Output-based permit allocations under uncertainty

Preliminary Draft

2014

Abstract

Output Based Allocations (OBA) have become a frequent ingredient in the design of permits market (EUETS, California, Australia...). We provide a new justification for the use of output based allocations. This justification is based on demand uncertainty and flexibility. With an OBA scheme, it is not feasible to control for the overall emissions cap since new permits are injected according to the level of production. With an uncertain demand, the total cap is then uncertain with OBA. It is shown that this flexibility has positive welfare implications. OBA can be used to (imperfectly) condition emissions on the level of economic activity since more permits are issued when demand is large and the emissions more valuable. Numerical simulations are used to quantify the size of this ‘flexibility’ component.

JEL Classification: D24, L13, H23, L74

Keywords: cap and trade, output-based allocation, climate policy,

1 Introduction

Introduction Emission trading schemes for GHGs mitigation with fixed caps are spreading (from the EU-ETS initiated in 2005 to the more recent CA-ETS in 2012).\footnote{For a survey of ETS see Hood (2010) Hood, 2010. For recent information on the CA-ETS see California Air Resources Board (2013) California Cap on Greenhouse Gas Emissions and Market-Based Compliance Mechanisms. \url{www.arb.ca.gov/cc/capandtrade/c-t-reg-reader-2013.pdf}} Since the seminal paper of \cite{Weitzman1974} it is known that under uncertainties fixed caps are a second best solution. Indeed the EU-ETS scheme has suffered from the unexpected economic crisis and the CO2 prices resulting from excessive fixed caps no longer incentivize firms to invest in abatement strategies.\footnote{There is already a large literature on the so-called Mechanism Stability Reserve (MSR) proposals to address this practical issue for the EU-ETS. See for instance Taschini, Luca (2013). Options for structural measures to improve the European Union Emissions Trading System: response to a European Commission consultation. Policy paper. Grantham Research.} In its design the CA-ETS introduced a mechanism for setting a price floor and a price cap to cope with uncertainties. This is in line with the hybrid instruments suggested in the literature \cite{RobertsSpence1976}.

In this paper we show that allocating a fraction of the emission rights through an output based mechanism (while auctioning the remaining part) would be socially optimal, even if...
a hybrid instrument were implemented. Output based allocations (OBA) are freely transferred to firms in proportion to their production using an industry benchmark. This result is important since, in practice, ETS schemes do incorporate OBA for leakage protection in carbon intensive and internationally traded sectors. The policy implication of our result is that uncertainty and leakage should be looked at simultaneously rather than independently. The introduction of OBA reduces somehow the imperfections of a fixed cap in face of uncertainty. Policy proposals in which the flexibility due to OBA (in exposed sectors) would be compensated by revising the amount of auctioned permits to satisfy a constant cap are counterproductive.

OBA schemes have been extensively studied in the literature (Böhringer and Lange; 2005; Böhringer et al.; 2010, 2012; Monjon and Quirion; 2011; Fischer and Fox; 2012). Typically abatement incentives are preserved while the carbon cost pass through is reduced. This is the traditional argument for using OBA to mitigate leakage since the competitiveness of regulated firms with unregulated imports is preserved. In absence of leakage OBA is clearly inefficient, it would induce excessive production. To tackle leakage, OBA schemes are outperformed by border tax adjustments. Some authors have argued that OBA may have some benefit under imperfect competition: reducing the pass through counteracts the restriction of production due to market power (Gersbach and Requate; 2004; Fischer; 2011; Fowlie et al.; 2012). We do not consider such an argument and look instead to the impact of OBA under uncertainty (say market uncertainty).

Assume a fraction of permits is auctioned and that the remaining part is allocated through an OBA scheme. The ex-post number of permits is no longer determined; if the market is high, it will be higher than if the market is low since the number of permits allocated through OBA depends on the unfolding of the uncertainties. This flexible cap goes in the right direction to increase welfare but it comes at the cost of overproduction. We shall determine the socially optimal fraction of permits that should be allocated through the OBA scheme, and show that this fraction remains strictly positive even in the presence of socially optimal price floor and price cap. To our knowledge this is a completely novel argument for the use of OBA.

These results are obtained using a standard three stage framework in which the regulator decides its policy design, then uncertainty unfolds, and finally firms take their decisions. We assume perfect competition, one sector and no imports. The presence of imports would not affect our analysis. This theoretical part suggests the conceptual relevance of OBA in face of uncertainty. A numerical illustration will complete the theoretical part to provide some order of magnitude and discuss the policy relevance of our results.

The paper is organized as follows: In the next section we present the model. Our theoretical argument is presented in Section 3. We analyze how OBA interacts with standard hybrid schemes in Section 5. In Section 4 (to be completed), we consider several sectors and provide numerical simulations.

---


4 The papers cited above do not introduce uncertainty. The only paper we can think of in which OBA and uncertainty are simultaneously considered is Meunier et al. (2014). In that paper it is again the leakage issue that is discussed and not the flexibility implication for the cap.
2 The model

Let us consider a homogeneous good, the demand for which is random. The inverse demand function is: \( p(q, \theta) \), where \( q \) is the total quantity consumed and \( \theta \) is a random parameter distributed over \([\theta_m, \theta_M]\) according to the cumulative distribution \( F \), a continuously differentiable function, with \( \mathbb{E}\theta = 0 \)\(^5\). The distribution of \( \theta \) can represent either risk or time variability of the demand. We assume that \( p \) is decreasing with respect to \( q \) and increasing with respect to \( \theta \). The corresponding consumer gross surplus is \( S(q, \theta) \) with \( S_q = p(q, \theta) \)\(^6\).

The production side is represented by a representative price taker firm. The production cost of the industry is \( C(q, e) \) in which \( e \) is the quantity of pollutant emissions. The cost function is positive and increasing with respect to the output quantity with \( C(0, e) = 0 \) for all \( e \). The marginal cost is decreasing with respect to the emission quantity: \( C_q e < 0 \). And the cost is convex:

\[
C_{qq} > 0; \; C_{ee} > 0; \; C_{qq} C_{ee} > C_{qe}^2. \tag{AC}
\]

The environmental damage depends on total actual emissions. It is assumed separable and is denoted \( D(e) \), a positive, increasing and convex function. Welfare in a state \( \theta \) is the difference between gross consumer surplus, production cost and environmental damage:

\[
w(q, e, \theta) = S(q, \theta) - C(q, e) - D(e); \tag{1}
\]

and expected welfare is

\[
W = \mathbb{E}[w(q, e, \theta)]. \tag{2}
\]

The environmental damage is not considered by firms when choosing their emissions. In order to correct for this market failure, the regulator implements a permits market. The total quantity of permits, a.k.a. the cap, comes from two sources: some permits are auctioned and some are freely allocated based on production. The regulator auctions \( \bar{e} \) permits, and issues new permits based on production at a rate \( \alpha \): for each unit produced a firm received \( \alpha q \) free permits. The rate \( \alpha \) is named the OBA rate in the rest of the article. The permit price is \( \sigma \). The total quantity of emissions is equal to the cap which is equal to the quantity of permits auctioned plus those freely allocated:

\[
e = \bar{e} + \alpha q.\]

The timing is the following:

- The regulator sets \( \bar{e} \) (permits auctioned) and \( \alpha \) (OBA rate);
- \( \theta \) is known and the firm decides how much to produce \( q \) and how much to emit \( e \). The prices \( p \) and \( \sigma \) clear the markets of output and permits.

\(^5\) The expectation operator is denoted \( \mathbb{E} \).
\(^6\) Partial derivatives are denoted by indexes: so \( S_q \) means \( \partial S/\partial q \).
The auctions are not precisely described, it is assumed that the price of auctioned permits is equal to the expected price of permits. The fact that these permits are sold before the realization of uncertainty is irrelevant: it is as if the regulator supply $\bar{e}$ permits in all demand states, or, alternatively allocates them for free in a lump-sum way.

The total quantity of emissions is not fixed but depends on the realization of the demand because of the output based allocations. What is fixed ex-ante is the quantity of auctioned permits, but the quantity of permits allocated through the OBA rule is random. The OBA rule introduces some flexibility in the permit market but at the cost of distorting the production and emissions decision. We investigate whether the former is worth the latter.

Several assumptions are made to make the analysis interesting and to avoid irrelevant complications. It worth producing in all demand states even taking into account environmental damage $P(0, \theta) > C_q(0, e)$ for $e$ such that $C_e(0, e) = D'(e)$, for all $\theta$.

**market equilibrium**

The profit of the representative firm, net of the cost of auctioned permits, is a function of prices and the regulatory variables:

$$\pi(p, \sigma, q, e, \bar{e}) = pq - C(q, e) - \sigma(e - \alpha q - \bar{e}).$$  \hspace{1cm} (3)

In a state $\theta$, the firm maximizes its profit and the two first order conditions are:

$$p + \alpha \sigma = C_q(q, e), \text{ and } -C_e(q, e) = \sigma.$$  \hspace{1cm} (4)

The marginal cost is equal to the output price plus the subsidy coming from the OBA rule, which is equal to the OBA rate times the permit price. The cost reduction obtained from an increase of emissions, the marginal abatement cost, is equal to the permit price. The net demand for auctioned permits is $e - \alpha q$ which is decreasing thanks to AC (cf Appendix A). The equilibrium price of permits clears the permit market, so that, if $\sigma$ is strictly positive then

$$e - \alpha q = \bar{e}.$$  \hspace{1cm} (5)

If $\alpha$ or $\bar{e}$ are large the permit price is null and $e - \alpha q < \bar{e}$.

Production and emissions are functions of $\theta$, and the regulatory variables $\bar{e}$ and $\alpha$. Some comparative statics are provided in the appendix. It is worth noting that in the general setting used if $\alpha$ is large there can be some counter-intuitive monotonicity. For instance, the influence of the permit price on the output supply is the sum of two opposite terms: there is a direct positive influence via the OBA rule that subsidize production, and there is an indirect negative effect via the reduction of emissions. For similar reasons, the influence of the number of permits auctioned on the output production and the emissions is a priori ambiguous for large $\alpha$. 

4
3 Optimal regulation

In this section the choice of the optimal number of auctioned permits is first described before considering the influence of the OBA rate. Indeed, the question is whether a positive OBA rate can enhance welfare even if the number of auctioned permits is optimally set.

Choice of the number of auctioned permits

Without the output based allocation ($\alpha = 0$), the choice of the number of auctioned permits is standard. Without output based allocation the quantity of emissions does not vary with the output demand, so the marginal environmental damage is constant thanks to our assumption of separability. And, at the optimal quantity of emissions the marginal environmental damage is equal to the expected price of permits.

Lemma 1 Without output based allocation ($\alpha = 0$), the optimal cap is such that the expected permit price is equal to the marginal environmental damage.

Proof. If $\alpha = 0$, the quantity of emissions is $\bar{e}$ in all demand states, then the derivative of welfare with respect to the number of auctioned permits is, by the envelop theorem:

$$\frac{dW}{d\bar{e}} = \mathbb{E}[-C_e - D'(\bar{e})] = \mathbb{E}[\sigma] - D'(\bar{e})$$

so, at the optimum $\mathbb{E}[\sigma] = D'(\bar{e})$. ■

If there is a positive OBA rate $\alpha > 0$, the production is no longer efficient so the influence of the number of auctioned permit on production matters when evaluating the effect of a change of the number of auctioned permit (the cross derivative $\partial W/\partial q$ is positive). Furthermore, because of the mechanism of free allocation, the effect of auctioned permits on total emissions in a state can be decomposed in two: the number of emissions increases directly via the increased quantity of auctioned permits and also indirectly via the increase of production and the associated freely allocated permits. Formally, the influence of the quantity of auctioned permits on welfare is

$$\frac{dw}{d\bar{e}} = -\alpha \sigma q_\bar{e} + (\sigma - D')e_\bar{e}$$

the first term of the right hand side is the efficiency cost of over-production times the sensitivity of production to the quantity of auctioned permits. The efficiency cost is precisely due to the OBA scheme. The second term is the effect of a change of emissions. Both derivatives are linked by $e_\bar{e} = 1 + \alpha q_\bar{e}$ so that,

$$\frac{dw}{d\bar{e}} = (\sigma - D') - \alpha D'q_\bar{e}$$

In this equation the first term casts the direct influence of an increase of auctioned permits and the second term the indirect influence on production. This second term encompasses
both the indirect effect on production and the associated increase of emissions due to free permits.

As mentioned the sign of \( q_e \) is not completely clear, a priori if more permits are auctioned
the production should be larger but this direct effect is partly compensated by a reduction
of the permit price and the subvention associated to output based free allocation. However
for small \( \alpha \), the first effect dominates and both \( q \) and \( e \) increase with \( \bar{e} \).

The choice of \( \alpha \) the output based rate

What needs to be determined is whether a positive OBA rate should be implemented when
the number of auctioned permits is optimally set.

Without uncertainty the first best allocation could be attained by auctioning the optimal
quantity of permits and no free allocation. If, without uncertainty, the number of auctioned
permits is suboptimal, then a positive OBA rate can enhance welfare. It is so because the
inefficiency cost of a small OBA rate is nearly null while the benefit from relaxing the cap is
strictly positive.

With uncertainty, the optimal ex-ante second best quantity of auctioned permits is never
optimal ex-post when the level of the demand is known. A positive OBA rate can then
increase welfare if it relaxes the cap more in demand states in which it is more beneficial to
do so.

In a demand state, the implementation of an OBA scheme has two effects: it modifies
the quantity of permits available and subsidize production. The former can be beneficial if
the number of permits is low, and the latter is always detrimental. Formally, the effect
of \( \alpha \) on short-term welfare is:

\[
\frac{dw}{d\alpha} = -\alpha\sigma q_\alpha + (\sigma - D')e_\alpha
\]  \hspace{1cm} (6)

There is a welfare loss from increasing production precisely because of the distortive
nature of the OBA scheme. This loss is potentially compensated by a gain emanating from
increasing emissions. Whether increasing emissions enhances or reduces welfare is reflected
in the comparison between the permit price and the marginal environmental damage.

The cost due to over-production is null for \( \alpha = 0 \). Even without uncertainty, if \( \bar{e} \) is such
that the permit price is above the marginal environmental damage the regulator can fix a
strictly positive OBA rate to relax the cap.

Lemma 2 Without uncertainty, the optimal scheme is to set \( \alpha = 0 \) and \( \bar{e} \) such that the
permit price is equal to the marginal environmental damage.

Proposition 1 With uncertainty, at the optimum scheme \( (\bar{e}^*, \alpha^*) \) the OBA rate is strictly
positive.

The optimal scheme satisfies the pair of equations:

\[
\mathbb{E}[\sigma - D'] = \alpha \mathbb{E}[D'q_e]
\]  \hspace{1cm} (7)

\[
\alpha = \text{cov}(\sigma - D', q) \times \left[ \mathbb{E}[D'q_\alpha] - \mathbb{E}[q] \mathbb{E}[D'q_e] \right]^{-1}
\]  \hspace{1cm} (8)
The proof is in Appendix [B]. It is instructive to consider the effect of a small OBA rate. If \( \alpha \) is null the quantity of auctioned permit should equalize the expected permit price with the marginal environmental damage (cf Lemma [1]). Then, the effect of a small increase of \( \alpha \) on the quantity of emissions is equal to the output production (taking the derivative of equation [5])

\[
e_{\alpha} = q \quad \text{so that the effect on expected welfare is}
\]

\[
E\left\{ \frac{dw}{d\alpha} \right\} = E\left\{ -\alpha \sigma q_{\alpha} + (\sigma - \delta)(q + \alpha q_{\alpha}) \right\} = E\left\{ (\sigma - D')q \right\} \quad \text{(for } \alpha = 0) \quad (9)
\]

Then, for \( \alpha = 0 \) the marginal environmental damage is constant and equal to the expected permit price so that

\[
E\left\{ (\sigma - D')q \right\} = E\left\{ (\sigma - D') \right\} E\left\{ q \right\} + \text{cov}(\sigma, q) = \text{cov}(\sigma, q) \quad (10)
\]

This equation casts the fact that there is a welfare gain with uncertainty because the OBA rate relaxes the cap precisely in states in which the permit price is larger than the environmental damage. It is so because we consider output demand uncertainty implying a positive correlation between production and the permit price. With uncertainty, implementing an OBA rate will relax the cap in all demand states, this will have a positive (resp. negative) effect in high (resp. low) demand states in which the permit price is higher (resp. lower) than the marginal environmental damage. Whether it will have a positive expected effect depends whether the increased of permits will be positively correlated with the permit price, and it is so because the increase of permits is proportional to production.

The choice of the optimal \( \alpha \) is represented by equation [8]. The first factor represents the gain from synchronizing the change of the number of permits with the difference between the permit price and the environmental damage. This factor would be null without uncertainty.

The second factor is related to the distorsive nature of the output based allocation compared to auctioned permits. With a positive OBA rate, both sources of permits have a production-efficiency cost due to their effect on production. However the influence of a change of \( \alpha \) can be decomposed in two: there is first the effect of the increased number of permits which is proportional to production and second the increase of the subsidy [7].

The number of auctioned permits should indeed be adjusted when the OBA is implemented. Equation [7] states that the expected permit price should be equal to expected environmental damage plus a term representing the cost associated to production inefficiency. Increasing auctioned permits increases production (if \( \alpha \) is not too large) which is costly because of the OBA scheme.

Remarks: for large \( \alpha \) production is increasing with the demand but the permit price is not necessarily because of the increase of free allocations due to higher production. For our proof it is sufficient that it is true for small \( \alpha \).

[7] Without uncertainty the second factor would be strictly positive because \( q_{\alpha} - qq_{\bar{e}} > 0 \).
4 Hybrid instruments

The need to complement permit market with flexibility measures is well known. OBA have such a flexibility property, and a question is whether they are still useful if other flexibility measures are already implemented.

In practice, a price cap and a price floor are such a flexibility ... In the academic debate, a permit market with a price cap and a price floor has been introduced and analyzed by Roberts and Spence (1976). Such a scheme is superior to both pure price and quantity instruments since it can mimic both.

Consider that the regulator decides to implement ex-ante a price ceiling and price floor on the permit market, these are respectively denoted $\bar{\sigma}$ and $\sigma$. These are equivalent to setting a penalty for not complying with the permit market and a subsidy for over-compliance. The timing is the same as in the preceding section. Both instruments are set ex-ante together with the number of auctioned permits and the OBA rate.

In the short-run if the demand for permits is high (resp. low), the price of permits will be equal to the ceiling (resp. floor). If the OBA rate is sufficiently small, the demand for permits is increasing with respect to the demand state $\theta$ (cf. Appendix A) and there are two demand states $\bar{\theta}$ and $\theta$ such that:

- If $\theta \geq \bar{\theta}$, then the permit price is equal to the ceiling: $\sigma = \bar{\sigma}$ and $e - \alpha q \geq \bar{e}$;
- If $\bar{\theta} < \theta < \bar{\theta}$, the permit price is between the floor and the ceiling $\sigma < \sigma < \bar{\sigma}$ and $e - \alpha q = \bar{e}$;
- If $\theta \leq \bar{\theta}$, then $\sigma = \sigma$ and $e - \alpha q < \bar{e}$.

We will proceed as in the previous section: the choice of the optimal design with a null OBA rate is described, and then, the effect of a marginal increase of the OBA rate is analyzed.

The implementation of a price floor and a price ceiling: The price ceiling influences production and emissions in high demand states. Since, when $\alpha = 0$ the production is optimal given the emissions, only the influence on emissions matters. The effect of the price ceiling on expected welfare is:

$$\frac{dW}{d\sigma} = \int_{\bar{\theta}}^{\theta_m} (\bar{\sigma} - D') e_\sigma dF(\theta)$$

Indeed, with a constant marginal environmental damage it would be optimal to implement a tax which amount to equalizing the ceiling with the floor with marginal environmental damage. With a strictly convex environmental damage, the ceiling can increase welfare because it relaxes the cap in high demand states in which the marginal abatement cost is
large. The optimal ceiling is equal to a weighted average of the marginal environmental
damage, the weights being the price sensitivity of emissions:

\[
\bar{\sigma} = \frac{E[D'e_{\theta}|\theta > \bar{\theta}]}{E[e_{\theta}|\theta > \bar{\theta}]}. 
\]

And a similar expressions hold for the choice of the optimal floor. Both instruments allow
to more closely make internalize the marginal environmental damage curve. Whereas this
curve was internalized via a straight line with a pure permit market, it is approximated by
kinked function (cf. Roberts and Spence [1976]).

The optimal number of auctioned permits is, in that case, chosen so that the environmen-
tal damage associated to it is equal to the expected permit price when the permit market is
in the price tunnel:

\[
D'(\bar{e}) = E[\sigma|\sigma < \theta < \bar{\sigma}] 
\]

Whether it is worth adding an OBA scheme to this permit market follows from a reasoning
similar to the previous reasoning. The derivative of expected welfare at \( \alpha = 0 \) is the effect
of \( \alpha \) on emissions (the production inefficiency being null for \( \alpha = 0 \)) so

\[
E \left[ \frac{dW}{d\alpha} \right] = E[(\sigma - D')e_{\alpha}]. 
\]

The OBA rate has a different influence in intermediary demand states than in extreme
demand states. In intermediary demand states, there is both a direct effect on emissions due
to the increased number of free permits and an indirect effect via the rise of production due
to a higher subsidy. In extreme demand states, the permit price is fixed, by the ceiling or the
floor, and free allocations alleviate the bill of the firm but not its emission choice. However,
there is still the subsidy channel that induces higher emissions due to higher production.

\[
E \left[ \frac{dW}{d\alpha} \right] = \int_{\theta_m}^{\bar{\sigma}} (\sigma - D')e_{\alpha}dF + \int_{\theta}^{\bar{\theta}} (\sigma - D'(\bar{e}))qdF + \int_{\theta}^{\bar{\theta}M} (\bar{\sigma} - D')e_{\alpha}dF 
\] (11)

The influence of the OBA rate is positive in the price tunnel for intermediary demand
states (second term above). The magnitude of this positive effect depends on the size of
the tunnel and the covariance between the output quantity and the permit price. The first
and third term have indeterminate signs that depends on the influence of the OBA rate on
emissions when the floor or the ceiling bind, and how this influence compares to the influence
of the permit price on emissions.

**Proposition 2** If the demand function is linear and the production cost \( C(q,e) \) is quadratic
a strictly positive OBA rate increases welfare.

**Proof.** With a linear demand function and a quadratic cost the derivatives \( e_{\theta} \) and \( \bar{e}_{\alpha} \)
are independent of the demand states. so the optimal price ceiling is such that

\[
\bar{\sigma} = E[D'|\theta > \bar{\theta}] 
\]
and the third term of the derivative of welfare with respect to $\alpha$ at zero in equation 11 is null. The same reasoning holds for the first term. Therefore, the derivative of welfare with respect to the OBA rate is positive and a strictly positive OBA rate can enhance welfare.

5 Multisector and numerical simulations

In this section we consider a multisector setting. We first provide a brief theoretical discussion of the correspondence with the mono sector case, before developing a linear specification. This linear specification is then used for a rough calibration based on Meunier et al. (2014) (to be done). In the linear specification we introduce carbon leakage, the main justification for the implementation of OBA. The optimal OBA rate can then be decomposed in several terms, a leakage one and a ‘flexibility’ term emphasized in the present work. The numerical simulations are used to get an idea of the size of the ‘flexibility’ component in the optimal OBA rate.

We then used our framework to consider whether the total emissions cap should be made flexible when a sector exposed to international competition is integrated.

5.1 Theory

Let us consider that there are two sectors 1 and 2. Production in sector $i$ is $q_i$ and emissions in sector $i$ are assumed equal to production $e_i = q_i$ (emissions rate normalized to 1). For each sector $i = 1, 2$, the price function of good $i$ is random $p_i(q_i, \theta_i)$ and corresponding surplus $S_i(q_i, \theta_i)$. Production costs are assumed null, so welfare in a state $\theta$:

$$w(q, \theta, e) = S_1(q_1, \theta_1) + S_2(q_2, \theta_2) - D(e)$$

A scheme is a triplet $(\bar{e}, \alpha_1, \alpha_2)$ of auctioned permits and OBA rates in each sector. The regulator first chooses the scheme and then the demand shocks are known and productions occur. Ther permit price is $\sigma$, and the total number of permits is

$$e = \bar{e} + \alpha_1 q_1 + \alpha_2 q_2$$

At the short-term market equilibrium $\sigma$ clears the market so that

$$q_1 + q_2 = \bar{e} + \alpha_1 q_1 + \alpha_2 q_2$$

(12)

In the short term, the price of good $i$ is equal to the permit price minus the OBA subsidy:

$$p_i(q_i, \theta_i) = (1 - \alpha_i)\sigma$$

So, the couple of equations satisfied by the two quantities $q_1$ and $q_2$ are:

$$p_1(q_1, \theta_1) = \frac{1 - \alpha_1}{1 - \alpha_2} p_2(q_2, \theta_2)$$

(13)

$$\alpha_1 q_1 + \alpha_2 q_2 = \bar{e}$$

(14)
Lemma 3  Any scheme \((\bar{e}, \alpha_1, \alpha_2)\) is equivalent (i.e. leads to the same quantities \(q_1\) and \(q_2\) in all demand states) to the scheme

\[ ((1 - \alpha_2)\bar{e}, \frac{\alpha_1 - \alpha_2}{1 - \alpha_2}, 0). \]

Proof. With the two schemes the same couple of equations (13) and (14) are satisfied which implies that the productions are identical with both schemes. 

If a scheme is implemented it can be replaced by an equivalent (from a welfare perspective) scheme in which only one sector receives free allocations. This is linked to the fact that production and sectoral emissions are equal. We can consider only one OBA rate, say \(\alpha_1\), fixing the other to zero. We can relax the positivity constraint on this rate since a theoretically negative OBA rate in sector 1 can be implemented by setting a positive OBA rate in the other sector.

Therefore, the two sectors case is very similar to the mono sector case considered. If one consider that the second sector is hidden in the abatement cost function, the tow sectors case correspond to the mono sector case with a random cost function. In that case, the covariance between sector 1 production and the permit price is not necessarily positive. When the demand in sector 2 is large so is the permit price but the production in sector 1 is small.

The choice of the OBA rate is related to similar arbitrages than in the mono-sector case. The inefficiency associated to OBA is a misallocation of the cap between the two sectors. The sector subject to the OBA produces and pollute too much, and the other not enough.

The influence of \(\alpha_1\) on short-term welfare is:

\[
\frac{dw}{d\alpha_1} = -\alpha_1\sigma \frac{\partial q_1}{\partial \alpha_1} + (\sigma - D') \left[ \frac{\partial q_1}{\partial \alpha_1} + \frac{\partial q_2}{\partial \alpha_1} \right]
\]

Furthermore, the two quantities are such that \((1 - \alpha_1)q_1 + q_2 = \bar{e}\) so

\[
\frac{\partial q_2}{\partial \alpha_1} = q_1 - (1 - \alpha_1) \frac{\partial q_1}{\partial \alpha_1}
\]

injecting in the welfare derivative:

\[
\frac{dw}{d\alpha_1} = (\sigma - D')q_1 - \alpha_1 D' \frac{\partial q_1}{\partial \alpha_1}
\]

(15)

5.2 specification

We use a linear specification to get explicit formula. The price functions are \(p_1 = a_1 + \theta_1 - q_1\), \(p_2 = a_2 + \theta_2 - q_2\) and a linear environmental damage \(D(e) = \delta e\).

Proposition 3 with the linear specification

\(^8\)With more than two sectors, only one sectoral OBA rate is redundant and can be set to zero.
• For an OBA rates $\alpha_1 > 0$ and $\alpha_2 = 0$, the optimal number of auctioned permits is:

$$\bar{e} = (1 - \alpha_1)(a_1 - \delta) + (a_2 - \delta)$$

• the riskier sector should have a larger OBA rate than the other sector;

• If sector 1 is riskier than 2, then the optimal OBA rates are $\alpha_2 = 0$ and

$$\alpha_1 = 1 - \frac{1}{2} \left[ (\Delta^2 + 4)^{1/2} - \Delta \right] (> 0)$$

where $\Delta = \frac{\text{var}(\theta_1) - \text{var}(\theta_2)}{\delta^2}$

• The optimal OBA rate in sector 1 is increasing with respect to the variance of demand in sector 1, and decreasing with respect to the variance of the demand in sector 2.

In Figure 1 a binary distribution of demand states with only sector random (the demand in sector 2 is not random), the uncertainty components is of non-negligible magnitude.

![Figure 1: The optimal oba rate in sector 1, if demand in sector 2 is certain, as a function of the ratio between the low demand state and the average.](image)

To be done (under progress): rough calibration and numerical simulations with some leakage in an exposed sector.

6 Conclusion

In this article the properties of output based allocations schemes has been studied. A new argument favorable to OBA has been presented. It is built on demand uncertainty and flexibility.

If the demand for a polluting is uncertain, the regulator has to fix the cap of the permit market under uncertainty. Ex-post, when the demand is known the cap is sub-optimal, being either too low or too high. An OBA scheme has indeed a distorsive nature since it amounts to subsidize production. Absent uncertainty the OBA rate should be null if the emissions
cap is optimally set. With uncertainty, an OBA scheme introduces some flexibility since the number of permits depends on the level of demand. An OBA scheme enhances welfare by allowing to condition the emissions cap with economic activity, though imperfectly. It is so because OBA relaxes more the cap when demand is high and emissions valuable (the permit price and the production are positively correlated). This feature is reduced if production cost are uncertain. Even if a price floor and ceiling are introduced an OBA scheme can still enhance welfare.

In a multisector setting, the OBA rate implemented in a sector is decreasing with respect to the uncertainty on the demand in the other sector. Indeed, in that if the demand in the other sector is high, the production in the sector subject to OBA is low and the flexibility of the cap due to the OBA is less effective. Numerical simulations shows that the flexibility component could be of the same order of magnitude than the leakage component (under progress).

References


Appendix

A Equilibrium

To describe the equilibrium in a demand state \( \theta \) we proceed as follow: we first consider a fixed permit price \( \sigma \) and show that the net demand for permits is decreasing with respect to \( \sigma \).

For a given \( \sigma \geq 0 \), the equilibrium is fully characterized by the two first order conditions \([4]\). We define \( f(\alpha, \sigma, \theta) \) and \( g(\alpha, \sigma, \theta) \) the unique couple of production and emissions respectively that solves the first order conditions.

Result: the net demand of auctioned permits \( g - \alpha f \) is decreasing with respect to \( \sigma \).

Proof. Take the derivative of the first order conditions \([4]\) with respect to \( \sigma \):

\[
\begin{bmatrix}
P_q - C_{qq} & -C_{qe} \\
-C_{qe} & -C_{ee}
\end{bmatrix}
\begin{bmatrix}
f_\sigma \\
g_\sigma
\end{bmatrix}
= \begin{bmatrix}
-\alpha \\
0
\end{bmatrix}
\]

(16)

So the effects of a change of the permit price on production and emissions are:

\[
\begin{bmatrix}
f_\sigma \\
g_\sigma
\end{bmatrix}
= \frac{1}{\Delta_1}
\begin{bmatrix}
C_{qe} + \alpha C_{ee} \\
(P_q - C_{qq}) - \alpha C_{qe}
\end{bmatrix}
\]

(17)

in which \( \Delta_1 = (C_{qq} - P_q)C_{ee} - C_{qe}^2 \) is strictly positive by assumption \([AC]\). The derivative of the net demand for permits is then:

\[
g_\sigma - \alpha f_\sigma = \frac{-1}{\Delta_1}
\left[
-P_q + C_{qq} - 2\alpha C_{qe} + \alpha^2 C_{ee}
\right]
\leq \frac{-1}{\Delta_1}
\left[
-P_q + C_{qq} - 2\alpha C_{qq}^{1/2} C_{ee}^{1/2} + \alpha^2 C_{ee}
\right]
\]

thanks to \([AC]\)

\[
\leq \frac{-1}{\Delta_1}
\left[
-P_q + (C_{qq}^{1/2} - \alpha C_{ee}^{1/2})^2
\right] < 0
\]

The equilibrium permit price depends on \( \alpha \) and \( \bar{e} \) and \( \theta \) it is \( \sigma(\alpha, \bar{e}, \theta) \) the unique solution of \( g(\alpha, \sigma, \theta) - \alpha f(\alpha, \sigma, \theta) = \bar{e} \).
The equilibrium quantities of output and emissions \( q(\alpha, \bar{e}, \theta) \) and \( e(\alpha, \bar{e}, \theta) \) are the (unique) solutions of the couple of equations:

\[
P(q, \theta) - C_q(q, e) - \alpha C_e(q, e) = 0
\]
\[
\alpha q - e = -\bar{e}
\]

To consider the comparative static of this equilibrium, first introduce

\[
\Delta_2 = -P_q + C_{qq} - 2\alpha C_{qe} + \alpha^2 C_{ee} > 0 \text{ (cf proof of result for the sign)}
\]

- the influence of \( \bar{e} \)

\[
\begin{bmatrix}
q_{\bar{e}} \\
e_{\bar{e}}
\end{bmatrix} = \frac{1}{\Delta_2}
\begin{bmatrix}
-C_qe - \alpha C_{ee} \\
q(C_{qq} + \alpha C_{qe} - P_q)
\end{bmatrix}
\]

for small \( \alpha \) both signs are positive. For large \( \alpha \) there is a counteracting effect via the subsidy and free allocations. With a larger number of auctioned permits the permit price is lower (cf the Result above) and so is the OBA subsidy, this effect can compensate the increase coming from a reduce cost of production and thus leads to a lower production.

- the influence of \( \alpha \)

\[
\begin{bmatrix}
q_{\alpha} \\
e_{\alpha}
\end{bmatrix} = \frac{1}{\Delta_2}
\begin{bmatrix}
-C_e - q(C_{qe} + \alpha C_{ee}) \\
-\alpha C_e + q(C_{qq} + \alpha C_{qe} - P_q)
\end{bmatrix}
\]

In both lines the first term comes from the subsidy component of the scheme, and the second term is the effect via the increase quantity of permits, it is equal to the effect of the quantity of auctioned permits times the quantity produced.

- the influence of \( \theta \):

\[
\begin{bmatrix}
q_{\theta} \\
e_{\theta}
\end{bmatrix} = \frac{1}{\Delta_2}
\begin{bmatrix}
P_{\theta} \\
\alpha P_{\theta}
\end{bmatrix}
\]

both quantities are increasing with the demand state. However, the monotonicity of the permit price is not that clear:

\[
\sigma_{\theta} = \frac{q_{\theta} - \alpha f_{\theta}}{-(g_{\theta} - \alpha f_{\theta})} = \frac{P_{\theta}}{\Delta_2}[-C_qe - \alpha C_{ee}]
\]

the first term in the brackets is positive but the second one is negative and comes from the OBA subsidy. If demand increases and \( \alpha \) is large the increase of permits coming from free allocations can compensate the pressure on marginal abatement cost coming from a higher production.
B Proof of proposition 1

The expected welfare should be written as a function of the two regulatory variables:

$$W(\bar{e}, \alpha) = \mathbb{E}[w(q(\alpha, \bar{e}, \theta), e(\alpha, \bar{e}, \theta), \theta)]. \quad (22)$$

The proof is organized as follow: Firstly, we show that at an optimal scheme $$(\bar{e}, \alpha)$$ the OBA rate is strictly positive. Secondly, we consider first order conditions to show that the optimal scheme satisfies equations (7) and (8). We have to disentangle the two steps because the sign of the right-hand side of (8) is not clear.

The first point is shown by contradiction. Assume that the optimal scheme encompasses $$\alpha = 0$$. According to Lemma (1), the quantity of auctioned permits is such that $$D'(\bar{e}) = \mathbb{E}[(\sigma - D'(\bar{e}))q]$$, the influence of a small increase of the OBA rate is then:

$$\frac{dW}{d\alpha} = \mathbb{E}[-\alpha \sigma q_{\alpha} + (\sigma - D'(q + \alpha q_{\alpha}))] = \mathbb{E}[(\sigma - D'(\bar{e}))q]$$

for $$\alpha = 0 \quad (23)$$

$$= \mathbb{E}[(\sigma - D'(\bar{e}))] \mathbb{E}[q] + \text{cov}(\sigma - D', q) \quad (24)$$

$$= \text{cov}(\sigma, q)$$

because $$\mathbb{E}[\sigma] = D' \quad (25)$$

The last step is to show that the covariance is positive. Both $$q$$ and $$\sigma$$ are increasing with respect to $$\theta$$ for $$\alpha = 0$$ from equations (20) and (21). Then, the covariance of two increasing function of a random variable is positive (proven below), so the derivative of expected welfare is strictly increasing and a small increase of $$\alpha$$ increases welfare which contradicts the optimality of the scheme.

**Result** If $$\phi$$ and $$\psi$$ are two real valued strictly increasing functions of $$\theta$$ then $$\text{cov}(\phi(\theta), \psi(\theta)) > 0$$.

**Proof.** Consider a second random variable $$\epsilon$$ independent from $$\theta$$ with the same distribution. Then

$$2\text{cov}(\phi(\theta), \psi(\theta)) = \text{cov}(\phi(\theta), \psi(\theta)) + \text{cov}(\phi(\epsilon), \psi(\epsilon)) = \text{cov}(\phi(\theta) - \phi(\epsilon), \psi(\theta) - \psi(\epsilon))$$

and

$$\text{cov}(\phi(\theta) - \phi(\epsilon), \psi(\theta) - \psi(\epsilon)) = \mathbb{E}[(\phi(\theta) - \phi(\epsilon)) \times (\psi(\theta) - \psi(\epsilon))]$$

and $$\forall(\theta, \epsilon) \in [\theta_{m}, \theta_{M}]^2$$ the product $$(\phi(\theta) - \phi(\epsilon)) \times (\psi(\theta) - \psi(\epsilon))$$ is positive and strictly so if $$\theta \neq \epsilon$$. 

We know prove that the two equations (7) and (8) are satisfied at an optimal scheme. At an optimal scheme the first order condition satisfied by the quantity of auctioned permits is:

$$0 = \frac{dW}{d\bar{e}} = \mathbb{E}[(\sigma - D' - \alpha D'q_{\bar{e}})] = \mathbb{E}[(\sigma - D')] - \alpha \mathbb{E}[D'q_{\bar{e}}]$$

which corresponds to equation (7). And for the optimal OBA rate, the first order condition
\[
0 = \frac{dW}{d\alpha} = \mathbb{E}[(\sigma - D')q - \alpha D'q_\alpha] \\
= \mathbb{E}[(\sigma - D')]\mathbb{E}[q] + \text{cov}(\sigma - D', q) - \alpha \mathbb{E}[D'q_\alpha] \\
= \text{cov}(\sigma - D', q) - \alpha (\mathbb{E}[D'q_\alpha] - \mathbb{E}[q]\mathbb{E}[D'q_\bar{e}])
\]

which gives equation (8).

\section*{C Linear specification}

\textbf{Calculations:}

the short-term equilibrium quantities are:

\[q_1 = (a_1 + \theta_1) - \frac{1 - \alpha_1}{(1 - \alpha_1)^2 + 1} [(1 - \alpha_1)(a_1 + \theta_1) + (a_2 + \theta_2) - \bar{e}]\]

\[q_2 = (a_2 + \theta_2) - \frac{1}{(1 - \alpha_1)^2 + 1} [(1 - \alpha_1)(a_1 + \theta_1) + (a_2 + \theta_2) - \bar{e}]\]

In both expressions the second term is the price of the good. The permit price is equal to the price of good 2:

\[\sigma = \frac{(1 - \alpha_1)(a_1 + \theta_1) + (a_2 + \theta_2) - \bar{e}}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}\]

the two quantities can be rewritten so as to isolate the random components:

\[q_1 = \mathbb{E}q_1 + \frac{\theta_1 - (1 - \alpha_1)\theta_2}{(1 - \alpha_1)^2 + 1}\]

\[q_2 = \mathbb{E}q_2 + \frac{(1 - \alpha_1)[(1 - \alpha_1)\theta_2 - \theta_1]}{(1 - \alpha_1)^2 + 1}\]

then if I use the equality obtained for the choice of the number of auctioned permits:

\[\delta\alpha_1 \mathbb{E} \frac{\partial q_1}{\partial \bar{e}} = \mathbb{E} [\sigma - \delta]\]

\[\delta [1 + \alpha_1 \frac{1 - \alpha_1}{(1 - \alpha_1)^2 + 1}] = \frac{1}{(1 - \alpha_1)^2 + 1} [((1 - \alpha_1)a_1 + a_2 - \bar{e}]\]

so

\[\delta [(1 - \alpha_1)^2 + 1 + \alpha_1 (1 - \alpha_1)] = (1 - \alpha_1)a_1 + a_2 - \bar{e}\]

so

\[\delta (2 - \alpha_1) = (1 - \alpha_1)a_1 + a_2 - \bar{e}\]
and the optimal number of auctioned permits as a function of this OBA rates is
\[
\bar{e}(\alpha_1) = (1 - \alpha_1)(a_1 - \delta) + (a_2 - \delta)
\] (26)
and the corresponding permit price:
\[
\sigma = \frac{(1 - \alpha_1)(\delta + \theta_1) + (\delta + \theta_2)}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}
\]
I can then come back to the choice of the optimal OBA rate. This is brutal calculation:

\[
\delta(\alpha_1 - \alpha_2)\mathbb{E}\frac{\partial q_1}{\partial \alpha_1} = \mathbb{E}[(p_2 - \delta)q_1] = \mathbb{E}[p_2 - \delta]q_1 + \text{cov}(p_2, q_1)
\]

(27)

Then,
\[
\mathbb{E}\frac{\partial q_1}{\partial \alpha_1} = \frac{(1 - \alpha_2)a_2 - \bar{e}}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2} + \frac{2(1 - \alpha_1)}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}q_1
\]

Thus,

\[
\delta(\alpha_1 - \alpha_2)\mathbb{E}\frac{\partial q_1}{\partial \alpha_1} - \mathbb{E}[p_2 - \delta]q_1 = \frac{\delta(\alpha_1 - \alpha_2)}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}[(1 - \alpha_2)a_2 - \bar{e} + (1 - \alpha_1)q_1]
\]

\[
= \frac{\delta(\alpha_1 - \alpha_2)(1 - \alpha_2)^2}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}[(1 - \alpha_2)a_2 + (1 - \alpha_1)a_1 - \bar{e}]
\]

\[
= \frac{\delta^2(\alpha_1 - \alpha_2)(1 - \alpha_2)^2}{(1 - \alpha_1)^2 + (1 - \alpha_2)^2}[(1 - \alpha_2) + (1 - \alpha_1)] \text{ injecting (26)}
\]

so, the optimal oba rates satisfy:

\[
\delta^2(\alpha_1 - \alpha_2)[(1 - \alpha_1) + (1 - \alpha_2)] = (1 - \alpha_1)(1 - \alpha_2)\Delta
\]

(28)
or with \( x = (1 - \alpha_1)/(1 - \alpha_2) \)

\[
(1 - x)(1 + x) = x\Delta
\]

(29)

with \( \Delta = [\text{var}(\theta_1) - \text{var}(\theta_2)]/\delta^2 \). Thus

\[
\alpha_1^* = 1 - (1 - \alpha_2)\frac{1}{2}[(\Delta^2 + 4)^{1/2} - \Delta]
\]