Existence of equilibrium in an economy with two distortions: Monopolistic competition and taxation

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Abstract

This paper proves the existence of equilibrium in an economy that has distortions due to both monopolistic competition and also linear commodity taxes. There are a finite number of industries, each of which is characterized by monopolistic competition. The degree of market power can vary from industry to industry. The government sets industry-specific commodity tax rates. Households are heterogeneous and are endowed with the only factor of production, labor. Existence is proved by applying Brouwer’s theorem to an appropriately constructed mapping.

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1 Introduction

A main message of this paper is that household heterogeneity need not be an obstacle to the existence of equilibrium in models with monopolistic competition and distortionary taxes. By extension, applied economists need not restrict themselves to representative household models.

Recent theoretical work on monopolistically competitive models has tended to focus on the representative household approach (e.g., Behrens and Murata 2007; Zhelobodko, Kokovin, Parenti, and Thisse 2012). Some authors have addressed heterogeneity among firms while maintaining identical households (e.g., Dhingra and Morrow 2014; Nocco, Ottaviano, and Salto 2014). Caplin and Nalebuff (1991) are at the other extreme with a continuum of heterogeneous households and parametric restrictions on the extent of heterogeneity. Although their model is one of strategic interaction between firms (unlike Chamberlinian monopolistic competition) they apply their results to the model of Dixit and Stiglitz (1977). Their proposition 14 proves existence of an equilibrium in the case of a representative household who has constant elasticity of substitution (CES) utility. As such, the richness of their heterogeneous framework is no longer present in their monopolistic competition application. None of these models include distortionary taxes.

The paper most similar to the research presented here is Arnold (2013). He establishes existence of equilibrium in Helpman and Krugman’s (1985) model of international trade with increasing returns and monopolistic competition. Arnold (2013) allows for some heterogeneity across households, but with limitations. In particular, his equation (1) requires that within each industry all households have constant elasticity of substitution (CES) preferences with the same elasticity for all households. The preferences of households differ in the way they combine their CES aggregators to determine their overall utility. By contrast, the model presented below not only relaxes the CES assumption but also allows different households to have different within-industry preferences. This extends considerably the range of preferences under consideration and breaks away from the highly stylized CES form.

In the proof of existence below, firms are infinitesimally small. As a result, they are too small to interact strategically with one another. While the lack of strategic interaction has been criticized (e.g., see the textbook discussion on pages 343–346 of Kreps 1990), this approach continues to be widely used in applied research. The Dixit–Stiglitz model and variations have been used quite extensively in international trade theory (e.g., Neary 2004) and in macroeconomics (e.g., Dixon and Rankin 1994), for instance.

With price setting firms and with tax wedges, the challenge in trying to prove existence of equilibrium is to construct an appropriate mapping of which the fixed points are equilibria. The proof in section 3.2 introduces an auxiliary optimization problem for households. This problem maps artificially constructed prices and incomes into optimal choices for leisure and consumption aggregates. The households in the “true” model do not actually solve the auxiliary problem. The problem is introduced to align households’ true first order conditions (that are impacted by the distortions) with conditions that are completely standard.
The paper proceeds as follows. Section 2 presents the model and the optimization problems of the households and firms. Section 3 gives the definition of equilibrium and then proves existence. A conclusion will follow in section 4 [but is not yet written].

2 Model

The economy has \( H \) households labeled \( h = 1, \ldots, H \) and \( I \) monopolistically competitive industries labeled \( i = 1, \ldots, I \). Within industry \( i \) the mass of firms is \( n_i > 0 \), and these firms are labeled \( j \in [0, n_i] \). Each firm produces a distinct variety of output with quantity denoted \( q_i(j) \), or simply \( q_{ij} \). Free entry in each of the \( I \) industries will drive profits to zero. Household \( h \) is endowed with \( L_h \) units of labor, the only factor of production; \( \ell^h \) is consumption of leisure so \( L^h - \ell^h \) is labor supply. The government taxes labor income at the rate \( t_0 < 1 \). The sales tax rate in industry \( i \) is \( t_i \); if negative, it is a subsidy but we impose \( t_i > -1 \). Each household pays a uniform head tax of \( T \) (subsidy if negative). The government provides \( g \) units of a pure public good (measured in units of labor).

2.1 Households

The preferences of the households are based on Dixit and Stiglitz (1977). Household \( h \) chooses leisure \( \ell^h \geq 0 \) and consumption \( q^h_i(j) \geq 0 \) (or simply \( q^h_{ij} \)) of the output of firm \( j \in [0, n_i] \) in industry \( i \geq 1 \) in order to

\[
\text{maximize } U^h(\ell^h, Y^h_1, \ldots, Y^h_I; g)
\]

\[
\text{subject to } Y^h_i := \int_0^{n_i} w^h_i(q^h_i(j))dj
\]

\[
\sum_{i=1}^I \int_0^{n_i} (1 + t_i)p_i(j)q^h_i(j) dj \leq (1 - t_0)w(L^h - \ell^h) - T + \pi^h
\]

where \( Y^h_i \) is an aggregator for industry \( i \) consumption; \( p_i(j) \) (or simply \( p_{ij} \)) is the price charged by firm \( j \) in industry \( i \); \( w \) is the wage rate; and \( \pi^h \) is dividend income which will be zero in a free entry equilibrium. Each household takes as given the values of all policy variables (taxes and also the public good \( g \)).

The following assumptions are imposed on the households’ utility functions.

Assumption 1

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1. A perfectly competitive industry can be approximated by having small fixed costs, constant marginal costs, and a large intra-industry elasticity of substitution.

2. One consequence of Dixit–Stiglitz type preferences is that households exhibit a love for variety. They get more utility if they can spread their consumption across more varieties. Appendix A relaxes this assumption by allowing for the possibility that some households may love variety while others might not.
(a) For each value of \( g \) and for all \( h \geq 1 \), the function \((\ell, Y_1, \ldots, Y_I) \mapsto U^h(\ell, Y_1, \ldots, Y_I; g)\) is defined on \( \mathbb{R}^{I+1}_+ \). It is a continuous function that is monotonically strictly increasing in each of its arguments. On \( \mathbb{R}^{I+1}_+ \), it is strictly quasi-concave and twice continuously differentiable with non-vanishing gradient.

(b) Suppose \((\ell, Y_1, \ldots, Y_I) \in \mathbb{R}^{I+1}_+ \) and \( 0 \in \{\hat{\ell}, \hat{Y}_1, \ldots, \hat{Y}_I\} \). Then for each value of \( g \) and for all \( h \geq 1 \), \( U^h(\ell, Y_1, \ldots, Y_I; g) > U^h(\hat{\ell}, \hat{Y}_1, \ldots, \hat{Y}_I; g) \).

Assumption 2 For all \( h \geq 1 \) and all \( i \geq 1 \), \( u^h_i \) is defined and continuous on \( \mathbb{R}_+ \) and is \( C^3 \) on \( \mathbb{R}_{++} \); \( u^h_i \) is strictly increasing and strictly concave; \( u^h_i(q) \uparrow \infty \) as \( q \downarrow 0 \) and \( u^h_i(q) \downarrow 0 \) as \( q \uparrow \infty \); \( u^h_i(0) = 0 \).

Assumption 3 For all \( h \geq 1 \) and all \( i \geq 1 \),

(a) \(-1 < \inf_{q > 0} \frac{qu^{h',\prime'}(q)}{u^h_i(q)} \leq \sup_{q > 0} \frac{qu^{h',\prime'}(q)}{u^h_i(q)} < 0 \), and

(b) \( q \mapsto qu^{h',\prime'}(q)/u^h_i(q) \) is continuous at \( q = 0 \).

Assumption [1]a) requires that the utility functions \( U^h \) satisfy standard conditions. Assumption [1]b) states that any indifference surface of \( U^h \) that has a non-empty intersection with the interior of the non-negative orthant is in fact contained entirely within the interior of the non-negative orthant. This will rule out corner solutions to utility maximization problems. Assumption [2] contains smoothness and concavity conditions for the functions \( u^h_i \). Assumption [3] will ensure that all households have elastic demand for all varieties of all goods, with the elasticity strictly bounded away from unity and infinity. This will be used when solving the profit maximization problems for the monopolistically competitive firms. Assumptions [2] and [3] are satisfied, for example, by the mixture of constant elasticity functions \( u^h_i(q) = A^h_i q^{1-1/\sigma^h_i} + q^{1-1/\rho^h_i} \) for \( A^h_i \geq 0 \), \( \sigma^h_i > 1 \), and \( \rho^h_i > 1 \).

Let \( M^h := (1 - t_0) w L^h - T + \pi^h \). This is household \( h \)'s wealth/income that it takes as given. Households also take tax rates and prices as given. In principle, we would like to solve the households’ problem for any possible configuration of producer prices \( p_{ij} \) for \( i \geq 1 \) and \( j \in [0, n_i] \). That is, since firms are free to choose their prices as they see fit, the households should be prepared to face any possible prices. We would then derive the households’ demand functions in the most general sense and solve for equilibrium prices. However, if we allow for completely arbitrary price functions \( j \mapsto p_{ij} \ (i \geq 1) \), the households’ optimization problems may be impossible to analyze. So for the sake of analytical tractability, we will need to impose

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Footnote 3: For this CES mixture example, if we drop the industry subscript \( i \) and the household superscript \( h \), we have the following two equivalent expressions: \(-qu''(q)/u'(q) = 1/\sigma + (1 - 1/\rho)/(1/\rho - 1/\sigma) / [A(1 - 1/\sigma)q^{1/\rho - 1/\sigma} + 1 - 1/\rho] \) and \(-qu''(q)/u'(q) = 1/\rho + A(1 - 1/\sigma)(1/\sigma - 1/\rho) / [A(1 - 1/\sigma) + (1 - 1/\rho)q^{1/\sigma - 1/\rho}] \). From the first of these expressions we see that if \( 1/\rho > 1/\sigma \) then \(-qu''(q)/u'(q) \) declines monotonically with limiting values of \( 1/\rho \) as \( q \to 0 \) and \( 1/\sigma \) as \( q \to \infty \). And from the second expression, if \( 1/\sigma > 1/\rho \) we also see that \(-qu''(q)/u'(q) \) declines monotonically, now with limiting values of \( 1/\sigma \) as \( q \to 0 \) and \( 1/\rho \) as \( q \to \infty \).
some structure on these price functions, but as little as possible. Assumption 5 in Appendix B formally deals with this issue. In short, we require the price functions \( j \mapsto p_{ij} \) to be measurable and integrable. This seems to strike a reasonable balance between tractability and generality. If we were to move further toward tractability and assume that prices \( p_{ij} \) were a continuous function of \( j \in [0, n_i] \), we would then be imposing enormous restrictions on firm behavior: no individual firm \( j' \) would be able to deviate and change its price away from \( p_{ij'} \) for this would break the continuity.

Let subscripts denote partial derivatives of \( U^h \). Specifically, \( U^h_{0i}(\ell, Y_1, \ldots, Y_I; g) := \frac{\partial}{\partial \ell} U^h(\ell, Y_1, \ldots, Y_I; g) \) and \( U^h_{i}(\ell, Y_1, \ldots, Y_I; g) := \frac{\partial}{\partial \ell} U^h(\ell, Y_1, \ldots, Y_I; g) \) for \( i \geq 1 \). If all after-tax prices and \( M^h \) are strictly positive, the first order necessary conditions for household \( h \)'s utility maximization problem are as follows\(^4\)

\[
\frac{U^h_{0i}(\ell^h, Y^h_1, \ldots, Y^h_I; g)}{(1-t_0)w} = \frac{U^h_{i}(\ell^h, Y^h_1, \ldots, Y^h_I; g)q_{ij}^h}{(1+t_i)p_{ij}} \quad i \geq 1, \quad \text{almost all } j \in [0, n_i] \tag{1}
\]

\[
(1-t_0)w\ell^h + \sum_{i=1}^{n_i} (1+t_i)p_{ij}q_{ij}^h \, dj = M^h. \tag{2}
\]

From \(^1\),

\[
q_{ij}^h = (u^{h'})^{-1} \left( \frac{(1+t_i)p_{ij} U^h_{0i}(\ell^h, Y^h_1, \ldots, Y^h_I; g)}{(1-t_0)w} \frac{U^h_{i}(\ell^h, Y^h_1, \ldots, Y^h_I; g)}{U^h_{0i}(\ell^h, Y^h_1, \ldots, Y^h_I; g)} \right) \quad i \geq 1, \quad \text{almost all } j \in [0, n_i]. \tag{3}
\]

This is household \( h \)'s demand for the output of firm \( ij \) conditional on the consumption aggregators and leisure. It will be an ingredient in the firm’s profit maximization problem.

In equilibrium we will focus attention on the symmetric case where, in each industry \( i \), all firms charge the same price. That is, \( p_{ij} = p_i \) for all \( j \in [0, n_i] \). Then under assumptions \(^1\) and \(^2\) any solution to equations \(^1\) and \(^2\) solves household \( h \)'s utility maximization problem.

\[2.2\] Firms

Since labor is the only factor of production, the firms’ technologies can be described with cost functions \( C_i(q) \) that give the labor input required to produce \( q \) units of output. Each of the \( n_i \) firms in industry \( i \) has the same technology.

**Assumption 4** For all \( i \geq 1 \), \( C_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). On \( \mathbb{R}_+ \) each \( C_i \) is twice continuously differentiable, strictly increasing, and convex. \( C_i(0) = 0, C_i(0^+) := \lim_{q \to 0} C_i(q) > 0 \), and \( d[qC_i'(q)/C_i(q)]/dq > 0 \) for all \( q > 0 \).

Assumption \(^4\) will help to ensure that the firms’ profit functions are well-behaved. The assumption of fixed costs, \( C_i(0^+) > 0 \), is consistent with monopolistic competition.\(^5\) The last condition states that the ratio

\[^4\]Since the households’ problem has a continuum of choice variables we should be careful with the conditions for optimality. Appendix B addresses these concerns.

\[^5\]The cost function \( C_i \) is discontinuous at \( q = 0 \). This might raise concerns about existence of equilibrium. However, the existence proof in section 3.2 establishes a lower bound \( q^*_i > 0 \) such that we can focus all attention on quantity choices \( q_i \geq q^*_i \): the firms in industry \( i \) will choose quantities that are bounded away from the discontinuity.
of marginal costs to average costs is increasing in output. These restrictions will be satisfied, for example, if
\[ C_i(q) = \begin{cases} 0 & \text{if } q = 0 \\ F_i + c_i q & \text{if } q > 0. \end{cases} \]

In order to solve the firms’ profit maximization problems we need to derive some properties of the demand curves that they face. In principle, those demand curves are determined by (3). A complication arises from the term “almost all” in (3). A particular firm \( j \) cannot know if it is a member of the exceptional set where demand is not given by the equation in (3). We will address this issue by assuming that firms behave as Bayesians. They maximize expected profits, with the expectation taken over the form of demand curve that they face (just as one might do in a Bayesian game). Since demand curves have the form shown in (3) with probability one, these expected profits coincide with the deterministic profits if we delete “almost all” from (3).

Since each firm is infinitesimally small, it has no influence on the consumption aggregators \( Y^h \). Then from (3) we can add across households and get the demand curve faced by firm \( j \) in industry \( i \):
\[ q_{ij} = \sum_{h=1}^{H} q_{ij}^h = \sum_{h=1}^{H} (u^{h^*})^{-1} \frac{(1 + t_i)p_{ij}^h U_0^h(\ell_i^h, Y^{h^*}_1, \ldots, Y^{h^*}_I; g)}{(1 - t_0)w U_i^h(\ell_i^h, Y^{h^*}_1, \ldots, Y^{h^*}_I; g)}, \tag{4} \]
and recall that \( p_{ij} \) is the producer price under the control of firm \( ij \). Due to the heterogeneity across households, the demand curves in (4) can be unruly and this can lead to complications in the analysis of the firms’ profit maximization problems. In principle we could take the profit function of firm \( ij \), \( p_{ij} q_{ij} - wC_i(q_{ij}) \), substitute for \( q_{ij} \) from (4), and maximize with respect to \( p_{ij} \). However, we get cleaner results by inverting the demand curve to get \( p_{ij} \) as a function of \( q_{ij} \) and then maximizing profits with respect to \( q_{ij} \). This requires some preliminary steps, which we now turn to.

Other than \( p_{ij} \), all other variables on the right hand side of (4) are beyond the control of the (infinitesimally small) firm. With this in mind, let
\[ \theta_i^h := \frac{1 + t_i}{(1 - t_0)w} \frac{U_0^h(\ell_i^h, Y^{h^*}_1, \ldots, Y^{h^*}_I; g)}{U_i^h(\ell_i^h, Y^{h^*}_1, \ldots, Y^{h^*}_I; g)}. \tag{5} \]
Each firm in industry \( i \) takes \( \theta_i^h \) as exogenous for each \( h \). Then (3) can be written as
\[ q_{ij}^h = (u^{h^*})^{-1}(\theta_i^h p_{ij}). \tag{6} \]

Thus each firm \( j \) in industry \( i \) faces the same demand curve from household \( h \). And since all firms in the same industry have identical technologies, without loss of generality we can drop the subscript \( j \). For notational

\[ ^6 \text{I.e., for any probability measure that is absolutely continuous with respect to Lebesgue measure.} \]

\[ ^7 \text{The two approaches — maximization with respect to } p_{ij} \text{ or with respect to } q_{ij} \text{ — are ultimately equivalent. The Bertrand versus Cournot distinction does not apply here since there are no strategic interactions between firms.} \]

\[ ^8 \text{From (4), household } h \text{'s elasticity of demand for good } ij, \ (p_{ij}/q_{ij}^h)(\partial q_{ij}^h/\partial p_{ij}), \text{ is } u^{h^*}(q_{ij}^h)/[q_{ij}^h u^{h''}(q_{ij}^h)]. \text{ The expression that appears in assumption } 3 \text{ is the reciprocal of this elasticity.} \]
convenience drop the subscript $i$ as well and, for now, focus on one industry. Define

$$v^h := (u^h)^{-1}$$

$$v(p; \Theta) := \sum_{h=1}^{H} v^h(\theta^h p)$$

where, for purposes of this definition, $\theta^1, \ldots, \theta^H$ are any positive numbers. In equilibrium, $\theta^h$ must coincide with its definition in (5). Then $q^h = v^h(\theta^h p)$ and $q = \sum_{h=1}^{H} q^h = v(p; \Theta)$. That is, $v$ is the demand curve for the firm’s output, written as a function of the price charged, $p$, and the exogenous parameter vector $\Theta$. Given these definitions, the following derivatives can be computed:

$$v^{hi}(x) \equiv \frac{1}{u^{hi} \circ (u^{hi})^{-1}(x)}$$

$$v^{hii}(x) \equiv -\frac{u^{hii} \circ (u^{hi})^{-1}(x)}{(u^{hi} \circ (u^{hi})^{-1}(x))^3}$$

$$\frac{\partial v}{\partial p}(p; \Theta) \equiv \sum_{h=1}^{H} \frac{\theta^h}{u^{hii} \circ (u^{hi})^{-1}(\theta^h p)}$$

$$\frac{\partial^2 v}{\partial p^2}(p; \Theta) \equiv -\sum_{h=1}^{H} \frac{(\theta^h)^2 \cdot u^{hii} \circ (u^{hi})^{-1}(\theta^h p)}{(u^{hi} \circ (u^{hi})^{-1}(\theta^h p))^3}.$$ 

Since each $u^h$ is strictly concave, it follows that $\partial v / \partial p < 0$. Thus, the demand curve $v$ can be inverted to yield the inverse demand curve $P(q; \Theta)$ which is defined implicitly by

$$v(P(q; \Theta); \Theta) \equiv q.$$ 

From this implicit definition, the following derivatives can be computed:

$$\frac{\partial P}{\partial q}(q; \Theta) \equiv \left[ \sum_{h=1}^{H} \theta^h \circ (u^{hi} \circ (u^{hi})^{-1}(\theta^h \cdot P(q; \Theta))) \right]^{-1}.$$  \hspace{1cm} (7)

$$\frac{\partial^2 P}{\partial q^2}(q; \Theta) \equiv \frac{\left( \sum_{h=1}^{H} \theta^h \circ (u^{hi} \circ (u^{hi})^{-1}(\theta^h \cdot P(q; \Theta))) \right)}{[\sum_{h=1}^{H} \theta^h \circ (u^{hi} \circ (u^{hi})^{-1}(\theta^h \cdot P(q; \Theta)))]^3}.$$  \hspace{1cm} (8)

These computations will now be used to verify that the firm’s profit maximization problem has an interior solution, and also to get an explicit form for the first and second order conditions.

Since the firm can observe the aggregate quantity demanded $q$ but it cannot observe the quantities demanded by individual households, it will be convenient to introduce the following notation:

$$Q^h(q; \Theta) := v^h(\theta^h P(q; \Theta)).$$  \hspace{1cm} (9)
These functions satisfy
\[
\sum_{h=1}^{H} Q^h(q; \Theta) \equiv q
\]
and
\[
\theta^h P(q; \Theta) \equiv u^{h'}(Q^h(q; \Theta))
\]
as a consequence of previous definitions. If we compare with (6), in equilibrium the functions \( Q^h(\cdot) \) will take on values that coincide with the quantities demanded by individual households.

Given that real marginal costs satisfy \( 0 < C'(0) < \infty \) and \( C'' \geq 0 \), it follows that if marginal revenue is infinite at zero quantity and zero at infinite quantity\(^9\), then there will be an interior solution to the profit maximization problem. Let \( MR \) denote marginal revenue. For \( q > 0 \),
\[
MR = P(q; \Theta) + q \frac{\partial P}{\partial q}(q; \Theta)
\]
\[
= P(q; \Theta) \left[ 1 + q \frac{P(q; \Theta)}{\sum_{h=1}^{H} u^{h'}(Q^h(q; \Theta))} \right]
\]
\[
= P(q; \Theta) \left[ 1 + 1 \frac{\sum_{h=1}^{H} Q^h}{q} \frac{u^{h'}(Q^h)}{Q^h u^{h''}(Q^h)} \right]
\]
(11)
where, on the last line\(^{10}\), \( Q^h \) is shorthand for \( Q^h(q; \Theta) \).

In order to examine the boundary properties of \( MR \), it will be necessary to examine the boundary properties of \( P(\cdot) \). To evaluate \( \lim_{q \to 0} P(q; \Theta) \) use the definition of \( P(\cdot) \) to get
\[
\lim_{q \to 0} v(P(q; \Theta); \Theta) = 0.
\]
Since \( v = \sum_h v^h \) and since each \( v^h \) is non-negative, we have \( \lim_{q \to 0} v^h(\theta^h P(q; \Theta)) = 0 \) for all \( h \). Since \( v^h := (u^h)^{-1} \); since \( u^h \) is strictly concave and satisfies Inada conditions (assumption 2 from section 2.1); and since \( \theta^h > 0 \) we conclude that
\[
\lim_{q \to 0} P(q; \Theta) = \infty.
\]
Similarly, if \( v \to \infty \) then there must be some \( h \) with \( v^h \to \infty \), so
\[
\lim_{q \to \infty} P(q; \Theta) = 0.
\]
\(^9\)This abuse of terminology is for convenience. These boundary conditions should be stated rigorously as limits.
\(^{10}\)The sum that appears on the last line of (11) is the elasticity of demand. It is the weighted average of each household’s elasticity. See footnote 8.
Under assumption 3 there exist uniform bounds which we can denote $B > b > 1$ such that $-B < \frac{u^h(q^h)}{q^h u^{h''}(q^h)} < -b$ for all $q^h > 0$ and for all $h$. Thus, from the last line of (11), we have $(1 - 1/b)P(q; \Theta) < MR < (1 - 1/B)P(q; \Theta)$ for all $q > 0$. This confirms that marginal revenue satisfies the desired boundary conditions: $MR = \infty$ at zero quantity and $MR = 0$ at infinite quantity. Hence, the profit maximization problem has an interior solution. The first order condition is

$$P(q; \Theta) \cdot \left[1 + \frac{1}{\sum_{h=1}^{H} Q^h \frac{u^h(Q^h)}{Q^h u^{h''}(Q^h)}} \right] - wC'(q) = 0. \quad (12)$$

The second order sufficient condition is $\frac{\partial}{\partial q} (MR - MC) < 0$. From the first line of (11), this is

$$2 \frac{\partial P}{\partial q} (q; \Theta) + q \frac{\partial^2 P}{\partial q^2} (q; \Theta) - wC''(q) < 0.$$  

Since $C$ is convex, the following condition is sufficient:

$$2 \frac{\partial P}{\partial q} (q; \Theta) + q \frac{\partial^2 P}{\partial q^2} (q; \Theta) < 0.$$  

Since $\partial P/\partial q < 0$ this sufficient condition can be written as

$$\frac{2 [P(q; \Theta)]^2}{q^2 [\partial P/\partial q]^2} + \frac{[P(q; \Theta)]^2 \cdot \partial^2 P/\partial q^2}{q [\partial P/\partial q]^3} > 0.$$  

Substitute for $\partial P/\partial q$ from (7) and $\partial^2 P/\partial q^2$ from (8), and use (10) to get

$$\sum_{h=1}^{H} Q^h \frac{u^h(Q^h)}{Q^h u^{h''}(Q^h)} \cdot \left[ \frac{Q^h u^{h''}(Q^h)}{wC'(q)} \right]^2 < 2 \left[ \sum_{h=1}^{H} Q^h \frac{u^h(Q^h)}{Q^h u^{h''}(Q^h)} \right]^2 \quad (13)$$

where, once again, $Q^h$ is shorthand for $Q^h(q; \Theta)$, and in equilibrium it must coincide with the quantity $q^h$ chosen by household $h$.\textsuperscript{11}

[To be completed.]

## 3 Equilibrium

We begin with the definition of equilibrium in section 3.1 then prove existence in section 3.2.

First, however, note that all of the $n_i$ firms in industry $i$ solve the same profit maximization problem so they all choose the same price $p_i$ and the same output quantity $q_i$, determined as in section 2.2. Since all firms in industry $i$ charge the same price, household $h$ demands the same quantity from each of them, $q^h_{ij} = q^h_i$. See [3]. With regard to exceptional sets of measure zero where the equality in (3) is violated, we search

\textsuperscript{11}If price, rather than quantity, is used as the firm’s choice variable, the first order condition is $v(p, \Theta) + [p - wC'(v(p; \Theta))] \frac{\partial}{\partial p} v(p; \Theta) = 0$ and the second order sufficient condition is $2 \frac{\partial v}{\partial p} + p \frac{\partial^2 v}{\partial p^2} - wC'' \cdot [\frac{\partial v}{\partial p}]^2 - wC' \cdot \frac{\partial^2 v}{\partial p^2} < 0$. Since $C'' \geq 0$, a stricter sufficient condition is the following: $2 \frac{\partial v}{\partial p} + p \frac{\partial^2 v}{\partial p^2} - wC' \cdot \frac{\partial^2 v}{\partial p^2} < 0$. If this is satisfied for all $p$, the first order condition uniquely determines the profit maximizing price. However, this sufficient condition mixes together demand and cost functions. Although we could impose assumptions on the model’s primitives to ensure that the condition is satisfied globally, such assumptions would be quite restrictive.
only for equilibria in which these sets are empty. Other equilibria may exist, but our task here is to prove existence in the symmetric case. Then the consumption aggregators for household $h$, $Y^h := \int_0^{n_i} u^h_i(q^h_j) dj$, reduce to $Y^h_i = n_i u^h_i(q^h_i)$. And a similar simplification applies to the integrals in the households’ budget equations (2).

We need to address one other preliminary issue. Since the government faces a budget constraint, it is not possible for all of the policy variables to be exogenous. We will address this by taking all of the tax variables $(t_0, t_1, \ldots, t_I, T)$ as exogenous and making the level of the public good $(g)$ endogenous. As a consequence we will need to ensure that the households’ utility adjusts smoothly to changes in $g$.

Assumption 1(c) For all $h \geq 1$ and for all values of $(\ell, Y_1, \ldots, Y_I)$, $U^h(\ell, Y_1, \ldots, Y_I; g)$ is a continuously differentiable function of $g \in \mathbb{R}$.

Assumption 1(c) allows for the possibility that $g < 0$. This is a technical quirk. Since $g$ is the only endogenous policy variable, and since the government must satisfy its budget, then by default we have no control over the value of $g$ in equilibrium. If the exogenous policy variables include a large fixed income guarantee for all households ($T$ negative and large in magnitude) we could end up with $g < 0$ in equilibrium. One could interpret $g < 0$ as an alarm bell that tells the government to change its tax policy.

3.1 Definition of equilibrium

The conditions for equilibrium are that households maximize utility, firms maximize profits, free entry drives profits to zero, the government satisfies its budget, and the labor market clears. All output markets clear automatically since each firm internalizes the households’ demand for its product when it chooses its output level.

For households, the conditions for utility maximization are

$$U^h_0(\ell^h, n_1 u^h_1(q^h_1), \ldots, n_I u^h_I(q^h_I); g) = U^h_h(\ell^h, n_1 u^h_1(q^h_1), \ldots, n_I u^h_I(q^h_I); g) u^h_i(q^h_i)$$

$$h = 1, \ldots, H$$

$$i = 1, \ldots, I$$

(14)

For firms, the conditions for profit maximization and free entry (zero profits) are

$$p_i + q_i \frac{\partial}{\partial q_i} P_i(q_i; \Theta_i) - w C_i(q_i) = 0$$

$$i = 1, \ldots, I$$

(16)

$$p_i q_i - w C_i(q_i) = 0$$

$$i = 1, \ldots, I$$

(17)

where $q_i := \sum_{h=1}^H q^h_i$ and where $\Theta_i$ and the derivative $\partial P_i/\partial q_i$ are derived from household behavior as in section 2.2. The government’s budget constraint is

$$HT + t_0 w (L - \ell) + \sum_{i=1}^I t_i p_i q_i n_i \geq wg$$

(18)
where $L := \sum_{h=1}^{H} h^b$ and $\ell := \sum_{h=1}^{H} \ell^h$. The resource constraint is

$$g + \ell + \sum_{i=1}^{I} n_i C_i(q_i) \leq L. \quad (19)$$

The government’s budget (18) will be satisfied automatically whenever (15), (17), and (19) hold since (17) implies that $\pi^h = 0$ for all $h$. This is just Walras’ Law. Therefore, (18) can be dropped from the equilibrium conditions. Also, (17) can be used to eliminate $p_i$ from the other equations, and (16) can be replaced with its long form, equation (12) from section 2.2. Then the equilibrium conditions are as follows:

$$U^h(q^h, 0, n_i u^b_i(q^h_i), \ldots, n_I u^b_I(q^h_I); g) = \frac{U^h(q^h, 0, n_i u^b_i(q^h_i), \ldots, n_I u^b_I(q^h_I); g) q_i u^{h'}_i(q^h_i)}{1 - t_0} \frac{(1 + t_i) C_i(q_i)}{1 - t_0} \quad h = 1, \ldots, H \quad i = 1, \ldots, I \quad (20)$$

$$(1 - t_0) q^h + \sum_{i=1}^{I} n_i (1 + t_i) q_i^h C_i(q_i) / q_i = (1 - t_0) h^b - T / w \quad h = 1, \ldots, H \quad (21)$$

$$1 + \left\{ \begin{array}{l}
\frac{H}{h=1} q_i^h \frac{u^{h'}_i(q^h_i)}{q_i^h u^{h''}_i(q^h_i)} = \frac{q_i C'_i(q_i)}{C_i(q_i)} \quad i = 1, \ldots, I
\end{array} \right. \quad (22)$$

$$g + \ell + \sum_{i=1}^{I} n_i C_i(q_i) \leq L \quad (23)$$

with $q_i$, $L$, and $\ell$ defined as above. Note that the wage rate $w$ cancels out from all equilibrium conditions other than the final term on the right hand side of (21) where it deflates the head tax $T$ into real terms. Since the head tax is exogenous and since $w$ appears nowhere else, we shall choose the normalization $w = 1$.

The equilibrium conditions constitute a system of $HI + H + I$ equations plus one weak inequality in the $HI + H + I + 1$ endogenous variables $q_i^h$, $\ell^h$, $n_i$, and $g$ with $I + 2$ exogenous policy variables $t_i, t_0, T$. In order for households to have positive income, the poll tax and the labor income tax rate must be chosen to satisfy

$$T < (1 - t_0) h^b \quad h = 1, \ldots, H \quad (24)$$

where we have used the normalization $w = 1$.

**Definition** An equilibrium is a vector $(q_1^1, \ldots, q_I^h, \ldots, q_I^H, \ell_1, \ldots, \ell^H, n_1, \ldots, n_I, g)$ that satisfies (20) through (23) for given values of $T, t_0, t_1, \ldots, t_I$ that satisfy (24).

### 3.2 Existence of equilibrium

The goal here is to construct a continuous function

$$F : (q_1^1, \ldots, q_I^h, \ldots, q_I^H, n_1, \ldots, n_I, g) \mapsto (\hat{q}_1^1, \ldots, \hat{q}_I^h, \ldots, \hat{q}_I^H, \hat{n}_1, \ldots, \hat{n}_I, \hat{g})$$

12
that maps a non-empty compact convex set $S$ into itself, and such that the fixed points of $F$ readily yield equilibria. For now, the set $S$ will simply be identified by

$$q_i^h \geq 0, \quad \sum_{h=1}^{H} q_i^h \in [q_i^-, q_i^+] \text{ with } q_i^- > 0, \quad n_i \in [0, n_i^+], \quad g \in [g^-, g^+] .$$

The superscripts $+$ and $-$ indicate bounds that will be specified later during the construction of $F$.

The construction begins with any point $((q_i^h), (n_i), g)$ in $S$. Use this point to define

$$q_i := \sum_{h=1}^{H} q_i^h \geq q_i^- > 0 \quad (25)$$

$$T_0^h := 1 - t_0 \quad (26)$$

$$T_i^h := \begin{cases} 
\frac{(1 + t_i)C_i(q_i)}{q_i u_i^h(q_i)} & \text{if } q_i^h \neq 0 \\
0 & \text{if } q_i^h = 0
\end{cases} \quad (27)$$

$$M^h := \sum_{i=1}^{I} n_i T_i^h [u_i^h(q_i) - q_i^h u_i^{ht}(q_i^h)] + T_0^h L^h - T. \quad (28)$$

In (28), if $q_i^h = 0$ then replace the indeterminate form $q_i^h u_i^{ht}(q_i^h)$ with its limiting value of zero.\footnote{From assumption 2, $u_i^h$ is concave and $u_i^h(0) = 0$ so $0 = u_i^h(0) \leq u_i^h(q) + (q - q)u_i^h(q)$ for all $q > 0$. Thus the term in square brackets in (28) is non-negative and if we take the limit as $q$ tends to zero we get $\lim_{q \to 0} qu_i^{ht}(q) = 0$ since, again, $u_i^h(0) = 0$.}

Note that the operations used to construct the $T$s and $M$s are all continuous on $S$, and $M^h > 0$ as a consequence of footnote 12 and condition (24) in section 3.1. Next, for each household $h$ solve the following auxiliary problem. The household chooses $(\ell^h, Y_1^h, \ldots, Y_I^h) \in \mathbb{R}^{I+1}$ to

maximize $U^h(\ell^h, Y_1^h, \ldots, Y_I^h; g)$

subject to $T_0^h \ell^h + \sum_{i=1}^{I} T_i^h Y_i^h \leq M^h$

$$Y_i^h \leq 1 + \left[ \sup_{q > 0} \frac{u_i^h(q)}{C_i(q)} \right] \cdot \left[ \frac{(1 - t_0)L - HT}{1 + t_i} \right]. \quad (29)$$

In order to be sure the problem has a solution an upper bound on $Y_i^h$ must be specified since $T_i^h$ could be zero. The motivation for the specific bound given above will be discussed later. The supremum that appears in that bound is finite since $u_i^h$ is concave — hence bounded from above by an affine function — while $C_i$ is convex and strictly increasing with $C_i(0^+) > 0$ — hence $C_i$ is bounded from below by an affine function with strictly positive intercept and strictly positive slope. For $\ell^h$, the budget constraint provides an upper bound: $\ell^h \leq M^h / T_0^h$. The auxiliary problem yields a unique solution

$$\ell^h(T^h, M^h; g) > 0, \quad Y^h(T^h, M^h; g) \gg 0$$
which is a continuous function of \((T^h, M^h; g)\), and hence a continuous function on \(S\). If the upper bounds on \(Y^h_i\) are not binding, then the first order conditions that characterize the solution will be

\[
T_i^h T_0^h(e^h, Y_1^h, \ldots, Y_l^h; g) = T_0^h T_i^h(e^h, Y_1^h, \ldots, Y_l^h; g)
\]

(30)

\[
T_0^h e^h + \sum_{i=1}^l T_i^h Y_i^h = M^h.
\]

(31)

Equations (30) and (31) will coincide with the equilibrium conditions (20) and (21) if \(Y_i^h = n_i u_i^h(q_i^h)\). (Recall the normalization \(w = 1\) in (21).)

The next step is to take the \(Y_i^h\) values that were produced by the auxiliary problem, and from these construct new values \(\hat{q}_i^h\) and \(\hat{n}_i\) that satisfy \(Y_i^h = \hat{n}_i u_i^h(\hat{q}_i^h)\) and also satisfy a version of the equilibrium condition (22). Then the output of the function \(F\) will be these new values \(\hat{q}_i^h\) and \(\hat{n}_i\), together with

\[
\hat{g} := L - \sum_{h=1}^H e^h(T^h, M^h; g) - \sum_{i=1}^l n_i C_i(q_i).
\]

(32)

This setting for \(\hat{g}\) will cause the resource constraint (23) to bind.

Proceed now by finding a value \(\hat{q}_i\) to satisfy

\[
\frac{\hat{q}_i C_i'(\hat{q}_i)}{C_i(\hat{q}_i)} = 1 + \left( \sum_{h=1}^H \frac{q_i^h}{\hat{q}_i} \frac{u_i^h(q_i^h)}{q_i^h u_i^h''(q_i^h)} \right)
\]

(33)

cf. (22). Notice that the new value \(\hat{q}_i\) appears only on the left hand side of (33). On the right hand side, if \(q_i^h = 0\), replace the corresponding term in the sum with its limiting value of zero. See assumption 3.

Equation (33) has a unique solution for \(\hat{q}_i\). The reason is as follows. Under assumption 3, the right hand side of (33) is strictly bounded between 0 and 1. The left hand side of (33) defines a differentiable, strictly increasing function of \(\hat{q}_i\) (under assumption 4), which approaches zero as \(\hat{q}_i\) approaches zero and increases toward a supremum of at least unity.\(^{13}\) Thus, not only is there a unique solution for \(\hat{q}_i\), but the solution is also a differentiable function of the \(q_i^h\) values, and furthermore, the boundedness of the right hand side of (33) yields bounds on \(\hat{q}_i\):

\[
\hat{q}_i \in (q_i^- , q_i^+ )
\]

with \(q_i^- > 0\).

Next, find a value \(\hat{n}_i\) that satisfies

\[
\sum_{h=1}^H (a_i^h)^{-1} (Y_i^h/\hat{n}_i) = \hat{q}_i
\]

(34)

where the \(Y_i^h\)s are the output of the auxiliary problem. Equation (34) is a consistency condition that requires individual household demands to sum to the aggregate demand in each industry. The left hand side of (34)

\(^{13}\)Under assumption 4, \(C_i\) is convex on \(R_+\). Therefore, for all \(q > 0\),

\[
C_i(0^+) \geq C_i(q) + (0 - q)C_i'(q) \quad \text{and} \quad C_i(q) \geq C_i(0^+) + (q - 0)C_i'(0^+) \geq qC_i'(0^+).
\]

where the last inequality follows from the assumption of fixed costs, \(C_i(0^+) > 0\). Combine these to get \(qC_i'(q)/C_i(q) \geq 1 - C_i(0^+)/C_i(q) > 1 - C_i(0^+)/[qC_i'(0^+)]\). Take limits to get \(\lim_{q \to \infty} qC_i'(q)/C_i(q) \geq 1\).
defines a function of \( \hat{n}_i \) which has limit value of zero as \( \hat{n}_i \) approaches infinity. It is a downward sloping function of \( \hat{n}_i \). Thus, as \( \hat{n}_i \) gets smaller, the function increases, eventually blowing up to infinity when \( Y^h_i/\hat{n}_i \) reaches the supremum of the range of \( u^h_i \) for any \( h \) (which might not occur until \( \hat{n}_i = 0 \)). Therefore, there is a unique \( \hat{n}_i > 0 \) that solves (34), and this solution is a differentiable function of \( \hat{q}_i \) and the \( Y^h_i \)'s. In addition, (34) yields the upper bound \( n^+_i \) since \( \hat{q}_i \) is bounded from below by \( q^-_i > 0 \) and the \( Y^h_i \)'s are explicitly bounded from above in the auxiliary problem.

The output of the function \( F \) is completed by specifying

\[
\hat{q}^h_i = (u^h_i)^{-1}(Y^h_i/\hat{n}_i)
\]  

which yields \( Y^h_i = \hat{n}_iu^h_i(\hat{q}^h_i) \). Also, \( \hat{q}^h_i > 0 \) and \( \sum_{h=1}^{H} \hat{q}^h_i \in (q^-_i, q^+_i) \) by construction.

The final ingredient in the definition of \( F \) is the bound for \( g \) (i.e., \( g^- \leq g \leq g^+ \)). From the auxiliary problems \( \ell^h \leq M^h/T^h_0 \), thus (32) yields

\[
L - \sum_{h=1}^{H} M^h/T^h_0 - \sum_{i=1}^{I} n^+_i C_i(q^+_i) \leq \hat{g} \leq L.
\]  

From (28),

\[
M^h \leq \sum_{i=1}^{I} n^+_i T^h_i u^h_i(q^+_i) + T^h_0 L^h - T
\]

and from (27)

\[
T^h_i \leq \frac{(1 + t_i)C_i(q^+_i)}{q^-_i u^h_i(q^+_i)}
\]

and we have the definition \( T^h_0 := 1 - t_0 \) from (26). Hence,

\[
M^h/T^h_0 \leq \sum_{i=1}^{I} n^+_i \frac{(1 + t_i)C_i(q^+_i)u^h_i(q^+_i)}{(1 - t_0)q^-_i u^h_i(q^+_i)} + L^h - T/(1 - t_0)
\]

and we can substitute into (36) to get bounds that we can impose on \( g \).

The function \( F \) has a fixed point as a consequence of Brouwer’s theorem. Fixed points are distinguished by the equality \( \hat{x} = x \) for all variables \( x \) that appear in the construction of \( F \). The goal now is to verify that the fixed points are equilibria. First note that these fixed points do not include values for \( \hat{\theta} \). However, values for \( \ell^h \) can easily be derived from the auxiliary problems.

There are four equilibrium conditions that must be checked: (20), (21), (22), and (23). Condition (23) is satisfied by the construction of \( \hat{g} \) in (32). Condition (22) is satisfied due to (33) since \( \hat{q}_i = \sum_{h=1}^{H} \hat{q}^h_i \) as a result of (34) and (35).

The task that remains is to check conditions (20) and (21). Both of these will be satisfied by the first order conditions (30) and (31) from the auxiliary problems, provided that the upper bounds on \( Y^h_i \) are not binding. Thus, the analysis that follows is directed toward verifying that these bounds do not bind at a fixed point.
For each $h$, the budget constraint in the auxiliary problem states that

$$T_0^h \ell^h + \sum_{i=1}^{I} T_i^h y_i^h \leq M^h.$$

Substitute for $M^h$ from its definition (28), and for $Y_i^h$ from (35), and rearrange to get

$$T_0^h \ell^h + \sum_{i=1}^{I} n_i T_i^h q_i^h u_i^h (q_i^h) \leq T_0^h L^h - T.$$

Next, substitute for the $T$s from their definitions in (26) and (27). Recall that $q_i = \sum_{h=1}^{H} q_i^h \geq q_i^- > 0$. Also, $q_i^h > 0$ from (35) since the value of $Y_i^h$ from the auxiliary problem is strictly positive. Thus,

$$(1 - t_0) \ell^h + \sum_{i=1}^{I} n_i (1 + t_i) C_i(q_i) q_i^h / q_i \leq (1 - t_0) L^h - T.$$

Sum this from $h = 1$ to $H$. Since all terms on the left hand side are non-negative, this provides the bound

$$n_i C_i(q_i) \leq \frac{(1 - t_0) L - HT}{1 + t_i}$$

which must hold at any fixed point. This bound will now be used to derive a bound on $Y_i^h$ that must hold at any fixed point. Since $q_i^h \leq q_i$, it follows that

$$Y_i^h \leq n_i u_i^h (q_i)$$

$$= \frac{u_i^h (q_i)}{C_i (q_i)} \cdot [n_i C_i (q_i)]$$

$$\leq \left[ \sup_{q > 0} \frac{u_i^h (q)}{C_i (q)} \right] \cdot \frac{(1 - t_0) L - HT}{1 + t_i}$$

where the last line follows from (37). Again, this bound must hold at any fixed point. Therefore, by comparing (29) with (38), it follows that at any fixed point the upper bound on $Y_i^h$ does not bind on the auxiliary problems. This completes the proof that an equilibrium exists.

4 Conclusion

[To be added.]
Appendix A Generalization of the extent of the love for variety

Since the functions $u^h_i$ satisfy assumption $2$ (strictly concave, and no utility if consumption is zero) we have the standard result that the consumption aggregators $Y^h_i := f^h_{n_i}(q^h_i(j))dj$ exhibit a love for variety. That is, if a total amount of consumption $Q$ is divided equally among the $n_i$ varieties so that $q^h_i(j) = Q/n_i$ and hence $Y^h_i = n_i u^h_i(Q/n_i)$, then $Y^h_i$ is an increasing function of $n_i$. The consumer gets more utility when the number of varieties in industry $i$ goes up, even as the total consumption in industry $i$ stays the same.

We can introduce more flexibility regarding the households’ love for variety if we generalize the consumption aggregators to the following:

$$Y^h_i := f^h_{n_i}(n_i) \int_0^{n_i} u^h_i(q^h_i(j))dj.$$  

The functions $f^h_i$ are continuously differentiable on $\mathbb{R}_{++}$ where they take on strictly positive values. We will impose a further condition below. With this specification for $Y^h_i$, household $h$ exhibits a love for variety in industry $i$ if $f^h_{n_i}(n_i) u^h_i(Q/n_i)$ is an increasing function of $n_i$, or equivalently if

$$1 - (Q/n_i)u^h_i'(Q/n_i)/u^h_i(Q/n_i) + n_i f^h_{n_i}'(n_i)/f^h_{n_i}(n_i) > 0.$$  

Conversely, the household has distaste for variety if this expression has the opposite sign.

The existence of equilibrium proof in section $3.2$ still applies, with only minor adjustments, if the $f^h_i$ functions satisfy an additional condition. In the symmetric equilibria under consideration, we have $q^h_i(j) = q^h_i$ for all varieties $j$ in industry $i$, so $Y^h_i = f^h_{n_i} n_i u^h_i(q^h_i)$ from which we get $q^h_i = (u^h_i)^{-1}(Y^h_i/[n_i f^h_{n_i}(n_i)])$, and aggregating across households,

$$q^h_i = \sum_{h=1}^H (u^h_i)^{-1}(Y^h_i/[n_i f^h_{n_i}(n_i))].$$

This is the generalization of (34) in the existence proof. We need this equation to have a unique solution for $n_i$, given the values of $q_i$ and $(Y^h_i)$, and this will be true if we have appropriate monotonicity and boundary conditions for the function $n \mapsto n f^h_i(n)$ for all households. So for each industry $i$, we assume that for all $h \geq 1$,

$$\lim_{n \to 0} n f^h_i(n) = 0, \quad \lim_{n \to \infty} n f^h_i(n) = \infty, \quad 1 + n f^h_{i}'(n)/f^h_i(n) > 0 \quad \text{for all } n > 0.$$  

For example, this is satisfied by $f^h_i(n) = n^{c^h_i-1}$ with $c^h_i > 0$ for all $h$. Furthermore, household $h$ will have distaste for variety in industry $i$ if $c^h_i$ is sufficiently small.

Appendix B Optimality conditions for the households’ problem

In this appendix we derive necessary conditions that must be satisfied by a solution to the households’ utility maximization problem. Recall the problem of household $h$. The choice variables are leisure $\ell^h \geq 0$ and the
quantity of consumption \( q^h_i(j) \geq 0 \) of the output of firm \( j \) in industry \( i \), where \( j \in [0, n_i] \) and \( i \in \{1, 2, \ldots, I\} \).

The problem is to

\[
\begin{align*}
\text{maximize} & \quad U^h(\ell^h, Y^h_1, \ldots, Y^h_I; g) \\
\text{subject to} & \quad Y^h_i := \int_0^{n_i} u^h_i(q^h_i(j))dj \\
& \quad \sum_{i=1}^I \int_0^{n_i} (1 + t_i)p_i(j)q^h_i(j)dj \leq (1 - t_0)w(L^h - \ell^h) - T + \pi^h.
\end{align*}
\]

Recall that \( (1 - t_0)w > 0, 1 + t_i > 0 \) for all \( i \geq 1 \), and \( M^h := (1 - t_0)wL^h - T + \pi^h > 0 \). We also make the following

**Assumption 5** For each \( i \geq 1 \), the price function \( j \mapsto p_i(j) \) is measurable, is strictly positive for almost all \( j \in [0, n_i] \), and is integrable: \( \int_0^{n_i} p_i(j)dj < \infty \).

When we address existence of equilibrium in section 3.2, we only consider the symmetric case where all firms in industry \( i \) charge the same price, in which case this assumption is satisfied trivially. However, at this stage, any restriction imposed on the price functions is effectively a restriction imposed on the ability of monopolistic firms to set prices, and these restrictions should be kept to a minimum.

It is implicit in the formulation of household \( h \)'s problem that all of the integrals converge. So for each \( i \), \( j \mapsto q^h_i(j) \) must be a measurable function of \( j \in [0, n_i] \) with \( \int_0^{n_i} u^h_i(q^h_i(j))dj < \infty \) and \( \int_0^{n_i} p_i(j)q^h_i(j)dj < \infty \).

Let \( \bar{\ell}^h, (\bar{q}^h_i(j)) \) denote a solution to household \( h \)'s utility maximization problem. For this solution, let \( \bar{E}^h_i := \int_0^{n_i} (1 + t_i)p_i(j)\bar{q}^h_i(j)dj \) be \( h \)'s total expenditure on the goods from industry \( i \) and let \( \bar{Y}^h_i := \int_0^{n_i} u^h_i(\bar{q}^h_i(j))dj \) be the optimal value of \( h \)'s consumption aggregator for industry \( i \). Since \( U^h \) is monotonically strictly increasing in \( \ell^h \) and in the consumption aggregators \( Y^h_i \) (assumption 1(a)), the usual two stage budgeting procedure applies: for each \( i \), \( (\bar{q}^h_i(j)) \) is a solution to

\[
\begin{align*}
\text{maximize} & \quad (\bar{q}^h_i(j))_{\geq 0} \\
\text{subject to} & \quad \int_0^{n_i} (1 + t_i)p_i(j)\bar{q}^h_i(j)dj \leq \bar{E}^h_i.
\end{align*}
\]

Our task is to derive a condition that characterizes the solution to this problem. We will draw on results from optimization theory (e.g., chapter 2 of Fleming and Rishel 1975), adapted to the problem at hand.

For ease of notation, drop the subscript \( i \) and the superscript \( h \). The characterization of optimality will

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\(^{14}\) In order to directly apply the results from chapter 2 of Fleming and Rishel (1975), we would have to assume \( j \mapsto p_i(j) \) is a continuous function of \( j \in [0, n_i] \) for each \( i \geq 1 \). As previously stated, this would be too strong an assumption.
apply on a subset of \( j \in [0, n] \) of full measure. Let
\[
S_{n1} := \left\{ j \in (0, n) : \frac{d}{dj} \int_0^j u(\bar{q}(j')) dj' = u(\bar{q}(j)) \right\}
\]
\[
S_{n2} := \left\{ j \in (0, n) : \frac{d}{dj} \int_0^j p(j') dj' = p(j) \right\}
\]
\[
S_{n3} := \left\{ j \in (0, n) : \frac{d}{dj} \int_0^j p(j') \bar{q}(j') dj' = p(j)\bar{q}(j) \right\}.
\]
All of these sets have full measure \( n \). See, e.g., corollary 3.33 of Folland (1999). Hence \( S_n := S_{n1} \cap S_{n2} \cap S_{n3} \) also has measure \( n \). Let \( K \) be a positive integer and let \( j_1, j_2, \ldots, j_K \) be points in \( S_n \). Without loss of generality, \( 0 < j_1 \leq j_2 \leq \cdots \leq j_K < n \). Also let \( q_1, q_2, \ldots, q_K \) be non-negative numbers. Given these values for the \( j_k \)s and the \( q_k \)s we introduce the following alternative consumption function:
\[
q(j; \delta_1, \delta_2, \ldots, \delta_K) := \begin{cases} 
q_1 & \text{if } j \in [j_1, j_1 + \delta_1) \\
q_2 & \text{if } j \in [j_2 + \delta_1, j_2 + \delta_1 + \delta_2) \\
\vdots & \\
q_K & \text{if } j \in [j_K + \delta_1 + \cdots + \delta_{K-1}, j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K) \\
\bar{q}(j) & \text{otherwise}
\end{cases}
\]
defined for \( j \in [0, n] \), and the \( \delta_s \) must lie in the simplex where \( \delta_k \geq 0 \) for all \( k \) and \( \delta_1 + \cdots + \delta_K \leq n - j_K \). Note that \( q(j; 0, 0, \ldots, 0) = \bar{q}(j) \).

Since the budgetary cost of the optimal consumption function is \( \int_0^n (1 + t)p(j)\bar{q}(j) dj = \bar{E} \), it follows that the budgetary cost of the alternative consumption function satisfies
\[
\int_0^n (1 + t)p(j)q(j; \delta_1, \ldots, \delta_K) dj - \bar{E}
\]
\[
= \int_{j_1}^{j_1 + \delta_1} (1 + t)p(j)(q_1 - \bar{q}(j)) dj + \cdots + \int_{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K}^{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K} (1 + t)p(j)(q_K - \bar{q}(j)) dj.
\]
and hence the alternative satisfies the budget constraint if and only if
\[
e(\delta_1, \ldots, \delta_K) := \int_{j_1}^{j_1 + \delta_1} (1 + t)p(j)(q_1 - \bar{q}(j)) dj + \cdots + \int_{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K}^{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K} (1 + t)p(j)(q_K - \bar{q}(j)) dj \leq 0. \tag{39}
\]
The difference between the alternative’s consumption aggregator and the optimal consumption aggregator is
\[
y(\delta_1, \ldots, \delta_K) := \int_0^n u(q(j; \delta_1, \ldots, \delta_K)) dj - \int_0^n u(\bar{q}(j)) dj
\]
\[
= \int_{j_1}^{j_1 + \delta_1} [u(q_1) - u(\bar{q}(j))] dj + \cdots + \int_{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K}^{j_K + \delta_1 + \cdots + \delta_{K-1} + \delta_K} [u(q_K) - u(\bar{q}(j))] dj. \tag{40}
\]
Since \( \bar{q}(j) \) is optimal for the household’s problem, it follows that if the alternative satisfies the budget constraint \( \text{[39]} \) then the alternative cannot deliver a strictly higher level of the consumption aggregator so
in that case we must have \( y(\delta_1, \ldots, \delta_K) \leq 0 \). We also know that when \((\delta_1, \ldots, \delta_K) = 0\), the alternative consumption function \( q(j; 0, \ldots, 0) \) coincides with the optimal consumption function \( \bar{q}(j) \), so \( e(0, \ldots, 0) = y(0, \ldots, 0) = 0 \). Thus \((\delta_1, \ldots, \delta_K) = 0\) solves the following nonlinear programming problem:

\[
\begin{align*}
\text{maximize} & \quad y(\delta_1, \ldots, \delta_K) \\
\text{subject to} & \quad e(\delta_1, \ldots, \delta_K) \leq 0 \\
& \quad \delta_k \geq 0 \text{ for all } k \text{ and } \delta_1 + \cdots + \delta_K \leq n - j_k.
\end{align*}
\]

Since \( j_k \in S_n \) for all \( k \), the following one sided partial derivatives exist for all \( k \): \( y_k(0, \ldots, 0) = u(q_k) - u(\bar{q}(j_k)) \) and \( e_k(0, \ldots, 0) = (1 + t)p(j_k)(q_k - \bar{q}(j_k)) \). Since \((\delta_1, \ldots, \delta_K) = 0\) is optimal for the nonlinear programming problem, there cannot exist a feasible ascent direction. That is, there cannot exist a vector \( v \) such that

\[
(0 v_1; \ldots; v_K) = 0
\]

for all \( v \). Thus there cannot exist a non-negative vector \((v_1, \ldots, v_K)\) such that

\[
y_1(0, \ldots, 0)v_1 + \cdots + y_K(0, \ldots, 0)v_K > 0 \text{ and } e_1(0, \ldots, 0)v_1 + \cdots + e_K(0, \ldots, 0)v_K < 0.
\]

Substitute for the partial derivatives and write this in vector form as

\[
\mathbb{B}(v_1, \ldots, v_K) \text{ with } v_k \geq 0 \text{ for all } k, \text{ such that } \sum_{k=1}^{K} v_k \begin{pmatrix} u(q_k) - u(\bar{q}(j_k)) \\ -(1 + t)p(j_k)(q_k - \bar{q}(j_k)) \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

where the vector inequality is component by component. Since the \( j_k \)s were chosen arbitrarily from \( S_n \), since the \( q_k \)s were arbitrarily chosen non-negative numbers, and since \( K \) was an arbitrarily chosen positive integer, we have the following result. Let \( C \) be the convex cone generated by all non-negative linear combinations of vectors of the form

\[
\begin{pmatrix} u(q) - u(\bar{q}(j)) \\ -(1 + t)p(j)(q - \bar{q}(j)) \end{pmatrix} \quad j \in S_n, \quad q \geq 0.
\]

Then there is no vector in \( C \) that has both of its components strictly positive. I.e., \( C \) is disjoint from \( \mathbb{R}_+^2 \) so we can separate these convex sets with a plane: There exists a non-zero vector \((\lambda_0, \lambda)\) such that \( \lambda_0 \cdot (y, e) \leq 0 \) for all \((y, e) \in C \) and \( \lambda_0 \cdot (v_1, v_2) \geq 0 \) for all \((v_1, v_2) \in \mathbb{R}_+^2 \). The latter condition tells us that \( \lambda_0 \geq 0 \) and \( \lambda \geq 0 \). Then the former condition, when applied to the vectors that generate \( C \), yields

\[
\lambda_0[u(q) - u(\bar{q}(j))] - \lambda(1 + t)p(j)(q - \bar{q}(j)) \leq 0 \quad \text{for all } j \in S_n, \text{ and for all } q \geq 0. \tag{41}
\]

We now demonstrate that \( \lambda_0 \neq 0 \). By contradiction, if \( \lambda_0 = 0 \) then \( \lambda \) must be strictly positive since \((\lambda_0, \lambda)\) is a non-zero vector. In this case, \(41\) reduces to

\[
-(1 + t)p(j)(q - \bar{q}(j)) \leq 0 \quad \text{for all } j \in S_n, \text{ and for all } q \geq 0. \tag{42}
\]

By assumption \(5\) \( p(j) > 0 \) for almost all \( j \in [0, n] \), and by Appendix C \( \bar{q}(j) > 0 \) for almost all \( j \in [0, n] \). Thus, there exists a \( j' \in S_n \) such that \( p(j') > 0 \) and \( \bar{q}(j') > 0 \). Apply this to \(42\) with \( q = 0 \):

\[
(1 + t)p(j')\bar{q}(j') \leq 0
\]
which is false. The assumption \( \lambda_0 = 0 \) has led to a contradiction.

We now know that \( \lambda_0 > 0 \). Due to the homogeneity of (41) there is no loss of generality in setting \( \lambda_0 = 1 \):
\[
  u(q) - u(q(j)) - \lambda(1 + t)p(j)(q - q(j)) \leq 0 \quad \text{for all } j \in S_n, \text{ and for all } q \geq 0
\]
where \( \lambda \geq 0 \). Since \( S_n \subset [0, n] \) has full measure, we can re-state this as
\[
  \bar{q}(j) \in \text{argmax}_{q \geq 0} \left[ u(q) - \lambda(1 + t)p(j)q \right] \quad \text{for almost all } j \in [0, n].
\]
Since \( \bar{q}(j) > 0 \) almost everywhere by Appendix C, this maximum condition yields its own first order condition,
\[
u' \left( \bar{q}(j) \right) = \lambda(1 + t)p(j) \quad \text{for almost all } j \in [0, n]
\]
where we can now say that \( \lambda \) is strictly positive since \( u' > 0 \) under assumption 2.

We now return to the full utility maximization problem of household \( h \), and we restore the subscript \( i \) and the superscript \( h \). Given the result from our analysis of the budget allocation problem for consumption in industry \( i \), there is no loss of utility if we add the following constraints to the utility maximization problem:
\[
u_i^h(q_i^h(j)) = \lambda_i^h(1 + t_i)p_i(j) \quad i \geq 1, \text{ almost all } j \in [0, n_i]
\]
and we add \( (\lambda_1^h, \ldots, \lambda_f^h) \), all strictly positive, as choice variables. That is, since we know that all maximizers satisfy these constraints anyway (for particular values of \( \lambda_i^h \)), the problem remains the same. Then household \( h \)'s problem is to choose \( \lambda_i^h > 0 \) for \( i \geq 1 \), \( \ell^h \geq 0 \), and \( q_i^h(j) \geq 0 \) for \( j \in [0, n_i] \) and \( i \geq 1 \) to
\[
\text{maximize } U^h(\ell^h, Y_1^h, \ldots, Y_f^h; g)
\]
subject to
\[
Y_i^h := \int_0^{n_i} u_i^h(q_i^h(j))dj
\]
\[
u_i^h(q_i^h(j)) = \lambda_i^h(1 + t_i)p_i(j) \quad i \geq 1, \text{ almost all } j \in [0, n_i] \quad (43)
\]
\[
\sum_{i=1}^f \int_0^{n_i} (1 + t_i)p_i(j)q_i^h(j) dj \leq (1 - t_0)w(L^h - \ell^h) - T + \pi^h.
\]

Let \( v_i^h \) denote the inverse of the marginal utility function, \( v_i^h := (u_i^h)^{-1} \). We can use the new constraint (43) to eliminate \( q_i^h(j) \) and replace it with \( v_i^h(\lambda_i^h(1 + t_i)p_i(j)) \) throughout the problem. The exceptional sets of measure zero (where the equalities in (43) fail to hold) have no effect on any of the integrals. Then the remaining choice variables are \( \lambda_i^h > 0 \) for \( i \geq 1 \) and \( \ell^h \geq 0 \), and they are chosen to
\[
\text{maximize } U^h(\ell^h, Y_1^h, \ldots, Y_f^h; g)
\]
subject to
\[
Y_i^h = \int_0^{n_i} u_i^h \circ v_i^h(\lambda_i^h(1 + t_i)p_i(j)) dj
\]
\[
\sum_{i=1}^f \int_0^{n_i} (1 + t_i)p_i(j)v_i^h(\lambda_i^h(1 + t_i)p_i(j)) dj \leq (1 - t_0)w(L^h - \ell^h) - T + \pi^h.
\]
This optimization problem has only a finite number of choice variables, as compared with household h’s original problem with a continuum of choice variables. Furthermore, standard techniques apply now. Appendix D demonstrates that we can differentiate with respect to \( \lambda_i^h \) under the integral sign since the required regularity conditions are satisfied. We proceed to derive the first order necessary conditions for optimality.

Since \( U^h \) is monotonically strictly increasing in \( \ell^h \) (assumption 1(a)), the budget constraint must bind at any optimum. Then

\[
\ell^h = L^h - \frac{T}{(1 - t_0)w} + \frac{\pi^h}{(1 - t_0)w} - \sum_{i=1}^{n_1} \int_0^{\pi_i} \frac{(1 + t_i)p_i(j)}{(1 - t_0)w} v_i^h(\lambda_i^h(1 + t_i)p_i(j))dj. \tag{45}
\]

By assumption 1(b), the constraint \( \ell^h \geq 0 \) will not bind at any optimum. In the utility function \( U^h \), substitute for \( \ell^h \) from (45) and for \( Y_i^h \) from (44). This is now an unconstrained problem. The first order necessary condition for the optimal choice of \( \lambda_i^h > 0 \) is \( U_0^h \partial \ell^h / \partial \lambda_i^h + U_i^h \partial Y_i^h / \partial \lambda_i^h = 0 \). Evaluate the partial derivatives to get

\[
U_i^h \int_0^{\pi_i} \frac{(1 + t_i)p_i(j)}{(1 - t_0)w} v_i^{h'}(\lambda_i^h(1 + t_i)p_i(j))(1 + t_i)p_i(j) dj
= U_i^h \int_0^{\pi_i} u_i^{h'} v_i^h(\lambda_i^h(1 + t_i)p_i(j))v_i^{h'}(\lambda_i^h(1 + t_i)p_i(j))(1 + t_i)p_i(j) dj.
\]

Since, by definition, \( v_i^h \) is the inverse function of \( u_i^{h'} \), on the right hand side we have \( u_i^{h'} v_i^h(\lambda_i^h(1 + t_i)p_i(j)) = \lambda_i^h(1 + t_i)p_i(j) \). Then we can pull out \( \lambda_i^h \) from this integral. And on the left hand side we can pull out \( 1/[(1 - t_0)w] \). Then

\[
\frac{U_i^h}{(1 - t_0)w} \int_0^{\pi_i} (1 + t_i)p_i(j)v_i^h(\lambda_i^h(1 + t_i)p_i(j))(1 + t_i)p_i(j) dj
= U_i^h \lambda_i^h \int_0^{\pi_i} (1 + t_i)p_i(j)v_i^{h'}(\lambda_i^h(1 + t_i)p_i(j))(1 + t_i)p_i(j) dj.
\]

The integral on the left hand side is identical to the integral on the right hand side, and both are strictly negative. Divide through by this common factor to get \( U_0^h / [(1 - t_0)w] = U_i^h \lambda_i^h \). Since \( \lambda_i^h \) satisfies (43) for almost all \( j \in [0, n_i] \) we have

\[
\frac{U_i^h}{(1 - t_0)w} = \frac{U_i^h u_i^{h'}(q_i^h(j))}{(1 + t_i)p_i(j)} \quad i \geq 1, \quad \text{almost all } j \in [0, n_i].
\]

This is (1), so we have confirmed that (1) is indeed a necessary condition for optimality for the utility maximization problem of household h.

**Appendix C** Households’ consumption levels are strictly positive almost everywhere

As in Appendix B, let \( \tilde{\ell}^h \geq 0 \) and \( \tilde{q}_i^h(j) \geq 0 \) (\( j \in [0, n_i], i \geq 1 \)) be a solution to the utility maximization problem of household h. Let \( \tilde{E}_i := \int_0^{n_i} (1 + t_i)p_i(j)\tilde{q}_i^h(j) dj \) be h’s optimal expenditure on industry i con-
sumption and let $\tilde{Y}_i^h := \int_0^{n_i} u_i^h(q_i^h(j))dj$ be $h$'s optimal consumption aggregator for industry $i$. For each $i$, we must have $\tilde{E}_i^h > 0$ since otherwise we would have $q_i^h(j) = 0$ almost everywhere in which case $\tilde{Y}_i^h = 0$. But this would be inconsistent with optimality under assumption 1(b).

From the two stage budgeting procedure, $(q_i^h(j))$ solves the following problem:

\[
\begin{align*}
\text{maximize} & \quad (q_i^h(j))_{\geq 0} \quad Y_i^h := \int_0^{n_i} u_i^h(q_i^h(j))dj \\
\text{subject to} & \quad \int_0^{n_i} (1 + t_i)p_i(j)q_i^h(j)dj \leq \tilde{E}_i^h.
\end{align*}
\]

We will now show that $q_i^h(j) > 0$ for almost all $j \in [0, n_i]$. For ease of notation, drop the subscript $i$ and the superscript $h$. Also, let $\mu$ denote Lebesgue measure on $\mathbb{R}$.

Let $A \subset [0, n]$ be the set where $\bar{q}(j) = 0$. By contradiction, suppose $\mu(A) > 0$. Since $p(j) > 0$ almost everywhere by assumption 3, the following interval of $\epsilon$ values is well defined: $0 \leq \epsilon < \bar{E}/[(1 + t) \int_A p(j)dj]$. On this interval define the function

\[ h_\epsilon(j) := \begin{cases} 
\epsilon & \text{if } j \in A \\
-\epsilon E^{-1}(1 + t) \int_A p(j')dj' \bar{q}(j) & \text{if } j \in [0, n] \setminus A.
\end{cases} \]

The budgetary cost of $h_\epsilon$ is

\[ \int_0^{n_i} (1 + t)p(j)h_\epsilon(j)dj = \epsilon \int_A (1 + t)p(j)dj - \left[ \epsilon E^{-1}(1 + t) \int_A p(j')dj' \right] \int_{[0, n] \setminus A} (1 + t)p(j)\bar{q}(j)dj \]

\[ = \epsilon(1 + t) \int_A p(j)dj - \left[ \epsilon E^{-1}(1 + t) \int_A p(j')dj' \right] \bar{E} \]

\[ = 0 \]

where the middle line follows from the definition of $\bar{E}$ since the integral that defines $\bar{E}$ vanishes on $A$. Since $h_\epsilon$ has a budgetary cost of zero, the following alternative consumption function,

\[ q_\epsilon(j) := \bar{q}(j) + h_\epsilon(j) = \begin{cases} 
\bar{q}(j) + \epsilon & \text{if } j \in A \\
[1 - \epsilon E^{-1}(1 + t) \int_A p(j')dj'] \bar{q}(j) & \text{if } j \in [0, n] \setminus A
\end{cases} \]

satisfies the household’s budget constraint and furthermore it is non-negative since we restricted $\epsilon$ to satisfy $0 \leq \epsilon < \bar{E}/[(1 + t) \int_A p(j)dj]$. Also, by definition, on $A$ we have $\bar{q} = 0$ so $q_\epsilon(j) = \epsilon$ if $j \in A$.

Let $Y(\epsilon)$ be the consumption aggregator derived from the alternative:

\[ Y(\epsilon) := \int_0^{n_i} u(\epsilon(j))dj = \int_A u(\epsilon) + \int_{[0, n] \setminus A} u\left(1 - \epsilon E^{-1}(1 + t) \int_A p(j')dj' \right) \bar{q}(j)dj. \]

Since $(\bar{q}(j))$ is optimal among all budget feasible consumption functions, and since $q_\epsilon \equiv \bar{q}$ if $\epsilon = 0$, we have $Y(\epsilon) \leq Y(0)$, so for all $\epsilon$ in the open interval $0 < \epsilon < \bar{E}/[(1 + t) \int_A p(j)dj]$, \[ 0 \geq \frac{Y(\epsilon) - Y(0)}{\epsilon} = \int_A \frac{u(\epsilon) - u(0)}{\epsilon}dj + \int_{[0, n] \setminus A} \frac{u\left(1 - \epsilon E^{-1}(1 + t) \int_A p(j')dj' \right) \bar{q}(j)}{\epsilon} - u(\bar{q}(j)) \]
Consider the expression \( u\left(1 - \epsilon \tilde{E}^{-1}(1 + t) \int_A p(j')dj'\right) \) that appears above. Under assumption 2, \( u \) is strictly concave with \( u(0) = 0 \). So for all \( \lambda \in (0, 1) \) and all \( q > 0 \), \( u(\lambda q + (1 - \lambda)0) > \lambda u(q) + (1 - \lambda)u(0) \) which yields \( u(\lambda q) > \lambda u(q) \). Hence \( u\left(1 - \epsilon \tilde{E}^{-1}(1 + t) \int_A p(j')dj'\right) \geq \left[1 - \epsilon \tilde{E}^{-1}(1 + t) \int_A p(j')dj'\right] u(\bar{q}(j)) \). Furthermore, again from strict concavity of \( u \), the difference quotient \( [u(\epsilon) - u(0)]/\epsilon \) is greater than \( u'(\epsilon) \).

Substitute these results into the above displayed equation and we have that for all \( \epsilon \) in the open interval \( 0 < \epsilon < \tilde{E}/[(1 + t) \int_A p(j) dj] \),

\[
0 \geq \frac{Y(\epsilon) - Y(0)}{\epsilon} > u'(\epsilon)\mu(A) - \left[\tilde{E}^{-1}(1 + t) \int_A p(j')dj'\right] \int_{[0,n]\setminus A} u(\bar{q}(j))dj.
\]

But this is impossible since, by assumption 2, \( \lim_{\epsilon \to 0} u'(\epsilon) = \infty \) while the integral of \( u(\bar{q}(j)) \) must be finite since otherwise there would be no solution to the household’s utility maximization problem. Thus the initial assumption that \( \mu(A) > 0 \) has led to a contradiction and we conclude that \( \mu(A) = 0 \), i.e., \( \bar{q}(j) > 0 \) for almost all \( j \in [0,n] \).

**Appendix D  Differentiation under the integral sign in the households’ problem**

The purpose of this appendix is to demonstrate that it is legitimate to differentiate with respect to \( \lambda^h_i \) under the integral sign in the following problem from Appendix B:

\[
\begin{align*}
\text{maximize } & \quad U^h(\ell^h, Y^h_1, \ldots, Y^h_f, g) \\
\text{subject to } & \quad Y^h_i = \int_0^{\lambda^h_i} u^h_i \circ v^h_i(\lambda^h_i(1 + t_i)p_i(j))dj \\
& \quad \sum_{i=1}^f \int_0^{\lambda^h_i} (1 + t_i)p_i(j)v^h_i(\lambda^h_i(1 + t_i)p_i(j))dj \leq (1 - t_0)w(L^h - \ell^h) - T + \pi^h.
\end{align*}
\]

This is a representation of the households’ utility maximization problem in which the choice variables are \( \ell^h \geq 0 \) and \( \lambda^h_i > 0 \) for \( i \geq 1 \).

Fix the value of \( i \) and focus attention on the two integrals that involve \( \lambda^h_i \). For ease of notation, drop the subscript \( i \) and the superscript \( h \). Then our task is to differentiate the following two functions of \( \lambda \):

\[
\begin{align*}
Y(\lambda) & := \int_0^n u \circ v(\lambda(1 + t)p(j))dj \\
E(\lambda) & := \int_0^n (1 + t)p(j)v(\lambda(1 + t)p(j))dj
\end{align*}
\]

where \( v = (u')^{-1} \). Furthermore, since our concern is the characterization of the optimality conditions for the utility maximization problem, we need only evaluate the derivatives of \( Y \) and \( E \) at the optimal value \( \bar{\lambda} \).

By stating that \( \bar{\lambda} \) is optimal, we are also implicitly stating that both integrals converge when \( \lambda = \bar{\lambda} \).
We can differentiate under the integral sign if the following conditions are satisfied. See, e.g., theorem 2.27 of Folland (1999). There exists \( \delta \in (0, \bar{\lambda}) \) such that

(i) the integrals that define \( Y(\lambda) \) and \( E(\lambda) \) both converge for all \( \lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \), and

(ii) the partial derivatives of the integrands with respect to \( \lambda \) are dominated as follows:

\[
\left| u' \circ v(\lambda(1+t)p(j))v'(\lambda(1+t)p(j))(1+t)p(j) \right| \leq y(j) \quad \text{for all } \lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \text{ and all } j \in [0, n]
\]

\[
\left| (1+t)p(j)v'(\lambda(1+t)p(j))(1+t)p(j) \right| \leq c(j) \quad \text{for all } \lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta] \text{ and all } j \in [0, n]
\]

where \( y \) and \( c \) are integrable functions of \( j \in [0, n] \).

We need to verify that (i) and (ii) are satisfied.

Since \( u' \circ v \) is the identity function, the absolute value terms in (ii) differ only by the multiplicative factor \( \lambda \). Since \( \lambda \) is explicitly bounded, if either of these terms is dominated then so is the other. We will analyze the first condition in (ii) and we will use \( v' = 1/u'' \) as well as assumption 3 which tells us that the absolute value of \( u'(q)/[qu''(q)] \) is uniformly bounded by some number \( B > 0 \):

\[
\Rightarrow \left| \frac{u' \circ v(\lambda(1+t)p(j))v'(\lambda(1+t)p(j))(1+t)p(j)}{v(\lambda(1+t)p(j))u'' \circ v(\lambda(1+t)p(j))} \right| \leq y(j) \quad \lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta], \quad j \in [0, n]
\]

\[
\Rightarrow B \cdot (1+t)p(j)v(\lambda(1+t)p(j)) \leq y(j) \quad \lambda \in [\bar{\lambda} - \delta, \bar{\lambda} + \delta], \quad j \in [0, n].
\]

Since \( v \) is monotonically strictly decreasing, \( y(j) := B \cdot (1+t)p(j)v(\bar{\lambda} - \delta)(1+t)p(j) \) can serve as a dominating function, provided it is integrable. And it indeed will be integrable if \( E(\bar{\lambda} - \delta) < \infty \) where \( E \) is the expenditure integral in condition (i) above. It follows that condition (ii) is actually subsumed within condition (i).

By monotonicity, condition (i) is satisfied if \( Y(\bar{\lambda} - \delta) < \infty \) and \( E(\bar{\lambda} - \delta) < \infty \) for some \( \delta \in (0, \bar{\lambda}) \). Recall that we know \( Y(\bar{\lambda}) < \infty \) and \( E(\bar{\lambda}) < \infty \). We will start with \( E(\bar{\lambda} - \delta) \) and we will find a value for \( \delta \) such that this integral converges. We again use \( v' = 1/u'' \) and the assumption that the absolute value of \( u'(q)/[qu''(q)] \)
is uniformly bounded by some number $B > 0$. Considering the integrand of $E(\tilde{\lambda} - \delta)$ we have\(^{15}\)

$$v((\tilde{\lambda} - \delta)(1 + t)p(j)) - v(\tilde{\lambda}(1 + t)p(j))$$

\[= -\delta(1 + t)p(j)v'(\tilde{\lambda}(j)(1 + t)p(j))\] for some $\tilde{\lambda}(j) \in [\tilde{\lambda} - \delta, \tilde{\lambda}]$

\[= -\delta(1 + t)p(j)\frac{u' \circ v(\tilde{\lambda}(j)(1 + t)p(j))}{\tilde{\lambda}(j)(1 + t)p(j)}\] since $u' \circ v =$ identity

\[= -\frac{\delta v'(\tilde{\lambda}(j)(1 + t)p(j))}{\tilde{\lambda}(j)} u' \circ v(\tilde{\lambda}(j)(1 + t)p(j))\frac{u'' \circ v(\tilde{\lambda}(j)(1 + t)p(j))}{v(\tilde{\lambda}(j)(1 + t)p(j))u'' \circ v(\tilde{\lambda}(j)(1 + t)p(j))}\]

\[\leq \frac{\delta v((\tilde{\lambda} - \delta)(1 + t)p(j))}{\lambda - \delta} B.\]

Now choose $\delta$ so that $\delta B/(\tilde{\lambda} - \delta) = 1/2$, i.e., $\delta = \tilde{\lambda}/(1 + 2B)$. Then the above calculation yields

$$v((\tilde{\lambda} - \delta)(1 + t)p(j)) - v(\tilde{\lambda}(1 + t)p(j)) \leq 0.5v((\tilde{\lambda} - \delta)(1 + t)p(j))$$

and hence

$$v((\tilde{\lambda} - \delta)(1 + t)p(j)) \leq 2v(\tilde{\lambda}(1 + t)p(j)).$$

If we apply this inequality (and this value of $\delta$) to the definition of the function $E$ we have $E(\tilde{\lambda} - \delta) \leq 2E(\tilde{\lambda})$. Since the latter integral converges, so must the former, and we have verified half of condition (i) above.

Finally, we verify that $Y(\tilde{\lambda} - \delta) < \infty$. We follow the same procedure as previously, use the same value of $\delta$, and apply the mean value theorem to the integrand in the definition of $Y$\(^{16}\)

$$u \circ v((\tilde{\lambda} - \delta)(1 + t)p(j)) - u \circ v(\tilde{\lambda}(1 + t)p(j))$$

\[= -\delta(1 + t)p(j)u' \circ v(\tilde{\lambda}(j)(1 + t)p(j))v'(\tilde{\lambda}(j)(1 + t)p(j))\] for some $\tilde{\lambda}(j) \in [\tilde{\lambda} - \delta, \tilde{\lambda}]$

\[= -\delta(1 + t)p(j)v(\tilde{\lambda}(j)(1 + t)p(j))\frac{u' \circ v(\tilde{\lambda}(j)(1 + t)p(j))}{v(\tilde{\lambda}(j)(1 + t)p(j))u'' \circ v(\tilde{\lambda}(j)(1 + t)p(j))}\]

\[\leq \delta(1 + t)p(j)v((\tilde{\lambda} - \delta)(1 + t)p(j)) B.\]

This yields

$$u \circ v((\tilde{\lambda} - \delta)(1 + t)p(j)) \leq u \circ v(\tilde{\lambda}(1 + t)p(j)) + \delta B(1 + t)p(j)v((\tilde{\lambda} - \delta)(1 + t)p(j))$$

and hence, drawing on our previous result,

$$Y(\tilde{\lambda} - \delta) \leq Y(\tilde{\lambda}) + \delta BE(\tilde{\lambda} - \delta) \leq Y(\tilde{\lambda}) + 2\delta BE(\tilde{\lambda}) < \infty.$$\(^{15}\)Since $v(0)$ is not defined, the calculation that follows is valid for almost all $j \in [0, n]$ where $p(j) > 0$ (assumption 5).

\(^{16}\)Again, the calculation that follows is valid for almost all $j \in [0, n]$ where $p(j) > 0$ (assumption 5).
References


