Incentive Compatible Networks and the Delegated Networking Principle

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Abstract

We construct a model of a principal-agent game of network formation (over layered networks) with asymmetric information and we consider the following two questions: (1) Is it possible for the principal to design a mechanism that links the reports of agents about their private information and the set of connections allowed and recommended by the principal via the mechanism in such a way that players truthfully reveal their private information to the principal and follow the recommendations specified by the mechanism. (2) An even more fundamental question we address is whether or not it is possible for the principal to achieve the same outcome (as that achieved via a mechanism and centralized reporting) by instead choosing a profile of sets of allowable ways to connect (here modeled as player-club specific sets - or catalogs - of networks) and then delegating connection choices to each pair of agents. We call this approach to network formation with incomplete information delegated networking and we show, under relatively mild conditions on our game-theoretic model of network formation, that strategic network formation with incomplete information, implemented via a mechanism and centralized reporting, is equivalent to implementation via delegated networking with monitoring.
1 Introduction

We consider the problem faced by a principal who seeks to structure incentives faced by a set of agents in forming a network of connections among themselves in such a way that each agent, in light of his private information, forms connections that are the best interest of the principal. Thus, the principal seeks to influence - if not control - not only who is interacting (i.e., which pairs of agent’s form connections) but also how they are interacting. In many networking situations, however, the principal, in addition to not being able to observe who is interacting and how, does not have complete information concerning the agent’s “type” (i.e., a parameter summarizing the agent’s basic characteristics). Thus, there is an adverse selection problem.

To address the issues raised above, we construct a principal-agent game of network formation (over layered networks) with asymmetric information and we consider the following two questions: (1) is it possible for the principal to design a mechanism that links the reports of agents’ about their private information and the set of connections allowed and recommended by the principal via the mechanism in such a way that players truthfully reveal their private information to the principal and follow the recommendations specified by the mechanism. (2) An even more fundamental question we address is whether or not it is possible for the principal to achieve the same outcome (as that achieved via a mechanism and centralized reporting) by instead choosing a profile of sets of allowable ways to connect (here modeled as node-pair specific sets - or catalogs - of arc types) and then delegating connection choices to each pair of players. We call this approach to network formation with incomplete information delegated networking and we show, under relatively mild conditions on our game-theoretic model of network formation, that strategic network formation with incomplete information, implemented via a mechanism and centralized reporting, is equivalent to implementation via delegated networking with monitoring. Thus, we show that the delegation principle of contracting theory holds for games of network formation with incomplete information.

By way of an application, we consider the problem faced by a regulator who seeks to incentivize the formation of a loan network that minimizes, to the extent possible, systemic risk. In addition to loan sizes and rates, a key characteristic of the loan network related to systemic risk is loan maturity structure. This will be our focus.

2 Layered Networks and Incomplete Information

2.1 Primitives

Assume the following:

(1) $\mathbb{N}$ is a set consisting of $n$ agents equipped with the discrete metric $\eta_{\mathbb{N}}$, and typical elements $i$ and $j$,

(2) $\mathbb{N} \times \mathbb{N}$ is the set consisting of $n^2$ possible pairs of agents (players), equipped with the discrete metric $\eta_{\mathbb{N} \times \mathbb{N}} := \eta_{\mathbb{N}} + \eta_{\mathbb{N}}$, and typical elements $ij$ (i.e., each player is given by a pair of agents, $ij \in \mathbb{N} \times \mathbb{N}$),

(3) $C$ is a compact metric space of clubs equipped with metric $\rho_C$ and typical element $c$, containing a special "no interaction" club $c_0$ (more on this below),

(4) $(S, \mathcal{B}(S))$ is the observable state space with typical element $s$ where $S$ is a complete, separable metric (Polish) space with metric $\rho_S$ and $\mathcal{B}(S)$ is the Borel $\sigma$-field,
(5) \( (T_i, B(T_i)) \) is the measurable space of agent types, with typical element \( t_i \) (representing agent \( i \)'s type), where \( T_i \) is a complete, separable metric (Polish) space with metric \( \rho_{T_i} \) and \( B(T_i) \) is the Borel \( \sigma \)-field.

(6) \( T_{ij} := T_i \times T_j \) is the space of \( ij \) player types, with typical element \( t_{ij} \), equipped with Borel product \( \sigma \)-field, \( B(T_{ij}) := B(T_i) \times B(T_j) \).

(7) \( T = \Pi_i T_i \) is the space of player type profiles (\( n \)-tuples) with typical element \( t \), equipped with Borel product \( \sigma \)-field, \( B(T) = \Pi_i B(T_i) \).

(8) \( (A_c, \rho_{A_c}) = \) the set of feasible arc types for club \( c \), a nonempty, weak star compact, convex subset of the separable norm dual, \( (E_c^*, \| \cdot \|') \), of a separable Banach space, \( (E_c, \| \cdot \|) \), equipped with compatible metric \( \rho_{w_c^*} \),

(9) \( A(\cdot, c) \) is club \( c \)'s feasible arc correspondence, a set-valued mapping from the set of all ordered pairs of agents, \( ij \), taking values in the collection, \( 2^{A_c} \), of \( \rho_{w_c^*} \)-closed subsets of \( A_c \) such that for each agent pair, \( ij \) (i.e., for each player \( ij \)),

\[
A(ijc) \subset A_c \subset E_c^*.
\]

Alternatively, the \( \cup c 2^{A_c} \)-valued correspondence, \( A(ij, \cdot) \), is player \( ij \)'s feasible arc correspondence across clubs.

We will refer to our list of primitives together with our assumptions as [A-1](\( \gamma \)), \( \gamma = 1, 2, \ldots, 9 \).

### 2.2 Connections and Networks

Connections are the fundamental building blocks of networks. While connections can be modeled in many different ways, all connections are made up of two basic ingredients: nodes and arcs. Our first objective is to construct a network model which allows for heterogenous categories of connections. For example in a banking network connections might be of two categories, for example data and information sharing or money transfers.

In order to treat the problem of incomplete information as well as to fully capture the layering of connection categories, our approach to the network formation problem will be to model connections between agents as a directed club network. In our club network - a bipartite network - each club \( c \in C \setminus \{c_0\} \) represents a category of connections (for example, in the case of interbank networks, connections might take the form of money transfers, data sharing, or loans with each club representing a maturity time point) and each potential club member is given by a pair of agents, \( ij \), where \( ij \) are the agents involved in the bilateral interaction. A typical connection in such a club network looks like the following:

![Figure 1: Player \( ij \) joins club \( c \) and takes action \( a \)](image)

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1 In our model, each player (node) in the network formation game is a pair of agents. In particular, because the basic strategic ingredients of our game of network formation are bilateral connections, it is useful to view each agent pair, \( ij \), as a player or a node in the connections game.
In Figure 1, agents $i$ and $j$, as player $ij$, enter into a category $c$ directed connection from $i$ to $j$ of type $a$, with
\[ a \in A(ijc) \subseteq A_c, \]
by joining club $c$ and taking action $a$. If player $ij$ joins the special no-interaction-club, $c_0$, then the only action that player can take is $a = 1$.

We will assume that if player $ij$ joins a club $c \neq c_0$ and takes action $a \in A(ijc)$, then agents $i$ and $j$ composing player $ij$ share their private information (i.e., $i$ knows $j$’s type $t_j$ and $j$ knows $i$’s type $t_i$) and this is common knowledge. Whereas if player $ij$ joins the no interaction club, $c_0$, and takes action 1 - the only action $ij$ can take - then agents $i$ and $j$ do not share their private information - and this too is common knowledge.

### 2.2.1 Ordered Pairs of Agents as Players

Our objective is to build a game theoretic model of endogenous network formation that makes clear how the interplay between the strategic behavior and network structure determine the payoff to the principal. Because the fundamental building blocks of such a network are bilateral interactions between agents, in our principal-agent game of network formation we will take as the set of players the set of all possible ordered pairs of agents, $ij := (i, j)$. Thus, the set of players is given by
\[ N \times N := \{ij : i \in N \text{ and } j \in N\}. \]

A group of players is given by a pre-network $g \subseteq N \times N$. We will call such a subset, a pre-network. The pre-network, consisting of a set of pre-connections, $(ij, c)$, specifies the structure of potential bilateral interactions between agents.

Because we view the problem of network formation with asymmetric information as a problem of club network formation, in addition to the set of node pairs representing players, $ij \in N \times N$, there is a set of nodes, $C$, representing clubs. Each club represents a category of bilateral interactions players $ij$ can enter into. Thus, if player $ij$ joins club $c$, then agent $i$ enters into a category $c$ interaction with agent $j$. Thus, for club networks, the set of nodes is given by,
\[ (N \times N) \cup C. \]

Because there are different clubs, in a club network representation of a network, the network is layered, with each layer being specified by a club.

Each club represents a particular category of bilateral interactions. For example, if player $ij$ - and therefore agent pair $ij$ - joins club $c$ this mean that agent $i$ enters into a category $c$ interaction with agent $j$, with the the specific type of category $c$ interaction from agent $i$ to agent $j$ being given by the arc label,
\[ a \in A(ijc) \subseteq A_c, \]
where $(ijc) \rightarrow A(ijc)$, is the feasible arc mapping from pre-connections into arc sets.

### 2.2.2 $\epsilon$-Connections and $\epsilon$-Networks

In our club network model, we will assume that each player, $ij \in N \times N$, can join multiple clubs, $c \in C$, and in each club player $ij$ takes a particular action $a$ from a feasible set of actions, $A(ijc)$, relevant to that club. Thus, in a club network a typical connection is given by
\[ (a, (ij, c)) \in A \times ((N \times N) \times C), \]
indicating that player $ij$ is in club $c$ and that in this club player $ij$ takes feasible action $a \in A(ijc)$.

We can think of the network as being layered with each layer being given by a particular club, $c$. As noted already, $(ij,c) \in (N \times N) \times C$, consisting of a player node, $ij$ and a club node, $c$, is called a pre-connection, and we will call

$$(a,(ij,c)) \in A_c \times ((N \times N) \times \{c\}),$$

a $c$-connection. A connection then is a pre-connection to which an arc from the appropriate category has been assigned. To describe connection $(a,(ij,c))$ in words we would say that an arc of type $a$ from category $c$ runs from node $ij$ to node $c$.

The connection, $(a,(ij,c))$, is feasible provided, $a \in A(ijc) \subset A_c$. Thus, a feasible connection is a pre-connection, $(ij,c)$, to which a feasible $c$-arc, $a \in A(ijc) \subset A_c$, has been assigned.

The set of all connections is given by

$$K := A \times ((N \times N) \times C),$$

while the set of all $c$-connections is given by

$$K_c := A_c \times ((N \times N) \times \{c\}).$$

We will equip $K_c$ with the sum metric,

$$\rho_{K_c} := \rho_{\omega_c} + \eta_N + \eta_N.$$

### 2.2.3 Club Networks

A club network $G$ is a nonempty subset of the set of connections, $A \times ((N \times N) \times C)$. Thus, a club network $G$ is of the form

$$G \subset A \times ((N \times N) \times C),$$

with typical element $(a,(ij,c))$. We will often be interested in the arc section of a club network $G$ at various preconnections. For example, at preconnection $(ij,c)$, the arc section of $G$ at $(ij,c)$ is given by

$$G(ijc) := \{a \in A(ijc) : (a,(ij,c)) \in G\}.$$

The arc section, $G(ijc)$, of club network $G$ at $(ij,c)$, lists the feasible set of arc used in connecting agent pair $ij$ to club $c$ in network $G$. The cardinality of the arc section, $|G(ijc)|$, gives the number of arcs used in connecting agent pair $ij$ to club $c$ in network $G$.

We have the following formal definition of a club network.

### Definition 1: (Club Networks)

Given arc set $A := \cup_c A_c$ and node sets, $N \times N$ and $C$, a club network is a nonempty subset, $G$, of $A \times ((N \times N) \times C)$ such that (i) $|G(ijc)| \leq 1$ and $|G(ijc)| = 1$ for some $c \in C$, and (ii) if for $c \in C$, $|G(ijc)| = 1$, then $(a,(ij,c)) \in G$ if and only if $a \in A(ijc) \subset A_c \subset A$. We will denote by $\mathcal{G}$ the collection of all feasible club networks. Thus,

$$\mathcal{G} := \{G \in P(A \times ((N \times N) \times C)) : \text{satisfying (i) and (ii)}\}.$$
The \( c \)-layer, \( G_c \in 2^K_c \), of club network \( G \), if it is nonempty, is given by a nonempty, \( \rho_{K_c} \)-closed subset of

\[
K_c := A_c \times ((N \times N) \times \{c\})
\]

such that (i) \( |G_c(ijc)| \leq 1 \) and (ii) if \( |G_c(ijc)| = 1 \), then \( (a, (ij, c)) \in G_c \) if and only if

\[
\{a\} = G_c(ijc) \subset A(ijc) \subset A_c \subset A.
\]

Here

\[
G_c(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G_c\}
\]

is the arc section of \( c \)-layer \( G_c \) at player \( ij \).

**\( c \)-Layer Representation of a Club Network**

Note that a club network \( G \) can be written as the union of its \( c \)-layers. Thus, each club network has a representation as the union of its constituent, individual club networks,

\[
G := \bigcup_{c \in C} G_c.
\]

Equipped with the Hausdorff metric \( h_{\rho_{K_c}} \), the collection of all possible \( \rho_{K_c} \)-closed subsets of \( c \)-connections, \( 2^K_c \), is a compact metric space. Thus, \( (2^K_c, h_{\rho_{K_c}}) \) the compact metric space contains all possible \( c \)-layers (including the empty layer). We will define the distance between two club networks, \( G^1 := \bigcup_{c \in C} G^1_c \) and \( G^2 := \bigcup_{c \in C} G^2_c \) as the sum of the distances between the \( c \)-layers which make up \( G^1 \) and \( G^2 \). Thus, the distance between club networks \( G^1 \) and \( G^2 \) is given by

\[
h_K(G^1, G^2) := \sum_c h_{\rho_{K_c}}(G^1_c, G^2_c),
\]

where \( h_{\rho_{K_c}}(G^1_c, G^2_c) \) is the Hausdorff distance in \( 2^K_c \).

The \( c \)-layer \( G_c \) of network \( G \) has as members, players, \( ij \), such that for some \( c \)-feasible action \( a \in A_c \),

\[
(a, (ij, c)) \in G_c.
\]

For each player \( ij \), the graph of the feasible arc correspondence,

\[
c \longrightarrow A(ijc),
\]

is given by

\[
GrA(ij, \cdot) := \{(c, a) \in C \times A : a \in A(ijc)\}.
\]

If \((c, a) \in GrA(ij, \cdot)\), then action \( a \) can be taken in club \( c \) by player \( ij \). Given club network \( G \), the arc section of \( G \) at \((ij, c)\), given by

\[
G(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G\},
\]

is such that for all \((ij, c)\),

\[
G(ijc) \subset A(ijc).
\]

Moreover,

\[
GrG(ij, \cdot) \subset GrA(ij, \cdot).
\]
**$ij$-Network Representation of a Club Network**  
Besides the $c$-layer representation, another useful representation of a club network $G$ is the $ij$-network representation. We can think of a club network $G$ as being composed of each individual player’s network, for example, $ij$’s network, $G_{ij} \subset A \times (\{ij\} \times C)$. Because each player $ij$ can join the no interaction club, $c_0$, (i.e., if agents $i$ and $j$ choose not to interact), then the $ij$-network $G_{ij}$ is given by

$$G_{ij} = \{(1, (ij, c_0))\}.$$ 

Thus, for each player, $ij$, $G_{ij}$ is a nonempty, closed subset of $A \times (\{ij\} \times C)$. Letting

$$K_{ij} := A \times (\{ij\} \times C),$$

each $ij$-network, $G_{ij}$, is contained in $P(K_{ij})$, the collection of all nonempty subsets of $K_{ij}$ (as opposed to being contained in $2^{K_{ij}}$, the collection of all closed subsets of $K_{ij}$ - including the empty set). The representation of club network $G$, via its constituent $ij$-networks, is given by

$$G = \bigcup_{ij} G_{ij}.$$ 

We will sometimes write $G$ as an $n^2$-tuple of $ij$-networks as follows,

$$G = (G_{11}, G_{12}, \ldots, G_{1n}, \ldots, G_{n1}, G_{n2}, \ldots, G_{nn})$$

rather than as a union of $ij$-networks

$$G = \bigcup_{ij} G_{ij}.$$ 

**$ij$-Network Domains**  
Let $P(ij \times C)$ denote the collection of all nonempty subsets of $\{(i, j)\} \times C$ where $\{(i, j)\} \times C$ is the set of all pre-connections for player $ij$ in the collection of all preconnections, $(N \times N) \times C$.

Given any club network, $G \subset A \times ((N \times N) \times C)$, the domain of $G$ is given by

$$D(G) := \{(ij, c) \in (N \times N) \times C : G(ijc) \neq \emptyset \}.$$ 

Similarly for any $ij$-network, $G_{ij} \subset A \times (\{ij\} \times C)$. The domain of $G_{ij}$ is given by the domain of the arc section mapping,

$$(ij, c) \mapsto G_{ij}(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G_{ij}\}$$

Thus,

$$D(G_{ij}) := \{(ij, c) \in ij \times C : G_{ij}(ijc) \neq \emptyset \}.$$ 

Because the set of clubs, $C$, includes the no-interaction club, $c_0$, each $ij$-network has a nonempty domain. Thus, $D(G_{ij}) \neq \emptyset$ for all players $ij$. In fact, we have for all $ij$-networks,

$$1 \leq |D(G_{ij})| \leq k + 1.$$ 

The collection of all feasible $ij$-networks that can be formed by player $ij$ is given by,

$$G_{ij} := \{G_{ij} \in P(K_{ij}) : |G_{ij}(ijc)| \leq 1 \text{ for all } c \text{ and } |G_{ij}(ijc)| = 1 \text{ for some } c\}.$$ 

We note that $G_{ij}(ijc) \in 2^{A(ijc)}$ the collection of all $\rho_{wz}$-closed subsets of $A(ijc)$ (including the empty set).
2.3 States and Their Network Representation

We will think of the state space as being given by \( \Omega := T \times S \), with typical element,

\[
(t, s) := (t_1, t_2, \ldots, t_n, s).
\]

If at time point \( k = 0, 1, 2, \ldots \), the prevailing state is \( (t_k, s_k) := (t_{1k}, t_{2k}, \ldots, t_{nk}, s_k) \), we will assume that each agent \( i \) knows \( (t_{ik}, s_k) \) - and that this is common knowledge. Thus, agent \( i \) knows his piece of the \( t \)-state at time point \( k \). Moreover, if two agents, \( i \) and \( j \), are connected at time point \( k \), then the agent pair (i.e., the player), \( ij \), know, \( (t_{ik}, t_{jk}, s_k) \) - but the pair, \( ij \), do not share the information about the types of those agents to whom \( i \) and \( j \) are directly connected at time point \( k \). Thus, if \( j \) and \( j' \) are connected and \( i \) and \( i' \) are connected, then the pair, \( ij \), does not know \( t_{jk} \) or \( t_{j'k} \).

Each player (agent pair), \( ij \), has a type given by

\[
t_{ij} := (t_i, t_j) \in T_i \times T_j := T_{ij}.
\]

While each agent pair knows their types, the principal only knows the agent type profile,

\[
t := (t_1, t_2, \ldots, t_n) \in \Pi T_i,
\]

up to a probability measure, \( \lambda(\cdot) \).

We will represent the state information possessed by each agent as a diagonal loop network \( I := T \cup S \), where

\[
T_i = \{i\} \times \{i\}
\]

is the collection of all \( t \)-connections, with typical element \( \tau_i \in T_i \). A \( t \)-network is an \( n \)-tuple

\[
\tau = (\tau_1, \ldots, \tau_n) \in T := \Pi_i T_i.
\]

The collection of all \( s \)-connections, with typical element \( \sigma_i \in S \),

\[
S := S \times \{i\} \times \{i\}.
\]

An \( s \)-network is an \( n \)-tuple

\[
\sigma = (\sigma_1, \ldots, \sigma_n) \in S,
\]

such that \( \sigma_i = s \in S \) for all \( i \).

It is clear that without confusion, we can represent each state information network

\[
\tau \cup \sigma \in I := T \cup S
\]

as \( \omega = (t, s) \). Equip \( T \) with the sum metric, \( \rho_T := \sum_i \rho_{T_i} \) and equip \( S \) with the metric, \( \rho_S := \rho_S \).

3 Mechanism Games vs Catalog Games

3.1 Games Over Mechanisms

Let \( \omega_{ij} = (t_{ij}, s) = (t_i, t_j, s) \) denote the part of the state, \( \omega = (t_1, t_2, \ldots, t_n, s) \) know to player \( ij \) (i.e., to agent pair \( ij \)). We begin with a formal definition.
Definition 2 (Direct $ij$-Network Formation Mechanism)

A direct (network formation) mechanism is a $(\mathcal{B}(\Omega_{ij}), \mathcal{B}(\mathbb{G}_{ij}))$-measurable mapping

$$\omega_{ij} \rightarrow G_{ij}(\omega_{ij}) \in \mathbb{G}_{ij}$$

from player $ij$’s state space, $\Omega_{ij}$, into the collection of all $ij$-networks, $\mathbb{G}_{ij}$, that player $ij$ can form.

Note that for $(ij, c) \notin \mathcal{D}(G_{ij}(\omega_{ij}))$, the set,

$$G_{ij}(\omega_{ij})(ijc) := \{a \in A(ijc) : (a, (ij, c)) \in G_{ij}(\omega_{ij})\},$$

is empty.

3.1.1 Incentive Compatible Mechanisms with Voluntary Nonparticipation

Let $G := (G_{ij}, G_{-ij}) \in \mathbb{G}$ be a feasible club network and let

$$u_{ij}(\omega, G) := u_{ij}(\omega_{ij}, \omega_{-ij}, G_{ij}, G_{-ij})$$

be the payoff to player $ij$ of type $t_{ij}$ if other player types are given by $t_{-ij}$ and if the profile of player club networks is given by $G := (G_{ij}, G_{-ij}) \in \mathbb{G}$.

We will assume that all payoff functions are Caratheodory - meaning that for each player $ij$, the payoff function

$$(\omega, G) \rightarrow u_{ij}(\omega, G)$$

is measurable in states on $\Omega$ and continuous in networks on $\mathbb{G}$. We will also assume that for any profile of $ij$-network formation mechanisms, $\omega \rightarrow G(\omega) := (G_{ij}(\omega_{ij}))_{ij \in N \times N}$, player $ij$’s payoff function,

$$\omega \rightarrow u_{ij}(\omega, G(\omega)),$$

is $\mathcal{B}(\Omega)$-measurable.

We will assume the following concerning each player’s payoff function, $u_{ij}(\cdot, \cdot)$.

[A-2] (Payoff functions are additively coupled)

We will assume that for each $(\omega, G) \in \Omega \times \mathbb{G}$,

$$u_{ij}(\omega, G) = u_{ij}(\omega_{ij}, G_{ij}) + \sum_{i'j' \neq ij} u_{ij'j'}(\omega_{ij}, \omega_{i'j'}, G_{i'j'}).$$

We have the following formal definition of an incentive compatible mechanism (or an IC mechanism).

Definition 3 (Incentive Compatible Network Formation Mechanism with Voluntary Nonparticipation)

We say that a network formation mechanism, $\omega \rightarrow G(\omega)$, is incentive compatible if for each player, $ij$, $ij$’s part of the mechanism, $\omega_{ij} \rightarrow G_{ij}(\omega_{ij})$, is such that for all $\omega_{ij}$ and $\omega'_{ij}$

$$\max\{u_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})), r_{ij}(\omega_{ij})\} \geq \max\{u_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})), r_{ij}(\omega_{ij})\}. \quad (1)$$

Denote by $M(\Omega, \mathbb{G})$ the set of all $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{G}))$-measurable functions and denote by $\mathcal{IC}$ the subset of $M(\Omega, \mathbb{G})$ consisting of functions that satisfy the IC inequalities (1).
Note that under the assumption that player payoff functions are additively coupled, in order for a mechanism, 

$$G(\cdot) \in M(\Omega, \mathbb{G})$$

to be incentive compatible, it suffices that each player’s part of the mechanism, $G_{ij}(\cdot) \in M(\Omega_{ij}, \mathbb{G}_{ij})$ be incentive compatible while allowing voluntary nonparticipation. In particular, under additively coupled payoffs, a mechanism, $G(\cdot) := (G_{ij}(\cdot), G_{-ij}(\cdot))$ is incentive compatible (allowing voluntary nonparticipation) provided that for each player, $ij$, and for each $\omega_{ij}$ and $\omega'_{ij}$,

$$\max \{ u_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})), r_{ij}(\omega_{ij}) \} \geq \max \{ u_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})), r_{ij}(\omega_{ij}) \} . \tag{2}$$

Here $r_{ij}(\omega_{ij}) := u_{ij}(\omega_{ij}, (1, (ij, c_0)))$ is player $ij$’s aggregate payoff - without the additive spillovers from other players’ interactions - when player $ij$ chooses not to participate in the mechanism and there is no information sharing between agents $i$ and $j$.

Letting

$$U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) := \max \{ u_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})), r_{ij}(\omega_{ij}) \},$$

We will assume that player $ij$ chooses nonparticipation if and only there is a positive gain from doing so. Therefore, if $G_{ij}(\cdot)$ satisfies the IC constraints with voluntary nonparticipation (2) and if for some $\Xi_{ij}$,

$$U_{ij}(\Xi_{ij}, G_{ij}(\Xi_{ij})) = r_{ij}(\Xi_{ij})$$

then

$$r_{ij}(\Xi_{ij}) > u_{ij}(\Xi_{ij}, G_{ij}(\Xi_{ij})),$$

and

$$r_{ij}(\Xi_{ij}) \geq u_{ij}(\Xi_{ij}, G_{ij}(\omega'_{ij})) \text{ for all } \omega'_{ij}.$$

Therefore, no amount of misreporting will make participation attractive.

If player $ij$ reports his private information to be $\omega'_{ij}$ then the mechanism recommends to player $ij$ that $ij$ form the $ij$-club network $G_{ij}(\omega'_{ij})$. Thus, if an $ij$-mechanism satisfies expression (1), then player $ij$ will participate in the mechanism and will have nothing to gain by misreporting his private information to the mechanism. The mechanism is incentive compatible with voluntary nonparticipation.

Assuming for the moment that all players report their information truthfully and choose the profile of $ij$-networks recommended by the mechanism, then the profile of $ij$-mechanisms will give rise to a feasible club network,

$$G(\omega) := (G_{ij}(\omega_{ij}), G_{-ij}(\omega_{-ij})) \in \mathbb{G},$$

with typical connection $(a_{ij}(\omega_{ij}), (ij, c))$ where preconnection, $(ij, c)$, is in the set of pre-connections, $D(G_{ij}(\omega_{ij}))$, specified by the mechanism and where for all $\omega_{ij} \in \Omega_{ij},$

$$a_{ij}(\omega_{ij}) \in A(ijc) \text{ for all } (ij, c) \in D(G_{ij}(\omega_{ij})).$$

We close this subsection on IC network formation mechanisms by noting that if for some type of player $ij$, with truthfully reported type, $\omega_{ij}$, $D(G_{ij}(\omega_{ij})) = \{(ij, c_0)\}$ so that $G_{ij}(\omega_{ij}) = \{ a_{ij}(\omega_{ij}) \} = \{1\}$, then player $ij$ of type $\omega_{ij} := (t_{ij}, s)$ will not interact and player $ij$ will have a payoff of

$$u_{ij}(\omega_{ij}, \omega_{-ij}, \{(1, (ij, c_0))\}, G_{-ij}(\omega_{-ij}))$$

$$\overset{G_{ij}(\omega_{ij})}{=}$$

$$u_{ij}(\omega_{ij}, (1, (ij, c_0))) + \sum_{ij} u_{ij}(\omega_{ij}, \omega_{-ij}, G_{-ij}(\omega_{-ij})).$$
Note that even though player \( ij \) engages in no interactions, player \( ij \)'s reservation payoff level,

\[ u_{ij}(\omega_{ij}, \omega_{-ij}; \{(1, (ij, c_0))\}, G_{-ij}(\omega_{-ij})) , \]

is still a function of the interactions of the other players via the other \( ij \)-networks, \( G_{-ij}(\omega_{-ij}) \).

### 3.1.2 The Principal's Problem over IC Mechanisms

Assume that the principal has payoff given by

\[ (\omega, G) \rightarrow V(\omega, G) := V(\omega, (G_{ij}, G_{-ij})), \]

over profiles of player types and profiles of player club networks. Under an incentive compatible network formation mechanism,

\[ (\omega_{ij}, \omega_{-ij}) \rightarrow (G_{ij}(\omega_{ij}), G_{-ij}(\omega_{-ij})) \]

the principal’s payoff becomes

\[ V(\omega, G(\omega)), \]

where

\[ G(\omega) := \begin{pmatrix} G_{11}(\omega_{11}) & \cdots & G_{1j}(\omega_{1j}) & \cdots & G_{1n}(\omega_{1n}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G_{nj}(\omega_{nj}) & \cdots & G_{nj}(\omega_{nj}) & \cdots & G_{nn}(\omega_{nn}) \end{pmatrix}_{n \times n} \]

In this section we will analyze the network formation problem under incomplete information as a principal-agent network formation game with adverse selection, assuming that the principal is allowed to design a profile of network formation mechanisms,

\[ \omega_{ij} \rightarrow (G_{ij}(\omega_{ij}))_{ij \in N \times N}, \]

so as to induce players to reveal their types and to follow the connection recommendations of the mechanism.

Given the principal’s probability beliefs, \( \lambda \), defined on the measurable state space, \( (\Omega, \mathcal{B}(\Omega)) \), the mechanism design problem, \( P1 \), faced by the principal is given by

\[ \max_{G(\cdot) \in M(\Omega, \mathcal{G})} \int_\Omega V(\omega, G(\omega))d\lambda(\omega) \quad \text{such that for all } ij, \text{ and for all } \omega_{ij} \text{ and } \omega'_{ij}, \]

\[ U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})). \quad (3) \]

Problem \( P1 \) can be restated as the following problem \( P2 \).

\[ \max_{G(\cdot) \in IC} \int_\Omega V(\omega, G(\omega))d\lambda(\omega) \quad (4) \]
3.2 Games over Catalogs

Our objective now is to characterize all incentive compatible network formation mechanisms via catalogs of $ij$-club networks. We call this characterization result the Delegated Networking Principle. The importance of the delegated networking principle in proving existence of an optimal network formation mechanism is that it allows us to convert the principal-agent network formation game over network-valued mechanisms with incentive compatibility constraints into an equivalent unconstrained principal-agent game over catalogs of $ij$-networks. With this conversion, we are able to avoid the difficult problem of searching for a topology for the function space of network-valued mechanisms making the subset of incentive compatible mechanisms compact, players’ payoff functions continuous, and the principal’s payoff function upper semicontinuous. Instead, our reformulation of the network formation game as a principal-agent game over catalogs of networks allows us to utilize the topology already present in the space of $ij$-networks to establish existence.

3.2.1 Catalogs of $ij$-Network

Preliminaries To begin, suppose that the principal, rather than offering each player, $ij$, a mechanism, $G_{ij}(\cdot)$, and requiring a report from player $ij$ concerning his type, $\omega_{ij}$, to determine the $ij$-network recommendation, instead offers each player $ij$ a catalog, $\mathcal{G}_{ij}$, of $ij$-networks and then observes the player $ij$’s choice from the $ij$-network catalog.

An $ij$-catalog, $\mathcal{G}_{ij}$, is a closed set of $ij$-networks $G_{ij}$. We must make precise what we mean by closed - as well as make precise what we mean by the distance between two catalogs. To begin, suppose $\mathcal{G}_{ij}^1$ and $\mathcal{G}_{ij}^2$ are two $ij$-network catalogs. Just as is the case with a club network $G$, any $ij$-network, $G_{ij}$, can be written as the union of its $c$-layers. Thus, we have

$$G_{ij} = \bigcup_c G_{ijc}$$

where for each $c$, $G_{ijc}$ is a subset of $\mathcal{A}(ijc) \times \{\{ij\} \times \{c\}\}$. Thus, the $c$-layer in any $ij$-network, $G_{ijc}$, is of the form

$$G_{ijc} \subset \mathcal{A}(ijc) \times \{\{ij\} \times \{c\}\}.$$

Because a player may not be a member of all clubs in network $G$, for some $c'$, player $ij$’s network, $G_{ijc'}$, may be empty. Hence,

$$G_{ijc} \in 2^{\mathcal{A}(ijc) \times \{\{ij\} \times \{c\}\}}.$$

Given two $ij$-networks, $G_{ij}^1$ and $G_{ij}^2$, the distance between them is given by

$$h_K(G_{ij}^1, G_{ij}^2) := \sum_c h_Kc(G_{ijc}^1, G_{ijc}^2),$$

and for each $c \in C$,

$$h_Kc(G_{ijc}^1, G_{ijc}^2) = \begin{cases} +\infty & \text{if } G_{ijc}^1 = \emptyset, G_{ijc}^2 \neq \emptyset, \\ \rho_{\omega}^c(a_{ijc}^1, a_{ijc}^2) & \text{if } G_{ijc}^1 \neq \emptyset, G_{ijc}^2 \neq \emptyset, \\ 0 & \text{if } G_{ijc}^1 = \emptyset, G_{ijc}^2 = \emptyset. \end{cases}$$

A $ij$-catalog $\mathcal{G}_{ij}$ is a $h_K$-closed, subset of $ij$-networks, where recall, the collection of all feasible $ij$-networks is given by,

$$\mathcal{G}_{ij} := \{G_{ij} \in \mathcal{P}(K_{ij}) : |G_{ij}(ijc)| \leq 1 \text{ for all } c \text{ and } |G_{ij}(ijc)| = 1 \text{ for some } c\}.$$
The Hyperspace of Catalogs of \(ij\)-Networks  
Recall that for each player \(ij\), \(P_{hK_\ell}(G_{ij})\) denotes the collection of nonempty \(hK_\ell\)-closed sets, \(G_{ij}\), where each set \(G_{ij} \subset G_{ij} \) consists of a collection of \(ij\)-networks, \(G_{ij} \in G_{ij}\), where each \(ij\)-network is the union, \(\cup_i G_{ijc}\), of \(\rho_{K_\ell}\)-closed subsets of \(A(ijc) \times (ij \times \{c\})\). The catalog, \(G_{ij}\), is closed in the sense that if \(\{G_{ij} \}_{n} \subset G_{ij}\) is a sequence such that
\[
G_{ij}^n \rightarrow_{hK} G_{ij}^n
\]
then \(G_{ij}^n \subset G_{ij}\). We want to equip the hyperspace of catalogs, \(P(hK_\ell(G_{ij}))\), with typical element \(G_{ij}\), with the Hausdorff metric, \(H_{hK_\ell}\), induced by the metric \(hK_\ell\) on \(G_{ij}\). Thus, in the hyperspace of catalogs \(P(hK_\ell(G_{ij})))\) if the catalog sequence \(\{G_{ij}^n\}_n\) is such that,
\[
G_{ij}^n \rightarrow_{H_{hK_\ell}} G_{ij}^n \text{ then } G_{ij}^n \in P(hK_\ell(G_{ij})).
\]
The Hausdorff metric \(H_{hK_\ell}\) is defined as follows: for \(G_{ij}^1\) and \(G_{ij}^2\) in \(P(hK_\ell(G_{ij})))\),
\[
H_{hK_\ell}(G_{ij}^1, G_{ij}^2) \ := \ max \left\{ e_{hK_\ell}(G_{ij}^1, G_{ij}^2), e_{hK_\ell}(G_{ij}^2, G_{ij}^1) \right\},
\]
where the excess of \(G_{ij}^1\) over \(G_{ij}^2\) is given by
\[
e_{hK_\ell}(G_{ij}^1, G_{ij}^2) \ := \ sup_{G_{ij}^1 \in G_{ij}^1} dist_{hK_\ell}(G_{ij}^1, G_{ij}^2)
\]
and
\[
\text{dist}_{hK_\ell}(G_{ij}^1, G_{ij}^2) \ := \ inf_{G_{ij}^2 \in G_{ij}^2} hK_\ell(G_{ij}^1, G_{ij}^2).
\]
Convergence in \(P(hK_\ell(G_{ij})))\) can be described as follows. Let \(\{G_{ij}^n\}_n\) be a sequence in \(P(hK_\ell(G_{ij})))\).
The limit inferior, \(Li\{G_{ij}^n\}_n\), of the sequence \(\{G_{ij}^n\}_n\) is defined as follows: \(G_{ij}^n \in Li\{G_{ij}^n\}_n\) if and only if there is a sequence \(\{G_{ij}^n\}_n\) in \(G_{ij}\) such that \(G_{ij}^n \in G_{ij}^n\) for all \(n\) and \(G_{ij}^n \rightarrow_{hK_\ell} G_{ij}^n\).
The limit superior, \(LS\{G_{ij}^n\}_n\), of the sequence \(\{G_{ij}^n\}_n\) is defined as follows: \(G_{ij}^n \in LS\{G_{ij}^n\}_n\) if and only if there is a subsequence, \(\{G_{ij}^{n_k}\}_n\) in \(G_{ij}\) such that \(G_{ij}^{n_k} \in G_{ij}^n\) for all \(k\) and \(G_{ij}^{n_k} \rightarrow_{hK_\ell} G_{ij}^n\).
The \(ij\)-catalog, \(G_{ij}^n\), is said to be the limit of the sequence, \(\{G_{ij}^n\}_n\), if
\[
Li\{G_{ij}^n\}_n = G_{ij}^n = LS\{G_{ij}^n\}_n.
\]
Moreover, \(H_{hK_\ell}(G_{ij}^n, G_{ij}^n) \rightarrow 0\) if and only if \(Li\{G_{ij}^n\}_n = G_{ij}^n = LS\{G_{ij}^n\}_n\) (see Aliprantis and Border, 2006).

3.2.2  The Player’s Problem

If the principle offers catalog \(G_{ij}\) to agents \(ij\) (i.e., player \(ij\)), then player \(ij\)'s problem is to choose an optimal \(ij\)-network, \(G_{ij}\), from the catalog \(G_{ij}\). Thus, player \(ij\)'s problem is given by
\[
U_{ij}^*(\omega_{ij}, G_{ij}) := \max_{G_{ij} \in U_{ij}} U_{ij}(\omega_{ij}, G_{ij})
\]
Under \([A-1]\) and \([A-2]\) it follows from Page (1992) that because \(G_{ij}\) is \(hK\)-compact, \(U_{ij}(\omega_{ij}, \cdot)\) is \(hK\)-continuous on \(G_{ij}\) (and measurable in \(\omega_{ij}\)) for each \(\omega_{ij}\), there exists an \(ij\)-network, \(G_{ij}^*(\omega_{ij})\). In fact, there exists an optimal measurable selection, \(\omega_{ij} \rightarrow G_{ij}^*(\omega_{ij}) \in G_{ij}\), such that
\[
U_{ij}(\omega_{ij}, G_{ij}^*(\omega_{ij})) = U_{ij}^*(\omega_{ij}, G_{ij}) \text{ for all } \omega_{ij}.
\]
By Proposition 4.1 in Page (1992), for each \( \omega_{ij} \), the \( ij^{th} \) player’s catalog payoff function, \( U_{ij}^*(\omega_{ij}, \cdot) \) is \( h_k \)-continuous on \( P(h_k f(G_{ij})) \) (i.e., is \( h_k \)-continuous in catalogs), and for each catalog, \( G_{ij}, U_{ij}^*(\cdot, G_{ij}) \) is \( \mathcal{B}(\Omega_{ij}) \)-measurable. Moreover, by Proposition 4.2 in Page (1992), for each \( \omega_{ij} \), the \( ij^{th} \) player’s best response correspondence

\[(\omega_{ij}, G_{ij}) \mapsto \Phi_{ij}(\omega_{ij}, G_{ij})\]

is \( \mathcal{B}(\Omega_{ij}) \times \mathcal{B}(\mathbb{R}) \)-measurable and for each \( \omega_{ij} \),

\[G_{ij} \mapsto \Phi_{ij}(\omega_{ij}, G_{ij}),\]

is \( h_k \)-upper semicontinuous on \( P(h_k f(G_{ij})) \).

### 3.2.3 The Principal’s Problem

The principal’s problem has two parts. First, for any given choice of a catalog profile

\[G := (G_{ij}, G_{-ij}) \in \prod_{ij} P(h_k f(G_{ij})),\]

together with players’ best response mappings,

\[\Phi(\omega, G) := \prod_{ij} \Phi_{ij}(\omega_{ij}, G_{ij})\]

the principal will first make \( ij \)-network recommendations to the players. The principal’s recommendation list is constructed by solving type by type for a given catalog profile, \( G \), the problem,

\[V^*(\omega, G) := \max_{(G_{ij}, G_{-ij}) \in \Phi(\omega, G)} V(\omega_{ij}, \omega_{-ij}, G_{ij}, G_{-ij}).\]

By Proposition 4.3 in Page (1992), \( V^*(\cdot, \cdot) \) is \( \mathcal{B}(\Omega_{ij}) \times \mathcal{B}(\mathbb{R}) \)-measurable and for each \( \omega \),

\[G \mapsto V^*(\omega, G),\]

is \( h_k^{n \times n} \)-upper semicontinuous on \( \prod_{ij} P(h_k f(G_{ij})) \).

With these technical preliminaries out of the way, the principal’s problem of finding an optimal catalog profile, \( G^* \), reduces to the following unconstrained problem:

\[
\max \ \left\{ \int_{\Omega} V^*(\omega, G) d\lambda(\omega) : G \in \prod_{ij} P(h_k f(G_{ij})) \right\}.
\] (7)

By Proposition 4.4 in Page (1992), there exists an optimal catalog profile, \( G^* \), such that

\[
\int_{\Omega} V^*(\omega, G^*) d\lambda(\omega) \geq \int_{\Omega} V^*(\omega, G) d\lambda(\omega)
\]

for all \( G \in \prod_{ij} P(h_k f(G_{ij})) \).

It is easy to see that if \( G^* \in \prod_{ij} P(h_k f(G_{ij})) \) solves the catalog problem, then the incentive compatible mechanism,

\[\omega \mapsto G^*(\omega) := (G_{ij}^*(\omega_{ij}), G_{-ij}^*(\omega_{-ij})) \in \Phi(\omega, G^*)\]
solves the principal’s problem over incentive compatible mechanisms given by

\[
\max_{G \in \mathcal{C}} \int_{\Omega} V(\omega, G(\omega)) d\lambda(\omega). \tag{8}
\]

By taking the $h_K$-closure of the ranges of the $ij$-mechanisms, $\omega_{ij} \rightarrow G^*_{ij}(\omega_{ij})$, making up the profile of mechanisms solving the principal’s mechanism problem, we can construct an $ij$-catalog profile that solves the principal’s problem over catalog profiles.\(^3\) Thus, if $\omega \rightarrow G^*(\omega)$ solves the principal’s mechanism problem, the catalog profile given by

\[
\left\{ G^*_{ij}(\omega_{ij}) : \omega_{ij} \in \Omega_{ij} \right\}^h \rightarrow \left( G^*_i(\Omega_{ij}) \right)_{ij}
\]

solves the principal’s catalog problem. Also note that for each $\omega_{ij}$, $G^*_{ij}(\omega_{ij})$ can be rewritten in the following way,

\[
G^*_{ij}(\omega_{ij}) := \bigcup_{c \in \mathcal{C}} G^*_{ijc}(\omega_{ij}) := (G^*_{ij0}(\omega_{ij}), G^*_{ij1}(\omega_{ij}), \ldots, G^*_{ijk}(\omega_{ij})). \tag{9}
\]

Note that if $(ij, c) \notin \mathcal{D}(G^*_{ij}(\omega_{ij}))$, then for the $c^{th}$ component of the $k + 1$-tuple in expression (9), we have

\[
G^*_{ijc}(\omega_{ij}) = \emptyset,
\]

while for all other $c$ (i.e., those such that $(ij, c) \in \mathcal{D}(G^*_{ij}(\omega_{ij})))$, $G^*_{ijc}(\omega_{ij})$ is a nonempty, $\rho_{K_{ij}}$-closed subset of

\[
A(ijc) \times (\{ij\} \times \{c\}),
\]

namely, $(a^*_{ijc}(\omega_{ij}), (ij, c))$.

### 4 The Delegated Networking Principle

We now formally state and prove the Delegated Networking Principle for a principal-multi-agent games of network formation with adverse selection (also see Page, 1989, 1992, 2010).

**Theorem 2 (The Delegated Networking Principle)**

Suppose assumptions [A-1] and [A-2] hold. The following statements are equivalent.

1. $G_{ij}(\cdot) \in M(\Omega_{ij}, \mathcal{G}_{ij})$, is incentive compatible, that is, for all $\omega_{ij}$ and $\omega'_{ij}$

   \[
   U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})).
   \]

   (2) $G_{ij}(\cdot) \in M(\Omega_{ij}, \mathcal{G}_{ij})$ is such that there exists a unique, minimal (by set inclusion) $ij$-catalog, $\mathcal{G}_{ij} \in \mathcal{P}_{h_K}(\mathcal{G}_{ij})$ satisfying

   \[
   G_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij}) \text{ for all } \omega_{ij},
   \]

   where

   \[
   G_{ij}(\omega_{ij}) \in \Phi_{U_{ij}}(\omega_{ij}, \mathcal{G}_{ij})
   \]

   if and only if,

   \[
   U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) = \max_{G_{ij} \in \mathcal{G}_{ij}} U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } \omega_{ij}.
   \]

\(^3\)Under the mechanism, $G^*_{ij}(\cdot)$, if player $ij$ reports his private information truthfully, say $\omega_{ij}$, then his intended $ij$-network is given by

\[
G^*_{ij}(\omega_{ij}) := \left\{ (a^*_{ijc}(\omega_{ij}), (ij, c)) \in A(ijc) \times (\{ij\} \times \{c\}) : (ij, c) \in \mathcal{D}(G^*_{ij}(\omega_{ij})) \right\}.
\]
Letting $\Sigma_{\Phi U_{ij}}(G_{ij})$ denote the collection of all (everywhere) measurable selections of the best response mapping, $\Phi U_{ij}(\cdot, G_{ij})$, with $ij$-catalog $G_{ij}$, the delegated networking principle can be stated compactly as follows:

$$G_{ij}(\cdot) \in \mathcal{IC}_{ij} \text{ if and only if } G_{ij}(\cdot) \in \Sigma_{\Phi U_{ij}}(G_{ij})$$

for some $ij$-catalog, $G_{ij}$.

Proof of the Delegated Networking Principle:

(1) $\implies$ (2) Let $G_{ij}(\cdot)$ be an incentive compatible $ij$-mechanism and consider the $ij$-catalog $G_{ij}(\Omega_{ij})^{h_K}$.

First, note that for each $\omega_{ij} \in \Omega_{ij}$,

$$U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } G_{ij} \in \overline{G_{ij}(\Omega_{ij})^{h_K}}.$$

Suppose not. Then for some $ij$ player type $\omega'_{ij}$ and $ij$-catalog, $G_{ij}(\omega'_{ij}) \in \overline{G_{ij}(\Omega_{ij})^{h_K}}$,

$$U_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})) > U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})).$$

Because $G_{ij}(\omega'_{ij}) \in \overline{G_{ij}(\Omega_{ij})^{h_K}}$, there exists a sequence $\{\omega^n_{ij}\}_n$ in $\Omega_{ij}$ such that $G_{ij}(\omega^n_{ij}) \rightarrow_{h_K} G_{ij}(\omega'_{ij})$. But now because

$$U_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij}))$$

by the $h_K$-continuity of $U_{ij}(\omega_{ij}, \cdot)$ the fact that $G_{ij}(\omega^n_{ij}) \rightarrow_{h_K} G_{ij}(\omega'_{ij})$ implies that for $n$ large enough,

$$U_{ij}(\omega_{ij}, G_{ij}(\omega^n_{ij})) > U_{ij}(\omega_{ij}, G_{ij}(\omega'_{ij})).$$

contradicting the fact that $G_{ij}(\cdot)$ is incentive compatible. Therefore, because

$$U_{ij}(\omega_{ij}, G_{ij}(\omega_{ij})) \geq U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } G_{ij} \in \overline{G_{ij}(\Omega_{ij})^{h_K}},$$

we conclude that

$$G_{ij}(\omega_{ij}) \in \Phi U_{ij}(\omega_{ij}, \overline{G_{ij}(\Omega_{ij})^{h_K}}) \text{ for all } \omega_{ij} \in \Omega_{ij}.$$

Now suppose that there exists another $ij$-catalog, $G_{ij}$, such that

$$G_{ij}(\omega_{ij}) \in \Phi U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } \omega_{ij} \in \Omega_{ij}$$

but that for some $G_{ij}(\omega'_{ij}) \in \overline{G_{ij}(\Omega_{ij})^{h_K}}$

$$G_{ij}(\omega'_{ij}) \notin G_{ij}$$

Again, because $G_{ij}(\omega'_{ij}) \in \overline{G_{ij}(\Omega_{ij})^{h_K}}$, there exists a sequence of player types $\omega^n_{ij}$ in $\Omega_{ij}$ such that $G_{ij}(\omega^n_{ij}) \rightarrow_{h_K} G_{ij}(\omega'_{ij})$. But now we have

$$G_{ij}(\omega^n_{ij}) \in \Phi U_{ij}(\omega_{ij}, G_{ij}) \text{ for all } n \text{ and } \Phi U_{ij}(\omega_{ij}, G_{ij}) \subseteq G_{ij}.$$ 

Because $G_{ij}$ is $h_K$-closed and $G_{ij}(\omega^n_{ij}) \rightarrow G_{ij}(\omega'_{ij})$, we must conclude that $G_{ij}(\omega'_{ij}) \in G_{ij}$, a contradiction.

(2) $\implies$ (1). The proof is straightforward. $\blacksquare$
5 The Equivalence of Mechanism Games and Catalog Games and Existence

Given \(ij\)-catalog, \(G_{ij}\), the type \(\omega_{ij}\) player is indifferent over the networks in \(\Phi_{U_{ij}}(\omega_{ij}, G_{ij})\), the principal will not be. In order to resolve the principal’s lack of indifference over \(\Phi_{U_{ij}}(\omega_{ij}, G_{ij})\), the principal will suggest or recommend a particular \(ij\)-network choice from \(\Phi_{U_{ij}}(\omega_{ij}, G_{ij})\). Any such recommendation will be followed by the type \(\omega_{ij}\) player because it is incentive compatible for the agent to do so provided the player is type \(\omega_{ij}\) and the \(ij\)-catalog offered is \(G_{ij}\). The principal’s optimal recommendation list is found by solving pointwise (i.e., player type by player type) the following problem

\[
\max_{(G_{ij})\in\Pi_{ij}\times N} \Phi_{U_{ij}}(\omega_{ij}, (G_{ij}), V(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij}\in\Pi_{ij}\times N)).
\]

Let

\[
V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij}\in\Pi_{ij}\times N) = \max_{(G_{ij})\in\Pi_{ij}\times N} \Phi_{U_{ij}}(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij}\in\Pi_{ij}\times N).
\]

By Theorem 2 in Himmelberg, Parthasarathy, and VanVleck (1976), given \(ij\)-catalogs, \((G_{ij})_{ij}\in\Pi_{ij}\times N\), there exists for each player \(ij\) a selection from \(\Phi_{U_{ij}}(\cdot, G_{ij})\), say

\[
G_{ij}^* (\cdot) \in \Sigma_{\Phi_{U_{ij}}}(G_{ij}),
\]

such that

\[
V(\omega_{ij}, \omega_{-ij}, (G_{ij}^* (\omega_{ij})), \in\Pi_{ij}\times N) = V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij}\in\Pi_{ij}\times N).
\]

Moreover, because

\[
\Phi(\cdot, \cdot) := \Pi_{ij}\times N \Phi_{U_{ij}}(\omega, \cdot)
\]

is measurable (i.e., \([B(\Omega) \times \Pi_{ij}\times N B(P_{hKf}(G_{ij}))]-measurable) and \(hK\)-compact-valued in \(\Pi_{ij}\times N P_{hKf}(G_{ij})\) and \(V(\cdot, \cdot)\) is measurable and \(hK\)-upper semicontinuous on

\[
\Pi_{ij}\times N P_{hKf}(G_{ij})
\]

for each \((\omega_{ij})_{ij}\in\Pi_{ij}\times N\), it follows from Theorem 2 in Himmelberg, Parthasarathy, and Van Vleck (1976) that \(V^*(\cdot, \cdot)\) is measurable. Moreover, because

\[
\Phi(\cdot, \cdot) := \Pi_{ij}\times N \Phi_{U_{ij}}(\omega_{ij}, \cdot)
\]

is \(hK\)-upper semicontinuous on \(\Pi_{ij}\times N P_{hKf}(G_{ij})\), it follows from Theorem 2 in Berge (1963) that \(V^*(\omega_{ij}, \omega_{-ij}, \cdot)\) is \(hK\)-upper semicontinuous on

\[
\Pi_{ij}\times N P_{hKf}(G_{ij}).
\]

The catalog game can now be stated very compactly as

\[
\max_{(G_{ij})\in\Pi_{ij}\times N} \Pi_{ij}\times N P_{hKf}(G_{ij}) \int_{\Omega} V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij}\in\Pi_{ij}\times N))d\lambda(\omega).
\]

The mechanism game can also be stated very compactly as

\[
\max_{(G_{ij})\in\Pi_{ij}\times N} \Pi_{ij}\times N \int_{\Omega} V(\omega_{ij}, \omega_{-ij}, (G_{ij}(\omega_{ij})))_{ij}\in\Pi_{ij}\times N)(\omega).
\]
Also, recall that
\[
\Sigma_{\Phi_{ij}(G_{ij})} := \{G_{ij}(\cdot): G_{ij}(\omega_{ij}) \in \Phi_{ij}(\omega_{ij}, G_{ij}) \text{ for all } \omega_{ij} \in \Omega_{ij}\}
\]
denotes the set of all measurable selections from \(\Phi_{ij}(\cdot, G_{ij})\) for a given \(ij\)-catalog \(G_{ij} \in P_{h_{K,F}}(G_{ij})\) and define
\[
\Sigma_{\Phi_{ij}} = \cup_{G_{ij} \in P_{h_{K,F}}(G_{ij})} \Sigma_{\Phi_{ij}}(G_{ij}).
\]
An alternative statement of the Delegation Principal is that
\[
\mathcal{IC}_{ij} = \Sigma_{\Phi_{ij}}.
\]

We now have our main result on the equivalence of mechanism games and catalog games. This result is an immediate consequence of the Delegation Principle.

**Theorem 3 (The Equivalence of Mechanism Games and the Catalog Games)**
Suppose assumptions \([A-1]\) and \([A-2]\) hold. Then the following statements are true.

1. If \((G^*_{ij}(\cdot))_{ij \in N \times N} \in \Pi_{ij \in N \times N} \mathcal{IC}_{ij}\) solves the network formation game over mechanisms given by
\[
\max_{(G_{ij}(\cdot))_{ij \in N \times N} \in \Pi_{ij \in N \times N} \mathcal{IC}_{ij}} \int_{\Omega} V(\omega_{ij}, \omega_{-ij}, (G_{ij}(\omega_{ij}))_{ij \in N \times N}) d\lambda(\omega),
\]
then \((G^*_{ij}(\Omega_{ij}^h))_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})\) solves the network formation game over catalogs \(\Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})\) given by
\[
\max_{(G_{ij})_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})} \int_{\Omega} V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij \in N \times N}) d\lambda(\omega),
\]

2. If \((G^*_{ij}(\cdot))_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})\) solves the network formation game over catalogs given by
\[
\max_{(G_{ij})_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})} \int_{\Omega} V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij \in N \times N}) d\lambda(\omega),
\]
then \((G^*_{ij}(\cdot))_{ij \in N \times N} \in \Pi_{ij \in N \times N} \Sigma_{\Phi_{ij}}(G^*_{ij})\)
such that
\[
V(\omega_{ij}, \omega_{-ij}, (G^*_{ij}(\omega_{ij})), (G_{ij})_{ij \in N \times N})
\]
\[
= \max_{(G_{ij})_{ij \in N \times N} \in \Pi_{ij \in N \times N} \Phi_{ij}(\omega_{ij}, G^*_{ij})} V(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij \in N \times N}).
\]
solves the network formation game over mechanisms given by
\[
\max_{(G_{ij}(\cdot))_{ij \in N \times N} \in \Pi_{ij \in N \times N} \mathcal{IC}_{ij}} \int_{\Omega} V(\omega_{ij}, \omega_{-ij}, (G_{ij}(\omega_{ij}))_{ij \in N \times N}) d\lambda(\omega).
\]

Because the principal’s optimal payoff function \(V^*(\omega_{ij}, \omega_{-ij}, \cdot)\) is upper semicontinuous on \(\Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})\) for all \((\omega_{ij}, \omega_{-ij})\) and because \(\Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})\) is an \(h_{K}\)-compact metric space, we easily obtain the following existence result.

**Theorem 4 (Existence of an Optimal Catalog)**
Suppose assumptions \([A-1]\) and \([A-2]\) hold. Then there exists an \(ij\)-catalog profile,
\[(G^*_{ij})_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij}),\]
solving the catalog game, that is, a catalog profile,
\[(G^*_{ij})_{ij \in N \times N},\]
such that
\[
\int_{\Omega} V^*(\omega_{ij}, \omega_{-ij}, (G^*_{ij})_{ij \in N \times N}) d\lambda(\omega)
\]
\[
= \max_{(G_{ij})_{ij \in N \times N} \in \Pi_{ij \in N \times N} P_{h_{K,F}}(G_{ij})} \int_{\Omega} V^*(\omega_{ij}, \omega_{-ij}, (G_{ij})_{ij \in N \times N}) d\lambda(\omega).
\]
Suppose that $(G_{ij}^*)_{ij \in \Pi \times \Pi} \in \Pi \times \Pi P_{h,k} f_((\mathbb{G}_{ij}))$ solves the catalog game (10) and

$$(G_{ij}^*(\cdot))_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \Sigma_{\Phi_{U ij}} (G_{ij}^*)$$

is such that

$$V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*(\omega_{ij})))_{ij \in \Pi \times \Pi} = \max_{(G_{ij})_{ij \in \Pi \times \Pi}} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*))_{ij \in \Pi \times \Pi}.$$ For all $(G_{ij})_{ij \in \Pi \times \Pi} \in \Pi \times \Pi P_{h,k} f_((\mathbb{G}_{ij}))$ and for all $(G_{ij}^*(\cdot))_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \Sigma_{\Phi_{U ij}} (G_{ij})$

$$\int_{\Omega} V^*(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*)_{ij \in \Pi \times \Pi}) d\lambda(\omega) = \int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega) \geq \int_{\Omega} V^*(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*)_{ij \in \Pi \times \Pi}) d\lambda(\omega) = \int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega),$$

where $(G_{ij}^*(\cdot))_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \Sigma_{\Phi_{U ij}} (G_{ij}^*)$ is such that

$$V((\omega_{ij})_{ij \in \Pi \times \Pi}, (G_{ij}^*(\omega_{ij})))_{ij \in \Pi \times \Pi} = \max_{(G_{ij})_{ij \in \Pi \times \Pi}} V((\omega_{ij})_{ij \in \Pi \times \Pi}, (G_{ij}^*))_{ij \in \Pi \times \Pi}.$$ Thus, we can conclude via the Delegated Networking Principle (i.e., via the fact that $\mathcal{C}_{ij} = \Sigma_{\Phi_{U ij}}$) that

$$\int_{\Omega} V^*(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*)_{ij \in \Pi \times \Pi}) d\lambda(\omega) = \int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega) = \max_{(G_{ij})_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \Sigma_{\Phi_{U ij}}} \int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega).$$

We have the following Corollary.

**Corollary (Existence of an Optimal Network Formation Mechanism)**

Suppose assumptions [A-1] and [A-2] hold. Then there exists a network formation mechanism $(G_{ij}^*(\cdot))_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \mathcal{C}_{ij}$ solving the mechanism game, that is, a mechanism $(G_{ij}^*(\cdot))_{ij \in \Pi \times \Pi}$ such that

$$\int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}^*(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega) = \max_{(G_{ij})_{ij \in \Pi \times \Pi} \in \Pi \times \Pi \mathcal{C}_{ij}} \int_{\Omega} V(\omega_{ij}, \omega_{\omega ij}, (G_{ij}(\omega_{ij})))_{ij \in \Pi \times \Pi} d\lambda(\omega).$$