a certain majority of votes in the legislature can be presented to a public vote as the final arbiter.1 For example, in a referendum in May 2011, Britain rejected a proposal to switch from a first-past-the-post election system to an alternative vote system. In March 2011, shortly after the fall of the Mubarek regime, Egypt approved in a widely contested referendum a series of constitutional reforms, including presidential term limits and election supervision mechanisms.

Motivated by these examples, I analyze the effect of institutional mechanisms to challenge agreed-upon legislation on the formation of these bills and the equilibrium payoffs to the parties. I start by recognizing that both exogenous factors and endogenous choices affect a party’s potential influence in a given post-bargaining stage. For example, a large literature, including Matsusaka (2005a) and (2005b), documents the surge in spending on referendum campaigns. Examples of such campaigns are advertising, media coverage or political rallies. Moreover, there exists growing evidence that the public is affected by these campaigns, as documented in de Figueiredo, Ji and Kousser (2011). With a new empirical approach that attempts to deal with the endogeneity of campaign spending, the authors find that spending both for and against a proposal influences the probability of its passage in the campaigners’ intended direction.2

Given the influence of these campaigns on voters, to what extent do the proposals introduced in a parliament reflect the parties’ public vote calculus? For instance, would the Egyptian constitutional reform package include more liberal propositions if the liberal faction were considered a more powerful player in the subsequent referendum?

1I do not consider referenda that are constitutionally-mandated regardless of the level of consensus in the parliament. For example, all but one US states require constitutional amendments to be approved in a referendum regardless of the level of congressional majority.

2The impact of campaign spending on referenda or citizen initiative outcomes has been studied in Gerber (1999) and Broder (2000). Empirical studies, including Bowler and Donovan (1998) that have treated campaign spending as exogenous find asymmetric effects of money of referendum outcomes: While spending against a proposal decreases its chances of passage, a similar effect does not exist when spending supports the proposal. Lupia and Matsusaka (2004) provides an overview and discussion of these results.
Specifically, how does a referendum process in which parties campaign to influence the probability of its outcome affect the contents of a legislative proposal? Under what conditions can the parties agree on a grand bargain that would obviate a referendum? Within the context of a referendum, the main goal of this paper is to study the consequences of a strategic post-bargaining stage on the equilibrium payoffs of the players endowed with varying degrees of “post-bargaining power”.

In order to address these questions, I build a one-period legislative bargaining model in which parties bargain over a bill with single-dimensional policy and distributive rent components. After the party with the most number of seats proposes both a policy and a rent allocation, other parties simultaneously vote on the proposal. If the proposal fails to win a simple majority, it is rejected and the game ends. Otherwise, the proposal passes. In the post-bargaining stage, parties can challenge the approved bill depending on its level of support in the parliament. I model the post-bargaining stage with a referendum in which parties can challenge the bill in a public vote only if it fails to receive a supermajority in the parliament. Once the challenge stage begins, parties campaign for or against the proposal to influence its outcome. The parties’ exogenous campaigning budgets characterize their post-bargaining power.

I define a political equilibrium for two and three-party parliaments and characterize it under the challenge model. I show that in the presence of looming institutional challenges, surplus coalitions are possible. Moreover, measures of post-bargaining power do not necessarily translate into higher equilibrium payoffs as the proposer party faces a trade-off between a higher probability of having its bill upheld in a post-bargaining challenge by including a “powerful” party in its coalition and proposing a bill that captures a high share of benefits for itself. In two-party parliaments, a grand bargain is more likely in equilibrium if the minority party commands a low status-quo payoff and the proposer has a large campaigning budget. Similarly, parties reach a grand bargain more easily in three-party parliaments when the smaller parties do not command high status quo payoffs or if all parties are ideologically close. Moreover, I find that the chances of a referendum are higher if the campaigning budgets of the smaller parties diverge widely. This is because in equilibrium, only the status-quo payoffs determine the proposer party’s utility from a grand bargain. Campaigning budgets matter only to the extent to which the proposer party can benefit from pitting one small party against

---

3 A two-party parliament can be considered as representing the outcome of a first-past-the-post election system, and a three-party parliament as the outcome of a proportional representation system.
another through coalition formation. More generally, a more asymmetric distribution of post-bargaining powers within a parliament incentivizes challenge procedures to the benefit of the proposer.

Having analyzed the factors that lead a dominant party to risk subsequent institutional challenges instead of inducing unanimity, I then study the composition of simple majority coalitions in three-party parliaments. In any political equilibrium, I show that the proposer party is more likely to partner with the party that has a lower status quo payoff or a closer ideal point. On the other hand, whether a large campaigning budget makes a party the preferred coalition partner depends on the type of political equilibrium. In the referendum model, this ambiguity result is a consequence of the proposer party’s essential trade-off that defines its decision-making: Although a rich partner and a poor opponent is desirable for increasing the probability that its bill is upheld, it comes at the expense of higher concessions to the rich partner in the bargaining stage. Which one of these effects dominates in equilibrium depends on the parameters of the model.

2 Related Literature

Building on the seminal work of Baron and Ferejohn (1989) and the uniqueness of payoffs result proved in Eraslan (2002), numerous models study the equilibrium consequences of different sources of bargaining power by treating the agreed-upon allocation as the final outcome. Some of these papers include Kalandrakis (2006) who studies proposal rights, McCarty (2000) who studies proposal and veto rights, Snyder, Ting and Ansolabehere (2005) who study weighted voting, and Yildirim (2007) who studies endogenous proposal power. Another branch of this literature studies bargaining models with stochastic surplus to be divided and includes Eraslan and Merlo (2002) and Diermeier, Eraslan and Merlo (2002). In addition, dynamic bargaining models such as Kalandrakis (2004), Duggan and Kalandrakis (2012), and Bowen, Chen and Eraslan (2014) consider situations in which the agreed-upon allocation becomes the new status-quo in the next bargaining period. However, these papers do not study institutions outside of the bargaining environment through which the agreed-upon outcome can be challenged. Veto-player models such as Winter (1996) are an exception for incorporating a post-bargaining stage in which bargaining outcomes can be overturned. Another example is Powell (1996), who considers a bargaining model in which players can impose outside settlements to
capture the whole pie, but this happens with pre-determined probabilities. In contrast to these exogenous sources of bargaining power, this paper introduces a new source of bargaining power that is generated from post-bargaining behavior.

The vote of confidence mechanism in legislatures, studied in Diermeier and Feddersen (1998), is an example of a post-bargaining institution that affects the bargaining equilibrium. The authors show that the existence of such a mechanism decreases the price of building coalitions in the legislature and results in equilibrium coalitions that are more cohesive and rewarded more handsomely. Similar to the veto procedure, their study is relevant to this paper through its explicit recognition of post-bargaining institutions directly affecting the outcomes of the bargaining game.

One of the main predictions of this paper is the formation of surplus coalitions even though minimum winning coalitions would be sufficient for the bills to formally pass in the legislature. Even though minimum winning coalitions were the main prediction of the baseline model of Baron and Ferejohn (1989), other papers have studied environments where this prediction fails to hold. For example, Goreclose and Snyder (1996) show that equilibrium coalitions will exhibit surplus members because such coalitions will be cheaper than minimum winning ones when certain bargaining protocol conditions are met.

The institutions of direct democracy, represented here by the post-bargaining referendum option, has been studied by both economists and political scientists from different angles. Romer and Rosenthal (1979) is one of the first models that deviate from the Downsian median voter prediction to study the voter’s choice between the status quo policy and some alternative proposed by a bureaucrat with agenda-setting power. Using the level of expenditures as the policy to be decided upon, they show that the actual level of expenditures will be at least as great as the one predicted in the Downsian model. Lupia and Matsusaka (2004) provide an overview of the political science literature with a focus on the effects of campaign money on the results of direct democracy exercises.

There also exists a large political science literature on the domestic ratification of international treaties using the two-level games approach, building on the seminal insight of Putnam (1988) that a smaller set of propositions that could get domestic approval increases the bargaining power of the negotiator at the international stage. Other relevant papers on two-level games include Iida (1996), Haller and Holden (1997), and Humphreys (2007). Although the sequence of the moves are similar to the referendum
model, with a public vote following a bargaining stage, this paper models the players with an eye toward the same public vote constraint as opposed to different domestic constituencies. Moreover, the constraint in the referendum model is not set exogenously by the median legislator or the median voter’s ideal point, but can be influenced through endogenous campaign spending.

The main results here have implications for public financing of issue campaigns. Papers such as Coate (2004) and Ashworth (2006) study the welfare effects of private campaign finance by interest groups. Since I do not model interest groups, this paper is silent on the impact of private campaign contributions. However, comparative statics on the parties’ exogenous campaigning budgets yield implications of public campaign finance for observed legislative outcomes.

Finally, I model the post-bargaining referendum option as a contest between the bargaining parties and therefore draw upon many results in contest theory. The most relevant of these theoretical papers are Baik (2008), who characterizes the equilibrium in contests with group-specific prizes, and Skaperdas and Vaidya (2012) who show how a Tullock contest function, which my model uses, can proxy voter behavior in referendums. Other relevant papers in contest theory include Dixit (1987), Hillman and Riley (1989) and Skaperdas (1996).

The rest of the paper is organized as follows: Section 3.3 introduces the model and defines a political equilibrium. Sections 3.4 and 3.5 respectively analyze equilibrium behavior in two and three-party parliaments. Section 3.6 discusses the implications of the equilibrium results on campaign finance policy and concludes.

3 The Model

I consider a situation of one-period legislative bargaining over a bill that consists of ideology and distributive components, followed by a referendum if the number of votes in the parliament falls within an institutionally designated interval.

Let $N$ denote the set of parties and $|N|$ the number of parties in the parliament. In this paper, only parliaments of two and three parties will be considered. The model consists of two stages: the bargaining stage and the challenge stage. In the bargaining stage, the party with the most number of seats proposes a bill and the other party (or parties) votes on it. In a three-party parliament, I assume that the two non-proposer
parties vote simultaneously on the bill. Let $x \in [0, 1]$ represent the ideological component of the proposal and let $\hat{x}_k$ denote party $k$’s ideal ideological point. In addition, let $y$ represent the proposed allocation of rents from the feasible set

$$Y = \{ y : \sum_{k=1}^{\lfloor N \rfloor} y_k \leq 1 \text{ and } y_k \geq 0 \ \forall \ k \},$$

where the fixed sum of rents is given by unity and $y_k$ denotes party $k$’s share. Hence, a proposal can be represented by $z \equiv (x, y) \in [0, 1] \times Y$. When the proposal is introduced to the parliament, there exists a status-quo bill $s \equiv (q, y^q)$, where $q \in [0, 1]$ denotes its ideological component and $y^q \in Y$ its rent allocation. I assume that party $k$’s preferences over a bill are represented by the quasi-linear utility function

$$u_k(z) = -(x - \hat{x}_k)^2 + \alpha y_k,$$

where $\alpha \in (0, 1)$ is some fixed weight.

After the proposer party makes an offer $z \in [0, 1] \times Y$ and the other party (or parties) votes on it, the proposal is accepted or rejected according to the following criteria: Let $k(z)$ denote the number of parties other than the proposer who support the bill $z$. If $k(z) = \lfloor N \rfloor - 1$, the proposal $z$ is unanimously accepted and becomes the law with no subsequent challenges. If $k(z) = 0$, the bill is automatically rejected in a three-party parliament. On the other hand, rejection without a challenge is not feasible in two-party parliaments, since the proposer party always commands a simple majority. Finally, if $k(z) = \lfloor N \rfloor - 2$, the proposal is temporarily accepted in the parliament to be challenged in a referendum. Any proposal that survives the challenge becomes the law.\(^4\ 5\)

If the proposal passes in the parliament without unanimous support, the dissenting party takes the bill to a referendum. I describe this challenge stage as a two-candidate competition in which the candidates are the proposal $z$ and the status quo $s$. Before the referendum takes place, each party $k$ simultaneously chooses a position $t \in \{Z, S\}$ and

\(^4\)This acceptance criteria represents the following general rule in parliamentary systems for important legislation or constitutional amendment proposals. Let $k$ denote the number of supportive legislators. If $k \leq \lfloor N \rfloor - 1$, where $\lfloor N \rfloor$ is odd, the proposal fails to win a simple majority and fails. If $k \geq \lambda(\lfloor N \rfloor - 1)$, where $\lambda \in (\frac{1}{2}, 1)$, it is accepted without a referendum. Finally, for all $k \in (\frac{\lfloor N \rfloor - 1}{2}, \lambda(\lfloor N \rfloor - 1))$, the proposal becomes law only if it is accepted in a referendum. Here, $\lambda$ represents the supermajority parameter for the parliament.

\(^5\)In a three party parliament, I assume without loss of generality that no party commands a majority of the seats and that two parties together cannot control a supermajority.
an irreversible campaign spending amount \( c \geq 0 \) to influence the voters (who will not be explicitly modeled). Position \( Z \) indicates a preference for the public acceptance of the proposal (yes vote on the referendum) and position \( S \) indicates a preference for its failure (no vote on the referendum).

Each party \( k \) is allocated an exogenously given campaigning budget \( w_k \in [0,1] \).\(^6\)

Upon observing the campaigns of each group, the public votes on the proposal in a referendum. If the proposal wins a simple majority of the public vote, it becomes the law. Otherwise, the status-quo bill prevails and all parties receive their status-quo payoffs. I assume that all the parameters of the model are common knowledge.

I model the referendum as a contest between the positions \( Z \) and \( S \) in which their winning prize is given respectively by \( z \) and \( s \). Hence, the winning prize constitutes a public good within each group of parties. Let \( C_t(z) \) denote the total campaign spending of parties aligned with position \( t \) when the proposed bill is \( z \) and let \( p_t(C_Z(z), C_S(z)) \) denote the probability that position \( t \) wins the referendum. I assume that the contest success function takes the Tullock lottery form so that the probability of winning for a party aligned with position \( t \) is given by

\[
p_t(C_Z(z), C_S(z)) = \begin{cases} \frac{C_t(z)}{C_Z(z) + C_S(z)} & \text{if } C_Z(z) + C_S(z) > 0 \\ \frac{1}{2} & \text{if } C_Z(z) = C_S(z) = 0 \end{cases} \tag{3}
\]

for \( t = Z, S \) and proposal \( z \). The above Tullock specification assumes that neither party has an inherent advantage in the contest. Moreover, it implies that a position’s winning probability is increasing in the spending of the parties aligned with it and decreasing in the spending for the other position.

The sequence of events can be summarized as follows:

- The proposer party offers a bill \( z \) to the parliament.
- The other party (or parties) votes on \( z \). If the vote(s) is such that the decision is not final, the challenge stage begins.
- Each party simultaneously and independently chooses a position \( t \) and an irreversible campaign spending \( c \) for the referendum.

\(^6\)Although private interest groups play an important role in financing referendum campaigns, I do not model them here in the interest of keeping the analysis tractable.
• The public votes in the referendum. If the proposal wins a simple majority of the public vote, it passes and becomes the law. If not, all players receive their status quo payoffs.

A pure bargaining strategy for party $k$ consists of a proposal $z \in [0, 1] \times Y$ if $k$ is the proposer party, and an acceptance rule $a_k : [0, 1] \times Y \rightarrow \{0, 1\}$ for the non-proposer parties $k$ such that $a_k(z) = 0$ indicates rejection of the proposal $z$ and $a_k(z) = 1$ indicates its acceptance.\footnote{\label{footnote:2}I assume that a party votes to accept a proposal when indifferent.} In addition, a pure challenge strategy for party $k$ consists of the following elements: a position rule $\rho_k : [0, 1] \times Y \rightarrow \{Z, S\}$ such that $\rho_k(z) = t$ indicates that party $k$ has chosen position $t$ for the referendum; and a campaign spending rule $\zeta_k : [0, 1] \times Y \rightarrow [0, w_k]$ such that $\zeta_k(z) = c$ yields the amount party $k$ spends on his chosen position’s campaign. Specifically, $\rho_k(z) = t$ indicates that party $k$ spends an amount $c = \zeta_k(z)$ for position $t$. A party jointly chooses its position and campaign spending amount.

Without loss of generality, fix party 1 as the proposer party. Let $\sigma \equiv (\sigma_1, \{\sigma_k\}_{k=2}^{|N|})$ denote a strategy profile, where $\sigma_1 = (z, \rho_1, \zeta_1)$ for the proposer party and $\sigma_k = (a_k, \rho_k, \zeta_k)$ for the non-proposer party (or parties) $k \neq 1$.

Let $N_Z = \{ k \in N : \rho_k(z) = Z \}$ and $N_S = \{ k \in N : \rho_k(z) = S \}$ respectively denote the set of parties that align themselves with positions $Z$ and $S$. Then, given a proposal $z$, the total campaign spending of each party group $N_t$ can be written as

$$C_t(z) = \sum_{k \in N_t} \zeta_k(z). \quad (4)$$

Given the equilibrium behavior of every other player, a political equilibrium to this game consists of optimal party strategies during both the bargaining and the challenge stages. Through backward induction, I solve for the Subgame-Perfect Nash equilibrium of this model, which is defined below:

**Definition 1.** A strategy profile $(\sigma_1, \{\sigma_k\}_{k=2}^{|N|})$ constitutes a political equilibrium if and only if the following conditions are satisfied:

\begin{enumerate}
  \item[(E1)] Given $z$ and $a_k(z)$ for $k \neq 1$ from the bargaining stage, and other parties’ challenge strategies $\rho_{-k}$ and $\zeta_{-k}$, party $k$’s position rule $\rho_k(z) = t$ and campaign spending
\end{enumerate}
\[ \text{rule } \zeta_k(z) = c \text{ solve} \]

\[
\max_{t \in \{Z,S\}, c \in [0,w_k]} u_k(s) + p_Z \left( \sum_{k \in N_t} c + \zeta_{-k}(z), C_{-t}(z) \right) [u_k(z) - u_k(s)] - c. \tag{5}
\]

(E2) For any given proposal \( z \), let

\[ V_k(z; \sigma) = u_k(s) + p_Z(C_{\rho_k(z)}(z), C_{-\rho_k(z)}(z))[u_k(z) - u_k(s)] - \zeta_k(z) \tag{6} \]

denote party \( k \)'s maximized expected payoff from the referendum when each party would be following its equilibrium challenge strategies. Then,

- If \(|N| = 2\), or \(|N| = 3\) and \( a_{-k}(z) = 1 \), \( a_k(z) = 1 \) if and only if \( u_k(z) \geq V_k(z; \sigma) \);
- If \(|N| = 3\) and \( a_{-k}(z) = 0 \), \( a_k(z) = 1 \) if and only if \( V_k(z; \sigma) \geq u_k(s) \).

(E3) Party 1’s proposal \( z \) solves

\[
\max_{z \in [0,1] \times Y} u_1(s) + p_Z(C_Z(z), C_S(z)) \cdot [u_1(z) - u_1(s)] - \zeta_1(z). \tag{7}
\]

Condition (E1) requires that each party’s position and campaign spending rules jointly maximize its expected payoff from the referendum. Condition (E2) rules out the use of weakly dominated strategies by the non-proposer party (or parties) during legislative voting. It requires that an acceptance vote is given to a proposal if and only if it is weakly preferred to voting to reject it. Finally, condition (E3) requires that given the subsequent optimal acceptance, position, and campaign spending rules of all the parties, the proposer party 1 makes an offer that maximizes its expected payoff. Before the bargaining stage begins, the referendum probabilities \( p_t \) for \( t = Z, S \) are within the control of party 1. Specifically, the proposer can induce any possible outcome by making the right offer.

Given the existence of equilibria in contests that describe the challenge stage of this model and the existence of a bargaining equilibrium for any profile of challenge strategies, a political equilibrium exists. In the following sections, I characterize the pure strategy Subgame Perfect Nash equilibria of this model respectively for parliaments of two and three parties.
4 Two-Party Parliaments

Two-party parliaments can be thought of as representing the outcome of a first-past-the-post election system. In this context, I assume that the proposer party 1 controls a simple majority, but not a supermajority, of the seats so that it needs the approval of the smaller party 2 in order to avoid a challenge stage.

I solve for the political equilibrium in a two-party parliament through backward induction. First, consider the parties’ equilibrium challenge strategies. If the game reaches this stage, the parties’ position choices for the referendum are trivial: On the equilibrium path, party 1 never campaigns against its own proposal so that $\rho_1(z) = Z$ always holds for any given proposal $z$. Similarly, if party 2 preferred a yes vote on the referendum, it would have accepted the proposal $z$ during bargaining in order to secure a sure outcome and not incur campaigning costs. Hence, $\rho_2(z) = S$ always holds as well on the equilibrium path.

Given the equilibrium position rules described in the above paragraph, the optimal campaign spending of the two parties for any given proposal $z$ from the bargaining stage are given by

$$\zeta_1(z) \in \arg \max_{c \in [0,w_1]} \frac{c}{c + \zeta_2(z)} u_1(z) + \frac{\zeta_2(z)}{c + \zeta_2(z)} u_1(s) - c; \quad (8)$$

$$\zeta_2(z) \in \arg \max_{c \in [0,w_2]} \frac{\zeta_1(z)}{\zeta_1(z) + c} u_2(z) + \frac{c}{\zeta_1(z) + c} u_2(s) - c. \quad (9)$$

For a more concise exposition in the following analysis, let

$$\epsilon_k(z) = |u_k(z) - u_k(s)| \quad (10)$$

represent party $k$’s stake from the challenge stage for any given proposal $z$, given by the difference in its utility from the two potential outcomes $z$ and $s$. Based on 8 and 9, the following lemma describes how the parties’ equilibrium campaign spendings respond to the bargaining outcome: 8

**Lemma 1.** Let $z$ and $z'$ be two proposals such that $\epsilon_k(z) \geq \epsilon_k(z')$ for party $k \in \{1, 2\}$. Then, $\zeta_k(z) \geq \zeta_k(z')$.

---

8 All proofs are in the Appendix.
referendum increases. For example, if one of two proposals implies a much lower payoff relative to the status-quo for party 2, then party 2 would fight harder for the failure of this proposal in the referendum. The larger difference between the winning and the losing prizes justifies a higher amount of equilibrium campaign spending compared to the proposal with the lower stakes.

In the following analysis, I first present the general characteristics of a political equilibrium in Proposition 1. Then, I focus on the parameter values that make a political equilibrium in which the challenge stage is reached on the equilibrium path more likely to be observed than one in which the parties settle in the parliament.

**Proposition 1.** In the political equilibrium of a two-party parliament,

1. The acceptance rule of party 2 is characterized as follows:
   - Party 2 rejects any offer $z$ for which $\epsilon_1(z)$ and $\epsilon_2(z)$ are such that $\zeta_2(z) = w_2$;
   - For the range of proposals $z$ for which $\epsilon_1(z)$ and $\epsilon_2(z)$ would imply a challenge stage equilibrium with $\zeta_1(z) = w_1$ and $\zeta_2(z) < w_2$ if rejected, party 2 accepts any offer $z$ such that $u_2(z) + w_1 \geq u_2(s)$;
   - For the range of proposals $z$ for which the implied challenge stage equilibrium is an interior one, party 2 accepts any offer $z$ that yields $u_2(z) \geq u_2(s)$.

2. If party 1 chooses to induce unanimity, it proposes $z$ such that $u_2(z) = u_2(s) - w_1$, where
   - $z$ features equal compromise on ideology, i.e. $x = \frac{x_1 + x_2}{2}$;
   - The difference between the parties’ rent shares, i.e. $y_1 - y_2$, increases as $u_2(s)$ decreases or $w_1$ increases;
   - For low (high) values of $\alpha$, party 1 may choose $y_1 = 0$ ($y_1 = 1$) and $x$ closer to $\hat{x}_1$ ($\hat{x}_2$).

3. If party 1 chooses to induce a referendum, it becomes more likely to do so by proposing $z = (\hat{x}_1, 1, 0)$ as opposed to any other proposal that yields a higher utility for party 2 as the two parties diverge ideologically.

The first part of Proposition 1 characterizes party 2’s equilibrium bargaining strategy. It indicates that any proposal that implies a sufficiently high stake for party 2 (either due
to a high status-quo payoff, a very unfavorable proposal, or both) so that it would fight by spending its entire campaigning budget in a subsequent challenge will be rejected by party 2. On the other hand, party 2 may be willing to settle for proposals that involve relatively lower stakes. For instance, if the proposal is such that neither party’s stake would justify exhausting its whole budget in a potential campaign, the typical criteria that party 2 accepts any proposal that leaves it at least as well-off as the status-quo applies. However, there may also exist situations in which the threat of a challenge allows the proposer to extract a surplus from party 2’s status-quo payoff in a settlement. Specifically, if the parameters of the model are not too extreme so that party 2 commands a sufficiently low status-quo and \( w_1 \) is not too large, party 2 will settle for less than its status-quo payoff. This is due to the threat a looming challenge stage poses for itself. With party 1 willing to exhaust its budget to defend its relatively higher stakes from the proposal, party 2’s meagre winning prize would not justify its counter campaign spending to defend the status-quo in this situation. Therefore, it is willing to pay a premium to party 1 in order to avoid this expensive challenge.

The second part of the proposition describes the optimal way to induce unanimity from party 1’s point of view. The proposition states that party 1 would extract a surplus of \( w_1 \) from party 2 in a settlement, reflecting the threat discussed in the above paragraph. The optimal proposal to induce this settlement involves an equal ideological compromise between the parties. However, if the status-quo ideology is such that party 2 would gain from this compromise, party 1 extracts these gains away in the form of a higher rent share.

The final part of the proposition focuses on the type of challenge equilibrium that would be preferred by party 1. The analysis indicates that the optimal proposal to induce a challenge in which party 2 exhausts its campaigning budget is the one that maximizes party 1’s winning prize, given by \( z = (\hat{x}_1, 1, 0) \). This is due to the fact that the probability of winning for party 1 is not affected by how much further party 2’s stake increases if party 2 is already spending its entire budget. For all other types of challenge stage equilibria in which \( \zeta_2(z) < w_2 \), the proposer faces the following trade-off: Even though a more favorable proposal for itself increases party 1’s winning prize, this comes at the expense of decreasing its winning probability as party 2 fights more aggressively by spending more. As \( \hat{x}_1 \) and \( \hat{x}_2 \) diverge, party 1’s expected payoff from this challenge may decrease sufficiently that a proposal short of \( z = (\hat{x}_1, 1, 0) \) is no longer justified.
More specifically, as the value of $\epsilon_2(z)$ increases due to this divergence, leading to higher spending by party 2, the proposal compromise that was made in the hopes of putting a check on party 2’s spending no longer pays off. In this situation, party 1 would be better-off offering $x = \hat{x}_1$ with all the rent allocated to itself, thereby provoking an all-out fight with $\zeta_1(z) = w_1$ and $\zeta_2(z) = w_2$.

Having described party 1’s incentives in choosing how best to realize a unanimity outcome in the parliament or to induce a challenge, it remains an open question which option party 1 will prefer. The following proposition takes up this task:

**Proposition 2.** Party 1 is more likely to prefer the unanimity outcome over a challenge for lower values of $u_2(s)$ and higher $w_1$. A lower $w_2$ incentivizes unanimity only if $w_2 > \alpha - u_1(s)$.

The intuition for why a smaller status-quo payoff for party 2 unambiguously contributes to a higher likelihood of observing unanimity is straightforward: Since party 1 offers $u_2(z) = u_2(s) - w_1$ to party 2 in order to get its acceptance, a lower $u_2(s)$ increases its sure payoff from the settlement. On the other hand, while a higher $w_1$ may contribute to a higher probability of winning for party 1 in a particular challenge, it also increases its unanimity payoff as $w_1$ is extracted from party 2. In equilibrium, the effect of $w_1$ on its unanimity payoff dominates the challenge stage effect, yielding the result in Proposition 2.

The conditional result in Proposition 2 on how $w_2$ affects party 1’s incentives between a settlement and a challenge illustrates another trade-off. In a challenge stage equilibrium with $(\zeta_1(z), \zeta_2(z)) = (w_1, w_2)$, changes in $w_2$ only affect party 1’s probability of winning in the referendum. On the other hand, if the equilibrium is such that $\zeta_1(z) < w_1$ and $\zeta_2(z) = w_2$, changes in $w_2$ affect not only party 1’s probability of winning, but also its campaign spending. Specifically, a higher $w_2$ unambiguously decreases party 1’s winning probability in this equilibrium, while it also decreases $\zeta_1(z)$ when the condition in Proposition 2 holds. When this is true, the marginal effect of lower campaign spending on party 1’s expected payoff from this challenge dominates the marginal effect of a lower winning probability, resulting in an increase in party 1’s expected challenge payoff. Thus, in this challenge stage equilibrium, sufficiently higher values of $w_2$ do not act as threat instruments due to their indirect effect on party 1’s campaign spending.

Based on this analysis, we would expect to observe a proposer party with a high campaigning budget work towards achieving unanimity by buying the smaller party out.
In contrast, a small party would act more aggressively by shunning a settlement if the stakes from the proposed bill are high enough. As the smaller party’s budget grows, this can initially act as a threat and therefore encourage unanimity. However, this effect may be reversed once a threshold is crossed. At this point, the smaller party’s budget starts to constitute an impediment to settlement.

5 Three-Party Parliaments

Studying a three-party parliament offers richer dynamics on coalition formation and incentives for a grand bargain in a non-cooperative framework than the two-party setting allowed. In this section, I assume that neither party controls a simple majority of the seats and that two parties together do not command a supermajority. Therefore, at least two parties must agree in order for a bill to pass in the parliament. A bill that has passed in the parliament with votes short of unanimity moves to the challenge stage. To abstract away from potential informational advantages, I assume that after party 1 makes an offer, the other two parties vote on it simultaneously.

When making a proposal, party 1 can induce one of the following four general outcomes: A grand bargain with unanimous agreement among all the three parties; rejection in the parliament; a challenge stage with party 2 as its partner and party 3 in the opposition; or a challenge stage with party 3 as its partner and party 2 in the opposition. Looking for a political equilibrium in a three-party parliament involves solving for the optimal offers that would induce each of the alternative outcomes and comparing party 1’s maximum expected payoffs from those outcomes.

Baik (2008) characterizes the pure-strategy Nash equilibrium of group contests in which the winning prize is a public good within each group. Since the winning probability in the referendum is a function of each party group’s total campaign spending, this characterization applies to the equilibrium of the challenge stage in this model. Specifically, since there are always two parties aligned with position Z in a challenge, the proposal z, which is the winning prize for members of group N_Z, constitutes a public good within this group.

To start characterizing the equilibrium of the challenge stage, first consider the parties’ position choices for a given proposal z. As in the two-party case, it can never be
optimal for the proposer to take a stand against its own bill so that we have $\rho_1(z) = Z$ on the equilibrium path for any given $z$. In order to have reached the challenge stage, it must have been the case that one party voted for the bill and one against it in the parliament. Let $h$ and $j$ denote these two non-proposer parties such that $a_h(z) = 1$ and $a_j(z) = 0$. If party $h$ preferred a no vote on the referendum, it would have voted to reject the proposal in the bargaining stage, leading to its defeat and thereby avoiding a costly and risky referendum. Therefore, $\rho_h(z) = Z$ on the equilibrium path. Similarly, if party $j$ preferred a yes vote on the referendum, it would have voted to accept the proposal during bargaining, leading to a unanimous agreement on $z$. Hence, it must be the case that $\rho_j(z) = S$ on the equilibrium path. Therefore, party $h$ for whom $a_h(z) = 1$ becomes party 1’s partner in the challenge stage and party $j$ for whom $a_j(z) = 0$ becomes its opponent.

In the challenge stage, each group $N_t$, $t \in \{Z, S\}$, decides on a total campaign spending $C = C_t(z)$, where $C_t(z)$ is as defined in 4. The members of a group do not act cooperatively; instead, campaign spending choices are made independently. For a given proposal $z$ and the total campaign spending of group $N_S$ given by $C_S(z) = \zeta_j(z)$, let $C^1_h(z)$ denote the best response total campaign spending of group $N_Z$ to $C_S(z)$ from the perspective of party 1 and let $C^h_Z(z)$ denote the same best response from the perspective of its partner party $h$. Specifically, define $C^1_Z(z)$ and $C^h_Z(z)$ such that

$$
C^1_Z(z) \in \arg \max_{C \in [0, w_1 + w_h]} \frac{C}{C + C_S(z)} u_1(z) + \frac{C_S(z)}{C + C_S(z)} u_1(s) - \zeta_1(z); 
$$

$$
C^h_Z(z) \in \arg \max_{C \in [0, w_1 + w_h]} \frac{C}{C + C_S(z)} u_h(z) + \frac{C_S(z)}{C + C_S(z)} u_h(s) - \zeta_h(z).
$$

As long as the proposal $z$ is such that $\epsilon_1(z) \neq \epsilon_h(z)$, party 1 and its partner have different opinions as to how they should best respond to $C_S(z)$. Moreover, since the winning prize $z$ is a public good for them, the decision on how the burden of the total spending $C_Z(z)$ will be shared in equilibrium is not trivial.

The following lemma, based on Baik (2008), characterizes how the total campaign spending $C_Z(z)$ of group $N_Z$ is determined and its burden is shared among parties 1 and $h$ in a Nash equilibrium. This lemma will then be used to characterize the challenge stage equilibrium.

**Lemma 2.** Suppose the proposal $z$ is such that $\epsilon_1(z) \geq \epsilon_h(z) > 0$. Then, taking the total
campaign spending $C_S(z) = \zeta_j(z)$ of group $N_S$ as given, parties 1 and $h$ choose their total equilibrium campaign spending $C_Z(z)$ and its allocation between $\zeta_1(z)$ and $\zeta_h(z)$ as follows:

1. If $C_Z^1(z) \leq w_1$, then $C_Z(z) = \zeta_1(z) = C_Z^1(z)$ and $\zeta_h(z) = 0$.

2. If $C_Z^h(z) \geq w_1 + w_h$, then $C_Z(z) = w_1 + w_h$, $\zeta_1(z) = w_1$, and $\zeta_h(z) = w_h$.

3. If $C_Z^1(z) > w_1$ and $C_Z^h(z) \leq w_1 + w_h$, then $C_Z(z) = \max\{C_Z^h(z), w_1\}$, $\zeta_1(z) = w_1$, and $\zeta_h(z) = \max\{0, C_Z^h(z) - w_1\}$.

Lemma 2 provides a full characterization of the equilibrium campaign spending decisions of the members of group $N_Z$. To gain some intuition, first note that the party with the higher stake from a challenge, determined by the proposal $z$ from the bargaining stage, will have a higher total campaign spending best response to group $N_S$ than its opponent. Part 1 of the lemma indicates that if the party with the higher stake can afford its best response total campaign spending using only its own resources, then it is the only member of group $N_Z$ that contributes to the campaign in equilibrium; its partner free-rides on its spending. This campaign more than meets the partner’s needs, obviating any spending on the partner’s part. On the other hand, if the total resources of the group cannot cover even the lower best response of the partner, then part 2 of the lemma indicates that each member exhausts its budget in equilibrium. There exists no free-riding in this situation. Finally, if the party with the higher stake cannot afford its best response total campaign spending with its own resources but the partner’s lower best response can be met with the total group budget, then the higher-stake party spends its entire budget on the campaign while its partner contributes the difference (if the difference is positive). In this scenario, the partner is at best a partial free-rider on the higher-stake party’s campaign spending.

In short, Lemma 2 shows that unless the stakes from a challenge are sufficiently high for both members of group $N_Z$, the party with the lower stake free-rides on its partner’s campaign spending that contributes positively to its probability of winning in the referendum. The following lemma uses the results of Lemma 2 in order to describe the general properties of a challenge stage equilibrium, which requires that group $N_Z$ is in equilibrium and that both groups are best-responding to each other:
Lemma 3. Let $z$ and $z'$ be two proposals such that $\epsilon_k(z) \geq \epsilon_k(z')$ for party $k \in \{1, 2, 3\}$. Then, $\zeta_k(z) \geq \zeta_k(z')$ in equilibrium. Moreover, for any given proposal $z$, the condition $\epsilon_1(z) \geq \epsilon_h(z)$ needs to hold in order for party $h \in N_Z$ to free-ride on $\zeta_1(z)$ in a challenge stage equilibrium.

Lemmas 2 and 3 together describe the properties of a challenge equilibrium for any proposal $z$ from the bargaining stage. Based on this challenge equilibrium, the political equilibrium of the model can be solved for via backward induction. The following propositions present general results on a political equilibrium. Following the same order of analysis as in the previous section, I study the structure of proposals that would respectively induce a grand bargain in the parliament and a subsequent challenge. Then, I focus in the remainder of the section on the conditions that make a grand bargain among the three parties more likely to be observed on the equilibrium path than a challenge.

Proposition 3. In the political equilibrium of a three-party parliament, the following are true about inducing a grand bargain among the parties:

1. Any proposal $z$ that would imply a challenge stage equilibrium with $\zeta_j(z) = w_j$ for $j \in N_S$ if rejected will move to a challenge.
2. In a unanimous agreement on a proposal $z$ that would otherwise lead to a challenge with free-riding in group $N_Z$, the party who would have been the free-rider partner is punished.
3. In party 1’s optimal unanimity-inducing offer $z$, its rent share $y_1$ increases in $y_1^q$, $w_1$, and $(q - \hat{x}_k)$ for $k = 2, 3$. Furthermore, its unanimity payoff $u_1(z)$ increases as the three parties get ideologically closer.

The first part of Proposition 3 presents a result on the structure of proposals on which a grand bargain is achievable. Specifically, it indicates that if an offer involves very high stakes for at least one party, either due to a high status-quo payoff or an unfavorable treatment in the proposal for that party, such that it would fight with all its budget in a potential challenge, unanimity is impossible to achieve. For this party, its certain payoff from unanimity is not high enough to justify foregoing the chance of regaining its status-quo payoff in a challenge. This mirrors the result in part 1 of Proposition 1 for a two-party parliament. In both cases, parties that have too much to lose from a proposal
will not settle.

Part 2 of Proposition 3 suggests that a proposal \( z \) on which a grand bargain is possible reflects the division of \( C_Z(z) \) among parties 1 and \( h \in N_Z \) that would be observed if \( z \) was instead rejected. For example, the proof shows that if an offer \( z \) implies a challenge stage equilibrium in which \( \zeta_1(z) = w_1 \) and \( \zeta_h(z) = 0 \), party 1 extracts a premium from party \( h \in \{2, 3\} \) equal to \( w_1 \) in a grand bargain. Likewise, if the opposite is true, party 1 needs to offer party \( h \) a premium of \( w_h \) in order to persuade it to join in the agreement.

The final part of Proposition 3 characterizes the properties of the optimal offer for party 1 that would induce unanimity. Not surprisingly, we observe that party 1 captures a higher share of the surplus as it becomes a more powerful player, either due to a higher status-quo or a higher campaigning budget. The intuition for these effects is as follows: A higher status-quo rent share for party 1 means that the other parties command less, thereby decreasing the amount they need to be compensated for in a grand bargain. Likewise, the more non-proposer parties are away from their ideal ideological points in the status-quo, the lower the compensation they require. On the effect of \( w_1 \) on \( y_1 \), the proof shows that party 1’s optimal unanimity-inducing offer \( z \) is such that if rejected, it would lead to a challenge equilibrium with \( \zeta_1(z) = w_1 \). Thus, \( w_1 \) can be interpreted as party 1’s reward for making an offer that “saves” the non-proposer parties the spending on their groups’ campaigns. Nonetheless, party 1 needs to compensate them for their ideological loss in the form of higher rent shares in proposal \( z \). Therefore, the results indicate that an ideologically-divided parliament always hurts party 1 in a grand bargain.

Having studied the structure of a unanimous agreement in a three-party parliament and how to best get there from party 1’s point of view, the following proposition focuses on the same questions for a referendum:

**Proposition 4.** In the political equilibrium of a three-party parliament, the following are true about inducing a challenge with party \( h \) as the partner and party \( j \) as the opponent of party 1:

1. For any challenge-inducing proposal \( z \), party 1’s expected payoff from a challenge increases as \( y_h^q \) decreases, \( (q - \hat{x}_h)^2 \) increases, and \( \hat{x}_1 \) and \( \hat{x}_h \) get closer.

2. For any challenge-inducing proposal \( z \) for which \( \zeta_h(z) > 0 \), a higher \( w_h \) decreases party 1’s expected payoff from the challenge if \( w_h \) and \( u_1(s) \) are sufficiently high;

3. All else constant, party 1 prefers to partner with party 2 instead of party 3 if
\[ u_2(s) \leq u_3(s); \]
\[ w_2 > w_3 \text{ for any proposal that implies a challenge stage equilibrium with } \zeta_h(z) = 0; \]
\[ w_2 \leq w_3 \text{ whenever } w_h \text{ and } u_1(s) \text{ are sufficiently high, and } w_2 > w_3 \text{ otherwise, for any proposal } z \text{ that implies a challenge stage equilibrium with } \zeta_h(z) > 0. \]

The results in Proposition 4 illustrate party 1’s incentives when deciding on the identity of its partner in a challenge. First, the proposition states that it necessarily increases party 1’s expected payoff from a challenge if its partner has a lower status-quo payoff. This is due to the fact that a party always requires at least its status-quo payoff in order to become party 1’s partner regardless of whether it will contribute to group \( N_Z \)’s campaign spending or become a free-rider in equilibrium. Thus, a lower status-quo payoff makes it more likely for a party to be designated as party 1’s partner in a challenge-inducing proposal.

To gain an intuition for why party 1’s decision on whether to partner with the high or the low-budget party depends on the type of challenge stage equilibrium considered and on the level of resources, note that the amount of a partner’s campaigning resources have two opposing effects on party 1’s expected challenge payoff: In an equilibrium with positive contributions from the partner, a higher \( w_h \) weakly increases the proposal’s winning probability. However, a party also demands a premium over its status-quo payoff from party 1 for agreeing to become an active partner. The analysis indicates that for proposals that imply a challenge with an active partner, the positive effect of a higher \( w_h \) on party 1’s expected challenge payoff due to a higher probability of winning is dominated by its negative effect due to a higher payment to the partner whenever \( w_h \) is too high or party 1’s stakes from the challenge are too low. In this case, a higher \( w_h \) overall decreases party 1’s expected payoff from such a challenge, because the high payment needed to persuade a rich party to become a partner does not justify the increase in party 1’s winning probability. On the other hand, for lower values of \( w_h \) and \( u_1(s) \) that imply high stakes from the challenge, the payment to the partner is justified. In this situation, party 1 would prefer the richer party as its partner.

However, Proposition 4 also indicates that this trade-off between a higher winning probability and a higher partner premium disappears once an equilibrium with a free-rider partner is considered. In these cases, a party can no longer demand a premium for agreeing to become a partner and its budget no longer affects the proposal’s probability.
of winning. However, the opponent’s budget \( w_j \) negatively affects party 1’s expected challenge payoff, giving party 1 the incentive to designate the low-budget party as its opponent.

Given the previous results in Proposition 3 on inducing a grand bargain and the above results on possible challenges, the following proposition presents the main result of this section on party 1’s choice between a grand bargain and a challenge outcome:

**Proposition 5.** In the political equilibrium of a three-party parliament, party 1 becomes more likely to prefer a grand bargain outcome over a challenge as

1. The non-proposer parties command lower status-quo payoffs;
2. The three parties get ideologically closer; and
3. The non-proposer parties’s campaigning budgets become more similar.

The first and the second parts of Proposition 5 are a direct implication of party 1’s unanimity payoff. To see why similar campaigning budgets between the non-proposer parties incentivizes a grand bargain, note that \( w_2 \) and \( w_3 \) do not affect party 1’s unanimity payoff, but determine the proposal needed to induce a given challenge equilibrium. In a challenge stage equilibrium in which the partner also contributes, the premium it demands increases as its resources become more similar to the opponent’s, because this increases the competitiveness of the referendum. Since this decreases party 1’s expected payoff from this challenge, it will be more likely to prefer a grand bargain.

The results on the proposer’s incentives between a grand bargain and a challenge in a three-party parliament mirror those in a two-party parliament. Specifically, the results in these sections indicate that lower status-quo payoffs of the non-proposer parties always incentivize unanimity. Moreover, both sections suggest that a partner’s higher budget can be a blessing in a challenge as long as it is not too high, a result that spans both types of parliaments. However, due to the presence of an additional party that the proposer can play against the other, the results on non-cooperative coalition formation are richer in the three-party parliament setting.
6 Concluding Remarks

This paper developed a model of legislative bargaining over a bill consisting of both an ideology and a distributive component followed by a challenge stage. I addressed the question of how an institutional challenge mechanism such as a referendum affects the parties’ optimal behavior in a parliament. The analysis of a proposer’s incentives between a grand bargain and a challenge indicates that post-bargaining power does not necessarily translate into higher equilibrium payoffs. Although the focus of the model is on legislative bargaining over proposals that can be subsequently challenged, its insights are applicable to other settings, including private sector organizational models. For example, the players in the model can be chosen to represent the board of directors of a corporation, with the chairman as the proposer and shareholders as the voters on proposals not approved with sufficient majority in the board room.

The results here have implications for campaign finance policies. Even though referenda can be both publicly and privately financed in most countries, this model is silent on this issue. The results for both two and three-party parliaments indicate that whether high or low campaigning budgets incentivize grand bargains depend on the parameters of the model. Therefore, if a planner’s goal is to propagate unanimously-approved deals in the parliament over costly challenge procedures, the appropriate campaign finance policy will depend on the status-quo commanded by each party and their current resources.

There exists a number of directions in which the model can be extended. For example, while I assumed that all the parameters on campaigning budgets, ideal ideological points, and status-quo payoffs are common knowledge, incorporating uncertainty with regards to either one of these parameters can be a natural extension. Although I believe that complete information is a more realistic setting in this model of a public interaction, incomplete information might be a better depiction of reality in private interaction models such as the corporate board example. Extending the model to N players for a more general setting or specifically modeling voters with ideological preferences may also yield interesting results on the dynamics of non-cooperative coalition formation.

Finally, this model does not entertain the possibility of new rounds of bargaining following a challenge stage. However, in reality, political processes might reconsider the same measures. Although I believe that introducing additional cycles of bargaining and challenge stages might make the model much less tractable with little additional
insights, it might be a useful endeavor for the purpose of capturing the dynamic aspects of similar political processes. Similarly, an additional stage of legislative elections would make voters strategic by giving them control over the identity of the proposer.

It is important to stress that I do not make any efficiency arguments in favor of one policy over another. For example, if the results suggest caps on campaign financing to incentivize grand bargains for certain ranges of parameters, this study can still not answer the question of how this policy would affect voter welfare. Any attempt to answer this question would require a normative exercise I refrain from.
7 Appendix A

Proof of Lemma 1. Based on 8 and 9, the first-order conditions for the parties’ optimal campaign spending choices are given by

\[
\epsilon_1(z) \left[ \frac{\zeta_2(z)}{(\zeta_1(z) + \zeta_2(z))^2} \right] - 1 \begin{cases} 
\geq 0 & \text{if } \zeta_1(z) > w_1 \\
= 0 & \text{if } \zeta_1(z) \in [0, w_1] \\
\leq 0 & \text{if } \zeta_1(z) = 0;
\end{cases}
\]

and

\[
\epsilon_2(z) \left[ \frac{\zeta_1(z)}{(\zeta_1(z) + \zeta_2(z))^2} \right] - 1 \begin{cases} 
\geq 0 & \text{if } \zeta_2(z) > w_2 \\
= 0 & \text{if } \zeta_2(z) \in [0, w_2] \\
\leq 0 & \text{if } \zeta_2(z) = 0.
\end{cases}
\]

Solving for \(\zeta_1(z)\) and \(\zeta_2(z)\) based on 13 and 14 implies that the unique pair of campaign spending rules \((\zeta_1(z), \zeta_2(z))\) is given by one of the following four equilibrium candidates, depending on the outcome of the bargaining stage:

1. \((\zeta_1(z), \zeta_2(z)) = (w_1, w_2)\) if and only if \(\epsilon_1(z) \geq \frac{(w_1+w_2)^2}{w_2} \) and \(\epsilon_2(z) \geq \frac{(w_1+w_2)^2}{w_1}\).

2. \((\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1 \epsilon_2(z)} - w_1)\) if and only if \(\epsilon_1(z) \geq \frac{w_1 \epsilon_2(z)}{\sqrt{w_1 \epsilon_2(z)} - w_1}\) and \(\epsilon_2(z) \leq \frac{(w_1+w_2)^2}{w_1}\).

3. \((\zeta_1(z), \zeta_2(z)) = (\sqrt{w_2 \epsilon_1(z)} - w_2, w_2)\) if and only if \(\epsilon_1(z) \leq \frac{(w_1+w_2)^2}{w_2}\) and \(\epsilon_2(z) \geq \frac{w_2 \epsilon_1(z)}{\sqrt{w_2 \epsilon_1(z)} - w_2}\).

4. \((\zeta_1(z), \zeta_2(z)) = \left( \frac{\epsilon_1(z)^2 \epsilon_2(z)}{[\epsilon_1(z) + \epsilon_2(z)]^2}, \frac{\epsilon_1(z) \epsilon_2(z)^2}{[\epsilon_1(z) + \epsilon_2(z)]^2} \right)\) if and only if \(\frac{\epsilon_1(z)^2 \epsilon_2(z)}{[\epsilon_1(z) + \epsilon_2(z)]^2} < w_1\) and \(\frac{\epsilon_1(z) \epsilon_2(z)^2}{[\epsilon_1(z) + \epsilon_2(z)]^2} < w_2\).

Notice that if the challenge stage equilibrium is such that \(\zeta_k(z) = w_k\), then \(\zeta_k(z)\) is constant in the value of \(\epsilon_k(z)\). On the other hand, if \(\zeta_k(z) = \sqrt{w_{-k} \epsilon_k(z)} - w_{-k}\) or if we have an interior equilibrium as characterized in item four above, then \(\zeta_k(z)\) is increasing in the value of \(\epsilon_k(z)\). This is straightforward to see for the first case. To see this for the interior challenge stage equilibrium, differentiate \(\zeta_k(z)\) as characterized in item four
with respect to the value of $\epsilon_k(z) \equiv \bar{\epsilon}_k$ to get
\[
\frac{(2\bar{\epsilon}_k \bar{\epsilon}_{-k})(\bar{\epsilon}_k + \bar{\epsilon}_{-k})^2 - (2\epsilon_k^2 \epsilon_{-k})(\bar{\epsilon}_k + \bar{\epsilon}_{-k})}{(\epsilon_k + \epsilon_{-k})^4},
\]
(15)
whose both numerator and denominator are positive. Hence, we conclude that the interior equilibrium level of campaign spending of each party $k$ is increasing in the value of $\epsilon_k(z)$. This completes the proof of Lemma 1.

\[\Box\]

\textit{Proof of Proposition 1.} Using backward induction, I first characterize the equilibrium acceptance strategy $a_2(z)$ of party 2 for any given proposal $z$.

If party 2 accepts party 1’s proposal $z$, its payoff would be given by $u_2(z)$ with certainty. Since it is risk-neutral, party 2 will accept any offer that yields a sure payoff of $u_2(z)$ that is at least as great as its expected payoff from the challenge stage equilibrium that would be observed based on $\epsilon_1(z)$ and $\epsilon_2(z)$.

Given $w_1$, $w_2$, and the status-quo bill $s$, suppose party 1 makes an offer $z$ such that $\epsilon_k(z) \geq \frac{(w_1 + w_2)^2}{w_{-k}}$ for both $k$. If rejected, this offer would imply $(\zeta_1(z), \zeta_2(z)) = (w_1, w_2)$. Therefore, given $\rho_1(z) = Z$ and $\rho_2(z) = S$ in any challenge stage equilibrium, this proposal $z$ implies an expected payoff for party 2 from the challenge stage given by
\[u_2(s) + \left(\frac{w_1}{w_1 + w_2}\right)[u_2(z) - u_2(s)] - w_2.\]
(16)

Comparing the sure payoff $u_2(z)$ with 16 implies that party 2 accepts $z$ if and only if $u_2(z) \geq u_2(s) - (w_1 + w_2)$, which can also be written as $\epsilon_2(z) \leq w_1 + w_2$. However, since the proposal $z$ under consideration is such that $\epsilon_2(z) \geq \frac{(w_1 + w_2)^2}{w_1}$ and $\frac{(w_1 + w_2)^2}{w_1} > w_1 + w_2$, the acceptance criteria can never be satisfied. Therefore, any proposal $z$ that would pave the way for a challenge stage with $(\zeta_1(z), \zeta_2(z)) = (w_1, w_2)$ if rejected will be rejected by party 2.

Second, suppose party 1 makes an offer $z$ such that the conditions for a challenge stage equilibrium in which $(\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1 \epsilon_2(z)} - w_1)$ as listed in item two in the proof of Lemma 1 are satisfied. This offer implies the following expected payoff for party 2 from the challenge stage:
\[u_2(s) + \sqrt{\frac{w_1}{\epsilon_2(z)}}[u_2(z) - u_2(s)] - \sqrt{w_1 \epsilon_2(z)} + w_1.\]
(17)
Comparing the sure payoff \( u_2(z) \) with \( 17 \) implies that party 2 accepts \( z \) if and only if

\[
 u_2(z) \geq u_2(s) + \frac{\sqrt{\epsilon_2(z)} - \sqrt{\epsilon_1(z)}}{\epsilon_2(z) - \sqrt{\epsilon_1(z)}},
\]

(18)

where the last term is negative since \( \zeta(z) = \sqrt{\epsilon_1(z)} - \epsilon_1(z) \). Re-arranging 18 yields

\[
 \epsilon_2(z) \leq \frac{(\sqrt{w_1\epsilon_2(z)} - w_1)\sqrt{\epsilon_2(z)}}{\epsilon_2(z) - \sqrt{\epsilon_1(z)}},
\]

(19)

which reduces to \( \epsilon_2(z) \leq w_1 \). Therefore, party 2 will accept any proposal \( z \) that would imply a subsequent challenge stage with \( (\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1\epsilon_2(z)} - w_1) \) as long as \( \epsilon_2(z) \leq w_1 \).

Third, suppose party 1 makes an offer \( z \) such that the equilibrium campaign spending if \( z \) were rejected is given by \( (\zeta_1(z), \zeta_2(z)) = (\sqrt{w_2\epsilon_1(z)} - w_2, w_2) \). The implied expected challenge stage payoff for party 2 is given in this case by

\[
 u_2(s) + \frac{\sqrt{w_2\epsilon_1(z)} - w_2}{\sqrt{w_2\epsilon_1(z)}}[u_2(z) - u_2(s)] - w_2.
\]

(20)

Then, party 2 accepts any offer \( z \) that yields a sure payoff of \( u_2(z) \) that is at least as great as 20, which reduces to the condition that \( z \) must satisfy \( \epsilon_2(z) \leq \sqrt{w_2\epsilon_1(z)} \).

However, since the proposal \( z \) under consideration is such that \( \epsilon_2(z) \geq \frac{w_2\epsilon_1(z)}{\sqrt{w_2\epsilon_1(z)} - w_2} \), the acceptance criteria can never be satisfied, because \( \sqrt{w_2\epsilon_1(z)} < \frac{w_2\epsilon_1(z)}{\sqrt{w_2\epsilon_1(z)} - w_2} \). Therefore, party 2 will reject all offers that would subsequently lead to a challenge stage with \( (\zeta_1(z), \zeta_2(z)) = (\sqrt{w_2\epsilon_1(z)} - w_2, w_2) \).

Finally, suppose party 1’s offer \( z \) is such that the equilibrium campaign spending in any challenge to \( z \) would be given by the interior equilibrium as listed in item four in the proof of Lemma 1. Constructing the expected payoff from the challenge stage as in the above cases yields the condition that \( z \) must satisfy \( \epsilon_2(z) \leq \zeta_1(z) + \zeta_2(z) \) in order to be accepted by party 2. Plugging in the equilibrium values of \( \zeta_1(z) \) and \( \zeta_2(z) \) into this condition yields

\[
 \epsilon_2(z) \leq \frac{\epsilon_1(z)\epsilon_2(z)}{\epsilon_1(z) + \epsilon_2(z)},
\]

(21)

which reduces to the condition that party 2 will accept any offer \( z \) for which \( u_2(z) \geq u_2(s) \) whenever the subsequent challenge stage equilibrium if \( z \) is rejected would be an interior
one.

Bringing together the above characterization of party 2’s acceptance rules for each possible challenge stage equilibrium, we observe that any proposal $z$ for which $\zeta_2(z) = w_2$ is rejected (although these are not the only offers that will be rejected). In addition, whenever $z$ is such that $\zeta_2(z) < w_2$, party 2 accepts any offer for which $\epsilon_2(z) \leq w_1$ if $\zeta_1(z) = w_1$ and any offer for which $u_2(z) \geq u_2(s)$ if $\zeta_1(z) < w_1$. This proves part 1 of Proposition 1.

Given the equilibrium acceptance strategy of party 2 for any proposal $z$, I now solve for party 1’s optimal proposals. In the rest of Proposition 1, part 2 solves for the best way to induce unanimity, whereas part 3 solves for the optimal proposal that would push the game to the challenge stage.

Suppose that party 1 will make an offer that will get party 2’s acceptance, thereby avoiding a challenge stage. The proof of part 1 indicated that there exist two methods with which party 1 can induce unanimity in the parliament: By offering $z$ such that a) $\epsilon_1(z) = \frac{w_1 \epsilon_2(z)}{\sqrt{w_1 \epsilon_2(z) - w_1}}$ and $\epsilon_2(z) \leq w_1$; or b) $\frac{\epsilon_k(z)^2 \epsilon_k(z)}{(\epsilon_1(z) + \epsilon_2(z))^2} < w_k$ for both $k$ and $u_2(z) \geq u_2(s)$. Since the first method implies that party 1 only needs to propose a $z$ for which $u_2(z) = u_2(s) - w_1$, whereas the second method requires $u_2(z) = u_2(s)$ for acceptance, party 1 would choose the first method if it wanted to induce unanimity.\(^9\)

To solve for the specifics of this offer, party 1 maximizes $u_1(z)$ subject to party 1’s acceptance constraint $u_2(z) \geq u_2(s) - w_1$ and the technical constraint $z \in [0, 1] \times Y$. The Lagrangian of this problem can be written as follows:

$$L = -(x - \hat{x})^2 + \alpha y_1 + \lambda_1[-(x - \hat{x})^2 + \alpha(1 - y_1) - u_2(s) + w_1]$$

$$+ \mu_1 x - \mu_2 (x - 1) + \gamma_1 y_1 - \gamma_2 (y - 1).$$

The first-order conditions for 22 are $x \in [0, 1], y_1 \in [0, 1], \lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \gamma_1 \geq 0, \gamma_2 \geq 0$,

$$-2(x - \hat{x}_1) - 2\lambda_1(x - \hat{x}_2) + (\mu_1 - \mu_2) \leq 0; \quad (23)$$

$$-2(x - \hat{x}_1) - 2\lambda_1(x - \hat{x}_2) + (\mu_1 - \mu_2) \geq 0; \quad (24)$$

\(^9\)To rule out extreme parameter cases such that no proposal $z$ would justify $\zeta_1(z) = w_1$, I impose the restriction that if $\epsilon_2(z) \leq \frac{(w_1 + w_2)^2}{w_1}$, then $\epsilon_1(z) \geq \frac{(w_1)^2(w_1 + w_2)^2}{w_1}$, which implies $\frac{\epsilon_1(z)}{\epsilon_2(z)} \geq (w_1)^2$. This assumption ensures that we can restrict our attention to the first of the two methods for inducing unanimity.
\[
\alpha - \lambda_1 \alpha + (\gamma_1 - \gamma_2) \leq 0; \quad (25)
\]

\[
[\alpha - \lambda_1 \alpha + (\gamma_1 - \gamma_2)]y_1 = 0; \quad (26)
\]

\[-(x - \hat{x}_2)^2 + \alpha(1 - y_1) - u_2(s) + w_1 \geq 0; \quad (27)
\]

\[-(x - \hat{x}_2)^2 + \alpha(1 - y_1) - u_2(s) + w_1] \lambda_1 = 0; \quad (28)
\]

along with \(\mu_1 x = 0; \mu_2 (1 - x) = 0; \gamma_1 y_1 = 0; \) and \(\gamma_2 (1 - y_1) = 0.\) An interior solution to this problem entails \(\mu_1 = \mu_2 = \gamma_1 = \gamma_2 = 0,\) and \(\lambda_1 = 1\) based on 26, yielding

\[x = \frac{\hat{x}_1 + \hat{x}_2}{2}.\] (29)

Solving for \(y_2\) using the fact that party 1 will not make an offer \(z\) that gives party 2 any higher utility than is needed for acceptance, \(u_2(z) = u_2(s) - w_1\) implies

\[y_2 = \alpha^{-1} \left[ \left( \frac{\hat{x}_1 + \hat{x}_2}{2} \right)^2 - (q - \hat{x}_2)^2 + \alpha y_2^q - w_1 \right]. (30)\]

Therefore, an equilibrium proposal \(z\) characterized by the ideology component in 29 and the rent component with \(y_2\) as given in 30 and \(y_1 = 1 - y_2\) induces an optimal unanimity outcome for party 1. Specifically, \(y_1 = 1 - y_2\) is given by

\[y_1 = \alpha^{-1} \left[ -\left( \frac{\hat{x}_1 + \hat{x}_2}{2} \right)^2 + (q - \hat{x}_2)^2 + \alpha y_1^q + w_1 \right]. (31)\]

Therefore, the difference between the rent shares of the two parties is given by

\[y_1 - y_2 = \alpha^{-1} \left[ -\left( \frac{\hat{x}_1 + \hat{x}_2}{2} \right)^2 + 2(q - \hat{x}_2)^2 + \alpha (y_1^q - y_2^q) + 2w_1 \right]. (32)\]

Notice that this difference increases as party 2’s status-quo payoff decreases and \(w_1\) increases.

Now consider possible corner solutions to this maximization problem. First, I claim that there exists no solution with \(x = 0\) or \(x = 1.\) To see this, first let \(\mu_1 > 0\) and \(\mu_2 = \gamma_1 = \gamma_2 = 0.\) This yields \(\lambda_1 = 1\) as before, resulting in the equality \(2\hat{x}_1 + 2\hat{x}_2 + \mu_1 = 0.\) Since this would imply \(\mu_1 < 0,\) the desired result is achieved. Second, let \(\mu_2 > 0\) and \(\mu_1 = \gamma_1 = \gamma_2 = 0.\) This situation yields the equality \(-4 + 2\hat{x}_1 + 2\hat{x}_2 - \mu_2 = 0,\) implying that \(\mu_2\) must be negative. Hence, we can conclude that the optimal ideology component
of $z$ must be such that $x \in (0, 1)$.

Second, I claim that solutions with $y_1 = 0$ or $y_1 = 1$ are possible for certain values of $\alpha$. Suppose $\gamma_1 > 0$ and $\gamma_2 = 0$. With $\mu_1 = \mu_2 = 0$, this yields $\alpha - \lambda_1 \alpha + \gamma_1 < 0$, or $\lambda_1 > \frac{\alpha + \gamma_1}{\alpha}$. Then, the condition $(x - \hat{x}_1) + \lambda_1 (x - \hat{x}_2) = 0$ implies

$$\frac{x - \hat{x}_1}{x - \hat{x}_2} = -\lambda_1 < -\left(1 + \frac{\gamma_1}{\alpha}\right),$$

which can hold for small values of $\alpha$, yielding $y_1 = 0$. In this situation, party 1 chooses $x$ closer to $\hat{x}_1$. Likewise, letting $\gamma_2 > 0$ implies

$$\frac{x - \hat{x}_1}{x - \hat{x}_2} = -\lambda_1 < -\left(1 - \frac{\gamma_2}{\alpha}\right),$$

which can hold for larger values of the parameter $\alpha$, yielding $y_1 = 1$. Here, party 1 chooses $x$ closer to $\hat{x}_2$ in order to secure party 2’s acceptance. This concludes the proof of part 2 of Proposition 1.

For part 3, suppose that party 1 will make an offer that will lead to a challenge on the equilibrium path. Note that of the four methods with which party 1 can push the bill into a challenge as summarized in part 1, two of these methods involve proposals that would imply $\zeta_2(z) = w_2$ in the referendum. In this case, the optimal $z$ is such that $x = \hat{x}_1$, $y_1 = 1$, and $y_2 = 0$. This is due to the fact that once party 2 starts spending a constant sum of $w_2$, the proposal $z$ no longer affects the probability of winning for party 1. Therefore, party 1 maximizes its expected payoff from the referendum by maximizing the value of $\epsilon_1(z)$.

To see when inducing a challenge stage equilibrium with $\zeta_2(z) < w_2$ would be preferred to one with $\zeta_2(z) = w_2$, I focus on the challenge stage equilibrium in which $(\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1 \epsilon_2(z) - w_1})$, which arises if the rejected proposal $z$ is such that $\epsilon_1(z) \geq \frac{w_1 \epsilon_2(z)}{\sqrt{w_1 \epsilon_2(z) - w_1}}$ and $\epsilon_2(z) \in \left(w_1, \left(\frac{w_1 + w_2}{w_1}\right)^2\right)$. For any proposal $z$ that satisfies these conditions, the expected payoff to party 1 from this challenge stage equilibrium is given by

$$u_1(s) + \sqrt{\frac{w_1}{\epsilon_2(z)}} \epsilon_1(z) - w_1,$$

maximizing which subject to the above conditions yields $x = \frac{\hat{x}_1 + \hat{x}_2}{2}$.

Party 1 prefers this challenge stage equilibrium with proposal $z$ to the one in which

\footnote{This is also justified by the parameter restriction imposed in Footnote 1.}
\((\zeta_1(z), \zeta_2(z)) = (w_1, w_2)\) whenever
\[
\sqrt{w_1 \epsilon_1(z)} \geq \frac{w_1}{w_1 + w_2} (\alpha - u_1(s)). \tag{36}
\]

Note that since \(z\) is such that \(\epsilon_2(z) \in \left( w_1, \frac{(w_1 + w_2)^2}{w_1} \right)\), the probability of winning is always at least as high for party 1 on the left-hand side of 36 as on the right-hand side of it. Therefore, this inequality needs the proposal \(z\) that would induce the challenge stage equilibrium with \((\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1 \epsilon_2(z)} - w_1)\) to be such that
\[
\epsilon_1(z) \in \left[ \frac{w_1}{w_1 + w_2} (\alpha - u_1(s)), (\alpha - u_1(s)) \right] \tag{37}
\]
in order to hold. Therefore, if the optimal proposal that would induce this challenge implies \(\epsilon_1(z) < \frac{w_1}{w_1 + w_2} (\alpha - u_1(s))\), party 1 prefers the challenge stage equilibrium with \(\zeta_2(z) = w_2\). Since the optimal proposal to induce a challenge with \(\zeta_2(z) < w_2\) involves equal compromise on ideology, \(\epsilon_1(z)\) decreases as \((\hat{x}_1 - \hat{x}_2)^2\) increases. This proves part 3 of Proposition 1.\(^{11}\)

Proof of Proposition 2. If party 1 induces unanimity by offering \(u_2(z) = u_2(s) - w_1\), its payoff is given by
\[
u_1(z) = -\left(\frac{\hat{x}_2 - \hat{x}_1}{2}\right)^2 - \left(\frac{\hat{x}_1 + \hat{x}_2}{2}\right)^2 + (q - \hat{x}_2)^2 + \alpha y_1^q + w_1. \tag{38}\]

Suppose the parties are sufficiently distant ideologically so that party 1 prefers a challenge stage equilibrium with \(\zeta_2(z) = w_2\). With the optimal proposal given by \(z = (\hat{x}_1, 1, 0)\), party 1’s maximum expected payoff from this challenge becomes
\[
u_1(s) + \frac{w_1}{w_1 + w_2} [\alpha - u_1(s)] - w_1 \tag{39}\]
if \(\zeta_1(z) = w_1\), and
\[
u_1(s) + \left(1 - \sqrt{\frac{w_2}{\alpha - u_1(s)}}\right) [\alpha - u_1(s)] - \sqrt{w_2(\alpha - u_1(s))} + w_2 \tag{40}\]
if \(\zeta_1(z) = \sqrt{w_2 \epsilon_1(z)} - w_2\).

\(^{11}\)Carrying out similar comparisons between other types of challenge stage equilibria yield similar results and hence are not repeated here.
Comparing 38 first with 39 suggests that party 1 becomes more likely to prefer a settlement over a challenge for low values of $u_2(s)$, and high values of $w_1$ and $w_2$. Comparing 38 with 40 confirms the relationship with $u_2(s)$ and $w_1$. However, differentiating 40 with respect to $w_2$ indicates that higher values of $w_2$ make settlement more likely to be preferred only if $w_2 < \alpha - u_1(s)$.

To complete the proof, suppose that the parties are ideologically closer so that party 1 would prefer a challenge stage equilibrium with $\zeta_2(z) < w_2$. Focusing on the equilibrium with $(\zeta_1(z), \zeta_2(z)) = (w_1, \sqrt{w_1 \epsilon_2(z)} - w_1)$, party 1’s expected payoff from this referendum is as given in 35, where $z$ is such that $\epsilon_1(z) \geq \frac{w_1 \epsilon_2(z)}{\sqrt{w_1 \epsilon_2(z)} - w_1}$ and $\epsilon_2(z) \in \left( w_1, \frac{(w_1 + w_2)^2}{w_1} \right)$. Comparing 38 with 35 confirms the above results on $u_2(s)$ and $w_1$. Therefore, we can conclude that a lower $u_2(s)$ and a higher $w_1$ unambiguously make settlement more likely to be observed. This completes the proof of Proposition 2.

**Proof of Lemma 2.** The proof is an application of the main result in Baik (2008) for players with a budget constraint.

Based on 11, $C_{Z}^1(z)$ satisfies

$$
\epsilon_1(z) \left[ \frac{C_S(z)}{(C_{Z}^1(z) + C_S(z))^2} \right] - 1 \begin{cases} 
\geq 0 & \text{if } C_{Z}^1(z) > w_1 + w_h \\
= 0 & \text{if } C_{Z}^1(z) \in [0, w_1 + w_h] \\
< 0 & \text{if } C_{Z}^1(z) = 0.
\end{cases} \tag{41}
$$

Similarly, based on 12, $C_{Z}^h(z)$ satisfies

$$
\epsilon_h(z) \left[ \frac{C_S(z)}{(C_{Z}^h(z) + C_S(z))^2} \right] - 1 \begin{cases} 
\geq 0 & \text{if } C_{Z}^h(z) > w_1 + w_h \\
= 0 & \text{if } C_{Z}^h(z) \in [0, w_1 + w_h] \\
< 0 & \text{if } C_{Z}^h(z) = 0.
\end{cases} \tag{42}
$$

Accordingly, the individual campaign spending of parties 1 and $h$ must satisfy the fol-
lowing first-order conditions in equilibrium for any given $C_S(z) = \zeta_j(z)$:

$$
\epsilon_1(z) \left[ \frac{C_S(z)}{(\zeta_1(z) + \zeta_h(z) + C_S(z))^2} \right] - 1 = \begin{cases} 
\geq 0 & \text{if } \zeta_1(z) > w_1 \\
= 0 & \text{if } \zeta_1(z) \in [0, w_1] \\
\leq 0 & \text{if } \zeta_1(z) = 0;
\end{cases}
$$

$$
\epsilon_h(z) \left[ \frac{C_S(z)}{(\zeta_1(z) + \zeta_h(z) + C_S(z))^2} \right] - 1 = \begin{cases} 
\geq 0 & \text{if } \zeta_h(z) > w_h \\
= 0 & \text{if } \zeta_h(z) \in [0, w_h] \\
\leq 0 & \text{if } \zeta_h(z) = 0.
\end{cases}
$$

First, suppose that $C^1_Z(z) \leq w_1$ so that solving 41 yields $C^1_Z(z) = \sqrt{\epsilon_1(z)C_S(z)} - C_S(z)$. Then, the individual best response of party 1 to its partner must also be less than or equal to $w_1$. Furthermore, it must equal $C^1_Z(z)$. By the assumption in Lemma 2 that $\epsilon_1(z) \geq \epsilon_h(z)$, it must be true that $C^1_Z(z) \geq C^1_Z(z)$. Therefore, the best response of party $h$ to party 1’s best response of choosing $C^1_Z(z)$ for any given $\zeta_h(z)$ is to spend a zero amount on the group’s campaign. In equilibrium, simultaneous best responding implies $\zeta_1(z) = C^1_Z(z)$ and $\zeta_2(z) = 0$ for a total equilibrium campaign spending of $C_Z(z) = C^1_Z(z)$. This proves part 1 of Lemma 2.

For part 2, suppose that $C^1_Z(z) \geq w_1 + w_h$, which implies $C^1_Z(z) \geq w_1 + w_h$. Then, the individual best response of each party to the other must be greater than its respective budget. This implies that we must have $\zeta_1(z) = w_1$ and $\zeta_h(z) = w_h$ in equilibrium, for a total campaign spending of $C_Z(z) = w_1 + w_h$. This proves part 2 of Lemma 2.

For the final part of the lemma, suppose the proposal $z$ is such that $C^1_Z(z) > w_1$ and $C^1_Z(z) \leq w_1 + w_h$. First, consider the case in which $C^1_Z(z) \leq w_1$. For any $\zeta_1(z) \in [C^1_Z(z), w_1]$, party $h$’s individual best response to party 1 is to choose a zero amount of campaign spending since $\zeta_1(z) \geq C^1_Z(z)$. This would imply a total campaign spending of $C_Z(z) \in [C^1_Z(z), w_1]$. However, since $C^1_Z(z) > w_1$, this cannot be optimal for party 1. Specifically, party 1 would have an incentive to increase its spending up to $w_1$. Similarly, for any $\zeta_1(z) < C^1_Z(z)$, party $h$ best responds by choosing $\zeta_h(z) = C^1_Z(z) - \zeta_1(z)$, resulting in a total campaign spending of $C_Z(z) = C^1_Z(z)$. However, since $C^1_Z(z) > C^1_Z(z)$, this also cannot be optimal for party 1. Therefore, the only equilibrium occurs at $\zeta_1(z) = w_1$ and $\zeta_2(z) = 0$, yielding $C_Z(z) = w_1$. In this case, party 1 cannot increase its individual spending since it is already exhausting its budget and does not have an incentive to
decrease it since $C^1_Z(z) > w_1$. Party $h$ does not have an incentive to increase its spending either since $C^h_Z(z) \leq w_1$. Therefore, if $C^1_Z(z) > w_1$ and $C^h_Z(z) \leq w_1$, the equilibrium is such that $\zeta_1(z) = w_1$ and $\zeta_2(z) = 0$.

Second, consider the case in which $w_1 < C^h_Z(z) \leq w_1 + w_2$. We know that any $\zeta_1(z) < w_1$ cannot be an equilibrium, since party $h$ would best respond to it by choosing $\zeta_h(z) = C^h_Z(z) - \zeta_1(z)$ and the resulting total campaign spending $C_Z(z) = C^h_Z(z)$ would be suboptimal from party 1’s point of view. Specifically, party 1 would have an incentive to increase its spending from $\zeta_1(z) < w_1$ to $w_1$. Therefore, the only equilibrium is such that $\zeta_1(z) = w_1$ and $\zeta_h(z) = C^h_Z(z) - w_1$, yielding the same $C_Z(z) = C^h_Z(z)$. This completes the proof of part 3 of the lemma.

Proof of Lemma 3. Lemma 2 characterized the optimal campaign spending of parties 1 and $h$ within the group $N_Z$. For any given $C_Z(z)$, the optimal campaign spending of the only member of group $N_S$, party $j$, is such that

$$\zeta_j(z) \in \arg \max_{C \in [0, w_j]} \frac{C_Z(z)}{C + C_Z(z)} u_j(z) + \frac{C}{C + C_Z(z)} u_j(s) - C. \quad (45)$$

The first-order conditions that the optimal $\zeta_j(z)$ needs to satisfy are given by

$$\epsilon_j(z) \left( \frac{C_Z(z)}{(C_Z(z) + \zeta_j(z))^2} \right) - 1 \begin{cases} \geq 0 & \text{if } \zeta_j(z) > w_j \\ = 0 & \text{if } \zeta_j(z) \in [0, w_j] \\ \leq 0 & \text{if } \zeta_j(z) = 0, \end{cases} \quad (46)$$

where $C_Z(z) = \zeta_1(z) + \zeta_h(z)$. Note that party $j$ cares only about $C_Z(z)$ and not about how its burden is shared among the members of group $N_Z$. Thus, for any given $C_Z(z)$, party $j$ best responds by choosing a campaign spending equal to either $w_j$ or $\sqrt{\epsilon_j(z)C_Z(z) - C_Z(z)}$, whichever is smaller.

To solve for the best response of group $N_Z$ to any given amount of $C_S(z)$, first suppose the proposal $z$ is such that $\epsilon_1(z) \geq \epsilon_h(z)$. Based on 41 and 42, the best response
\( C_Z(\zeta) \) in this case is given by

\[
C_Z(\zeta) = \begin{cases} 
\sqrt{\epsilon_1(\zeta)C_S(\zeta) - C_S(\zeta)} & \text{if } C^1_Z(\zeta) < w_1 \\
w_1 + w_h & \text{if } C^h_Z(\zeta) \geq w_1 + w_h \\
\max\{w_1, C^h_Z(\zeta)\} & \text{if } C^1_Z(\zeta) > w_1 \text{ and } C^h_Z(\zeta) \leq w_1 + w_h.
\end{cases}
\]

(47)

On the other hand, if the proposal \( z \) is such that \( \epsilon_h(z) \geq \epsilon_1(z) \), then the best response \( C_Z(\zeta) \) to any given \( C_S(\zeta) \) becomes

\[
C_Z(\zeta) = \begin{cases} 
\sqrt{\epsilon_h(z)C_S(\zeta) - C_S(\zeta)} & \text{if } C^h_Z(\zeta) < w_h \\
w_1 + w_h & \text{if } C^1_Z(\zeta) \geq w_1 + w_h \\
\max\{w_h, C^1_Z(\zeta)\} & \text{if } C^h_Z(\zeta) > w_h \text{ and } C^1_Z(\zeta) \leq w_1 + w_h.
\end{cases}
\]

(48)

Thus, solving for the unique pure-strategy equilibrium of the challenge stage in which both groups are simultaneously best responding to each other yields the following candidates for the equilibrium triplet \((\zeta_1(z), \zeta_h(z), \zeta_j(z))\):

1. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\sqrt{\epsilon_1(z)w_j - w_j}, 0, w_j)\) if and only if \( \epsilon_1(z) \geq \epsilon_h(z) \); \( \epsilon_1(z) \leq \frac{(w_1 + w_j)^2}{w_j} \); and \( \epsilon_j(z) \geq \frac{\epsilon_1(z)w_j}{\sqrt{\epsilon_1(z)w_j - w_j}} \).

2. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (w_1, w_h, w_j)\) if and only if \( \epsilon_1(z) \geq \frac{(\sum_k w_k)^2}{w_j} \); \( \epsilon_h(z) \geq \frac{(\sum_k w_k)^2}{w_j} \); and \( \epsilon_j(z) \geq \frac{(\sum_k w_k)^2}{w_j} \).

3. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (w_1, \max\{\sqrt{\epsilon_h(z)w_j - w_j} - w_1, 0\}, w_j)\) if and only if \( \epsilon_1(z) \geq \epsilon_h(z) \); \( \epsilon_1(z) \geq \frac{(w_1 + w_j)^2}{w_j} \); \( \epsilon_h(z) \leq \frac{(\sum_k w_k)^2}{w_j} \); and \( \epsilon_j(z) \geq \max\{\frac{\epsilon_h(z)w_j}{\sqrt{\epsilon_h(z)w_j - w_j}}, w_j\} \).

4. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (0, \sqrt{\epsilon_h(z)w_j - w_j}, w_j)\) if and only if \( \epsilon_h(z) \geq \epsilon_1(z) \); \( \epsilon_h(z) \leq \frac{(w_1 + w_j)^2}{w_j} \); and \( \epsilon_j(z) \geq \frac{\epsilon_h(z)w_j}{\sqrt{\epsilon_h(z)w_j - w_j}} \).

5. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\max\{\sqrt{\epsilon_1(z)w_j - w_j} - w_h, 0\}, w_h, w_j)\) if and only if \( \epsilon_h(z) \geq \epsilon_1(z) \); \( \epsilon_1(z) \leq \frac{(\sum_k w_k)^2}{w_j} \); \( \epsilon_h(z) \geq \frac{(w_1 + w_j)^2}{w_j} \); and \( \epsilon_j(z) \geq \max\{\frac{\epsilon_1(z)w_j}{\sqrt{\epsilon_1(z)w_j - w_j}}, \frac{(w_1 + w_j)^2}{w_h}\} \).

6. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\sqrt{\epsilon_1(z)^2 \epsilon_h(z)} \frac{\epsilon_1(z)w_j}{(\epsilon_1(z) + \epsilon_j(z))^2}, 0, \sqrt{\epsilon_1(z)^2 \epsilon_h(z)} \frac{\epsilon_1(z)w_j}{(\epsilon_1(z) + \epsilon_j(z))^2})\) if and only if \( \epsilon_1(z) \geq \epsilon_h(z); \)

\[\frac{\epsilon_1(z)^2 \epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2} < w_1; \text{ and } \frac{\epsilon_1(z)w_j}{(\epsilon_1(z) + \epsilon_j(z))^2} < w_j.\]
7. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (w_1, w_h, \sqrt{\epsilon_j(z)(w_1 + w_h) - w_1 - w_h})\) if and only if \(\sqrt{e(z)} \geq \frac{\sqrt{(w_1 + w_h)e_j(z)}}{\epsilon_j(z)}\); and \(\epsilon_j(z) \leq \frac{(\sum w_g)^2}{w_1 + w_h}\), where \(\bar{e}(z) = \max\{\epsilon_1(z), \epsilon_h(z)\}\).

8. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = \left( w_1, \max\{\frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2}, 0\}, \max\{\frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2}, \sqrt{\epsilon_j(z)w_1 - w_1}\} \right)\)
if and only if \(\epsilon_1(z) \geq \epsilon_h(z)\); \(\frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2} > w_1; \frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2} < w_1 + w_h;\) and \(\max\{\frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2}, \sqrt{\epsilon_j(z)w_1 - w_1}\} < w_j\).

9. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = \left( 0, \frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2}, \frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2}\right)\)
if and only if \(\epsilon_h(z) \geq \epsilon_1(z)\); \(\frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2} < w_h; \frac{\epsilon_h(z)^2\epsilon_j(z)}{(\epsilon_h(z) + \epsilon_j(z))^2} < w_j\).

10. \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = \left( \max\{\frac{\epsilon_1(z)^2\epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2}, 0\}, w_h, \max\{\frac{\epsilon_1(z)^2\epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2}, \sqrt{\epsilon_j(z)w_h - w_h}\} \right)\)
if and only if \(\epsilon_h(z) \geq \epsilon_1(z)\); \(\frac{\epsilon_1(z)^2\epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2} > w_h; \frac{\epsilon_1(z)^2\epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2} < w_1 + w_h;\) and \(\max\{\frac{\epsilon_1(z)^2\epsilon_j(z)}{(\epsilon_1(z) + \epsilon_j(z))^2}, \sqrt{\epsilon_j(z)w_h - w_h}\} < w_j\).

In each of these equilibrium candidates, it can be observed that \(\zeta_k(z)\) is increasing as the value of \(\epsilon_k(z)\) increases. In addition, party \(h\) free-rides on party 1’s campaign spending only if the proposal \(z\) is such that \(\epsilon_1(z) \geq \epsilon_h(z)\). This can be seen by inspecting the above candidates in which \(\zeta_h(z) = 0\) and \(\zeta_h(z) = C^h_Z(z) - \frac{1}{2}\). This completes the proof of Lemma 3.

\(\square\)

\textit{Proof of Proposition 3.} Consistent with backward induction, I first focus on the acceptance strategies \(a_2(z)\) and \(a_3(z)\) of the non-proposer parties 2 and 3 for any given proposal \(z\). Each party’s payoff from voting to accept or reject the proposal depends on the other party’s vote. First, for any given proposal \(z\), if \(a_k(z) = 1\) for both \(k\), then each party \(k\) gets a sure payoff of \(u_k(z)\). Second, if \(a_2(z) = 1\) and \(a_3(z) = 0\), then the bill moves to a challenge stage in which \(\rho_1(z) = \rho_2(z) = Z\) and \(\rho_3(z) = S\), with the associated equilibrium campaign spending of each party given by \((\zeta_1(z), \zeta_2(z), \zeta_3(z))\). In this case, each party’s receives an expected payoff determined by the specified challenge. Third, if \(a_2(z) = 0\) and \(a_3(z) = 1\), the challenge stage features \(\rho_1(z) = \rho_3(z) = Z\) and \(\rho_2(z) = S\), along with each party’s associated equilibrium campaign spending. Finally, if \(a_k(z) = 0\) for both parties, then each party \(k\) receives its status-quo payoff \(u_k(s)\).

For any given proposal \(z\), \(a_2(z) = 1\) is a dominant strategy for party 2 if a) \(u_2(z)\) is at least as great as its expected payoff from a challenge with \(\rho_1(z) = \rho_3(z) = Z\) and \(\rho_2(z) = S\); and b) its expected payoff from a challenge with \(\rho_1(z) = \rho_2(z) = Z\) and
\[ \rho_3(z) = S \] is at least as great as \( u_2(s) \). Similarly, \( a_3(z) = 1 \) is a dominant strategy for party 3 if a) \( u_3(z) \) is at least as great as its expected payoff from a challenge with \( \rho_1(z) = \rho_2(z) = Z \) and \( \rho_3(z) = S \); and b) its expected payoff from a challenge with \( \rho_1(z) = \rho_3(z) = Z \) and \( \rho_2(z) = S \) is at least as great as \( u_3(s) \).

In order for party 1 to induce unanimity as the unique equilibrium outcome of the game, the proposal \( z \) must be such that the following conditions based on the non-proposer parties’ acceptance strategies hold:

- \( u_k(z) \) is at least as great as party \( k \)'s expected payoff from a challenge with \( \rho_k(z) = S \) for \( k = 2, 3 \);
- The following two conditions for \( k = 2, 3 \) do not simultaneously hold: \( u_k(s) \) is at least as great as party \( k \)'s expected payoff from a challenge with \( \rho_k(z) = Z \).

To solve for the optimal proposal \( z \) from party 1’s perspective that would induce unanimity in the parliament by satisfying the above conditions, we need to find the unanimity-inducing offer \( z \) for each of the possible challenge stage equilibrium candidates identified in Lemma 3 and compare party 1’s unanimity payoff for all such offers.

I first focus on the first five equilibrium candidates listed in the proof of Lemma 3 in which \( \zeta_j(z) = w_j \). Consider a proposal \( z \) such that \( \epsilon_1(z) \leq \frac{(w_1+w_j)^2}{w_j}, \) \( \epsilon_1(z) \geq \epsilon_h(z), \) and \( \epsilon_j(z) \geq \frac{\epsilon_1(z)w_j}{\epsilon_1(z)w_j - w_j}, \) which would give rise to a challenge stage equilibrium listed in item one if rejected. Suppose party 1 chooses \( h = 2 \) and \( j = 3 \) so that if rejected, this proposal would imply a challenge stage equilibrium with \( \rho_2(z) = Z, \rho_3(z) = S, \) and \( (\zeta_1(z), \zeta_2(z), \zeta_3(z)) = (\sqrt{\epsilon_1(z)w_3} - w_3, 0, w_3) \). Party 3’s expected payoff from this challenge is given by

\[
\left( \frac{\sqrt{\epsilon_1(z)w_3} - w_3}{\sqrt{\epsilon_1(z)w_3}} \right) u_3(z) + \left( \frac{w_3}{\sqrt{\epsilon_1(z)w_3}} \right) u_3(s) - w_3. \tag{49}
\]

Then, party 3 accepts this offer if and only if \( u_3(z) \) is at least as great as 49, which implies that we must have \( \epsilon_3(z) \leq \sqrt{\epsilon_1(z)w_3} \).

To derive party 2’s acceptance condition, suppose that party 1 now chooses \( h = 3 \) and \( j = 2 \) so that this offer goes to a challenge in which \( \rho_2(z) = S \) and \( \rho_3(z) = Z \). In this scenario, the equilibrium levels of campaign spending are given by \( (\zeta_1(z), \zeta_2(z), \zeta_3(z)) = \)

---

\[ \text{36} \]
that simultaneously satisfies \( z \) together with the parties’ acceptance criteria implies the following: Party 2 accepts any \( \epsilon \) requires that \( \epsilon \geq 0 \).

As a result, we can conclude that any proposal \( z \) above. In the interest of space, each of these analyses will not be presented separately.

In order for unanimity to be realized, the additional condition that \( u_k(s) \) is not at least as great as party \( k \)’s expected payoff from a challenge with \( \rho_k(z) = Z \) for both \( k = 2, 3 \) needs to be met. To check for this, construct party \( k \)’s expected payoff from a challenge with \( \rho_k(z) = Z \) for \( k = 2, 3 \):

\[
\left( \frac{\sqrt{\epsilon_1(z)w_{-k} - w_k}}{\sqrt{\epsilon_1(z)w_{-k}}} \right) u_k(z) + \left( \frac{w_{-k}}{\sqrt{\epsilon_1(z)w_{-k}}} \right) u_k(s).
\]  

(51)

The condition that \( u_k(s) \) is at least as great as 51 reduces to \( u_k(s) \geq u_k(z) \) for \( k = 2, 3 \). Thus, unanimity requires that \( u_2(s) \geq u_2(z) \) and \( u_3(s) \geq u_3(z) \) are not simultaneously true for proposal \( z \).

First, suppose without loss of generality that the proposal \( z \) is such that \( u_2(s) < u_2(z) \). Then, the conditions that need to hold for unanimity are \( u_2(z) \geq u_2(s) \) and \( u_3(z) \geq u_3(s) - \sqrt{\epsilon_1(z)w_3} \). Since the challenge stage equilibrium under consideration requires that \( \epsilon_2(z) \leq \epsilon_1(z) \leq \frac{(w_1 + w_3)^2}{w_3} \) and \( \epsilon_3(z) \geq \frac{\epsilon_1(z)w_3}{\sqrt{\epsilon_1(z)w_3 - w_3}} \), bringing these conditions together with the parties’ acceptance criteria implies the following: Party 2 accepts any proposal \( z \) such that \( u_2(z) \in [u_2(s), u_2(s) + \frac{(w_1 + w_3)^2}{w_3}] \). However, there exists no proposal \( z \) that simultaneously satisfies \( \epsilon_3(z) \geq \frac{\epsilon_1(z)w_3}{\sqrt{\epsilon_1(z)w_3 - w_3}} \) and party 3’s acceptance criteria. Second, suppose the proposal \( z \) is such that \( u_3(s) \geq u_3(z) \). Carrying out the same analysis as above this time yields the result that party 2’s acceptance criteria cannot be reconciled with the equilibrium conditions on \( z \). Therefore, any proposal \( z \) that would imply a subsequent challenge stage equilibrium with \( (\zeta_1(z), \zeta_3(z), \zeta_j(z)) = (\sqrt{\epsilon_1(z)w_j - w_j}, 0, w_j) \) if rejected cannot induce unanimity in the parliament.

Carrying out the same analysis for equilibrium candidates numbered two through five in the proof of Lemma 3 yields the same result as the first equilibrium candidate above. In the interest of space, each of these analyses will not be presented separately.

As a result, we can conclude that any proposal \( z \) for which \( \zeta_j(z) = w_j \) will be rejected by party \( j \in N_3 \). This proves part 1 of Proposition 3.

Consider the sixth equilibrium candidate listed in the proof of Lemma 3 in which the
In addition, it can offer the following rent shares: 

First, party 1 can choose \( x = \frac{\hat{x}_1 + \hat{x}_3}{2} \), thereby compromising ideologically with party 3. In addition, it can offer the following rent shares:

\[
y_1 = \alpha^{-1} \left[ \alpha y_1^q + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right], \tag{52}
\]

\[
y_2 = \alpha^{-1} \left[ \alpha y_2^q - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 \right], \tag{53}
\]

\[
y_3 = \alpha^{-1} \left[ \alpha y_3^q - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right]. \tag{54}
\]

Second, it can choose \( x = \frac{\hat{x}_1 + \hat{x}_2}{2} \), compromising ideologically with party 2, and offer

\[
y_1 = \alpha^{-1} \left[ \alpha y_1^q + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right], \tag{55}
\]

\[
y_2 = \alpha^{-1} \left[ \alpha y_2^q - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right], \tag{56}
\]

\[
y_3 = \alpha^{-1} \left[ \alpha y_3^q - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 \right]. \tag{57}
\]

Having solved for the optimal way to induce unanimity with a proposal that would induce a challenge stage equilibrium as listed in item six if rejected, now consider the seventh equilibrium candidate characterized by \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (w_1, w_h, \sqrt{\epsilon j(z)}(w_1 + w_h) - w_1 - w_h)\). Based on their expected payoffs, the acceptance criteria of parties \( k = 2, 3 \) become \( \epsilon_k(z) \leq w_1 + w_{-k} \). Moreover, the final unanimity condition implies that we must have \( \epsilon_k(z) \geq w_h \sqrt{\frac{\epsilon_k(z)}{w_1 + w_h}} \) for at least one \( k \in \{2, 3\} \). Without loss of generality,
suppose that this condition holds for party 2. Then, the unanimity conditions yield
c_3(z) \leq w_1 + w_2 and c_2(z) \leq w_2. Confirming that there exist proposals z that can
simultaneously satisfy these and the equilibrium conditions for item seven, party 1 max-
imizes u_1(z) by choosing z subject to the above two constraints. Solving this program
yields the following two alternative unanimity-inducing offers: First, party 1 can choose
\( x = \frac{\hat{x}_1 + \hat{x}_2}{2} \) and offer the following rent shares:

\[
y_1 = \alpha^{-1} \left[ \alpha y_1^0 + w_1 + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right],
\]

(58)

\[
y_2 = \alpha^{-1} \left[ \alpha y_2^0 + w_2 - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 \right],
\]

(59)

\[
y_3 = \alpha^{-1} \left[ \alpha y_3^0 - (w_1 + w_2) - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right].
\]

(60)

Second, party 1 can choose \( x = \frac{\hat{x}_1 + \hat{x}_2}{2} \) and offer

\[
y_1 = \alpha^{-1} \left[ \alpha y_1^0 + w_1 + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right],
\]

(61)

\[
y_2 = \alpha^{-1} \left[ \alpha y_2^0 - (w_1 + w_3) - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right],
\]

(62)

\[
y_3 = \alpha^{-1} \left[ \alpha y_3^0 + w_3 - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 \right].
\]

(63)

Now consider the eighth equilibrium candidate characterized by \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (w_1, \max\{\frac{\epsilon_h(z)\zeta_j(z)}{(\epsilon_h(z)+\epsilon_j(z))^2} - w_1, 0\}, \max\{\frac{\epsilon_h(z)\zeta_j(z)}{(\epsilon_h(z)+\epsilon_j(z))^2}, \sqrt{\epsilon_j(z)}w_1 - w_1\})\). A similar analysis
suggests that there again exist two unanimity-inducing offers corresponding to two dif-
fert acceptance criteria: First, \( u_2(z) \geq u_2(s) - w_1 \) and \( u_3(z) \geq u_3(s); \) and second
\( u_2(z) \geq u_2(s) \) and \( u_3(z) \geq u_3(s) - w_1 \). Checking that there exist proposals z that satisfy
both the acceptance criteria and the equilibrium conditions, we can proceed with party
1’s maximization problem. If party 1 chooses a unanimity-inducing offer z based on the
first acceptance criteria, the offer z involves \( x = \frac{\hat{x}_1 + \hat{x}_2}{2}, y_1 \) as given in 58, and

\[
y_2 = \alpha^{-1} \left[ \alpha y_2^0 - w_1 - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 \right].
\]

(64)
\[ y_3 = \alpha^{-1} \left[ \alpha y_3 q - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right]. \quad (65) \]

On the other hand, if it chooses the offer based on the second acceptance criteria, the offer \( z \) now involves \( x = \frac{\hat{x}_1 + \hat{x}_2}{2}, \) \( y_1 \) as given in 61, and

\[ y_2 = \alpha^{-1} \left[ \alpha y_2 q - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 - \hat{x}_2}{2} \right)^2 \right], \quad (66) \]

\[ y_3 = \alpha^{-1} \left[ \alpha y_3 q - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 \right]. \quad (67) \]

The ninth equilibrium candidate is similar to the sixth candidate in the sense that the partners completely free-ride in both cases and the groups fight unconstrained against each other. The only difference is the identity of the partner. Thus, partner party \( h \)'s acceptance criteria is stricter, requiring a higher premium from party 1. Thus, this way to induce unanimity will never be optimal for party 1.

Finally, consider the tenth equilibrium candidate, which implies the following alternative acceptance criteria: First, \( u_3(z) \geq u_3(s) \) and \( u_2(z) \geq u_2(s) + w_2, \) and second \( u_3(z) \geq u_3(s) + w_3 \) and \( u_2(z) \geq u_2(s). \) If party 1 chooses a unanimity-inducing offer \( z \) based on the first acceptance criteria, the offer \( z \) involves \( x = \frac{\hat{x}_1 + \hat{x}_3}{2}, \)

\[ y_1 = \alpha^{-1} \left[ \alpha y_1 q - w_2 + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_3}{2} \right)^2 \right], \quad (68) \]

\[ y_2 = \alpha^{-1} \left[ \alpha y_2 q + w_2 - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 + \hat{x}_3 - 2\hat{x}_2}{2} \right)^2 \right], \quad (69) \]

\[ y_3 = \alpha^{-1} \left[ \alpha y_3 q - w_1 - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 \right]. \quad (70) \]

On the other hand, if it chooses this offer based on the second acceptance criteria, then the offer \( z \) involves \( x = \frac{\hat{x}_1 + \hat{x}_3}{2}, \)

\[ y_1 = \alpha^{-1} \left[ \alpha y_1 q - w_3 + (q - \hat{x}_2)^2 + (q - \hat{x}_3)^2 - \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 - \left( \frac{\hat{x}_1 - \hat{x}_2}{2} \right)^2 \right], \quad (71) \]
\[ y_2 = \alpha^{-1} \left[ \alpha y_2^q - (q - \hat{x}_2)^2 + \left( \frac{\hat{x}_1 - \hat{x}_2}{2} \right)^2 \right], \quad (72) \]

\[ y_3 = \alpha^{-1} \left[ \alpha y_3^q + w_3 - (q - \hat{x}_3)^2 + \left( \frac{\hat{x}_1 + \hat{x}_2 - 2\hat{x}_3}{2} \right)^2 \right]. \quad (73) \]

The above analysis indicates that the utility each party settles for in a grand bargain reflects its strength in the post-bargaining stage. To see this, first consider equilibrium candidate seven in which both members of group $N_Z$ fight against $N_S$ with all their resources. In this case, the two parties who would belong to $N_Z$ if the proposal $z$ is rejected can each extract a premium equal to their campaigning budgets from the party that would belong to $N_S$ in a grand bargain. In the equilibrium candidate eight, the non-proposer partner party is at least partially free-riding on party 1’s campaign spending. Thus, party 1 is able to extract from its partner an amount equal to its campaigning budget when inducing a settlement. Equilibrium candidate ten demonstrates the reverse of this situation with party 1 free-riding on its partner’s campaign spending. This proves part 2 of Proposition 3.

Part 3 of the proposition describes the optimal way for the proposer to induce unanimity. Comparing the maximum value of $u_1(z)$ from a unanimous agreement in each of the cases considered above, it can be observed that party 1 can secure the maximum payoff from unanimity with a proposal $z$ that satisfies the equilibrium conditions of items seven or eight. Although the optimal $z$ that induces unanimity in these two cases is different, they both imply the same sure-payoff for party 1. Specifically, for each of these cases, party 1 can induce unanimity by proposing either $x = \frac{\hat{x}_1 + \hat{x}_3}{2}$ and 58 for itself, or $x = \frac{\hat{x}_1 + \hat{x}_2}{2}$ and 61 for itself. Its rent share in either of these cases indicates that it is increasing in $y_1^q$, $w_1$, $(q - \hat{x}_2)$, and $(q - \hat{x}_3)$. Moreover, since each party gets compensated for their ideological utility loss in the grand bargain through its rent share as can be observed in 59, 60, 62, and 63, party 1’s unanimity payoff strictly increases as the three parties get ideologically closer. This completes the proof of Proposition 3.

**Proof of Proposition 4.** In order to analyze the optimal proposals to get to a given challenge stage equilibrium for party 1, I first focus on the non-proposer parties’ voting strategies. In order for a proposal $z$ to induce a unique challenge stage equilibrium with $\rho_h(z) = Z$ and $\rho_j(z) = S$ for $h, j \in \{2, 3\}$ and $h \neq j$, the following conditions must hold:

- Party $h$’s expected payoff from a challenge with $\rho_h(z) = Z$ must be at least as
great as $u_i(s)$;

- Party $j$’s expected payoff from a challenge with $\rho_j(z) = S$ must be at least as great as $u_j(z)$;

- The following conditions do not simultaneously hold: Party $h$’s expected payoff from a challenge with $\rho_h(z) = S$ is at least as great as $u_h(z)$; and party $j$’s expected payoff from a challenge with $\rho_j(z) = Z$ is at least as great as $u_j(s)$.

Consider the challenge stage equilibrium candidate listed in item one in the proof of Lemma 3. In order to induce a challenge stage equilibrium with $\rho_h(z) = Z$, $\rho_j(z) = S$, and $(\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\sqrt{\epsilon_1(z)}w_j - w_j, 0, w_j)$, party 1’s proposal $z$ must meet the corresponding equilibrium conditions, satisfy party $h$’s acceptance criteria, and violate party $j$’s acceptance criteria. The analysis in Proposition 3 indicated that party $j$ will reject any offer $z$ that would give rise to this equilibrium if rejected. In addition, among the range of proposals that would give rise to this challenge if rejected in the parliament, party $h$ will accept any $z$ such that $u_h(z) \in [u_h(s), u_h(s) + \frac{(w_1 + w_j)^2}{w_j}]$.

In equilibrium, party 1 will not offer any higher surplus to party $h$ than is required to get its acceptance. Thus, the optimal $z$ to induce this challenge will be such that $u_h(z) = u_h(s)$. Moreover, since $\zeta_j(z) = w_j$ for any proposal $z$ in this range, party 1 cannot influence the amount of $C_S(z)$. Thus, the proposal $z$ need not worry about party $j$’s rejection as long as it satisfies the equilibrium conditions. The Lagrangian of this problem can be written as

$$L = -(x - \hat{x}_1)^2 + y_1 - 2\sqrt{w_j\epsilon_1(z) + w_j + (\lambda_1 - \lambda_2)[-(x - \hat{x}_h)^2 + 1 - y_1 - u_h(s)] + \lambda_2 \frac{(w_1 + w_j)^2}{w_j}}$$

$$+ \lambda_3 \left[ -(x - \hat{x}_j)^2 - \frac{\epsilon_1(z)w_j}{\sqrt{\epsilon_1(z)w_j - w_j}} \right].$$

With $y_j = 0$, solving this program for $x$, $y_1$, and $y_h = 1 - y_1$ yields $x = \frac{\hat{x}_1 + \hat{x}_h}{2}$,

$$y_1 = y_1^q + y_j^q + (q - \hat{x}_h)^2 - \left(\frac{\hat{x}_1 - \hat{x}_h}{2}\right)^2,$$

$$y_h = y_h^q - (q - \hat{x}_h)^2 + \left(\frac{\hat{x}_1 - \hat{x}_h}{2}\right)^2.$$
the result that the optimal proposal $z$ to induce any challenge stage equilibrium involves offering $x = \frac{\hat{x}_1 + x_h}{2}$. Since party $h$ requires at least $u_h(s)$ in order to become party 1’s partner in a challenge regardless of how much it will spend, it can be observed from 75 and 76 that party 1’s winning prize increases as $y_q^h$ decreases, $(q - \hat{x}_h)^2$ increases, and it gets ideologically closer to party $h$. Moreover, since $\epsilon_h(z)$ increases as $u_h(s)$ decreases for any proposal $z$, Lemma 3 indicates that $\zeta_h(z)$ would be weakly higher, thus weakly increasing the proposal’s winning probability. Therefore, party 1’s expected payoff would increase. This proves part 1 of Proposition 4.

Part 2 of Proposition 4 is concerned with how party 1’s expected payoff from a challenge is affected by its partner’s campaigning budget. In the interest of brevity, I do not present here the solutions for the optimal proposals that would induce each possible challenge stage equilibrium. Instead, I focus on two examples that demonstrate party 1’s different incentives with regards to the other parties’ campaigning budgets.

Consider the equilibrium candidate listed in item two in the proof of Lemma 3. Solving for the optimal proposal to induce this particular challenge equilibrium yields $x = \frac{\hat{x}_1 + x_h}{2}$, $y_j = 0$,

$$ y_1 = y_1^q + y_j^q + (q - \hat{x}_h)^2 - \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 - \frac{(w_1 + w_h + w_j)^2}{w_j}, \quad (77) $$

$$ y_h = y_h^q - (q - \hat{x}_h)^2 + \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 + \frac{(w_1 + w_h + w_j)^2}{w_j}. \quad (78) $$

As a result, party 1’s maximum expected payoff from this type of challenge becomes

$$ \left( \frac{w_1 + w_h}{w_1 + w_h + w_j} \right) \left[ y_j^q + \sum_{k=1,h} (q - \hat{x}_k)^2 - 2 \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 - \frac{(w_1 + w_h + w_j)^2}{w_j} \right] + u_1(s) - w_1. \quad (79) $$

Differentiating 79 with respect to $w_h$ yields

$$ \frac{(w_j)^2 \epsilon_1(z) - 2(w_1 + w_h)(w_1 + w_h + w_j)}{(w_1 + w_h + w_j)^2 w_j} \quad (80) $$

where $\epsilon_1(z)$ is calculated using the optimal proposal and equals the expression in brackets in 79. The sign of this expression depends on the parameters of the model. Specifically,
it is negative if
\[ \epsilon_1(z) < \frac{2(w_1 + w_h)(w_1 + w_h + w_j)}{(w_j)^2}, \] (81)
and positive otherwise. Thus, we conclude that a higher \( w_h \) decreases party 1’s expected payoff from the type of challenge in item 2 of Lemma 3 if \( \epsilon_1(z) \) is sufficiently small, which happens if \( u_1(s) \) is large, or if \( w_1 \) or \( w_h \) are high. Analyzing other equilibrium candidates in which \( \zeta_h(z) > 0 \) indicates that this relationship holds more generally. This proves part 2 of Proposition 4.

For Part 3, consider party 1’s decision on the identity of party \( h \). Since a lower \( u_h(s) \) and higher \( u_j(s) \) necessarily increase party 1’s expected payoff in any challenge equilibrium, it follows that holding everything else constant, party 1 would prefer to partner with the party that commands the lower status-quo payoff.

To see how the partner decision is affected by the parties’ campaigning budgets, consider an alternative challenge equilibrium in which the proposal \( z \) is such that \( \zeta_h(z) = 0 \). Since it has already been analyzed, I focus on the equilibrium given in the first item in Lemma 3. With the proposal \( z \) given by \( x = \frac{\hat{x}_1 + \hat{x}_h}{2}, \ y_j = 0, \ y_1 \) as in 75, and \( y_h \) as in 76, party 1’s maximum expected payoff from this challenge becomes
\[
\left[ 1 - \sqrt{\frac{w_j}{\epsilon_1(z)}} \right] \epsilon_1(z) + u_1(s), \tag{82}
\]
where \( \epsilon_1(z) = -2 \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 + y_j^2 + (q - \hat{x}_1)^2 + (q - \hat{x}_h)^2 \). It can be observed from 82 that it does not depend on \( w_h \) and depends negatively on \( w_j \). Furthermore, this relationship holds in other challenge equilibrium candidates in which \( \zeta_h(z) = 0 \). Thus, party 1 would prefer to have as its opponent the party with the lower campaigning budget in such challenge equilibria. The rest of the proposition follows from the analysis for part 2. Thus, this completes the proof of Proposition 4.

**Proof of Proposition 5.** Analyzing party 1’s incentives between inducing a grand bargain and a challenge requires comparing its maximum payoff from each of the two outcomes. However, since the type of challenge equilibrium that will maximize party 1’s expected payoff depends on different conditions on the parameters of the model, I only present here the relevant results from comparing party 1’s maximum unanimity payoff with certain types of challenge stage equilibria for the sake of brevity.

First, consider the challenge stage equilibrium listed in item one in the proof of
Lemma 3, where \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\sqrt{\epsilon_1(z)}w_j - w_j, 0, w_j)\). Given the optimal proposal \(z\) characterized in the proof of Proposition 4 that would give rise to this challenge equilibrium, party 1’s maximized payoff from this challenge is as given in 82, where \(\epsilon_1(z) = -2 \left( \frac{\hat{x}_1 - \hat{x}_j}{2} \right)^2 + y_j^q + (q - \hat{x}_j)^2 + (q - \hat{x}_j)^2\). The proof of Proposition 3 characterized the optimal proposal \(z\) to induce unanimity, which involves \(x = \frac{\hat{x}_1 + \hat{x}_j}{2}\) and

\[
y_1 = \alpha^{-1} \left[ \alpha y_1^q + w_1 + (q - \hat{x}_h)^2 - \left( \frac{\hat{x}_1 + \hat{x}_j - 2\hat{x}_h}{2} \right)^2 \right].
\]

Therefore, party 1’s maximum payoff from unanimity is given by

\[
\alpha y_1^q + w_1 + \sum_{k=2,3} (q - \hat{x}_k)^2 - \left( \frac{\hat{x}_1 + \hat{x}_j - 2\hat{x}_h}{2} \right)^2 - 2 \left( \frac{\hat{x}_1 - \hat{x}_j}{2} \right)^2. \tag{83}
\]

Comparing 83 with the maximum expected payoff from the considered challenge indicates that party 1 prefers a grand bargain over this challenge if

\[
w_1 - \left( \frac{\hat{x}_1 + \hat{x}_j - 2\hat{x}_h}{2} \right)^2 - 2 \left( \frac{\hat{x}_1 - \hat{x}_j}{2} \right)^2 \geq -2 \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 + y_j^q - \sqrt{w_j\epsilon_1(z)} - (q - \hat{x}_1)^2, \tag{84}
\]

where \(\epsilon_1(z)\) is given as before. Condition 84 is more likely to hold if the non-proposer parties \(h\) and \(j\) each commands a lower status-quo payoff. Moreover, this relationship carries over to other types of challenge equilibria. This proves Part 1 of Proposition 5.

Notice that condition 84 becomes more likely to hold as

\[
- \left( \frac{\hat{x}_1 + \hat{x}_j - 2\hat{x}_h}{2} \right)^2 - 2 \left( \frac{\hat{x}_1 - \hat{x}_j}{2} \right)^2 + 2 \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 \tag{85}
\]

increases, which, when manipulated, suggests that all parties need to be ideologically close for unanimity to be preferred. This is also a relationship that carries over to other types of challenge equilibria. This proves Part 2 of Proposition 5.

Proposition 4 indicated that the individual roles \(w_h\) and \(w_j\) might play on party 1’s incentives between a grand bargain and a challenge are ambiguous and depend on the particular challenge equilibrium considered. However, to see how party 1’s incentives respond to the relative budgets of the non-proposer parties, consider a challenge equilibrium such as item five in the proof of Lemma 3 in which \((\zeta_1(z), \zeta_h(z), \zeta_j(z)) = (\max\{ \sqrt{\epsilon_1(z)}w_j - w_j - w_h, 0 \}, w_h, w_j)\). Solving for the optimal proposal \(z\) that would
lead to this challenge yields $x = \frac{\hat{x}_1 + \hat{x}_h}{2}$, $y_j = 0$,

$$y_1 = y_1^q + (q - \hat{x}_h)^2 - \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 \frac{(w_h + w_j)^2}{w_j}, \quad (86)$$

$$y_h = y_h^q - (q - \hat{x}_h)^2 + \left( \frac{\hat{x}_1 - \hat{x}_h}{2} \right)^2 \frac{(w_h + w_j)^2}{w_j}. \quad (87)$$

The condition obtained by comparing the maximum expected payoff from this challenge and 83 is more likely to hold as $\frac{w_h}{w_j}$ increases. Note that if $w_h > w_j$, this would require the two parameters to diverge, whereas if $w_h < w_j$, they must become more similar. However, since this is a challenge equilibrium in which the low-budget party is more likely to become the partner based on Proposition 4, it is more likely that $w_h < w_j$. Thus, more similar budgets decrease the payoff from this challenge. This completes the proof of Proposition 5.
References


