Towards an Understanding of the Political Economy of Multidimensional Poverty Measurement

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Abstract
In a game theoretic framework, we study how scalar multidimensional poverty measures affect the strategic interactions of ministers responsible for reducing deprivations in the measure’s dimensions. The ministers share a common interest in reducing (measured) multidimensional poverty, but also have preferences over alternate uses of their allocated budgets; because the multidimensional poverty measure is a scalar that depends on the contributions of each minister, it is a public good among ministers who can free ride on each other’s antipoverty spending. The allocation of resources across ministers and the measure’s parameters (the weights assigned to each dimension and the extent of deprivation depth aversion) affect equilibrium free riding and antipoverty spending. We consider two policy objectives for the agent allocating budgets and choosing parameters: “maximize resources devoted to the poor” (the goal of a benevolent government) and “maximize measured poverty reduction” (the goal of a government concerned with looking good). The two policy objectives are often in conflict: a reallocation of resources that improves measured poverty reduction, for instance, always decreases total antipoverty spending in equilibrium, for common parameterizations. Increasing deprivation depth aversion may increase or decrease the resources actually spent on the poor, depending on whether disparities across dimensions are due mostly to the number of deprived households, or to their average deprivation. We illustrate using data from Mexico, the first country to adopt an official multidimensional poverty measure.

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1 Introduction

Poverty is a multidimensional phenomenon. Although “inadequate income is a strong predisposing condition for an impoverished life,” Sen (1999, p. 87) asserts that “poverty must be seen as the deprivation of basic capabilities rather than merely as lowness of incomes.” The recent debate about whether multiple dimensions of poverty can be credibly aggregated in a single scalar measure has revived scholarly interest in multidimensional poverty measures. On one hand, Ravallion (2011b, p. 235) argues that “we should aim for a credible set of multiple indices rather than a single multidimensional index;” on the other, Deaton (2011, p. 14) argues that multidimensional measures are “required,” and that they “need to be calculated from surveys that collect multiple measures for each respondent” due to the correlation of deprivations across dimensions. Meanwhile, scalar indices that capture multiple dimensions of poverty in a single axiomatically consistent measure—pioneered by Chakravarty et al. (1998), Tsui (2002), and Bourguignon and Chakravarty (2003)—are gaining significant policy traction.

The measure proposed by Alkire and Foster (2011a), which combines information about joint deprivations in multiple dimensions and is decomposable by subgroup and dimension, is becoming part of many countries’ official poverty measurement and antipoverty program targeting methods. In 2009, Mexico replaced its official (unidimensional) poverty measure with a multidimensional one, based on Alkire and Foster’s (AF). Colombia adopted the AF class of multidimensional poverty measures in its official poverty measurement in 2011, and is also using it to design and target its social programs. Bhutan, Chile, and the Philippines have also implemented an official multidimensional poverty measure based on AF; twenty-seven other countries are considering adopting one, participating in the Multidimensional Poverty Peer Network. At the subnational level, Brazil’s São Paulo and Minas Gerais states, China’s Wu Ling Mountain region, and Vietnam’s Ho Chi Minh city have all decided to measure multidimensional poverty using AF, and in Minas Gerais the measure is being used extensively in targeting a statewide poverty reduction program.

While the implications of using Alkire and Foster’s (2011a) measures for targeting antipoverty programs have been studied (Alkire and Seth, 2013; Azevedo and Robles, 2013; Duclos et al., 2014), the political economy implications of measuring multi- rather than unidimensional poverty have not. Given that multidimensional poverty measures will likely be “used by policymakers in the allocation of funds to reduce poverty in an efficient and equitable way” (Thorbecke, 2011, p. 485), it is important to study how the adoption of a multidimensional poverty measure might affect interactions between ministries, allocations of budgets, and overall antipoverty spending. Although the proponents of multidimensional poverty measures stress the importance of decomposing the overall scalar measure “to show how each of the components moved and impacted the poor” (Alkire et al., 2011, p. 504), it is only natural that the single scalar would be used as an
influential communication mechanism. Hence, a scalar measure à la AF will impact the strategic incentives of government agents. In this paper, we address these issues in a simple game theoretic framework.

Specifically, multidimensional poverty in our model is measured using AF. Ministries can spend their allocated budgets on two things: poverty reduction and private consumption, which can be thought of as corruption or some other form of spending that does not benefit the other ministries. Poverty reduction is a public good among government ministries because all government members look good when measured multidimensional poverty falls. Hence, the game has a similar structure to the classic contribution to a public good game (Bergstrom et al., 1986). The ministers play a simultaneous game in which, taking their budgets and the parameters of the AF measure as given, they maximize a function that is increasing in both the amount of multidimensional poverty reduction achieved—which is, in turn, a function of all ministries’ spending—and in private spending. The parameters of the AF measure (the value \(\alpha\) characterizing deprivation depth aversion and the weights assigned to each dimension) change the production technology of the public good and thus change the strategic incentives ministers have to reduce poverty.

After determining equilibrium antipoverty expenditures by each ministry, we consider the political economy factors that might influence three choices: 1) the budgets allocated to each ministry, 2) the dimensions and associated weights in the multidimensional poverty measure, and 3) the value of \(\alpha\), which measures aversion to the depth of deprivations. We assume that these choices are made by outside agents prior to the simultaneous game between ministries—for example, budget allocations might be chosen by the ministry of finance, prime minister, or president, while the measure’s dimensions and weights might be selected by a specially appointed committee.

First, with respect to budget allocations, when a country’s allocation of government funds is influenced by its official poverty measure, “the setting or redesigning of the poverty methodology” (e.g., by replacing the official unidimensional poverty measure by a multidimensional one as Mexico has done) “can have serious...
budgetary implications” (Foster 2007, p. 3). We explore these potential implications using two possible policy objectives for the agent choosing budgets: (i) the total spent on poverty reduction by the ministries and (ii) measured poverty reduction. Our results tell a cautionary tale because the objectives often conflict: the committee can change budgets in order to marginally increase measured poverty reduction (and, hence, government prestige), while at the same time reducing total anti-poverty expenditures. This occurs for commonly used parameterizations of the AF measure.

Second, with respect to weights, there are a number of ways in which these might be selected by the commission (for an overview, see Decanq and Lugo 2013). Often, equal or nested equal weights are used due the lack of a “reliable basis” for instead weighting dimensions based on their relative importance to individuals (Mayer and Jencks 1989, p. 96). A potential flaw in using equal weights for a multidimensional index is that “it is the relative weights on its dimensions that matter most,” and relative weights depend not only on the weights themselves but also on the extent of normalized deprivations in each dimension (Ravallion 2011a, p. 477, his emphasis). Alternatively, Alkire and Foster (2011b, p. 310) recommend choosing weights based on “recurrent participatory and deliberative processes,” although one of Ravallion’s (2011b) critiques is that these processes are unlikely to occur in practice. Indeed, it is a fundamental disagreement about weights that lies at the heart of the debate over multidimensional poverty measurement (Lustig 2011). Here, we consider a political economy process for setting weights in which, prior to the simultaneous poverty reduction game between ministries, another agent chooses weights to maximize either antipoverty spending or measured poverty reduction.

As pointed out by Sen (1987), the choice of dimensions is an elementary aspect of the choice of weights, since excluding a dimension from the measure is equivalent to assigning it zero weight. We derive conditions under which the largest equilibrium measured poverty reduction obtains at a corner solution in the selection of weights, i.e., when one dimension is assigned a weight of zero.

Third, we explore the incentives underlying the choice of α, the parameter of aversion to the depth of deprivations. In countries such as Colombia that present multidimensional poverty using various values of α, it is important to understand what might lead the government to select a particular value of α for their “primary” or most emphasized results. The importance of α goes beyond normative considerations: as emphasized by Chakravarty and Silber (2008), α changes the substitutability of deprivations across dimensions within the index. This, in turn, can affect the strategic interactions between ministries in nuanced ways.

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6As an extreme example, consider an index of multidimensional poverty that is sensitive to the depth of deprivations and in which one or more of the indicators is dichotomous. Since the normalized gap in the dichotomous dimension equals one for all poor that are deprived in that dimension, as the parameter of deprivation depth aversion increases, the relative weight assigned to the dichotomous indicator increases (see, e.g., Foster 2007; Battiston et al. 2013).

7For simplicity, we do not consider the possibility of heterogeneous weights across population subgroups. Ravallion (2012) critiques the use of homogeneous weights across individuals, with the reasoning that “for example, the weight on access to a school must depend on whether the household has children” (p. 9). Although Bellani (2013) proposes a methodology for incorporating heterogeneous weights that vary across groups, the multidimensional measures used in practice have always fixed the weights across individuals.
While we make several simplifying assumptions, introduced as we describe our formal model in the next section and discussed further in our Conclusion, the model has a number of interesting results. We begin by showing that there always exists a unique equilibrium to the model, which is useful when dealing with more complicated cases that can only be numerically simulated. For the simpler cases of \( \alpha = 0 \) and \( \alpha = 1 \), we derive explicit solutions that express each ministry’s equilibrium contributions to poverty reduction, along with the resulting overall poverty reduction, as a function of each ministry’s budget, propensity to spend on private consumption, and weight in the initial AF index. For \( \alpha = 2 \), we find that in a symmetric setup where the two ministries have the same budgets and elasticities of private consumption, measured poverty reduction (prestige) is maximized when a weight of zero is assigned to the dimension with lower contribution to the initial AF index. In other words, governments may have an incentive to continue using unidimensional poverty measures to appear to be reducing poverty by more than if a multidimensional measure is used. Furthermore, this incentive is especially strong for relatively poor governments: even if forced to use a multidimensional measure, they always prefer a distribution of weights as uneven as possible.

In contrast, for any \( \alpha \), when ministries have similar budgets and preferences, asymmetries in each dimension’s contribution to initial poverty reduce total anti-poverty efforts: if deprivation cutoffs or weights are chosen so that different dimensions have significantly different relative weights in the poverty measure, the strategic interaction between ministries is such that overall spending on poverty reduction is reduced.

With respect to the choice of \( \alpha \), increasing \( \alpha \) can increase or decrease total antipoverty spending, depending on the breadth and depth of initial deprivations in each dimension. Because the comparative statics are highly complex, we make additional simplifying assumptions and obtain the following result: when the average shortfall of those who are deprived is similar across dimensions but one dimension has more deprived individuals, poverty-reducing efforts are increasing in \( \alpha \); on the other hand, when the number of deprived individuals is similar across dimensions but their average shortfall is higher in one dimension than another, total antipoverty spending is decreasing in \( \alpha \). Intuitively, increasing \( \alpha \) has two countervailing effects. First, it increases the complementarity across dimensions; this reduces free riding and tends to increase anti-poverty expenditures. Second, increasing \( \alpha \) can exacerbate the initial asymmetry in relative dimension weights, with the negative consequences earlier described. This effect does not arise when shortfalls are the same across dimensions, which is why our results with respect to \( \alpha \) go in different directions.

The rest of the paper is organized as follows. The next section presents the model and provides the intuition underlying its variables. Section 3 describes the equilibrium of the game between ministers as well as the optimal allocation of budgets and selection of weights and parameters. Section 4 illustrates our results.

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8E.g., if ministries have different propensities for corruption, different budgets, unequal weights, and if \( \alpha \) is not necessarily equal to its commonly used integer values.
and their implications through a detailed example using Mexican data. Section 5 concludes.

2 The model

Preferences and strategic interaction. We consider a simple economy with two ministers, each in charge of one of the two dimensions relevant for the measurement of poverty. Minister \( j \)'s utility function \((j = 1, 2)\) has the Cobb-Douglas formulation \( U_j(P, s_j) = \ln(P) + \beta_j \ln(s_j) \), where \( P \) is a prestige term that depends on poverty reduction, \( s_j \) is private consumption—which can be thought of as spending on corruption or any other ministry-specific spending that is not enjoyed by the other ministries—and \( \beta_j \) is a parameter affecting the marginal rate of substitution between private consumption and prestige. Each minister \( j \) divides her total financial resources \( r_j \) between an investment in a poverty reduction program and private consumption. For simplicity, we conduct our analysis with multiplicative policies and we denote with \( x_j \) the financial resources that minister \( j \) devotes to poverty reduction. In particular, spending of \( x_j \) reduces poverty in dimension \( j \) by 100\(x_j \) percent; if \( x_j = 1 \), then poverty is eliminated in dimension \( j \).

The strategic interaction among policymakers is modeled as a simultaneous public good provision game in which prestige is a public good enjoyed by the ministers.\(^9\) Since the quantity of public good depends on the reduction of the multidimensional poverty index, we indicate it as a function of \( x_1 \) and \( x_2 \). Therefore, taking as given \( x_2 \), minister 1 maximizes for \( x_1 \in [0, \min\{r_1, 1\}] \) the function

\[
U_1(x_1, x_2) = \ln(P(x_1, x_2)) + \beta_1 \ln(r_1 - x_1),
\]

and minister 2 solves an identical problem taking \( x_1 \) as given.

Poverty measurement and production technology. The way poverty is measured enters into the production function for prestige. To make our main point in the simplest possible framework, we assume poverty is measured with an extremely simplified version of an AF index over two dimensions. Consider what Atkinson (2003) terms the union approach (that is, if an agent is deprived in any one dimension, then that agent is poor) and, following the notation and terminology in Alkire and Foster (2011a), begin from the matrix of normalized gaps \( g^1 \), in which columns represent dimensions relevant for poverty and rows represent individuals; if person \( i \) is not deprived in dimension \( j \), \( g^1_{ij} = 0 \).\(^{10}\) Now, denote with \( g^1_{i,j} \) the \( j \)th

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\(^9\)This description favors simplicity over realism but it is sufficient to highlight the main forces we are interested in. Clearly, refining the description of how government policy is formed is important. However, as long as multiple agents are responsible for policy and these agents cannot write complete and enforceable binding contracts with one another, then a public good problem among policymakers arises.

\(^{10}\)More specifically, the normalized gaps \( g^1_{ij} \) are defined as follows. Let \( y_{ij} \) indicate individual \( i \)’s achievement in dimension \( j \), and \( z_j \) be a deprivation cutoff for dimension \( j \) such that individuals with \( y_{ij} < z_j \) are considered deprived in dimension \( j \). The normalized gap \( g^1_{ij} = (z_j - y_{ij})/z_j \) if \( y_{ij} < z_j \) and 0 otherwise (Alkire and Foster, 2011a). To extend this framework to
column of $g^1$ and let $\mu(\cdot)$ be the mean operator. Also, for $\alpha \geq 1$, let $g^\alpha_{*j}$ be the column vector derived from $g^1_{*j}$ by raising to the power $\alpha$ every element of $g^1_{*j}$; for $\alpha = 0$, $g^0_{*j}$ is the $j$th column of $g^0$ whose typical element $g^0_{ij} = \mathbb{I}(g^1_{ij} > 0)$, where $\mathbb{I}(\cdot)$ is the indicator function taking a value of 1 if its argument is true and 0 otherwise. Using the dimension-specific decomposition, and weighing each dimension with $w$ index is $w$ element $g^\alpha >$ for multidimensional poverty AF indices become $\geq$ greater than some poverty cutoff $k$, for more than two dimensions where an individual is poor if the weighted sum of the dimensions in which she is deprived is the intersection approach where an individual is poor only if she is deprived in both dimensions, or an intermediate approach for more than two dimensions where an individual is poor if the weighted sum of the dimensions in which she is deprived is greater than some poverty cutoff $k$, let $g^1$ be the censored matrix of deprivations, with $g^1_{ij} = 0$ for all $j$ if $i$ is non-poor.

Using the dimension-specific decomposition, and weighing each dimension with $w_1$ and $w_2 = 1 - w_1$, the Alkire–Foster multidimensional poverty index $M_\alpha$ is calculated as

$$M_\alpha = \left( \frac{w_1}{n} \sum_{i=1}^{n} g^\alpha_{i1} + \frac{w_2}{n} \sum_{i=1}^{n} g^\alpha_{i2} \right) = w_1 \mu(g^\alpha_{*1}) + w_2 \mu(g^\alpha_{*2}).$$

We assume that resources spent $x_j$ produce a multiplicative reduction in poverty: all elements of the vector $g^1_{*j}$ of normalized gaps are reduced by a fraction $x_j$. In other words, minister $j$ can reduce by $100x_j$ percent all normalized gaps of the dimension within her control. In the case of $\alpha = 0$, this would result in no poverty reduction (for $x_j < 1$) since the depth of deprivations is not taken into account by $M_0$; hence, for $\alpha = 0$ we instead assume that the proportion of the population that is deprived in dimension $j$ is reduced proportionally by $x_j$. Therefore, applying the previous formulation, we have that the after-intervention multidimensional poverty index AF indices become

$$M'_\alpha = w_1 \mu(g^\alpha_{*1}) (1 - x_1)^\alpha + w_2 \mu(g^\alpha_{*2}) (1 - x_2)^\alpha$$

for $\alpha > 0$ and $M'_0 = w_1 \mu(g^0_{*1}) (1 - x_1) + w_2 \mu(g^0_{*2}) (1 - x_2)$. Hence, the percentage improvement in the index is

$$\% \Delta M_\alpha(x_1, x_2) = \frac{M_0 - M'_\alpha}{M_\alpha} = \pi_1(\alpha) (1 - (1 - x_1)^\alpha) + \pi_2(\alpha) (1 - (1 - x_2)^\alpha)$$

for $\alpha > 0$, where $\pi_j(\alpha)$ is the fraction of the initial index value that is due to dimension $j$: $\pi_j(\alpha) = w_j \mu(g^\alpha_{*j}) / M_\alpha$, for $j = 1, 2$. Note as well that $\pi_1(\alpha) + \pi_2(\alpha) = 1$. For $\alpha = 0$ the structure is similar to that of the $\alpha = 1$ case: $\% \Delta M_0(x_1, x_2) = \pi_1(0) x_1 + \pi_2(0) x_2$.

In fact, the results for the $\alpha = 0$ analysis with multiplicative reductions in the proportion who are deprived follow immediately from the $\alpha = 1$ case with multiplicative reductions in normalized gaps. For the $\alpha = 0$ case, we replace the matrix of normalized gaps $g^1$ with the matrix of deprivations $g^0$ whose typical element $g^0_{ij}$ equals 1 if individual $i$ is deprived in dimension $j$ and 0 otherwise, then calculate $\% \Delta M_1(x_1, x_2)$ on the $g^0$ matrix (assuming multiplicative reductions in each element of $g^0$ just like we would for $\alpha = 1$).

To see why this works, consider the column vector of dimension $j$ deprivations $g^0_{*j} = (1, 1, 0, 1, 0)^\top$: three individuals are deprived in dimension $j$ and two are not. We are assuming that antipoverty spending of $x_j$
will (for the $\alpha = 0$ analysis) reduce the number of individuals deprived in dimension $j$ by a proportion $x_j$, so if $x_j = 2/3$, we will have a post-intervention vector of deprivations in which one individual is deprived and four are not, so $\mu(g'_{xj}) = 1/5$. The same result obtains if we were to reduce each element of $g'_x$ by the proportion $x$, which is what we are doing when $\alpha \geq 1$: we would then have $g'_x = (1/3, 1/3, 0, 1/3, 0)\top$ and $\mu(g'_x) = 1/5$.

Example 1 in the Appendix illustrates our procedure and demonstrates a potential obstacle: the same poverty reduction efforts generate different percentage reductions in two multidimensional poverty indices for two different values of $\alpha$. Therefore, while it is reasonable to assign prestige based on the percentage reduction in the poverty index, it is opportune to find a prestige function that takes $\%\Delta M_\alpha (x_1, x_2)$ as its argument and produces an outcome that is, as much as possible, not affected by $\alpha$, since $\alpha$ could be a choice variable. At a minimum, this independence of $\alpha$ should hold in symmetric environments: if ministers take the same action (i.e., $x_1 = x_2 = x$), then the prestige prize they receive should be not affected by $\alpha$. A very simple solution is to invert $M(x, x)$ and “normalize” using $P(z) = 1 - (1 - z)^{1/\alpha}$. Indeed, note that $P(\%\Delta M_\alpha (x, x)) = x$, which is independent of $\alpha$. The function $P(z)$ is said to be normalized because if no effort is exerted $P$ equals zero, while if maximum useful effort is exerted (i.e., for $x_1 = x_2 = 1$), then $P = 1$. Of course, other normalizations are possible and, moreover, as soon as $x_1 \neq x_2$, the value of $\alpha$ does affect the value of prestige. We then obtain

$$P(x_1, x_2) \equiv P(\%\Delta M_\alpha (x_1, x_2)) = 1 - (\pi_1 (\alpha) (1 - x_1)^\alpha + \pi_2 (\alpha) (1 - x_2)^\alpha)^{1/\alpha}. \quad (3)$$

Finally, note that that the marginal product of $x_1$ is calculated as

$$\frac{\partial P}{\partial x_1} = \pi_1 (\alpha) \cdot \left( \frac{1 - P}{1 - x_1} \right)^{1-\alpha} > 0, \quad (4)$$

and that

$$\frac{\partial^2 P}{\partial x_1^2} = (1 - \alpha) \frac{\pi_1 (\alpha) \pi_2 (\alpha) (1 - P)^{1-2\alpha} (1 - x_2)^\alpha}{(1 - x_1)^{2-\alpha}};$$

so, the prestige function is concave in $x_1$ as soon as $\alpha \geq 1$. (A similar analysis applies for $x_2$.) With the exception of the case $\alpha = 0$, which, as explained above, is handled exactly like the $\alpha = 1$ case, we assume $\alpha \geq 1$ for the rest of the paper. It is worth noting that the introduction of this prestige function is intended to allow us to analyze the effects of different choices of $\alpha$; we could instead use $\%\Delta M_\alpha (x_1, x_2)$ in the ministers’ utility functions for the parts of the paper where $\alpha$ is fixed, without any change to the fact that the prestige function is increasing and concave in antipoverty spending nor any change to the results from Propositions 1, 3 or Corollaries 1, 2 below. Furthermore, when considering a fixed $\alpha = 1$ (or $\alpha = 0$),
\[ P(x_1, x_2) = \% \Delta M_{\alpha} (x_1, x_2). \]

### 3 Equilibrium

We now characterize equilibrium for this game, beginning with minister 1, who maximizes the utility in (1) by choosing \( x_1 \) for given \( x_2 \). Because we are interested in comparative statics exercises, we focus on an interior solution, i.e., one such that \( 0 < x_1 < 1 \). This involves the usual equality of marginal rates of substitution and transformation:

\[ \frac{\partial U_1}{\partial P} \cdot \frac{\partial P}{\partial x_1} = \frac{\partial U_1}{\partial s_1}; \tag{5} \]

corner (noninterior) solutions are ruled out by the sufficient conditions in the following Lemma, with proof in the Appendix.

**Lemma 1 (Interior solutions).** Let \( \alpha > 1 \). If for \( j = 1, 2 \) we have

\[ \begin{align*}
  i) & \quad \pi_j(\alpha) \frac{1}{2} (r_j - 1) < \beta_j \\
  ii) & \quad \pi_j(\alpha) r_j \left( 1 - \frac{1}{\alpha} \right)^{(1-\alpha)} > \frac{1}{\alpha} \beta_j,
\end{align*} \]

then the optimal choice of both ministers is interior. The same conclusion holds if \( \alpha = 1 \) or \( \alpha = 0 \), \( r_j - 1 < \beta_j \), and \( \pi_j(\alpha) r_j > \beta_j \), for \( j = 1, 2 \).

Unless otherwise stated, for the rest of the paper we focus on parameter values such that \( x_1 \) and \( x_2 \) are interior. Using (4), equation (5) is

\[ \pi_1(\alpha) \left( \frac{1-P}{1-x_1} \right)^{1-\alpha} (r_1 - x_1) = P \beta_1, \tag{6} \]

and with similar steps we obtain minister 2’s condition:

\[ \pi_2(\alpha) \left( \frac{1-P}{1-x_2} \right)^{1-\alpha} (r_2 - x_2) = P \beta_2. \tag{7} \]

Equilibrium requires both (6) and (7) to be satisfied. We then obtain the following result, with proof in the Appendix.

**Proposition 1 (Existence and uniqueness).** There exists a unique solution to (6) and (7).

Three very important cases in practice are \( \alpha = 0 \), \( \alpha = 1 \), and \( \alpha = 2 \).
3.1 Equilibrium and comparative statics for $\alpha = 0$ and $\alpha = 1$

If $\alpha = 1$, then the first order conditions (FOC) for an interior solution simply yield $x_j = r_j - P(\cdot) \beta_j / \pi_j(1)$, for $j = 1, 2$. We then obtain that the equilibrium level of prestige is

$$P^* = \frac{\pi_1(1) r_1 + \pi_2(1) r_2}{1 + \beta_1 + \beta_2},$$

and the equilibrium contributions are

$$x^*_1 = r_1 - \beta_1 \frac{\pi_2(1) r_2}{1 + \beta_1 + \beta_2}, \quad x^*_2 = r_2 - \beta_2 \frac{\pi_1(1) r_1 + r_2}{1 + \beta_1 + \beta_2}.$$  

Clearly, comparative statics results are readily analyzed using the above expressions for $\alpha = 1$. For instance, for a fixed amount of resources $r_1 + r_2$, a government that maximizes prestige should redirect resources towards the minister responsible for the largest fraction $\pi_j(1)$ of poverty, as (8) shows. However, if the objective is to increase total resources devoted to poverty reduction and $\pi_1(1) > \pi_2(1)$, then $r_1$ should be decreased. To see this, using (9) and substituting the value of equilibrium prestige in (8), we have

$$x_1 + x_2 = (r_1 + r_2) - \left(\frac{\beta_1}{\pi_1(1)} + \frac{\beta_2}{\pi_2(1)}\right) P^*.$$  

In other words, when allocating a fixed budget across ministers with $\alpha = 1$, the policy objectives “minimize measured poverty” and “maximize resources spent on the poor” are always in conflict.

We now consider the choice of weights. Note that a marginal increase in $\pi_1(1)$ increases $P^*$ if and only if $r_1 > r_2$, as we see from (8). As far as the total amount spent on the poor, assuming $\pi_1(1)$ is such that both $x_1$ and $x_2$ are strictly positive and using (10), we have:

$$\frac{\partial (x_1 + x_2)}{\partial \pi_1(1)} = \frac{\pi_2^2(1) r_2 \beta_1 - \pi_1^2(1) r_1 \beta_2}{(1 + \beta_1 + \beta_2) \pi_1^2(1) \pi_2(1)}$$

$$= \frac{(1 - \pi_1(1))^2 r_2 \beta_1 - \pi_1^2(1) r_1 \beta_2}{(1 + \beta_1 + \beta_2) \pi_1^2(1) (1 - \pi_1(1))}.$$  

from the above expression we see that $x_1 + x_2$ is a single-peaked function of $\pi_1(1)$: if the derivative is negative at some level of $\pi_1(1)$, it remains negative for all larger values of $\pi_1(1)$. Therefore, the global maximum for $x_1 + x_2$ is achieved either at the boundary values for $\pi_1(1)$, i.e., the value that makes $x_1 = 0$ or the value
that makes $x_2 = 0$, or, setting (11) equal to 0, at

$$\pi_1 (1) = \frac{\sqrt{r_2 \beta_1}}{\sqrt{r_1 \beta_2} + \sqrt{r_2 \beta_1}} = \frac{1}{1 + \frac{r_1 \beta_2}{\sqrt{r_2 \beta_1}}} \equiv \pi_1^*.$$  \hspace{1cm} (12)

Note that $\pi_1^*$ decreases in $r_1$; moreover, assuming a symmetric propensity to divert resources away from the poor (i.e., $\beta_1 = \beta_2$), $\pi_1^* > (1 - \pi_1^*) \equiv \pi_2^*$ if and only if $r_1 < r_2$. If the agent deciding weights is interested in maximizing total antipoverty spending and optimal weights are interior, the minister with the lower budget will be assigned a higher relative weight in the multidimensional poverty measure. This potentially counterintuitive result is due to the fact that ministers do not maximize the total sum spent on the poor; rather, they are interested in prestige. And, as equation (10) again makes clear, these two objectives often conflict.

The above assumes that parameter values are such that both ministers contribute to poverty-reducing programs, but not enough to fully eliminate any dimension’s contribution to poverty. To complete the characterization, we now highlight how to proceed if parameter values are such that one agent does not contribute in equilibrium—similar considerations apply for the corner solution where one minister contributes all of her resources to fighting poverty ($x_j = 1$). Without loss of generality, consider $x_2 = 0$. In this case, we have

$$P^* = \frac{\pi_1 (1) r_1}{1 + \beta_1}, \quad x_1^* = \frac{r_1}{1 + \beta_1},$$

and $x_2 = 0$ is rational if the appropriate inequality for the FOC holds, that is

$$\frac{\pi_1 (1) r_1 \beta_2}{(1 + \beta_1) \pi_2 (1)} \geq r_2;$$  \hspace{1cm} (13)

it is worth pointing out that (13) cannot hold if the conditions in Lemma 1 are verified.\footnote{To see this, note that $\frac{\pi_1 (1) r_1 \beta_2}{(1 + \beta_1) \pi_2 (1)} < \frac{\pi_1 (1) r_1}{\pi_2 (1) \beta_2} < \pi_1 (1) r_2 \leq r_2$, where the first inequality follows from $r_1 < 1 + \beta_1$ and the second by $\beta_2 < \pi_2 (1) r_2$.}

Equation (13), along with the symmetric condition for $x_1 = 0$, can now be used to determine when $\pi_1^*$ in (12) is actually a strict global maximum for $x_1 + x_2$. This is important because, when $\pi_1^*$ is a global maximum and the commission chooses weights to maximize antipoverty spending, using a multidimensional index strictly increases the amount of resources devoted to the poor (relative to a unidimensional index that assigns no weight to one of the two dimensions). At the optimal interior value in (12), we have

$$\frac{\pi_1 (1)}{\pi_2 (1)} = \sqrt{\frac{r_2 \beta_1}{r_1 \beta_2}};$$  \hspace{1cm} (14)
thus, substituting into (13) and its homologous for $x_1$ we see that, as long as
\[
\frac{\sqrt{\beta_1 \beta_2}}{(1 + \beta_2)} \leq \sqrt{\frac{r_1}{r_2}} \leq \frac{(1 + \beta_1)}{\sqrt{\beta_1 \beta_2}},
\]
then $\pi_1^*$ in (12) is a strict global maximum. Note that condition (15) is satisfied strictly for symmetric setups.

For $\alpha = 0$, if we assume that antipoverty spending results in a multiplicative decrease in the proportion of individuals who are deprived rather than in the depth of their deprivations (and since we are assuming a union approach to poverty identification), the structure is exactly the same as if we (i) replace the deprivation matrix $g^1$ with $g^0$, a matrix of zeros and ones indicating whether individual $i$ is deprived in dimension $j$, (ii) calculate $M_1$ on this new matrix $g^0$, (iii) reduce each element of each column $g^0_j$ by the proportion $x_j$ to obtain $g^0_0$, and (iv) calculate $\%\Delta M_1(x_1, x_2)$ on $g^0$ and $g^0_0$. Hence, all of the results presented above for $\alpha = 1$ also apply for $\alpha = 0$ after replacing $\pi_j(1)$ with $\pi_j(0)$ throughout.

### 3.2 Equilibrium and comparative statics for $\alpha = 2$

For $\alpha = 2$, it is possible to explicitly solve for the $x_j^*$ as done above. However, even focusing on an interior solution in which both ministers are contributors, it is not possible to derive an explicit expression for $P^*$, unless one considers the symmetric case in which $r_1 = r_2 = r$, $\beta_1 = \beta_2 = \beta$, and $\pi_1 = \pi_2 = 1/2$. Then, we have
\[
P^* = \frac{r}{1 + 2\beta}, \quad x_1^* = x_2^* = \frac{r}{1 + 2\beta}.
\]

Since explicit solutions are usually hard to reach for $\alpha \neq 1$ or in asymmetric setups, Proposition 1 is especially important, because it guarantees the validity of a computational approach and of simulations. Nonetheless, it is possible to obtain interesting analytical results for an asymmetric environment and $\alpha = 2$.

For $\alpha = 2$ and both ministers contributing a positive amount, the FOCs require
\[
x_j = \frac{1 + r_j}{2} - \frac{1}{\pi_j} t \left( \pi_j, \frac{1 - r_j}{2}, \beta_j P(1 - P) \right),
\]
where $t(a, b, c) \equiv \sqrt{(ab)^2 + ac}$, an increasing and concave function of $a$, for $c$ larger than zero. Substituting $x_j$ above in the definition of $P$, we obtain the following equilibrium condition, in which $P$ is an implicit

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12 To simplify notation, in this section we omit the argument from $\pi_i$, since $\alpha$ is here fixed at 2.

13 Using equation (16), we now see that $x_j \geq 0$ is guaranteed by $r_j \geq \frac{\beta_j}{\pi_j} P(1 - P)$, a sufficient condition for which is $\frac{r_j}{\beta_j} \pi_j \geq \frac{1}{4}$. 14
function of only exogenous quantities:

\[(1 - P)^2 = \sum_{j=1,2} \ell \left( \pi_j, \frac{1 - r_j}{2}, \beta_j P(1 - P) \right), \tag{17}\]

where \(\ell(a, b, c) = 2ab^2 + c + 2bt(a, b, c)\).

By Proposition 1, there is a unique solution to (17). We see the left-hand side (LHS) of (17) is strictly decreasing in \(P\). The exact shape of the right-hand side (RHS) depends on the values of \(r_1\) and \(r_2\), but it turns out that the RHS is inverse U-shaped, because it is an increasing function of \(P(1 - P)\). We depict both the LHS and RHS in Figure 1 with thin and thick lines, respectively. Equilibrium prestige is denoted with \(P^*\).

Condition (17) can be used to answer policy-relevant questions and to better understand the meaning of percentage improvements in indices. For instance, even fixing \(\alpha = 2\), freedom remains in constructing an index by assigning weights \(w_1\) and \(w_2\) (and therefore \(\pi_1\) and \(\pi_2\)) to each of the dimensions of poverty. In an otherwise symmetric environment, i.e., for \(\beta_1 = \beta_2 = \beta\) and \(r_1 = r_2 = r\), should one choose \(w_1\) so that \(\pi_1 = \pi_2 = \frac{1}{2}\) to maximize \(P^*\)? It turns out the answer depends on whether \(r\) is larger or smaller than 1, that is, on whether a minister has sufficient resources to eliminate deprivations in her dimension of competence.

**Proposition 2** (Choice of weights in symmetric environments). Let \(\alpha = 2\), \(\beta_1 = \beta_2 = \beta\), and \(r_1 = r_2 = r\). Consider \(\pi_1 < \frac{1}{2}\). If \(r < 1\), then a marginal decrease in \(\pi_1\) increases \(P^*\). If \(r > 1\), then a marginal decrease in \(\pi_1\) decreases \(P^*\).
Proof of Proposition 2. Under symmetry, the RHS of (17) is
\[ 2 \left( \frac{1 - r}{2} \right)^2 + 2 \beta P (1 - P) + 2 (1 - r) \frac{t \left( \pi_1, \frac{1 - r}{2}, \beta P (1 - P) \right) + t \left( 1 - \pi_1, \frac{1 - r}{2}, \beta P (1 - P) \right)}{2}, \]
and we recall \( t \) is concave in its first argument. Therefore, viewing \( x \) and \( 1 - x \) as the two equally likely outcomes of a random variable, if \( y < x < 1/2 \), second-order stochastic dominance yields
\[ \frac{1}{2} t(y, b, c) + \frac{1}{2} t(1 - y, b, c) < \frac{1}{2} t(x, b, c) + \frac{1}{2} t(1 - x, b, c). \]
Therefore, if \( r < 1 \), the RHS (17) decreases as \( \pi_1 \) decreases, point-by-point for any value of \( P \). And since the LHS of (17) decreases in \( P \), as one clearly sees in Figure 1 \( P^* \) increases as \( \pi_1 \) decreases. The opposite implication is true for \( r > 1 \).

Several economic forces are at play in Proposition 2 from issues of free riding between ministers to technological considerations; their interaction is complex. In keeping with the focus of this paper on the measurement of multidimensional poverty, here we discuss the technological aspects. When \( \alpha = 2 \), there is some degree of complementarity between the ministers’ efforts, as opposed to the perfect substitutes case implied by \( \alpha = 1 \). Ceteris paribus, this complementarity pushes towards equal weights. In our positive analysis, minister 1 takes advantage of this complementarity inasmuch as the marginal product of \( x_1 \) is enhanced by \( x_2 \). For \( \alpha = 2 \), equation (4) yields
\[ \frac{\partial P}{\partial x_1} = \frac{\pi_1 (1 - x_1)}{[\pi_1 (1 - x_1)^2 + \pi_2 (1 - x_2)^2]^2}, \]
and the cross-partial is
\[ \frac{\partial^2 P}{\partial x_1 \partial x_2} = \frac{\pi_1 \pi_2 (1 - x_1)(1 - x_2)}{[\pi_1 (1 - x_1)^2 + \pi_2 (1 - x_2)^2]^2}. \]
(Analogous conditions apply, reversing the roles of the two ministers, for the extent to which the marginal product of \( x_2 \) is enhanced by \( x_1 \).) Clearly, the value in (18) is strictly positive, thus confirming that complementarity exists. Moreover, (18) shows that the values of \( x_1 \) and \( x_2 \) determine how strong complementarity is. While in general complex, we can gather intuition on how the strength of complementarity changes with resources in the fully symmetric case in which \( \pi_1 = \pi_2 \). Symmetry implies \( x_1 = x_2 = x \), and (18) reduces to
\[ \frac{\partial^2 P}{\partial x_1 \partial x_2} = \frac{1}{4(1 - x)} . \]
Since contributions to poverty reductions are a normal good for ministers, an increase in resources \( r \) increases
by (19), increases complementarity. Therefore, complementarity is more important for relatively rich
governments, in accord with Proposition 2.

Proposition 2 provides a first step to answer the question of how to choose weights to maximize equilib-
rium prestige: relatively poor governments look better with unequal weights. However, Proposition 2 only
considers interior solutions for \( x_1 \) and \( x_2 \), and does not explore what occurs for values of \( \pi_1 \) that are so low or
high that either \( x_1 = 0 \) or \( x_2 = 0 \) in equilibrium. The following result, with proof in the Appendix,
demonstrates that if \( r < 1 \), then the above reasoning reaches the conclusion that \( P^* \) is maximized with \( \pi_1 = 0 \),
that is, for a unidimensional measure of poverty. Proposition 2 leaves open the possibility that relatively rich
government may look better with a multidimensional measure (indeed, a corollary is that prestige has a local
maximum at \( \pi_1 = 1/2 \) in symmetric environments). However, when \( \pi_1 \) is low enough that the equilibrium \( x_1 \)
is no longer interior, prestige could be higher than at this interior local maximum. The following result shows
this to be the case: a relatively rich government also maximizes prestige by assigning zero weight to one
dimension and maintaining a unidimensional poverty measure. The difference for relatively rich governments
is that, if political constraints restrict their choices of weights to those that guarantee an interior solution,
they maximize measured poverty reduction by assigning equal relative weights.

**Proposition 3** (Choice of weights in symmetric environments: corner solution). Let \( \alpha = 2 \), \( \beta_1 = \beta_2 = \beta \),
and \( r_1 = r_2 = r \). \( P^* \) is maximized for \( \pi_1 = 0 \).

Proposition 3 shows that, if possible, a prestige-motivated government unable to rein in internal free
riding would choose a unidimensional poverty measure. Of course, this does not imply anything about how
the welfare of the poor changes for multidimensional vs. unidimensional indices, but just that when \( \alpha = 2 \)
governments facing internal free riding look better with a unidimensional measure rather than a multidimen-
sional one, in terms of percentage improvements in poverty, *ceteris paribus*. Even if governments are forced
to work in a truly multidimensional framework, Proposition 2 shows that relatively poor governments would
reduce the importance of one dimension as much as allowed. However, a richer (\( r_1 > 1 \), \( r_2 > 1 \))
government has at its disposal an interior (local) maximum which, in symmetric environments, obtains at
\( \pi_1 = 1/2 \).

We now generalize our results on this constrained interior maximum to include asymmetric environments
in our analysis. We proceed along the lines of the proof of Proposition 2: if one wants to maximize \( P^* \), then
one should choose \( \pi_1 \) to minimize the value of the RHS of (17), for any possible \( P \), as is clearly seen from
Figure 1. The following Lemma, with proof in the Appendix, is very useful for the rest of the analysis.

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14In the next section, we examine how weights affect total antipoverty spending (rather than measured poverty reduction).
Lemma 2 (Properties of \ell). Consider the function \( \ell \left( \pi_j, \frac{1-r_j}{2}, \beta_j P(1-P) \right) \) in equation (17). We have

i) \( \ell_{\pi_j\pi_j} > 0 \) if \( r_j > 1 \),  
ii) \( \ell_{\pi_j\beta_j} < 0 \) if \( r_j > 1 \),  
iii) \( \ell_{\pi_j\beta_j} < 0 \),  
iv) \( \ell_{\pi_j r_j} < 0 \) if \( r_j \leq 1 \), and  
v) \( \ell_{r_j r_j} > 0 \),

where subscripts indicate partial derivatives.

Proof. The proof proceeds as that for Corollary 1, using convexity as established in part v) of Lemma 2.

Focusing on \( r_1 > 1, r_2 > 1 \), we now consider how the “optimal” choice of weights is affected by asymmetries and obtain the following results.

Corollary 1 (Optimal interior weights in asymmetric environments). Let \( r_1 > 1 \) and \( r_2 > 1 \). The pair \((\pi_1^*, \pi_2^*)\) is an interior maximizer of \( P^* \) if and only if \( \ell_{\pi_1}(\pi_1^*, \frac{1-r_1}{2}, \beta_1 P(1-P)) = \ell_{\pi_2}(\pi_2^*, \frac{1-r_2}{2}, \beta_2 P(1-P)) \).

Proof. For \( r_1 > 1 \) and \( r_2 > 1 \), the RHS of (17) is a convex function of \((\pi_1, \pi_2)\), by Lemma 2 and \( \ell_{\pi_1,\pi_2} = 0 \). Weights must satisfy the linear constraint \( \pi_1 + \pi_2 = 1 \). Therefore, the FOC \( \ell_{\pi_1} = \ell_{\pi_2} \) identifies a minimum for the RHS of (17), as desired. \( \square \)

An immediate consequence of Lemma 2 and Corollary 1 is that, if \( r_1 = r_2 > 1 \) and \( \beta_1 > \beta_2 \), then \( \pi_1^* > \pi_2^* \): a minister with a larger utility for diverting funds away from poverty reduction should be assigned a higher weight and responsibility in the determination of the multidimensional index.\(^{15}\) Again focusing on the technological aspects of the intuition, note that the higher weight on dimension 1 counterbalances the larger free riding incentive for minister 1 that otherwise would result in \( x_1 \) much smaller than \( x_2 \). By concavity of the prestige production function, the positive effect on \( x_1 \) tends to be larger than the negative effect on \( x_2 \), ceteris paribus.

We conclude this section by fixing the parameters that define the index, and considering how a total amount of resources \( R > 0 \) should be divided to maximize \( P^* \).

Corollary 2 (Choice of resource allocation). The pair \((r_1^*, r_2^*)\) such that \( r_1^* + r_2^* = R \) is an interior maximizer of \( P^* \) if and only if \( \ell_{r_1}(\pi_1, \frac{1-r_1}{2}, \beta_1 P(1-P)) = \ell_{r_2}(\pi_2, \frac{1-r_2}{2}, \beta_2 P(1-P)) \).

Proof. The proof proceeds as that for Corollary 1 using convexity as established in part v) of Lemma 2. \( \square \)

Consider a situation in which total resources are limited and each minister receives \( r_j < 1 \). Corollary 2 implies that, if \( \pi_1 < \pi_2 \) but \( \beta_1 = \beta_2 \), then \( r_1^* < r_2^* \): a minister in charge of a dimension that is less important in the overall index should receive less funds.\(^{16}\) Quite intuitively, since the marginal product of minister 1’s contributions tends to be lower, more resources should be devoted to minister 2. Less intuitively, we also obtain that, if \( \pi_1 = \pi_2 \) but \( \beta_1 > \beta_2 \), then \( r_1^* > r_2^* \): a minister with a larger incentive to divert funds should

\(^{15}\)The result follows by noticing that, by Lemma 2, \( \ell_{\pi_j\beta_j} < 0 \) and \( \ell_{\pi_j r_j} > 0 \) for \( r_j > 1 \): if \( \pi_1 \leq \pi_2 \), then the LHS of the characterizing equation in Corollary 1 would be smaller than its RHS.

\(^{16}\)The proof follows using the derivatives in Lemma 2 very much as in footnote 15.
receive more resources. However, as discussed earlier, note that the larger resources for minister 1 reduce the marginal benefit of private consumption and thus counterbalance the free riding incentive for minister 1 that otherwise would result in \( x_1 \) much smaller than \( x_2 \). By concavity of the prestige production function, the positive effect on the smaller \( x_1 \) tends to be larger than the negative effect on the larger \( x_2 \), ceteris paribus, even if a fraction of the extra resources will be diverted.\(^{17}\)

It is worth pointing out that Corollary 1 does not state how to choose \( \pi_1 \) and \( \pi_2 \) to maximize the total resources devoted to poverty reduction, and, similarly, Corollary 2 does not state how to choose \( r_1 \) and \( r_2 \) to accomplish this objective. In general, maximizing \( P \) does not coincide with maximizing \( x_1 + x_2 \), as we earlier demonstrated for the \( \alpha = 0 \) and \( \alpha = 1 \) cases in Section 3.1. To see that they do not necessarily coincide when \( \alpha = 2 \), consider \( \pi_1 = 0.35, \beta_1 = \beta_2 = 1 \), and \( r_1 + r_2 = 1.5 \). The maximum \( P \) is obtained at \( r_1 \approx 0.604, r_2 \approx 0.896 \), with \( x_1 + x_2 \approx 0.44299 \) and \( P \approx 0.25563 \). However, by considering a small change to \( r_1 = 0.61 \) and \( r_2 = 0.89 \) one obtains \( x_1 + x_2 \approx 0.44346 \) and \( P \approx 0.25562 \): an increase in total expenditure on poverty reduction at the price of a lower percentage improvement in the index.\(^ {18}\) More formally, maximizing measured poverty reduction is achieved by minimizing \( \sum_j \ell \left( \pi_j, \frac{1 - r_j}{2}, \beta_j P(1 - P) \right) \) in (17), so that the inverse-u-shaped curve in Figure 1 moves downwards and the equilibrium \( P^* \) moves to the right. In contrast, maximizing antipoverty spending is achieved by minimizing \( \sum_j \frac{1}{r_j} \ell \left( \pi_j, \frac{1 - r_j}{2}, \beta_j P(1 - P) \right) \) from (16); these two different objectives sometimes, but not always, conflict.

### 3.3 The choice of \( \alpha \)

We now investigate the consequences of different values of \( \alpha \). Because of the complexities involved, we specialize the model to the case \( r_1 = r_2 = 1, \beta_1 = \beta_2 = \beta \), but leave open the possibility that \( \pi_1 (\alpha) \neq \pi_2 (\alpha) \). For these parameter values, summing the FOCs in (6) and (7) gives

\[
(1 - P)^{1-\alpha} (\pi_1 (\alpha) (1 - x_1)^\alpha + \pi_2 (\alpha) (1 - x_1)^\alpha) = 2\beta P.
\]

Therefore, using (3), the equilibrium value of prestige simplifies to \( P = 1/(1 + 2\beta) \), a quantity that does not depend on \( \alpha \). Further substitution into the FOC yields

\[
\pi_j (\alpha) (1 - x_j)^\alpha = \frac{1}{2} \left( \frac{2\beta}{1 + 2\beta} \right)^\alpha.
\]

\(^ {17}\)Otherwise stated, note that \( P \) implies that the marginal rate of substitution between \( x_1 \) and \( x_2 \) is \( \pi_1 (1 - x_1)/(\pi_2 (1 - x_2)) \): i.e., the dimension with the lower investment generates the larger marginal benefit.

\(^ {18}\)Similarly, a change in weights can cause \( P \) and \( x_1 + x_2 \) to move in opposite directions. Continuing with the above example with optimally chosen budgets for fixed weights \( \pi_1 = 0.35, \pi_2 = 0.65 \), an increase in \( \pi_1 \) to 0.36 and corresponding decrease in \( \pi_2 \) decreases prestige to \( P \approx 0.25486 \) but increases total antipoverty spending to \( x_1 + x_2 \approx 0.44999 \).
While the equilibrium prestige prize is independent of $\alpha$ in this setup, other quantities of interest change with $\alpha$. Using (2) and (3), the percentage improvement in the measure of poverty is $\%\Delta M_{\alpha} = 1 - (1 - P)^{\alpha}$. Therefore, since $(1 - P)$ is constant in $\alpha$ and smaller than 1, $(1 - P)^{\alpha}$ decreases in $\alpha$, so the percentage improvement in the measure of poverty increases in $\alpha$.

We now turn attention to how $\alpha$ affects the total resources expended to fight poverty. Using (20), if $\pi_j(\alpha) = 1/2$, then $1 - x_j = 2\beta/(1 + 2\beta)$, a quantity independent of $\alpha$. Otherwise, if $\pi_j(\alpha) \neq 1/2$, then (20) implies that the sum of private consumptions is

$$
(1 - x_1) + (1 - x_2) = \left(\frac{2\beta}{1 + 2\beta}\right) \left(\frac{1}{2}\right)^{\frac{\beta}{\pi}} \left[\left(\frac{1}{\pi_1(\alpha)}\right)^{\frac{\beta}{\pi}} + \left(\frac{1}{\pi_2(\alpha)}\right)^{\frac{\beta}{\pi}}\right].
$$

(21)

Consistently with the implications of Corollary 1 when applied to a symmetric environment, equation (21) implies that, for fixed $\alpha \geq 1$, asymmetries in $\pi_1$ and $\pi_2$ reduce the total efforts aimed at fighting poverty. In other words, poverty reduction effort is maximized when weights $w_1$ and $w_2$ are set so that each dimension’s contribution to the original multidimensional poverty index are equal. Intuitively, the result follows because, since both ministers benefit in the same way from a unique multidimensional index, they have an equilibrium incentive to equalize the ratio of marginal product and marginal cost of efforts across dimensions. The marginal product is well-known to equal $\pi_j(1 - x_j)^{\alpha - 1}$. In the setup of this subsection, the marginal cost is $1/(1 - x_j)$. Thus, in equilibrium, there is perfect equalization of the contribution to poverty improvement by dimension, as described in (20). Since the marginal benefit of prestige is the same for each minister and prestige is constant in equilibrium, a 10% increase in $\pi_j$ reduces $(1 - x_j)^{\alpha}$ by 10%. But, since $\alpha \geq 1$, if $\pi_1 < \pi_2$, then an increase in $\pi_1$ results in an increase in $x_1$ that more than compensates for the reduction in $x_2$, because $x_1$ is initially smaller.

Since $\pi_j(\alpha) = w_j \mu(g_{\alpha j}^*)/M_{\alpha}$, this implies that the dimension with initially lower average (raised to the $\alpha$ power) normalized shortfalls should receive a higher weight. Therefore, our positive argument corresponds with what Decanq and Lugo (2013) call frequency-based weights, where a dimension’s weight is an inverse function of the average deprivation in that dimension. In the literature, the normative logic behind frequency-based weights “is that if owning a refrigerator is much more common than owning a dryer, a greater weight should be given to the former indicator so that if an individual does not own a refrigerator, this rare occurrence will be taken much more into account in computing the overall degree of poverty than if some individual does not own a dryer” (Deutsch and Silber, 2005, p. 150).

How do total efforts aimed at fighting poverty depend on $\alpha$? Since $\pi_j(\alpha)$ changes with $\alpha$ in nontrivial ways, further simplification is needed to gain more insight into this question. Suppose that $n_j$ agents are

\[19\]To see this formally, consider the total resources not devoted to poverty reduction described by (21), and note that the term in square brackets is convex in $\pi_j$, so it achieves a minimum for $\pi_1 = \pi_2 = 1/2$. 

17
We then have
\[
\pi_j(\alpha) = \frac{n_j \cdot (g_j)^\alpha}{n_1 \cdot (g_1)^\alpha + n_2 \cdot (g_2)^\alpha},
\] (22)
for \(j = 1, 2\). The next result, with proof in the Appendix, considers two special, but instructive, cases.

**Proposition 4** (Choice of \(\alpha\) in symmetric environments). Consider \(\pi_1 \neq \pi_2\). 1) If \(n_1 = n_2 = n\), but \(g_1 \neq g_2\), then poverty-reducing efforts are decreasing in \(\alpha\). 2) In contrast, if \(g_1 = g_2\), but \(n_1 \neq n_2\), then poverty-reducing efforts are increasing in \(\alpha\).

The intuition for part 1) of Proposition 4 is simple. As (22) shows, if \(n_1 = n_2\) then the asymmetry between \(\pi_1\) and \(\pi_2\) is exacerbated by an increase in \(\alpha\). Therefore, the result follows from our earlier discussion following equation (21). As for part 2), if \(g_1 = g_2\), then \(\pi_1\) and \(\pi_2\) are independent of \(\alpha\), therefore the above force does not enter into play. What matters is that, as \(\alpha\) increases, the two dimensions become more complementary. Indeed, thinking of \(P\) in (3) as a production function and using (4), one sees that the marginal rate of transformation between \(x_1\) and \(x_2\) is
\[
\frac{\pi_1}{\pi_2} \left(\frac{1 - x_1}{1 - x_2}\right)^{\alpha-1},
\]
so the slope of the isoquants above approaches that for a Leontief production function as \(\alpha\) grows large. And the larger the complementarity, the smaller the incentive of a minister to free ride on the other, since one’s shortfall cannot easily be made up by another’s increased contribution. Therefore, total contributions tend to increase with \(\alpha\).

## 4 Illustration

We illustrate using data from Mexico, the first country to adopt an official multidimensional poverty measure based on AF. We use indicators from Mexico’s official measure and the same data that is used by the Mexican government. We restrict ourselves to two indicators since our theoretical model has two dimensions. Because all of the dimensions except income are dichotomous, we focus on \(\alpha = 0\), a case described in Section 3.1.

Our illustration accomplishes three objectives. First, it illustrates the usefulness of our framework even when one or more of the dimensions of poverty is measured ordinally. Since many of the governments implementing multidimensional poverty use ordinal variables in their measure, they often use \(M_0\); hence, it is important to accommodate this possibility in our framework. Second, it illustrates many of the theoretical results from Section 3 for \(\alpha = 0\), in particular the strength of the conflict between maximizing equilibrium measured poverty reduction and maximizing resources devoted to the poor. In particular, a government
that seeks to maximize prestige will reallocate resources to the minister with the higher relative weight in the index, while a government that seeks to maximize total antipoverty spending will do just the opposite. And a committee choosing weights with the objective to maximize prestige will assign higher weight to the ministry with a larger budget, while one that seeks to maximize antipoverty spending will choose weights based on (12) for interior solutions: these weights tend toward equality but, in the equal propensities for private consumption case, require assigning less weight to the minister with the larger budget. Third, by illustrating that ministers with a low relative weight in the index (due to a low absolute weight or low deprivations relative to other dimensions), a high propensity for diverting funds, or a low relative budget have great incentives to free ride when multidimensional poverty is measured using a scalar measure, it points to the importance of decomposing multidimensional poverty by dimension and prominently publishing its partial indices as well as the overall measure.  

4.1 Data and Mexico’s multidimensional poverty measure

We use the 2010 Encuesta Nacional de Ingresos y Gastos de Hogares (National Income and Expenditure Survey; ENIGH). The survey was modified in 2008 to include the Módulo de Condiciones Socioeconómicas (Socioeconomic Conditions Module; MCS), which was specifically designed to enable the measurement of multidimensional poverty using the indicators decided on by the Mexican government’s Consejo Nacional de Evaluación de la Política de Desarrollo Social (National Council for the Evaluation of Social Development Policy; CONEVAL). The 2010 survey includes 30,169 households, and is representative at the national, urban, rural, and state levels. The government is required by law to produce multidimensional poverty measures for the country as a whole and for each state every two years, as well as for all municipalities every five years (CONEVAL, 2010). Mexico’s official poverty measurement uses the ENIGH MCS data and identifies seven dimensions: income, education, access to health care, access to social security, housing quality, basic housing services, and food security.

Income in our illustration is defined identically to the definition used in Mexico’s official poverty measurement: it includes income from labor, self-employment, capital, public and private transfers, the imputed value of own production, regular but not extraordinary in-kind payments, and regular but not extraordinary gifts received. The imputed value of owner-occupied housing is not included in the official income concept, with the argument that households cannot use this “income” component to meet their basic needs. Two deprivation cutoffs are used in the income dimension, resulting in two reported levels of multidimensional poverty in Mexico. The first cutoff, called the minimum wellbeing line, corresponds to extreme poverty and

20This, in turn, stresses the importance of the dimensional decomposability axiom; such decompositions and the publishing of partial indices have been advocated by proponents of the AF measure (e.g., Alkire et al., 2011).
is calculated as the per-adult-equivalent cost of a minimum basket of food, which in turn is based on caloric intake requirements and observed consumption patterns of households whose members approximately meet the caloric intake minimums. The second cutoff, called the wellbeing line, also incorporates the cost of basic non-food necessities based on consumption patterns of the same families (those who approximately meet the caloric intake minimums).\footnote{21}

Of the seven dimensions included in Mexico’s multidimensional poverty measure, we choose to include income as one of the two dimensions of our illustration because it is given a much larger weight than other dimensions: the weight on income deprivations is 1/2, while the weight on each of the remaining six dimensions is 1/12. This accords with Foster’s (2007, p. 9) point that “a weight on income that is higher than the equal-weight case, but lower than the full-weight case, represents a reasonable compromise between a traditional ‘economic’ view of poverty and a more inclusive multidimensional view.” In addition, it is important to include income as a dimension due to its fungibility and usefulness in reducing deprivations in other dimensions and its salience in discussions about poverty (Foster, 2007).

With respect to the non-income dimensions, the indicators and their deprivation cutoffs were chosen based on rights guaranteed in the Mexican Constitution and other laws; this method of selecting dimensions and cutoffs is praised by Alkire and Foster (2011b). In our illustration, we focus on access to healthcare as our second dimension. In Mexico, healthcare is not universally provided by the government; a household is defined as deprived in access to healthcare if they do not have any type of medical insurance (including the government-provided insurance for the uninsured \textit{Seguro Popular}, insurance through the Mexican Institute of Social Security which is provided to formal sector workers, insurance for state employees, or private insurance). There are a number of reasons we chose health as the second dimension for our illustration. First, Birdsall (2011) argues that multidimensional poverty measures have independent value compared to Ravallion’s (1996, p. 1332) “multiple-indicator approach” if they reveal additional information about poverty dynamics; lack of health insurance in Mexico has indeed been shown to increase the vulnerability of income-poor households to becoming poorer (López Calva and Ortiz Juárez, 2009). Second, there is a ministry that is clearly tied to this dimension (the Health Ministry). Although there is also a ministry clearly tied to the dimension of education, our assumption that spending results in proportional reductions in the number of deprived individuals seems more adequate for access to healthcare than for the education dimension since educational deprivation in Mexico’s measure is a stock variable for those older than 15.\footnote{22}

\footnote{21}The cutoffs are calculated separately for urban and rural areas. The minimum wellbeing line for 2010 equals 1125.42 pesos per adult equivalent per month in urban areas and 800.26 pesos in rural areas. The wellbeing line equals 2328.82 pesos in urban areas and 1489.78 in rural areas. Adult equivalence and economies of scale are taken into account as follows: the household head is assigned a weight of 1, additional adults ages 19 and above are assigned a weight of 0.9945, adolescents ages 13–18 are assigned a weight of 0.7057, children ages 6–12 are assigned a weight of 0.7382, and children ages 0–5 are assigned a weight of 0.7031.

\footnote{22}Specifically, deprivations in education are defined as follows. For children ages 3–15, those who neither attend a formal
4.2 Illustration using official dimensions and $\alpha = 0$

We denote income as dimension $j = 1$ and health as $j = 2$. Using the minimum wellbeing line (MWL), $\mu(g^0_1) = 0.194$, while using the wellbeing line (WL), $\mu(g^0_1) = 0.520$. In other words, 19% of the Mexican population had income below the MWL in 2010 and was thus unable to afford a basic food basket based on a minimum caloric intake for each of its members. Just over half of the population had income below the cost of a basket of basic needs including both food and nonfood needs. In access to health insurance, $\mu(g^0_2) = 0.292$; 29% of Mexicans did not have access to some form of health care despite the fact that access is, in theory, guaranteed by law.

Using equal weights between the two dimensions and the MWL as the deprivation cutoff in the income dimension, $M_0 = 0.243$; using the WL, $M_0 = 0.406$. Another weighting scheme we consider is assigning a larger weight to income than to health, which is in line with Foster’s (2007) recommendations and reflects the assignment of weights in practice in Mexico. Specifically, we consider the case where the weight on income is six times larger than the weight on health, which reflects the ratio of weights in Mexico’s official multidimensional poverty measure. Finally, we consider a weighting scheme in the tradition of frequency-based weights, where dimensions in which deprivations are higher receive less weight because indicators in which very few are deprived might imply that these deprivations have a larger negative impact on wellbeing. Specifically, we consider setting weights so that each dimension contributes equally to the initial level of multidimensional poverty (in other words, $w_1$ and $w_2 = 1 - w_1$ are chosen so that $\pi_1 = \pi_2$).

Table II shows the results of our model using these different weighting schemes for various budget sizes and propensities to spend on private consumption. In Panel A, the ministries have the same budgets and elasticities of private consumption. Their budgets are insufficient to eliminate deprivations in their respective dimensions, since $r_1 = r_2 = 0.5 < 1$. Both have a high propensity for antipoverty spending and a low propensity for private spending, as seen by their low values of $\beta_j$. Nevertheless, utility is concave in poverty reduction, which explains why they do not spend all of their resources fighting poverty. In the equal weights case using the MWL (column I), the ministry responsible for income poverty—which for simplicity we will refer to as the social ministry—spends 80% (0.396/0.5) of its budget fighting poverty in equilibrium, while the health ministry spends 86% (0.431/0.5).

Comparing column I to columns II and III, it is obvious that the choice of weights in the multidimensional measure impacts the total spent to fight poverty. Column II corresponds to $w_1 = 6/7, w_2 = 1/7$, reflecting the ratio of weights assigned to the income and health dimensions in Mexico’s actual measure, and column III

---

education institution nor have completed the minimum basic secondary education are considered deprived. For those older than 15 born after 1982, those who did not complete the required basic secondary education are considered deprived; for those born before 1982, those who did not complete the former requirement of primary education are considered deprived.

However, when $\alpha = 0$ and $r_1 = r_2$, it follows from [8] that government prestige is unaffected by the choice of weights.
Table 1: Illustration with Mexican data

<table>
<thead>
<tr>
<th></th>
<th>Minimum wellbeing line</th>
<th>Wellbeing line</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(I)</td>
<td>(II)</td>
</tr>
<tr>
<td>$w_1$</td>
<td>0.500</td>
<td>0.857</td>
</tr>
<tr>
<td>$w_2$</td>
<td>0.500</td>
<td>0.143</td>
</tr>
<tr>
<td>$\mu_1(g_0^1)$</td>
<td>0.194</td>
<td>0.194</td>
</tr>
<tr>
<td>$\mu_2(g_0^2)$</td>
<td>0.292</td>
<td>0.292</td>
</tr>
<tr>
<td>$M_0$</td>
<td>0.243</td>
<td>0.208</td>
</tr>
<tr>
<td>$\pi_1(0)$</td>
<td>0.399</td>
<td>0.799</td>
</tr>
<tr>
<td>$\pi_2(0)$</td>
<td>0.601</td>
<td>0.201</td>
</tr>
</tbody>
</table>

Panel A: $r_1 = r_2 = 0.5$, $\beta_1 = \beta_2 = 0.1$

<table>
<thead>
<tr>
<th></th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$x_1^* + x_2^*$</th>
<th>$M_0^*$</th>
<th>$P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^*$</td>
<td>0.396</td>
<td>0.448</td>
<td>0.417</td>
<td>0.435</td>
<td>0.454</td>
</tr>
<tr>
<td>$x_2^*$</td>
<td>0.431</td>
<td>0.292</td>
<td>0.417</td>
<td>0.384</td>
<td>0.014</td>
</tr>
<tr>
<td>$x_1^* + x_2^*$</td>
<td>0.826</td>
<td>0.740</td>
<td>0.833</td>
<td>0.819</td>
<td>0.468</td>
</tr>
<tr>
<td>$M_0^*$</td>
<td>0.142</td>
<td>0.121</td>
<td>0.136</td>
<td>0.237</td>
<td>0.284</td>
</tr>
<tr>
<td>$P^*$</td>
<td>0.417</td>
<td>0.417</td>
<td>0.417</td>
<td>0.417</td>
<td>0.417</td>
</tr>
</tbody>
</table>

Panel B: $r_1 = 0.55$, $r_2 = 0.45$, $\beta_1 = \beta_2 = 0.1$

<table>
<thead>
<tr>
<th></th>
<th>$x_1^*$</th>
<th>$x_2^*$</th>
<th>$x_1^* + x_2^*$</th>
<th>$M_0^*$</th>
<th>$P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^*$</td>
<td>0.448</td>
<td>0.495</td>
<td>0.467</td>
<td>0.483</td>
<td>0.500</td>
</tr>
<tr>
<td>$x_2^*$</td>
<td>0.382</td>
<td>0.230</td>
<td>0.367</td>
<td>0.331</td>
<td>0.000</td>
</tr>
<tr>
<td>$x_1^* + x_2^*$</td>
<td>0.830</td>
<td>0.725</td>
<td>0.833</td>
<td>0.814</td>
<td>0.500</td>
</tr>
<tr>
<td>$M_0^*$</td>
<td>0.144</td>
<td>0.116</td>
<td>0.136</td>
<td>0.232</td>
<td>0.265</td>
</tr>
<tr>
<td>$P^*$</td>
<td>0.408</td>
<td>0.442</td>
<td>0.417</td>
<td>0.428</td>
<td>0.457</td>
</tr>
</tbody>
</table>

The choice of deprivation cutoffs, which we treated as given in our theoretical model, also influence total antipoverty spending by influencing $\mu_j(g_0^0)$ and hence $\pi_j(\alpha)$. Using the WL as the income deprivation cutoff increases asymmetries between $\pi_1$ and $\pi_2$ for both the $w_1 = w_2$ and $w_1 = 6w_2$ cases, which causes a decrease in antipoverty spending (compare columns I and II to columns IV and V). The reduction is particularly drastic when $w_1 = 6w_2$. Using the WL implies that nearly twice as many individuals are deprived in the income dimension as those deprived in access to healthcare. When, on top of having a larger value of $\mu_j(g_0^0)$, the income dimension is also assigned a much larger weight $w_1 = 6w_2$, the impact of the health dimension on the overall poverty measure becomes very small: $\pi_2 = 0.086$, indicating that less than 9% of initial multidimensional poverty is explained by the health dimension (column V). Thus, the marginal benefit (in terms of measured poverty reduction) of each dollar spent by the health minister on poverty reduction is
low: indeed, it decides to spend less than 3% of its budget on poverty reduction despite having a fairly low utility from private consumption.

This finding has policy implications: if the dimension of a particular ministry contributes very little to the initial level of poverty, that ministry will have little incentive to spend on poverty reduction. In Mexico, the six non-income dimensions each have a low weight of 1/12 in the official multidimensional poverty measure. In 2010, the quality housing dimension, for example, contributed an extremely small amount to the multidimensional poverty index ($\pi_j = 0.013$). This also provides an argument for frequency-type weights.

The line of reasoning given by Deutsch and Silber (2005) for frequency-based weights, adapted to the case of Mexico, would be that since only 15% of the population was deprived in the quality housing dimension in 2010, while 61% was deprived in access to social security, an individual who is one of the few lacking quality housing is likely more deprived (ceteris paribus) than one of the many lacking access to social security. Another justification arises from examining the political economy of multidimensional poverty reduction: if a higher absolute weight is not assigned to the quality housing dimension, the housing ministry will not have much incentive to spend on reducing deprivations in its dimension.

Having illustrated some potential implications of different weights in the multidimensional poverty measure, we now look at the implications of changing budgets. In Panel B, we allocate slightly more of the budget to minister 1: $r_1 = 0.55, r_2 = 0.45$. In column V we see that minister 2 now contributes nothing to poverty reduction. This occurs because her resources and the weight assigned to dimension 2 are too low: using (13), we have $\pi_1 r_1 \beta_2 / ((1 + \beta_1) \pi_2) \approx 0.534 > r_2 = 0.45$. Concentrating instead on the scenarios in which an interior equilibrium obtains and comparing Panels A and B, we know from (8) that reallocating resources from minister 2 to minister 1 increases measured poverty reduction if $\pi_1 > \pi_2$, as we have in columns II and IV; for the same reason, measured poverty reduction decreases in column I, since equal weights and larger deprivations in health than income using the MWL imply $\pi_1 < \pi_2$. It also follows from (8) and (10) that when weights are chosen to force $\pi_1 = \pi_2$, a reallocation of budgets—as long as $r_1 + r_2$ is held fixed—does not affect $P^*$ or $x_1^* + x_2^*$, as seen in columns III and VI.

The effects of increasing minister 1’s budget on total antipoverty spending are distinct from the effects on measured poverty reduction: in all cases except for column V, which involves a corner solution, the two move in opposite directions (as equation (10) shows). For example, in the equal weights WL case (column IV), increasing the budget of the ministry responsible for the income dimension in which 52% of the population is deprived increases measured poverty reduction from 41.7% to 42.8%. However, total antipoverty spending decreases slightly, from 81.9% to 81.4% of budgets. Of course, the social minister increases her absolute antipoverty spending, but this is more than offset by a decrease in poverty spending by the health minister, who is able to free ride more now that her relative budget is lower.
Since \( r_1 \neq r_2 \) in Panel B, we can also observe how weights affect measured poverty reduction and antipoverty spending when resources differ across ministers. From (8), \( P^* \) is maximized when assigning the highest weight possible to the ministry with the largest available antipoverty budget. Since ministry 1 has a higher budget in Panel B, \( P^* \) is highest when weights are chosen to make \( \pi_1 \) much larger than \( \pi_2 \), as is the case when \( w_1 = 6w_2 \). This highlights a perverse incentive that some governments may have to continue measuring poverty unidimensionally: if the social ministry has more funds available to fight poverty than, say, the education or health ministry\(^{24}\), continuing to measure income poverty only (which is equivalent to setting \( w_1 = 1 \) and thus \( \pi_1 = 1 \)) will make the government appear to be reducing poverty by more than it would if it changes to a multidimensional measure.

When \( r_1 \neq r_2 \), the optimal weights to maximize antipoverty spending are no longer those that generate equal relative weights: although total antipoverty spending is higher in columns III and VI than the other columns in Table 1, an even higher antipoverty spending can be achieved with the weights derived in (12) for interior equilibria. From (12), \( \pi_1^* \approx 0.475 \), which gives slightly higher antipoverty spending than in the \( \pi_1 = \pi_2 \) case: \( x_1^* + x_2^* \approx 0.834 \) when \( \pi_1 \approx 0.475 \). For the budgets and elasticities of private spending in Panel B, it is easy to verify that (15) holds, so this local interior maximum is a global maximum.

5 Conclusion

We use a simple game theoretic framework to analyze the political economy issues that can come into play when poverty is measured multidimensionally. When different deprivations are combined into a scalar measure of poverty, and prestige is jointly bestowed on ministers for improvements in this measure, a contribution to a public good game arises between government ministries in charge of reducing deprivations in different dimensions. Each minister has preferences over antipoverty spending and private spending; the latter can be thought of as corruption or simply other spending that doesn’t reach the poor. Each minister’s antipoverty spending benefits all ministries because it reduces multidimensional poverty and increases government prestige. The weights assigned to each dimension, the ministers’ propensity for private spending, their budgets, and the measure’s \( \alpha \) parameter of deprivation depth aversion all affect the ministers’ ability to reduce multidimensional poverty and their incentives to free ride on the contributions of other ministers.

A number of interesting results emerge from our model. For \( \alpha = 0 \) (the most common choice by governments in practice since it allows dimensional achievements to be measured ordinally) and \( \alpha = 1 \), the allocation of resources across ministries depends critically on the policy objective of the agent assigning bud-

\(^{24}\) Although the health and education ministries usually have higher overall budgets, it is not unrealistic for them to have a lower budget that could politically be spent to fight poverty without alienating the middle class who also demands health and education services.
gets. If this agent is concerned with the government “looking good” in terms of measured poverty reduction, he assigns as much budget as possible to the minister whose dimension makes a larger contribution to initial poverty. If, on the other hand, he wants to maximize total antipoverty spending by the ministers, he does the opposite, assigning as much budget as possible to the minister whose dimension has lower importance. A technical committee assigning weights for the measure will give as much weight as possible to the dimension that corresponds to the ministry with the larger budget if its objective is to maximize government prestige. In contrast, if the committee’s objective is to maximize poverty-reducing effort by the ministers, weights should be chosen so that each dimension’s contribution to initial poverty is much more similar. While the exact values of the weights depend on the ministers’ budgets and propensity for private spending, if ministers have identical preferences, then the minister with the lower budget should receive a larger weight.

Ministries with a low relative weight in the multidimensional poverty measure have large incentives to free ride even if they have a sizeable budget and low elasticity of private consumption. In our illustration with Mexican data, when we use the actual ratio of absolute weights in Mexico’s official measure and the higher of the two official income deprivation cutoffs, the health dimension has a low relative weight and, as a result, the health ministry has a relatively low incentive to spend on the poor. The choice of $\alpha$ can have nuanced implications for antipoverty spending: it increases complementarities across dimensions which reduces the potential to free ride, but also exacerbates initial asymmetries in relative dimension weights, with the negative consequences on antipoverty spending outlined above.

Our study has a number of limitations, and is only a first attempt at understanding the complicated political economy issues at play when poverty is measured multidimensionally. Most obviously, our model of the political process is extremely simplified. Myriad factors and restrictions that are not included in our model affect the amount that different ministries spend to reduce poverty. Furthermore, agents may be able to increase antipoverty spending by cooperating to some extent. Nevertheless, as long as multiple agents are responsible for the dimensions of a scalar measure of multidimensional poverty and these agents cannot write complete and enforceable binding contracts with one another, then a public good problem among policymakers arises. Coalition governments in which coalitions are not very stable might be especially susceptible to free riding concerns since it will be harder for ministers in such governments to cooperate and achieve higher total antipoverty spending than in the non-cooperative equilibrium considered here.

We make a number of simplifying assumptions regarding the measurement of multidimensional poverty as well. First, we restrict the analysis to the Alkire and Foster (2011a) class of multidimensional poverty measures, which do not allow for varying complementarity or substitutability between deprivations in different dimensions. It has been argued that for this reason Bourguignon and Chakravarty’s (2003) class of measures
are “probably better” if the number of dimensions under consideration is small (Silber 2011, p. 480). Nevertheless, we view this restriction as justified since the AF specification is the one being adopted by governments around the world, as well as by the United Nations Development Program for over 100 countries in their 2010 Human Development Report (UNDP 2010; see also Alkire and Santos 2014). Second, we focus on the simplified case in which the multidimensional poverty index has two dimensions. While this is unlikely to be the case in practice, it is a common simplification that allows us to gain intuition about the relevant interactions between ministries. Third, our two-dimensional AF measure uses the union approach to identify the poor: an individual is poor if she is deprived in at least one dimension. Using the intersection approach, where an individual is poor only if she is deprived in both dimensions, should only increase the incentives for ministries to free ride because the other minister’s actions can completely eliminate an individuals’ poverty status. When the model is extended beyond two dimensions and Alkire and Foster’s (2011a) dual cutoff approach is used, where a cutoff $k$ is chosen such that an individual is identified as poor if $\sum_j w_j g^0_{ij} \geq k$, there should still be more free riding than under the union approach.

We also make a number of simplifying assumptions regarding the preferences of the ministers and the poverty-reducing implications of their spending. First, we restrict the ministers’ preference functions to be Cobb-Douglas; it is worth noting that if instead they are quasilinear in multidimensional poverty reduction, the amount allocated to poverty reduction by minister 2 does not enter into the FOC of minister 1’s maximization problem and vice versa, so the political economy dynamics of using a multidimensional poverty measure become less interesting. Second, we assume that antipoverty spending in each dimension causes proportional reductions in individuals’ deprivations in that dimension, or in the case of $\alpha = 0$ which is insensitive to the depth of deprivations, we assume spending results in a proportional reduction in the number of individuals deprived in that dimension. If, instead of assuming this constant marginal product technology for poverty reduction, we considered a more realistic concave production function, corner solutions would be less common. Third, we assume away “spillovers” across dimensions: spending by minister $j$ reduces deprivations in dimension $j$ only. Including such spillover effects would affect the details of our analysis, but would only exacerbate the free riding incentives we analyze, and would also cause them to persist even if the multidimensional measure is decomposed by dimension. Relaxations of our simplifying assumptions

25 Other characteristics of Alkire and Foster’s (2011a) measure have been criticized as well. One is that under the strong focus axiom, an increase in the attainment of a poor individual in one of her non-deprived dimensions does not reduce poverty, while it may be the case that attainments in non-deprived dimensions can, to some degree, be substituted for deprivations in other dimensions (Permanyer 2014). The AF class is also not continuous in its arguments at each deprivation cutoff (except when the union approach is used to identify the poor) due to the dual cutoff approach. Furthermore, a regressive transfer in a certain dimension can decrease poverty even when $\alpha \geq 1$ if it causes the less poor individual to no longer be deprived in that dimension and as a result to no longer be poor (Permanyer 2014, footnote 9).

26 For example, Kanbur (1987) assumes two sectors when studying the political economy of unidimensional poverty reduction, and Duclos et al. (2014) assume two dimensions for multidimensional poverty targeting.

27 This ignores, for example, the potential of increased income to reduce health deprivations (e.g., Pritchett and Summers 1996) or food insecurity (Sen 1981) and the causal impact of schooling on health (Conti et al. 2010).
are the subject of ongoing research.

A number of policy implications emerge. To reduce free riding and increase overall antipoverty spending, governments can use a measure that satisfies the dimensional decomposability axiom and prominently publish the partial indices for each dimension in conjunction with the overall index. Our line of reasoning reinforces what has been advocated by proponents of the AF measure (e.g., Alkire et al., 2011) and what is indeed done by some governments (e.g., Colombia and Mexico). This also stresses the importance of the dimensional decomposability axiom, which is not met by some measures proposed in the literature.28

Since relative weights depend on both the absolute weights and vector of deprivations, the sometimes arbitrary selection of weights and cutoffs can have important implications for antipoverty spending. If the objective is to maximize antipoverty spending and ministers have similar budgets and propensities for private spending, these should be chosen so that each dimension initially contributes to an approximately equal proportion of total multidimensional poverty. Our model also sheds light on how the selection of $\alpha$ (in cases where the different dimensions are measured with continuous variables) affects antipoverty spending: if deprivations are larger or more disperse in one dimension, using a higher $\alpha$ can exacerbate asymmetries across dimensions which increases free riding, but will also increase the complementarity of dimensions which decreases free riding. The resulting change in antipoverty spending is ambiguous and depends on the specifics of the situation. Finally, the policy objectives of total antipoverty spending and government prestige often conflict, which is an important result to keep in mind when evaluating the process used to select the multidimensional poverty measure’s parameters or the allocation of budgets across ministries.

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28Our paper thus contributes to the debate about which axioms should be required of a multidimensional poverty measure, especially in light of Alkire and Foster’s (2014) impossibility result that no measure can satisfy both the axiom of dimensional decomposability and Bourguignon and Chakravarty’s (2003) axiom of non-decreasing poverty under correlation-increasing switch (also called “dimensional transfer”).
Appendix

**Example 1** (An illustration of AF Index).

Consider the matrix of normalized gaps
\[
\begin{bmatrix}
0 & 0 \\
\frac{1}{3} & 0 \\
\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{3}
\end{bmatrix}.
\]

In this economy, out of 4 individuals the first agent is not poor, the second agent is deprived in the first dimension, with a normalized gap of 1/3 of the dimension-1-specific poverty line, the third agent is deprived in both dimensions, with a normalized gap of 1/2 of each dimension-specific poverty line, and the fourth agent is deprived only in the second dimension, with a normalized gap of 1/3. For this economy, if \(w_1 = w_2 = 1/2\), we have

\[
M_1 = \frac{1}{2} \left( \frac{0 + \frac{1}{3} + \frac{1}{2} + 0}{4} + \frac{0 + 0 + \frac{1}{2} + \frac{1}{3}}{4} \right) = \frac{5}{24}, \\
M_2 = \frac{1}{2} \left( \frac{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2}{4} \right) = \frac{13}{144},
\]

and one can proceed in a similar fashion for larger values of \(\alpha\).

Consider now an expenditure of resources described by \(x_1 = \frac{1}{3}\) and \(x_2 = \frac{1}{3}\). We then have that the new values of the multidimensional poverty indexes are

\[
M_1 = \frac{1}{2} \left( \frac{\frac{1}{3} \left(1 - \frac{1}{3}\right) + \frac{1}{2} \left(1 - \frac{1}{3}\right) + \frac{1}{2} \left(1 - \frac{1}{3}\right) + \frac{1}{3} \left(1 - \frac{1}{3}\right)}{4} \right) = \frac{5}{36},
\]

and

\[
M_2 = \frac{1}{2} \left( \frac{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}{4} \right) = \frac{13}{324},
\]

resulting in a 33% reduction for \(M_1\) and a 56% reduction for \(M_2\).

**Proof of Lemma** We conduct the proof for \(\alpha > 1\) and consider Condition i), which rules out \(x_1 = 1\). The rest of the proof follows similarly and is here omitted. A corner solution in which \(x_1 = 1\) would require

\[
\frac{\partial U_1}{\partial P} \cdot \frac{\partial P}{\partial x_1} \geq \frac{\partial U_1}{\partial s_1}; \quad (23)
\]

Condition i) ensures that this inequality is violated. To see this, note first that, if \(\alpha > 1\) and \(x_2 < 1\), then \(1\) implies \(\frac{\partial P}{\partial x_1} = 0\), and \(23\) is violated. If \(\alpha > 1\) and \(x_2 = 1\), then \(\frac{\partial P}{\partial x_1} = \pi_1^{\frac{1}{\alpha}}\) and \(P = 1\). Therefore, we
have
\[ \frac{\partial U_1}{\partial P} \cdot \frac{\partial P}{\partial x_1} = \frac{1}{P} \cdot \pi_1^\frac{1}{1} = \pi_1 = \frac{\beta_1}{r_1 - 1} = \frac{\partial U_1}{\partial s_1}, \]

where the strict inequality follows from Condition i). The above violates (23). \(\square\)

**Proof of Proposition 1.** First, notice that sufficiency of the interior solutions for the FOCs in (6) and (7) follows by concavity, given \(\alpha \geq 1\). Similarly, existence of an equilibrium follows by standard continuity arguments. We now establish uniqueness along the lines of Cornes and Hartley (2007a,b) by restating the problem in a simpler, “aggregate” fashion. Rewrite (6) and (7) as
\[
\pi_j(\alpha) \left(1 - x_j\right)^\alpha \alpha \left(1 - P\right) = \left(1 - x_j\right) \beta_j r_j - x_j \cdot P, \tag{24}
\]
for \(j = 1, 2\). Therefore, we can interpret the left-hand side of (24) as the fraction of \((1 - P)^\alpha\) that is accounted for by dimension \(j\). Defining this quantity as \(f_j(P)\), we have
\[
\pi_j(\alpha) \left(1 - x_j\right)^\alpha = \frac{\pi_j(\alpha)}{(1 - P)^\alpha} \cdot f_j(P), \tag{25}
\]
Equation (25) also allows to express \(r_j - x_j = (r_j - 1) + (1 - x_j)\) as a function of only \(f_j\) and \(P\), along with the exogenous parameter \(r_j\), so that equation (24) becomes
\[
\frac{f_j(P)}{\pi_j(\alpha)} = \frac{1}{\pi_j(\alpha)} \cdot \frac{\left(\frac{f_j(P)}{\pi_j(\alpha)}\right)^\frac{1}{\pi_j(\alpha)}}{r_j - 1 + \left(\frac{f_j(P)}{\pi_j(\alpha)}\right)^\frac{1}{\pi_j(\alpha)}} \cdot (1 - P) \cdot P.
\]
Finally, defining \(\left(\frac{f_j(P)}{\pi_j(\alpha)}\right)^\frac{1}{\pi_j(\alpha)} = y_j\), we obtain that the FOC for agent \(i\) can be restated as
\[
Q(y_j, P) \equiv \pi_j(\alpha) \cdot y_j^\alpha \cdot (r_j - 1 + y_j \cdot (1 - P)) - y_j \cdot \beta_j \cdot P = 0. \tag{26}
\]
Trivially, \(\frac{\partial Q}{\partial P} < 0\). Moreover,
\[
\frac{\partial Q}{\partial y_i} = \pi_j(\alpha) \cdot \alpha y_j^{\alpha - 1} \cdot (r_j - 1 + y_j \cdot (1 - P)) + \pi_j(\alpha) \cdot y_j(1 - P) - \beta_j \cdot P
= (\alpha - 1)\beta_j P + \pi_j(\alpha) \cdot y_j(1 - P) > 0,
\]
where the second equality follows from (26). Therefore, we see that, when (26) holds, \(y'(P) > 0\) by the implicit function theorem. This implies \(f_j'(P) > 0\), which ensures uniqueness of equilibrium, since equilibrium requires \(f_1(P) + f_2(P) = 1\) and \(f_1 + f_2\) is a strictly increasing function when it is positive. Once this
equilibrium $P$ is determined, say $P^*$, equation \([26]\) and \(\frac{\partial Q}{\partial y^j} > 0\) can be used to find the unique value of $y_j(P^*)$ in equilibrium; therefore, $f_j(P^*)$ is determined as well. Finally, $x_j(P^*)$ is determined through the definition of $f_j$ evaluated at $P^*$.

**Proof of Proposition 3** Consider first $r \leq 1$. The argument in Proposition 2 covers the case in which $\pi_1$ is still large enough to imply $x_1 > 0$. Any further lowering of $\pi_1$ that results in $x_1 \geq 0$ is also covered by that proposition. So, the only thing to consider is what happens by lowering $\pi_1$ when $x_1 = 0$. Equation \([16]\) yields the value for $x_2$, which substituted into the definition of $P$ gives the following analogue of \([17]\) :

\[
(1 - P)^2 = 1 - \pi_2 + 2\pi_2 \left(\frac{1 - r_2}{2}\right)^2 + \beta_2 P (1 - P) + (1 - r_2) \sqrt{\left(\frac{\pi_2}{2}\right)^2 + \pi_2\beta_2 P (1 - P)}.
\]  

(27)

The derivative of the RHS with respect to $\pi_2$ is

\[
-1 + 2 \left(\frac{1 - r_2}{2}\right)^2 + \frac{1}{\pi_2} \left(1 - r_2\right) \frac{2 \left(\pi_2 \frac{1 - r_2}{2}\right)^2 + \beta_2 P (1 - P)}{\sqrt{\left(\frac{\pi_2}{2}\right)^2 + \pi_2\beta_2 P (1 - P)}}
\]

\[
< -1 + 2 \left(\frac{1 - r_2}{2}\right)^2 + \frac{1}{\pi_2} \left(1 - r_2\right) \frac{\left(\pi_2 \frac{1 - r_2}{2}\right)^2 + \pi_2\beta_2 P (1 - P)}{\sqrt{\left(\frac{\pi_2}{2}\right)^2 + \pi_2\beta_2 P (1 - P)}}
\]

\[
= -1 + 2 \left(\frac{1 - r_2}{2}\right)^2 + \frac{\left(1 - P\right)^2 - (1 - \pi_2) - 2\pi_2 \left(\frac{1 - r_2}{2}\right)^2 - \beta_2 P (1 - P)}{\pi_2}
\]

using \([27]\).

\[
\frac{(1 - P)^2 - 1 - \beta_2 P (1 - P)}{\pi_2} < 0,
\]

Therefore $P^*$ is increasing in $\pi_2$, because the LHS of \([27]\) is decreasing in $P$. Thus, if $r \leq 1$, then $P^*$ is maximized for $\pi_2 = 1$.

Consider now $r > 1$. The argument in Proposition 2 shows that, among all values of $\pi_1$ large enough to imply $x_1 > 0$, $P^*$ is maximized for $\pi_1 = 1/2$. By continuity, this covers the boundary value $\tilde{\pi}_1$ defined as the infimum of all $\pi_1$ such that $x_1 > 0$. By continuity again, $\tilde{\pi}_1$ is also the maximum of all $\pi_1$ such that $x_1 = 0$. To analyze what happens for these values of $\pi_1$ (and therefore $\pi_2$), we now find the minimum of the RHS of \([27]\). Note that this expression is concave in $\pi_2$, since its second derivative with respect to $\pi_2$ is

\[
\sqrt{\left(\frac{\pi_2}{2}\right)^2 + \pi_2\beta_2 P (1 - P)} \cdot \frac{1 - r}{\pi_2^2} < 0.
\]

Therefore, the RHS of \([27]\) is minimized either at $\pi_2 = 1$ or at $\pi_2 = \tilde{\pi}_1$. This last value is already covered by the previous discussion, so there are only two candidates for the largest $P^*$: using $\pi_1 = 1/2$ or $\pi_1 = 0$. Direct calculations show that the former yields $P^* = r/(1 + 2\beta)$, while the latter yields $P^* = r/(1 + \beta)$, thus
concluding the proof.

Proof of Lemma 2 i) We see that \( \ell_{\pi_j, \pi_j} = (1 - r_j)t_{\pi_j, \pi_j} \). Since \( t \) is a concave function of its first argument, we have \((1 - r_j) < 0 \) and \( t_{\pi_j, \pi_j} < 0 \), therefore part i) is proven. ii) Follows from

\[
\ell_{\pi, \beta_j} = \frac{(1 - r_j)P^2(1 - P)^2}{4} \left[ \frac{t(\pi_j, \frac{1-r_j}{2}, \beta_j P(1 - P))}{3} \right]^3.
\]

iii) Follows from

\[
\ell_{r_j, \beta_j} = -\frac{\pi_j^3 P^2(1 - P)^2}{2} \left[ \frac{t(\beta_j P(1 - P))}{3} \right]^3.
\]

iv) Follows from

\[
\ell_{r_j, \pi_j} = (r_j - 1) - \frac{\pi_j \left( \frac{1-r_j}{2} \right)^2}{3} \left[ \frac{t(\pi_j, \frac{1-r_j}{2}, \beta_j P(1 - P))}{3} \right]^3 (\pi_j \beta_j P(1 - P)) - \frac{2\pi_j \left( \frac{1-r_j}{2} \right)^2}{t(\pi_j, \frac{1-r_j}{2}, \beta_j P(1 - P))}.
\]

v) We have

\[
\ell_{r_j, r_j} = \pi_j + \frac{1 - r_j}{2} \left( \pi_j \right)^3 \left( \frac{\pi_j (r_j - 1)^2 + 6P (1 - P) \beta_j}{4 \left[ \pi_j \left( \frac{1-r_j}{2} \right)^2 + \pi_j \beta_j P(1 - P) \right]^{\frac{1}{2}}} \right) < 0,
\]

we have

\[
\ell_{r_j, r_j} > \pi_j + \frac{1 - r_j}{2} \left( \pi_j \right)^3 \left( \frac{\pi_j (r_j - 1)^2}{4 \left[ \pi_j \left( \frac{1-r_j}{2} \right)^2 \right]^{\frac{1}{2}}} \right) = 0,
\]

thus concluding the proof.

Proof of Proposition 4 After substitution of (22) into (21), the total resources not devoted to fighting poverty are

\[
(1 - x_1) + (1 - x_2) = \frac{2\beta}{1 + 2\beta} \left[ \frac{1}{g_1} \left( \frac{n_1 \cdot (g_1)^{\alpha} + n_2 \cdot (g_2)^{\alpha}}{2 \cdot n_1} \right)^{\frac{1}{\alpha}} + \frac{1}{g_2} \left( \frac{n_1 \cdot (g_1)^{\alpha} + n_2 \cdot (g_2)^{\alpha}}{2 \cdot n_2} \right)^{\frac{1}{\alpha}} \right].
\]

(28)
To prove part 1 of the proposition, after further assuming that \( n_1 = n_2 = n \), we have

\[
(1 - x_1) + (1 - x_2) = \frac{2\beta}{1 + 2\beta} \cdot \left[ \frac{1}{g_1} \left( \frac{(g_1)^\alpha + (g_2)^\alpha}{2} \right)^{\frac{1}{\alpha}} + \frac{1}{g_2} \left( \frac{(g_1)^\alpha + (g_2)^\alpha}{2} \right)^{\frac{1}{\alpha}} \right]
\]

\[
= \frac{2\beta}{1 + 2\beta} \cdot \frac{g_1 + g_2}{g_1 \cdot g_2} \cdot \left[ \left( \frac{(g_1)^\alpha + (g_2)^\alpha}{2} \right)^{\frac{1}{\alpha}} \right]
\]

\[
= \frac{2\beta}{1 + 2\beta} \cdot \frac{g_1 + g_2}{g_1 \cdot g_2} \cdot \exp \left( \frac{1}{\alpha} \ln \left( \frac{(g_1)^\alpha + (g_2)^\alpha}{2} \right) \right),
\]

so, as \( \alpha \) changes, the total amount of resources not devoted to poverty reduction increases if and only if the term in square brackets above increases. We have

\[
\frac{\partial}{\partial \alpha} \left[ \frac{1}{\alpha} \ln \left( \frac{(g_1)^\alpha + (g_2)^\alpha}{2} \right) \right] = \frac{\ln(2) + \frac{z}{\alpha + w} \ln(z) + \frac{w}{\alpha + w} \ln(w) - \ln(z + w)}{\alpha^2},
\]

where \((g_1)^\alpha = z\) and \((g_2)^\alpha = w\), and we can rewrite the numerator of the above displayed equation as

\[
\ln(2) + y \ln(y) + (1 - y) \ln(1 - y), \tag{29}
\]

where \( y = z/(z + w) \), so that \( y \in (0, 1) \). We now conclude the proof of this first part of the proposition by showing that \( \text{\ref{29}} \) is positive. Indeed, \( \text{\ref{29}} \) is strictly convex for \( y \in (0, 1) \), so it has a strict minimum at \( y = \frac{1}{2} \). Therefore, as long as \( g_1 \neq g_2 \), \( y \neq 1/2 \), so \( \text{\ref{29}} \) is strictly larger than

\[
\ln(2) + \frac{1}{2} \ln \left( \frac{1}{2} \right) + \frac{1}{2} \ln \left( \frac{1}{2} \right) = 0.
\]

To prove the second part of the proposition, note that equation \( \text{\ref{28}} \), if \( g_1 = g_2 \) but \( n_1 \neq n_2 \), yields

\[
(1 - x_1) + (1 - x_2) = \frac{2\beta}{1 + 2\beta} \cdot \left[ \left( \frac{n_1 + n_2}{2 \cdot n_1} \right)^{\frac{1}{\alpha}} + \left( \frac{n_1 + n_2}{2 \cdot n_2} \right)^{\frac{1}{\alpha}} \right].
\]

As in part 1, as \( \alpha \) changes, the total amount of resources not devoted to poverty reduction increases if and only if the term in square brackets above increases. We have

\[
\frac{\partial}{\partial \alpha} \left[ \left( \frac{n_1 + n_2}{2 \cdot n_1} \right)^{\frac{1}{\alpha}} + \left( \frac{n_1 + n_2}{2 \cdot n_2} \right)^{\frac{1}{\alpha}} \right] = -\frac{1}{\alpha^2} \left[ \left( \frac{n_1 + n_2}{2 \cdot n_1} \right)^{\frac{1}{\alpha}} \ln \left( \frac{n_1 + n_2}{2 \cdot n_1} \right) + \left( \frac{n_1 + n_2}{2 \cdot n_2} \right)^{\frac{1}{\alpha}} \ln \left( \frac{n_1 + n_2}{2 \cdot n_2} \right) \right]
\]

and, letting \( \frac{n_1}{n_2} = \nu \), we see that the proof is complete if

\[
f(\nu) = \ln \left( \frac{1}{2\nu} + \frac{1}{2} \right) \left( \frac{1}{2\nu} + \frac{1}{2} \right)^{\frac{1}{\alpha}} + \ln \left( \frac{1}{2\nu} + \frac{1}{2} \right) \left( \frac{1}{2\nu} + \frac{1}{2} \right)^{\frac{1}{\alpha}} > 0.
\]

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To see that $f(\nu) > 0$, first note that $f(\nu) = f\left(\frac{1}{\nu}\right)$, so if one shows that $f(\nu) > 0$ for $\nu < 1$, then the case $\nu > 1$ follows as well. So, we now fix $\nu < 1$. Then, by concavity of the logarithm, we have

$$
\ln\left(\frac{1}{2\nu} + \frac{1}{2}\right) - \ln(1) > \frac{1}{2\nu + \frac{1}{2}} \left(\frac{1}{2\nu} + \frac{1}{2} - 1\right) = \frac{1 - \nu}{1 + \nu},
$$

so

$$
\ln\left(\frac{1}{2\nu} + \frac{1}{2}\right) > \frac{1 - \nu}{1 + \nu}. \tag{30}
$$

Also by concavity, we have

$$
\ln(1) - \ln\left(\frac{1}{2\nu} + \frac{1}{2}\right) < \frac{1}{2\nu + \frac{1}{2}} \left(1 - \left(\frac{1}{2\nu} + \frac{1}{2}\right)\right) = \frac{1 - \nu}{1 + \nu},
$$

so

$$
\ln\left(\frac{1}{2\nu} + \frac{1}{2}\right) > -\frac{1 - \nu}{1 + \nu}. \tag{31}
$$

Hence

$$
f(\nu) > \frac{1 - \nu}{1 + \nu} \left(\left(\frac{1}{2\nu} + \frac{1}{2}\right)^{\frac{1}{\nu}} - \left(\frac{1}{2\nu} + \frac{1}{2}\right)\right) > 0,
$$

where the first inequality follows by (30) and (31), while the second follows by $\nu < 1$. Therefore, the proof of part 2 is complete and increasing $\alpha$ increases total effort. \qed
References


