Hybrid Invariance and Oligarchic Structures

Susumu Cato*
Institute of Social Science, The University of Tokyo

September 25, 2014

Abstract

This paper addresses the problem of Arrovian preference aggregation. In this paper, we do not impose any assumptions of social rationality; social preferences are allowed to be any binary relation (possibly incomplete and intransitive). We introduce the axiom of hybrid invariance, which is an interprofile axiom. We characterize the decisive structure associated with a collective choice rule that satisfies hybrid invariance.

Keywords: Social choice; Arrow’s impossibility theorem; Filter; Hybrid invariance; Oligarchy
JEL classification: D71

---

*Tel: +81-3-5841-4904, Fax :+81-3-5841-4905, susumu.cato@gmail.com, Institute of Social Science, University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.
1 Introduction

A decisive coalition is a group of individuals that has the power to determine social rankings. In the case of the method of majority decision (MMD), any group including more than half the individuals is decisive. According to the Condorcet paradox, the MMD may generate a social preference cycle, and thus, it reveals an irrational property. An implication of the paradox is that the decisive structure under the MMD is not compatible with social rationality. This observation suggests that the decisive structure is associated with social rationality.

The seminal works of Kirman and Sondermann (1972) and Hansson (1976) provide a formal analysis of the relationship between social rationality and the decisive structure.1 A benchmark result is Hansson’s theorem. If a quasi-transitive collective choice rule satisfies weak Pareto and independence of irrelevant alternatives, then the collection of decisive coalitions forms a filter (Hansson 1976, Theorem 3). Given a filter, we can construct a quasi-transitive collective choice rule satisfying the three axioms such that the corresponding collection of decisive groups is the filter (Hansson 1976, Theorem 4). An essential implication of these results is that the intersection of two decisive groups is decisive. Hansson’s theorem clarifies the closed connection between the oligarchic structure and quasi-transitive rationality.2 Following his approach, Brown (1974), Fishburn (1987), and Cato (2012, 2013ab) study the correspondence between the degree of social rationality and the types of decisive structures.3

Various authors, including Buchanan (1954) and Shapiro (2009), argue that rationality is not necessary for the political process. In particular, Buchanan (1954) states that “[r]ationality or irrationality as an attribute of the social group implies the imputation to that group of an organic existence apart from that of its individual components” (Buchanan, 1954, page 116). In this paper, we do not impose any kind of social rationality, but instead require the political processes to be stable. Stability of the political process has a positive effect on economic growth, and thus, it has instrumental value. Moreover, it has intrinsic value for a modern democratic society.

This paper introduces an axiom of social stability, which we call the concept of hybrid invariance. This axiom is concerned with two preference profiles that generate the same social preference. A hybrid preference profile of the two is a profile in which each individual has a preference under either of the two profiles. Hybrid invariance requires that the asymmetric part of a social preference under the hybrid extend the original profiles. Roughly speaking, a collective decision process is invariant for the hybridization of two profiles that yield the same social preference. We show that if a collective choice rule satisfies weak Pareto, independence of irrelevant alternatives, and hybrid invariance, then the collection of decisive coalitions forms a filter. The oligarchic structure is established without the coherence property of social preferences.

We also introduce a weakening and a strengthening of hybrid invariance. It is shown that the collection of decisive coalitions forms a prefILTER under weak hybrid invariance. This result is an appropriate counterpart of Brown’s theorem (Brown, 1974, 1975). We also show that if a collective choice rule is restricted to a certain class, the collection of decisive coalitions

---

1The pioneering contribution of Fishburn (1970) stimulates their works. He shows the existence of an Arrovian social welfare function under an infinite population by focusing on the decisive structure.

2The classical works on oligarchy are provided by Gibbard (2014) and Guha (1972). See also Iritani, Kamo, and Nagahisa (2013).

3See also Mihara (1997), Bossert and Suzumura (2011), and Noguchi (2011).
forms an ultrafilter under strong hybrid invariance. The result implies that Arrow’s theorem is recovered without social rationality.

The nature of hybrid invariance is crucially different from quasi-transitivity. Quasi-transitivity is an intra-profile condition that states a restriction for one profile with a certain property. On the other hand, hybrid invariance is an inter-profile condition that restricts social preferences under several profiles with certain connections. Our result implies that axioms that are different in nature can have the same implication on the decisive structure.

There are a number of works on the Arrovian approach without social rationality. As a response to Buchanan (1954), Arrow (1963) introduces the concept of path independence, which states that a global choice problem can be decomposed into smaller problems. Plott (1973) provides a formal analysis of path independence, which can be regarded as a postulate on stability. However, its implication is crucially different from hybrid invariance. As shown by Plott (1973), path independence is an intraprofile axiom that is directly related to quasi-transitive rationality.

Denicolò (1993) and Sen (1993, Theorem 3) establish Arrow’s theorem without any kind of social rationality by employing the choice-functional approach. Both authors introduce modified versions of independence that work effectively for a social choice function. Their independence axioms are essentially stronger than the standard independence axiom.鱼 Fishburn (1987) emphasizes that independence axioms are interprofile conditions. Their works and ours are similar in spirit at this point. However, the meaning of hybrid invariance is quite different from that of independence axioms, which are conditions on the use of information.

The rest of this paper is organized as follows. Section 2 introduces our definitions. Section 3 presents our main results on hybrid invariance. Section 4 examines the strong version of hybrid invariance. Section 5 discusses the relationship between hybrid invariance and social rationality. Section 6 concludes the paper.

2 Preliminaries

Let $X$ be the set of alternatives. A binary relation $R$ on $X$ is a subset of $X \times X$. The symmetric part is defined by

$$I(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \in R\}.$$ 

The asymmetric part is defined by

$$P(R) = \{(x, y) \in X \times X : (x, y) \in R \text{ and } (y, x) \notin R\}.$$ 

The dual of $R$ is defined as follows:

$$R^d = \{(x, y) \in X \times X : (y, x) \in R\}.$$ 

The complement of $R$ is defined as follows:

$$R^c = \{(x, y) \in X \times X : (x, y) \notin R\}.$$ 

4 Denicolò (1998) explicitly shows that Sen’s independence is stronger than standard independence.
$R$ is transitive if for all $x, y, z \in X$,
\[ [(x, y) \in R \text{ and } (y, z) \in R] \Rightarrow (x, z) \in R. \]

$R$ is total if for all $x, y, z \in X$,
\[ [(x, y) \in R \text{ or } (y, x) \in R]. \]

Let $\mathcal{B}$ be the set of binary relations on $X$ and $\mathcal{R}$ be the set of weak orders on $N$. The restriction of $R$ to $Y \subseteq X$ is denoted by $R|_Y$.

Let $N$ be a (finite or infinite) set of individuals. A preference profile $\mathbf{R}$ is a list of individual weak orders $(R_i)_{i \in N}$: each preference profile is an element of $\mathcal{R}^N$. A collective choice rule is a function $f$ from $\mathcal{R}^N$ to $\mathcal{B}$. A coalition $A \subseteq N$ is decisive for $f$ if $\bigcap_{i \in A} P(R_i) \subseteq P(f(\mathbf{R}))$ for all $\mathbf{R} \in \mathcal{R}^N$.

Now, we introduce the Arrovian axioms for $f$.

**Weak Pareto (WP):** $N$ is decisive for $f$.

**Independence of Irrelevant Alternatives (IIA):** For all $x, y \in X$, and all $\mathbf{R}, \mathbf{R'} \in \mathcal{R}^N$, if $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ for all $i \in N$, then $f(\mathbf{R})|_{\{x,y\}} = f(\mathbf{R'})|_{\{x,y\}}$.

Here, we introduce some mathematical concepts. Given a set $V$, a filter $\mathcal{F}$ on $V$ is a collection of subsets of $V$ that satisfies the following:

(i) $V \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$;

(ii) if $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$;

(iii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An ultrafilter on $N$ is a filter with the following property: $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. A prefilter $\mathcal{F}$ on $V$ is a collection of subsets of $V$ that satisfies (i), (ii), and (iii') for all $k \in \{0, 1, \ldots, K\}$, if $A^0, A^1, \ldots, A^K \in \mathcal{F}$, then $\bigcap_{k \in \{0, 1, \ldots, K\}} A^k = \emptyset$.

Note that every filter must be a prefilter. The fundamental properties of filters and ultrafilters are found in Willard (1970).

### 3 Hybrid Invariance and Oligarchic Structures

In this section, we introduce the concept of hybrid invariance and examine its implications. Given two profiles $\mathbf{R}^\alpha, \mathbf{R}^\beta \in \mathcal{R}^N$, define
\[
\mathcal{C}(\mathbf{R}^\alpha, \mathbf{R}^\beta) = \{ \mathbf{R} \in \mathcal{R}^N : R_i = R^\alpha_i \text{ or } R_i = R^\beta_i \text{ for all } i \in N \}.
\]

$\mathcal{C}(\mathbf{R}^\alpha, \mathbf{R}^\beta)$ is a collection of “hybrid” preference profiles that are combinations of $\mathbf{R}^\alpha$ and $\mathbf{R}^\beta$.

The following axiom requires that if social preferences under two profiles are the same, then social preferences under hybrid profiles extend the asymmetric part of the social preferences under the original profiles.

**Hybrid Invariance (HI):** For all $\mathbf{R}, \mathbf{R'} \in \mathcal{R}^N$, if $f(\mathbf{R}) = f(\mathbf{R'})$, then $P(f(\mathbf{R})) \subseteq P(f(\mathbf{R'}))$ for all $\mathbf{R}'' \in \mathcal{C}(\mathbf{R}, \mathbf{R'})$.

The following theorem states that the collection of decisive coalitions forms a filter under WP, IIA, and HI.
Theorem 1. If a collective choice rule $f$ satisfies WP, IIA, and HI, then $D_f$ is a filter on $N$.

Proof. (i) By WP, $N \in D_f$ and $\emptyset \notin D_f$.

(ii) Let $A \in D_f$. Take $B \supseteq A$. If $(x, y) \in P(R_i)$ for all $i \in B$, then $(x, y) \in P(R_i)$ for all $i \in A$. Since $A \in D_f$, $(x, y) \in P(f(R))$.

(iii) Take $A, B \in D_f$. Fix $R^* \in L$ with $(x, y) \in R^*$. Let $R \in \mathcal{R}^N$ be such that $R^*_i = R_i$ for all $i \in A$. Since $A \in D_f$, we have $f(R) = R^*$. Let $R' \in \mathcal{R}^N$ be such that $R'_{i} = R_i$ for all $i \in B$. Since $B \in D_f$, we have $f(R') = R^*$. Let $R''$ be such that $R''_{i} = R'_i$ for all $i \in A \cap B$; $R''_{i} = R'_i$ for all $i \in A \setminus B$; $R''_{i} = R_i$ for all $i \in B \setminus A$; and $R''_{i} = R'_i$ for all $i \notin A \cup B$.

Note that $R'' \in \mathcal{C}(R, R')$.

Since $f(R) = f(R')$, HI implies that $P(R^*) \subseteq P(f(R''))$.

Thus, $(x, y) \in P(f(R''))$. Note that each individual in $A \cap B$ prefers $x$ to $y$ and that the preferences of the individuals outside of $A \cap B$ are not identified. IIA implies that $A \cap B$ is decisive over $(x, y)$ for $f$. ■

The following result implies that the converse of Theorem 1 must be true.

Theorem 2. Given a filter $\mathcal{F}$ on $N$, there exists a collective choice rule $f$ satisfying WP, IIA, and HI such that $D_f = \mathcal{F}$.

Proof. Define $f$ as follows:

$$f(R) = \left( \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R_i) \right)^{cd}.$$ 

Thus, $(x, y) \in f(R)$ if there exists no $A \in \mathcal{F}$ such that $(y, x) \in P(R_i)$ for all $i \in A$.

(i) Both WP and IIA are satisfied.

(ii) We now show that HI is satisfied. Suppose that $f(R) = f(R')$. Then,

$$\bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R_i) = \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R'_i).$$

Suppose that $(x, y) \in P(f(R)) = P(f(R'))$. Then,

$$(x, y) \in \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R_i) = \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R'_i).$$
\{i \in N : (x, y) \in P(R_i)\} \in \mathcal{F} \text{ and } \{i \in N : (x, y) \in P(R'_i)\} \in \mathcal{F}$. Take any $R'' \in \mathcal{C}(R, R')$. Since $R'' = R_i$ or $R'' = R_i$ for all $i \in N$, $(x, y) \in P(R''_i)$ for all $i \in \{i \in N : (x, y) \in P(R_i)\} \cap \{i \in N : (x, y) \in P(R'_i)\}$.

$$\{i \in N : (x, y) \in P(R''_i)\} \in \mathcal{F}.$$  

We have $\{i \in N : (y, x) \in P(R''_i)\} \notin \mathcal{F}$. To the contrary, suppose that $\{i \in N : (y, x) \in P(R''_i)\} \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, the finite intersection property implies that

$$\{i \in N : (y, x) \in P(R''_i)\} \cap \{i \in N : (x, y) \in P(R''_i)\} = \emptyset,$$

which contradicts the fact that $\emptyset \notin \mathcal{F}$.

(iii) We now show that $\mathcal{D}_f = \mathcal{F}$. Assume that $A \notin \mathcal{F}$. This implies that $B \notin \mathcal{F}$ for all $B \subseteq A$. We have $(x, y) \notin f(R)$ when $(x, y) \in P(R_i)$ for all $i \in A^c$. Thus, $A \notin \mathcal{D}_f$. Assume that $A \in \mathcal{F}$. If $(x, y) \in P(R_i)$ for all $i \in A$, then $(x, y) \notin f(R)$ because $A \in \mathcal{F}$. We have $\{i \in N : (y, x) \in P(R_i)\} \notin \mathcal{F}$ because $\mathcal{F}$ is a filter. Thus, $(y, x) \notin f(R)$. Thus, $A \in \mathcal{D}_f$. ■

Theorems 1 and 2 are counterparts of Hansson’s theorems. Note that the constructed collective choice rule in the proof of Theorem 2 is also quasi-transitive. Given a filter $\mathcal{F}$ on $N$, there exists a quasi-transitive collective choice rule $f$ satisfying WP, IIA, and HI such that $\mathcal{D}_f = \mathcal{F}$.

Subsequently, we introduce a weakening of HI. We consider a hybrid of $K$ preference profiles.

Define

$$\mathcal{C}_K(R^0, R^1, \ldots, R^K) = \{R \in \mathcal{R}^N : \forall i \in N, \exists k \in \{0, 1, \ldots, K\}, R_i = R^k_i\}$$

A hybrid preference in $\mathcal{C}_K(R^0, R^1, \ldots, R^K)$ can be defined in a recursive way: $R \in \mathcal{C}^*(R^0, \ldots, R^K)$ if and only if there exists a list of preference profiles $(\hat{R}^0, \hat{R}^1, \ldots, \hat{R}^K)$ such that

$$\hat{R}^0 = R^0, \hat{R}^k \in \mathcal{C}(\hat{R}^{k-1}, R^k) \text{ for all } k \in \{1, \ldots, K\}, \text{ and } \hat{R}^K = R.$$  

The following axiom is weaker than HI.

**Weak Hybrid Invariance (WHI):** For all $K \in \mathbb{N}$ and all $R^0, R^1, \ldots, R^K \in \mathcal{R}^N$, if $f(R^k) = f(R^\ell)$ for all $k, \ell \in \{0, 1, \ldots, K\}$, then $P(f(R^0)) \subseteq f(R)$ for all $R \in \mathcal{C}_K(R^0, R^1, \ldots, R^K)$.

The following result is an appropriate counterpart of Theorem 1.

**Theorem 3.** If a collective choice rule $f$ satisfies WP, IIA, and WHI, then $\mathcal{D}_f$ is a prefilter on $N$.

**Proof.** (i) WP implies that $N \in \mathcal{D}_f$ and $\emptyset \notin \mathcal{D}_f$.

(ii) By the definition of $\mathcal{D}_f$, if $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

(iii) Take $A^0, A^1, \ldots, A^K \in \mathcal{D}_f$. Let $R^k \in \mathcal{R}^N$ be such that

$$R_i = R^*_i \text{ for all } i \in A^k;$$

$$R_i = (R^*)^d \text{ for all } i \notin A^k.$$  

Since $A^k \in \mathcal{D}_f$, we have $f(R^k) = R^*$. Thus, $f(R^k) = f(R^\ell)$ for all $k, \ell \in \{0, 1, \ldots, K\}$.  

6
Let \( R \in \mathcal{R}^N \) be such that
\[
R_i = R^* \text{ for all } i \in \bigcap_{k \in \{0,1,\ldots,K\}} A^k;
\]
\[
R_i = (R^*)^d \text{ for all } i \notin \bigcap_{k \in \{0,1,\ldots,K\}} A^k.
\]

Note that \( R \in \mathcal{C}_K(\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_K) \). From WHI, it follows that \( P(R^*) \subseteq f(R) \). If \( \bigcap_{k \in \{0,1,\ldots,K\}} A^k = \emptyset \), then WP implies that \( P(f(R)) = P((R^*)^d) \). This is a contradiction. Thus, it follows that \( \bigcap_{k \in \{0,1,\ldots,K\}} A^k \neq \emptyset \).

A prefilter is sufficient to construct \( f \) satisfying WP, IIA, and WHI.

**Theorem 4.** Given a prefilter \( \mathcal{F} \) on \( N \), there exists a collective choice rule \( f \) satisfying WP, IIA, and WHI such that \( D_f = \mathcal{F} \).

**Proof.** Define
\[
f(R) = \left( \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R_i) \right)^{cd}.
\]

(i) Both WP and IIA are satisfied.

(ii) We now show that WHI is satisfied. Let \( R^0, R^1, \ldots, R^K \in \mathcal{R}^N \) such that \( f(R^k) = f(R^\ell) \) for all \( k, \ell \in \{0,1,\ldots,K\} \). By way of contradiction, suppose that \( (x, y) \in P(f(R^\eta)) \), which implies that \( (x, y) \in \bigcap_{i \in A^*} P(R^k_i) \) for some \( A^0 \in \mathcal{F} \). For all \( k \in \{1,\ldots,K\} \), there exists \( A^k \in \mathcal{F} \) such that \( (x, y) \in \bigcap_{i \in A^k} P(R^k_i) \). This implies that for all \( R \in \mathcal{C}_K(R^0, R^1, \ldots, R^K) \), \( (x, y) \in P(R_i) \) for all \( i \in \bigcap_{k \in \{0,1,\ldots,K\}} A^k \). Note that \( (x, y) \notin f(R) \) only if \( (y, x) \in \bigcap_{i \in A^*} P(R^0_i) \) for some \( A^* \in \mathcal{F} \). This implies that \( A^* \bigcap \bigcap_{k \in \{0,1,\ldots,K\}} A^k = \emptyset \). Since \( \mathcal{F} \) is a prefilter on \( N \), there exists no \( A^* \in \mathcal{F} \) such that
\[
A^* \bigcap \bigcap_{k \in \{0,1,\ldots,K\}} A^k = \emptyset.
\]
This is a contradiction.

(iii) Following the same procedure as in the proof of Theorem 2, we can show that \( D_f = \mathcal{F} \).

4 Ultrafilter and Strong Hybrid Invariance

The strong version of HI is formulated as follows.

**Strong Hybrid Invariance (SHI):** For all \( R, R' \in \mathcal{R}^N \), if \( f(R) = f(R') \), then \( f(R) = f(R'') \) for all \( R'' \in \mathcal{C}(R, R') \).

Neither the Pareto extension rule nor the Pareto rule satisfies SHI. Thus, SHI is not satisfied under the “proper” oligarchic structures.

The collection of decisive coalitions might not form an ultrafilter under WP, IIA, and SHI. To see this, consider the following CCR \( \hat{f} \):
\[
\hat{f}(R)|_{\{x, y\}} = \begin{cases} R_1|_{\{x, y\}} & \text{if } \{x, y\} = \{x^*, y^*\}; \\
R_2|_{\{x, y\}} & \text{otherwise.}
\end{cases}
\]
\(\hat{f}\) satisfies WP, IIA, and SHI. \(D_{\hat{f}}\) is not an ultrafilter.

There exists a one-to-one relationship between an ultrafilter and SHI when a class of rules is restricted in a certain manner. Given \(F \subseteq 2^V\), define \(f_F\) as follows:

\[
f_F(R) = \left( \bigcup_{A \in F} \bigcap_{i \in A} P(R_i) \right)^{cd}.
\]

**Theorem 5.** Let \(F \subseteq 2^N\). A collective choice rule \(f_F\) satisfies WP, IIA, and SHI if and only if \(F\) is an ultrafilter on \(N\).

**Proof.** ‘If.’ Suppose that \(F\) is an ultrafilter on \(N\). Since both WP and IIA are obviously satisfied, it suffices to show that SHI is satisfied. Let \(R, R' \in R^N\) be such that

\[
f(R) = f(R').
\]

This is equivalent to the following:

\[
\bigcup_{A \in F} \bigcap_{i \in A} P(R_i) = \bigcup_{A \in F} \bigcap_{i \in A} P(R_i')
\]

Take \(R'' \in C(R, R')\). It suffices to show that \(P(f_F(R)) \supseteq P(f_F(R''))\). By way of contradiction, assume that \((x, y) \in \bigcup_{A \in F} \bigcap_{i \in A} P(R_i')\) and \((x, y) \notin \bigcup_{A \in F} \bigcap_{i \in A} P(R_i)\). Then, \((x, y) \in \bigcap_{i \in B} P(R_i'')\) for some \(B \in F\) and \((x, y) \notin \bigcap_{i \in C} P(R_i')\) for no \(C \in F\). The latter implies that \(\{i \in N : (x, y) \notin P(R_i')\} \notin F\). \(\{i \in N : (x, y) \in P(R_i')\} \in F\) but \(\{i \in N : (x, y) \in P(R_i)\} \notin F\). Since \(F\) is an ultrafilter on \(N\), we have \(\{i \in N : (y, x) \in R_i\} \in F\). Similarly, \(\{i \in N : (y, x) \in R_i'\} \in F\). By the finite intersection property, \(B \cap \{i \in N : (y, x) \in R_i\} \cap \{i \in N : (y, x) \in R_i'\} \in F\). Let

\[
\hat{A} = \{i \in N : (y, x) \in R_i\} \text{ and } \hat{B} = \{i \in N : (y, x) \in R_i'\}.
\]

Since \(R'' \in C(R, R')\), we have \(\{i \in N : (y, x) \in P(R_i')\} \subseteq \{i \in N : (x, y) \in P(R_i)\} \cup \{i \in N : (x, y) \in P(R_i')\}\). Note that \(\{i \in N : (x, y) \in P(R_i)\} = \hat{A}^c\) and \(\{i \in N : (x, y) \in P(R_i')\} = \hat{B}^c\). It is obvious that \(B \cap \hat{A} \cap \hat{B} \subseteq \{i \in N : (x, y) \in P(R_i')\} \cap \hat{A} \cap \hat{B} \subseteq (\hat{A}^c \cup \hat{B}^c) \cap \hat{A} \cap \hat{B}\). By De Morgan’s laws,

\[
(\hat{A} \cap \hat{B})^c \cap (\hat{A} \cap \hat{B}) = \emptyset.
\]

Thus, \(B \cap \hat{A} \cap \hat{B} = \emptyset\). This is a contradiction.

‘Only if.’ Suppose that a collective choice rule \(f_F\) satisfies WP, IIA, and SHI. As shown in Theorem 2, the collection of decisive coalitions for \(f_F\) is identical with \(F\). Theorem 1 implies that \(F\) is a filter. By way of contradiction, suppose that there exists \(A \subseteq N\) such that \(A \notin F\) and \(A^c \notin F\). Take any linear order \(R^*\). Let \(R \in R^N\) be such that

\[
R_i = R^* \text{ for all } i \in A, \quad R_i = (R^*)^c \text{ for all } i \in A^c.
\]

By definition, \(f(R) = X \times X\).

Let \(R' \in R^N\) be such that

\[
R_i' = (R^*)^c \text{ for all } i \in A, \quad R_i' = R^* \text{ for all } i \in A^c.
\]

8
5 Hybrid Invariance and Social Rationality

Now, we discuss the relationship between hybrid invariance and social rationality. The analysis in the preceding sections suggests that hybrid invariance restricts the decisive structure in a similar manner to social rationality. The following result implies that we can observe the direct connection between hybrid invariance and social rationality for a certain class of collective choice rules.

**Theorem 6.** Let \( F \subseteq 2^N \) with \( N \in F \) and \( \emptyset \in F \).

(i) A collective choice rule \( f_F \) satisfies HI if and only if it is quasi-transitive.

(ii) A collective choice rule \( f_F \) satisfies SHI if and only if it is transitive and complete.

(iii) A collective choice rule \( f_F \) satisfies WHI if and only if it is acyclic.

**Proof.** (i) ‘If.’ Suppose that \( f_F \) is quasi-transitive. We first show that if \( A, B \in F \), then \( A \cap B \in F \).

Quasi-transitivity is not satisfied. Thus, if \( A, B \in F \), then \( A \cap B \notin F \). Let \( R \in R^N \) be such that

\[
\{(x,y),(y,z),(x,z)\} \subseteq P(R_i) \text{ for all } i \in A \cap B,
\{(x,y),(z,x),(z,y)\} \subseteq P(R_i) \text{ for all } i \in A \setminus B,
\{(z,x),(y,z),(y,x)\} \subseteq P(R_i) \text{ for all } i \in B \setminus A,
\{(z,y),(y,x),(z,x)\} \subseteq P(R_i) \text{ for all } i \notin A \cup B.
\]

This implies that \( \bigcup_{A \in F} \bigcap_{i \in A} P(R_i') = \bigcup_{A \in F} \bigcap_{i \in A} P(R_i'') \). Take \( R''' \in C(R',R'') \). We prove that \( P(f_F(R')) \subseteq P(f_F(R''')) \). It suffices to show that

\[
\forall x,y \in X : (x,y) \in \bigcup_{A \in F} \bigcap_{i \in A} P(R_i') \Rightarrow (x,y) \in \bigcup_{A \in F} \bigcap_{i \in A} P(R_i''').
\]
Note that \((x, y) \in \bigcap_{i \in B} P(R'_i)\) for some \(B \in \mathcal{F}\) and \((x, y) \in \bigcap_{i \in C} P(R''_i)\) for some \(C \in \mathcal{F}\). Since \(\mathbf{R}''' \in C(\mathbf{R}', \mathbf{R}'')\), \((x, y) \in \bigcap_{i \in B \cap C} P(R'_i)\). We have \(B \cap C \in \mathcal{F}\). Thus, it follows that \((x, y) \in \bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} P(R''_i)\). HI is satisfied.

‘Only if.’ Suppose that \(f_{\mathcal{F}}\) satisfies HI. Since \(N \in \mathcal{F}\) and \(\emptyset \in \mathcal{F}\), WP is satisfied. By the construction of \(f_{\mathcal{F}}\), IIA is also satisfied. Theorem 1 implies that the family of decisive coalitions is a filter on \(N\). By Theorem 2, the family is identical to \(\mathcal{F}\). Thus, \(\mathcal{F}\) is a filter on \(N\).

Suppose that \((x, y) \in P(f_{\mathcal{F}}(\mathbf{R}))\) and \((y, z) \in P(f_{\mathcal{F}}(\mathbf{R}))\). Then, there exist \(A, B \in \mathcal{F}\) such that

\[
(x, y) \in \bigcap_{i \in A} P(R_i) \quad \text{and} \quad (y, z) \in \bigcap_{i \in B} P(R_i).
\]

Since \(\mathcal{F}\) is a filter, \(A \cap B \in \mathcal{F}\). Note that \((x, z) \in \bigcap_{i \in A \cap B} P(R_i)\). By definition of \(f_{\mathcal{F}}\), we have \((x, z) \in P(f_{\mathcal{F}}(\mathbf{R}))\).

(ii) ‘If.’ Suppose that \(f_{\mathcal{F}}\) is transitive and complete. By (i), it is true that if \(A, B \in \mathcal{F}\), then \(A \cap B \in \mathcal{F}\). We now show that if \(A \notin \mathcal{F}\), then \(A^c \in \mathcal{F}\). Suppose that \(A \notin \mathcal{F}\) and \(A^c \notin \mathcal{F}\). Let \(\mathbf{R} \in \mathbf{R}^N\) be such that

\[
\{(x, y), (z, y), (x, z)\} \subseteq \bigcap_{i \in A} P(R_i),
\]

\[
\{(y, x), (y, z), (x, z)\} \subseteq \bigcap_{i \in A^c} P(R_i).
\]

Since \(A \notin \mathcal{F}\) and \(A^c \notin \mathcal{F}\), \((x, y) \in I(f_{\mathcal{F}}(\mathbf{R}))\) and \((y, z) \in I(f_{\mathcal{F}}(\mathbf{R}))\). Since \(N \in \mathcal{F}\), \((x, z) \in P(f_{\mathcal{F}}(\mathbf{R}))\). Transitivity is not satisfied. Thus, if \(A \notin \mathcal{F}\), then \(A^c \in \mathcal{F}\).

We show that SHI is satisfied. Let \(\mathbf{R}', \mathbf{R}'' \in \mathbf{R}^N\) be such that

\[
f_{\mathcal{F}}(\mathbf{R}') = f_{\mathcal{F}}(\mathbf{R}'').
\]

By following a method similar to the ‘if’ part of Theorem 5, we can prove that \(f_{\mathcal{F}}(\mathbf{R}'') = f_{\mathcal{F}}(\mathbf{R}''')\) for all \(\mathbf{R}''' \in C(\mathbf{R}', \mathbf{R}'')\).

‘Only if.’ Suppose that \(f_{\mathcal{F}}\) satisfies SHI. Since \(N \in \mathcal{F}\) and \(\emptyset \in \mathcal{F}\), WP is satisfied. IIA is also satisfied. Then, \(\mathcal{F}\) is an ultrafilter on \(N\). To show that transitivity is satisfied, suppose that \((x, y) \in f_{\mathcal{F}}(\mathbf{R})\) and \((y, z) \in f_{\mathcal{F}}(\mathbf{R})\). Then,

\[
(y, x) \in \bigcap_{i \in A} P(R_i) \quad \text{for no} \ A \in \mathcal{F} \quad \text{and} \quad (z, y) \in \bigcap_{i \in B} P(R_i) \quad \text{for no} \ B \in \mathcal{F}.
\]

Thus, \(\{i \in N : (y, x) \in P(R_i)\} \notin \mathcal{F}\) and \(\{i \in N : (z, y) \in P(R_i)\} \notin \mathcal{F}\).

\[
\{i \in N : (x, y) \in R_i\} \notin \mathcal{F} \quad \text{and} \quad \{i \in N : (y, z) \in R_i\} \notin \mathcal{F}.
\]

\[
\{i \in N : (x, z) \in R_i\} \notin \mathcal{F}.
\]

This implies that \(\{i \in N : (z, y) \in P(R_i)\} \notin \mathcal{F}\). Then, \((x, z) \in f_{\mathcal{F}}(\mathbf{R})\). Completeness is obviously satisfied.

(iii) Since the proof is similar to that of (i), we omit it.

Finally, we show the relationship between HI and Plott’s path independence. Given \(A \subseteq X\) and \(\mathbf{R} \in \mathbf{R}^N\), the set of maximal elements is defined as follows:

\[
M_f(A, \mathbf{R}) = \{x \in A : (y, x) \in P(f(\mathbf{R})) \text{ for no } y \in A\}.
\]
If $f$ is not acyclic, $M_f(A, R)$ is empty for some $A \subseteq X$ and some $R \subseteq \mathcal{R}^N$.

$f$ is said to be path independent when

$$M_f(A \cup B, R) = M_f(M_f(A, R) \cup M_f(B, R), R)$$

for all $A, B \subseteq X$ and $R \in \mathcal{R}^N$.

As a corollary to Theorem 6, we have the following result.

**Corollary 1.** Let $\mathcal{F} \subseteq 2^N$ with $N \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$. A collective choice rule $f_\mathcal{F}$ satisfies HI if and only if it is path-independent.

**Proof.** It suffices that $f_\mathcal{F}$ is quasi-transitive if and only if it is path independent. If $f_\mathcal{F}$ is quasi-transitive, then $f$ is path independent. This follows from Plott (1972, Theorem 4).

Suppose that $f_\mathcal{F}$ is not quasi-transitive. There exist $R \in \mathcal{R}^N$ and $x, y, z \in X$ such that $(x, y) \in P(f_\mathcal{F}(R))$, $(y, z) \in P(f_\mathcal{F}(R))$, and $(x, z) \notin P(f_\mathcal{F}(R))$. Let us denote $M_{f_\mathcal{F}}(A, R)$ by $M(A)$. Since $M(\{x, y, z\}) = \{x\}$, $M(\{x, y\}) = \{x\}$, and $M(\{x, z\}) = \{x, z\}$, $M(\{x, y, z\}) \neq M(M(\{x, y\}) \cup M(\{z\}))$. Thus, $f$ is not path independent.

Under certain conditions, HI is equivalent to quasi-transitive rationality.

## 6 Concluding Remarks

There exist many possible formulations of stability. In this paper, we introduced the concept of hybrid invariance, which requires that social preferences be stable under the hybridization of individual preference profiles. We examined (i) how hybrid invariance restricts the decisive structure and (ii) how hybrid invariance is related with social rationality.

We considered three versions of hybrid invariance: hybrid invariance, weak hybrid invariance, and strong hybrid invariance. They are systematically associated with the decisive structure. The family of decisive coalitions forms a filter under hybrid invariance, it forms a prefilter under weak hybrid invariance, and it forms an ultrafilter under strong hybrid invariance. We found that each concept of hybrid invariance corresponds to some degree of social rationality. Thus, our stability concept serves as an alternative rationale for social rationality.

## Acknowledgements

This paper was financially supported by Grant-in-Aids for Young Scientists (B) from the Japan Society for the Promotion of Science and the Ministry of Education, Culture, Sports, Science and Technology (30553101).

## References


