An Allocation Rule for Hypernetwork Game

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Cooperation

- We study agents aiming at a collective or individual goal
- Network are pervasive to model such situations
The setting

Cooperation

- We study agents aiming at a collective or individual goal
- Network are pervasive to model such situations
- Jackson-Wolinsky (1996) The value a set of agents generates depends explicitly of the network structure
A Friendship Network
Scientific Collaboration
What is common to all these situations:

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How to distribute the value generated by an hypernetwork?
Hypernetworks in the literature:

- Davis, Gardner and Gardner (1941): "Southern women study". Sample of 14 social events. Two women are connected if they attended the same event.
Bipartite network and Hypernetwork representations
Galaskiewicz (1985) analyses of CEO’s interaction in country clubs in Chicago during the 70’s.
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The ”Six Degrees of Kevin Bacon” in the network of collaborating film actors, where nodes = actors and groups = casts of films
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Other possible applications: scientific collaboration with a coauthorship hypernetwork, international trade with multilateral trade agreements, etc.
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Both papers generalize the **Shapley Value** for cooperative game when communication must use special structures.

Drawback : Two different structures may generate the same value.
Jackson and Wolinsky (1996): the value from cooperation depends explicitly on the network structure. They characterize the Myerson value.
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Caulier (2010), Slikker and van den Nouweland (2012): characterization of the position value as the Shapley value for links in a network game.
The current paper:

- Cooperation captured by a hypernetwork: **Hypernetwork Game**

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An allocation rule for hypernetwork games
The current paper:

- Cooperation captured by a hypernetwork: **Hypernetwork Game**
- Design axiomatically a way to distribute the value of any hypernetwork: **Allocation Rule**
- By adapting the classical Shapley axioms, we obtain a generalization of the **Position value** for hypernetwork games

An allocation rule for hypernetwork games

- Side dish: we show how to compute the Shapley Value in any distributive lattice.
Notations

- $N = \{1, \ldots, n\}$ finite set of players/agents who may form coalitions
- $2^N$ is the set of non-empty coalitions:
  \[ 2^N = \{S \mid S \neq \emptyset, S \subseteq N\} \]
- An **hypernetwork** is a pair $(N, H)$ with $H = \{S_1, \ldots, S_K\}$ such that for all $k \neq k'$: $S_k \neq S_{k'}$, for all $k$: $S_k \in 2^N$. 
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- **Inclusion**: $H \subseteq H' \iff [S \in H \Rightarrow S \in H']$
- $N(H)$ is the set of players involved in at least one coalition of $H$. $N \setminus N(H)$ is the set of isolated players. Nota : $i$ isolated not to be confused with $\{i\} \in H$ the singleton coalition.
We denote $2^N$ the **complete** hypernetwork and $\emptyset$ the **empty** hypernetwork.
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$|H|$ denotes the number of coalitions in $H$ with $|\emptyset| = 0$ by convention.
Set of hypernetworks on $N = \{1, 2\}$ ordered by inclusion

\[
\left\{\left\{1\right\}, \left\{2\right\}, \left\{1, 2\right\}\right\}
\]
Notations

Let $S \subseteq N$ and $H \in \mathcal{H}$. We denote $H(S) = H \cap 2^S$ the collection of coalitions in $H$ involving only players of coalition $S$. Two players $i$ and $j$ are connected if there exists a path joining them: A path in $H$ between $i$ and $j$ is a sequence $(S_1, \ldots, S_t)$ such that $i \in S_1$, $j \in S_t$ and for all $k \in \{1, \ldots, t\}$, $S_k \in H$ and for all $k \in \{1, \ldots, t-1\}$, $S_k \cap S_{k+1} \neq \emptyset$. The notion of connectedness in $H$ forms a partition $\Pi(H)$ of $N$. The set of components of $H$ is $C(H) = \{(S, H(S)) | S \in \Pi(H)\}$. 

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The notion of connectedness in $H$ forms a partition $\Pi(H)$ of $N$. The set of components of $H$ is $C(H) = \{(S, H(S)) \mid S \in \Pi(H)\}$. 
We define a value function $v : \mathcal{H} \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of value functions is $\mathcal{V}$.

A value function is **component additive** if

$$
 v(H) = \sum_{(S, H(S)) \in C(H)} v((H(S))
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A value function is **component additive** if

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An **hypernetwork game** is a pair $(\mathcal{N}, \nu)$
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A value function is **component additive** if

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An **hypernetwork game** is a pair $(N, v)$.

An **allocation rule** is a function $Y : \mathcal{H} \times \mathcal{V} \rightarrow \mathbb{R}^n$ with $Y_i(H, v)$ the share received by player $i$ in hypernetwork $H$ with value function $v$. 

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To present the allocation rule we propose, we need the following:

**Definition (Unanimity hypernetwork games)**

For any hypernetwork $H \in \mathcal{H}$, we define $u_H \in \mathcal{V}$ the **unanimity value function** satisfying:

$$
\begin{align*}
    u_H(H') &= \begin{cases} 
        1 & \text{if } H \subseteq H' \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
$$

The set of unanimity hypernetwork games forms a linear basis for $\mathcal{V}$, hence any game can be written as

$$v = \sum_{H \in \mathcal{H}} \Delta_H(v) u_H.$$
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The set of unanimity hypernetwork games forms a linear basis for $\mathcal{V}$, hence any game can be written as

$$v = \sum_{H \in \mathcal{H}} \Delta^H(v) u_H$$
\[ v(H) = \sum_{H' \in \mathcal{H}} \Delta^{H'}(v) u_{H'}(H) \]

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\Delta^{H}(v) = v(H) - \sum_{H' \subset H} \Delta^{H'}(v)
\]

The Harsanyi Dividends
The allocation rule: Generalized position value

\[ PV_i(H, v) = \sum_{H' \subseteq H} \sum_{S \in H'_i} \frac{\Delta_{H'}(v)}{|S||H|} \]

with \( H_i : \{ S \in H \mid i \in S \} \)
Axiom (Additivity)

An allocation rule $\mathcal{Y} : \mathcal{H} \times \mathcal{V}$ is additive if

$$\mathcal{Y}(H, \nu_1 + \nu_2) = \mathcal{Y}(H, \nu_1) + \mathcal{Y}(H, \nu_2)$$

for all $\nu_1, \nu_2 \in \mathcal{V}$. 
Properties

Definition (Superfluous coalition)
A coalition $S \in H$ is superfluous for hypernetwork $H$ in hypernetwork game $(N, v)$ if

$$v(H^{'}) = v(H^{' \cup} S)$$

for all $H^{'} \subseteq H \setminus S$
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Definition (Superfluous coalition)
A coalition $S \in H$ is superfluous for hypernetwork $H$ in hypernetwork game $(N, \nu)$ if

$$\nu(H') = \nu(H' \cup S)$$

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Axiom (Superfluous coalition property)
An allocation rule $Y : \mathcal{H} \times \mathcal{V}$ has the superfluous coalition property if

$$Y(H, \nu) = Y(H \setminus S, \nu)$$

for all $H \in \mathcal{H}$, $\nu \in \mathcal{V}$ et $S$ superfluous for $H$ in $(N, \nu)$. 
Properties

Axiom (Component Efficiency)

An allocation rule $Y : \mathcal{H} \times \mathcal{V}$ is component efficient if for all $\nu$ component additive and all components $(S, H(S)) \in C(H)$, $H \in \mathcal{H}$:

$$\sum_{i \in S} Y_i(H, \nu) = \nu(H(S))$$
Definition (Coalition anonymity)

A value function $\nu \in \mathcal{V}$ is **coalition anonymous** on $H$ if, for all $H' \subseteq H$ and $H'' \subseteq H$ such that $|H'| = |H''|$, we have:

$$\nu(H') = \nu(H'')$$
Properties

Definition (Coalition anonymity)
A value function $v \in \mathcal{V}$ is coalition anonymous on $H$ if, for all $H' \subseteq H$ and $H'' \subseteq H$ such that $|H'| = |H''|$, we have:

$$v(H') = v(H'')$$

Definition (Hypernetwork Power)
The power of a player $i$ in an hypernetwork $H$ is

$$\psi_i(H) = \sum_{S \in H_i} \frac{1}{|S|}$$
Axiom (Coalition anonymity)

An allocation rule $Y : \mathcal{H} \times \mathcal{V}$ is coalition anonymous if for all $H \in \mathcal{H}$ and $\nu$ coalition anonymous for $H$, there exists an $\alpha \in \mathbb{R}$ such that:

$$Y_i(H, \nu) = \alpha \cdot \psi_i(H)$$
Theorem

An allocation rule $Y : \mathcal{H} \times \mathcal{V}$ satisfies additivity, the superfluous coalition property, component efficiency and is coalition anonymous if and only if $Y$ is the position value $PV :$

$$PV_i(H, v) = \sum_{H' \subseteq H} \sum_{S \in H'_i} \frac{\Delta^{H'}(v)}{|S||H|}$$
Conclusion:

- Hypernetworks generalize networks, coalitions, partitions, etc.
- By restricting the value taken by the value function, we get TU games, network games, partition function form games, etc.
- The axioms adapt those from the Shapley value and generalize those proposed for the position value.
- If we restrict the size of each possible coalition to 2, we characterize the position value for network games also.
- Our axioms can easily be translated to characterize a position value in a distributive lattice.