Public education expenditure and economic growth: a V-shaped relationship *

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Abstract

This paper examines the effects of increasing public educational expenditure on the fertility rate and the long-run economic growth rate analytically. Considering the public education policy, individuals decide the number of children and the level of education investment in their children. This paper find that there exists a V-shaped relationship between public education expenditure and the long-run economic growth rate. In addition, if the life expectancy is sufficiently large, a rise in the labor income tax rate to finance public education expenditure can promote the long-run economic growth rate and increase the fertility rate.

Keywords Endogenous fertility, Public education, Human capital, Longevity

JEL-Classification J13, I22, H52

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A large number of endogenous growth literature concludes that the public education expenditure promotes human capital accumulation and consequently promotes the economic growth. But, the relationship between the size of public education expenditure and the economic growth rate is unclear in the empirical literature. Some recent literature on the endogenous growth theory insist that a rise of the public education expenditure not only stimulates economic growth but also have some offsetting effects.

This paper also explain why does the public education expenditure have the mixed effect on the long-run economic growth from the new view point. I assume that households decide both fertility and private education decisions and therefore they face the quality and quantity trade-off of children. Financing public education expenditure affect their private educational investment per child directly but also their fertility choice indirectly because households change their decisions about the quality and quantity of children in respond to the public education policy as Figure 1.

The theory is based on a closed three-period-lived OLG economy populated by indi-

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2For example, Blankenau and Simpson (2004) and Basu and Bhattacharai (2012) emphasize that the way of financing the public education expenditure or the tax burden to finance the public education expenditure can have a negative effect on the economic growth.

3The data for the public education expenditure as a percentage of GDP in 2011 and the private education enrollment rate in 2012 is from OECD (2014). The private education enrollment rate represents the sum of the enrollment rate in the government-depended and private -depended private schools. The data for total fertility rate in 2011 is from ?.
iduals who live for three periods: childhood, adulthood and old-age. Considering the public education policy, households decides not only their consumption schedules but also the number of children and the children’s quality respectively. In this analysis, differently from Zhang (1997), the public education investment is in-kind benefits, which is measured in units of teachers employed by the government or instruction hours in public schools.

I show that there exist two education cases depending on the public education policy. The one is the mixed education case in which both households and the government invest children’ education and the other is the complete public education case in which the government only invest their children. Financing the public education expenditure has different effects on the long-run economic growth depending on these two educational cases. In the mixed education case, the public education expenditure impedes the economic growth because expanding the public education expenditure increase fertility indirectly and crowd out the private education expenditure significantly. On the contrary, in the complete public education case, the public education expenditure promotes economic growth simply because financing the public education expenditure increases the total educational investment.

Previous literature examine the effects of the public education expenditure on the private education and the long-run economic growth rate with the endogenous fertility. Azarnert (2008) and Azarnert (2010) suggest the possibility that the free public education accelerates the poverty and slows the economic growth because of the household’s endogenous fertility decision. But they does not consider how the government finances the public education expenditure and what effects financing the public education expenditure has on household’s optimal decisions. Zhang (1997) find that the public education expenditure in the form of the educational subsidy not only increases the private education expenditure directly but also increases fertility indirectly because the public education subsidies lowers the cost of educating children relative to the marginal cost of rearing a child. Both direct and indirect effects promotes the long-run economic growth. In contrast, in this paper, the public education expenditure has two conflict effects on the long-run economic growth. First, the public education expenditure increases the per-child public education investment directly and promotes the long-run economic growth. Second, the public education expenditure increases the fertility rate and crowds out the private education investment because the public education expenditure decreases the marginal cost of rearing a child relative to the marginal cost of educating children. The indirect increase in fertility and decrease in the public education investment decreases the per-child public

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4These two educational cases are observed in reality. In some East-Asian countries such as Japan and Korea, a large number of households leave their children’s education go to both public and private schools. In contrast, in some European countries, households mainly leave their children’s education to public schools. From OECD (2014), the percentage of students in private-dependent and government-dependent upper secondary school stand at 31% in Japan, 44% in Korea, on the contrary, 2% in Denmark and 8% in Germany.
educational investment and impedes the long-run economic growth.

In addition, this paper find that the growth maximizing education policy in BGP depends on the life expectancy of old-age-period individuals. The life-expectancy of old age generations influences on individual’s incentive to have children and, therefore, per child public education investment. If the life expectancy is sufficiently high, the government can maximize the long-run economic growth by increasing the public education expenditure. This result has the implication about the public education policy in countries that face the problem with the declining birth rate and long longevity. For example, some East-Asian countries such as Japan and Korea face the low birth rate and the high longevity in which households bear a major part of education\(^5\). In these countries, financing the public education expenditure can not only promote the long-time economic growth but also improve the population aging problem.

The rest of paper is organized as follows: Section 2 presents the model used in this paper. Section 3 examines fertility rate and per capita output growth rate in the BGP. Section 4 devotes to show the effects of public education expenditure on the economic growth rate in the BGP and derive the growth-maximizing public education policy. The last section offers concluding remarks.

\section{The model}

In this section, I build the model used in this paper. Time is discrete and goes from zero to infinite. I consider a closed-economy that is populated by overlapping generations of individuals who live for three periods; childhood, adulthood and old-age. All decisions are made in the adulthood period of life. Individuals go to private and public schools in the childhood period passively. They decide their consumption schedule, the number of children and the quality of children in the adulthood period. They retire and consume their saving in the old-age period. Individuals face uncertainly about their lifetime at the beginning of old-age period.

\subsection{Individuals}

Individuals care about the adulthood consumption \(c_t\), the old-age consumption \(d_{t+1}\), their number of children \(n_t\), and the quality of children \(h_{t+1}\). Preferences of any individual in generation \(t\) are given by

\begin{equation}
U_t = \log c_t + p \log d_{t+1} + \gamma \log n_t h_{t+1},
\end{equation}

\footnote{From \cite{source}, the total fertility rates in Japan and Korea in 2012 are 1.39 and 1.24 respectively, while the average total fertility rate in OECD is 1.70.}
where $c_t, d_{t+1}$ are consumptions of generation $t$. The parameter $p \in (0,1)$ is the probability that individuals alive at the beginning of the old-age period and $\gamma \in (0,1)$ is an impure altruism factor for children. The budget constraints for the adulthood period and the old-age period are

$$
c_t = (1 - \tau)(1 - \phi n_t)w_t h_t - w_t h_t n_t e_t - s_t, \quad (2)
$$

$$
d_{t+1} = (1 + r_{t+1})s_t. \quad (3)
$$

Generally speaking, child-rearing is a labor intensive in parental time especially in mother’s time. Therefore I assume that raising one child takes a constant fraction $\phi \in (0,1)$ of an adult’s time.

The human capital of generation $t+1$ is accumulated by three factors; the human capital of their parents $h_t$, the per-child public educational investment $E_t$ and the per-child private education investment $e_t$

$$
h_{t+1} = B(E_t + \psi e_t)\eta h_t, \quad (4)
$$

where $B > 0$ is an efficiency parameter and $\eta \in (0,1)$ measures the elasticity of children’s human capital accumulation to the total educational investments. The private and public education investments are measured in units of time of teachers employed by individuals and the government. I assume that private education investment is perfectly substitutes to the private educational investment but $\psi \geq 1$ times as more efficient as public educational investment. And the initial old-age individuals acquire the initial labor efficiency $h_0$.

Given the wage rate $w_t$, the rental price $r_{t+1}$, the per-child child-rearing time $\phi$, the labor income tax rate $\tau$ and the per-child public education expenditure $E_t$, individuals during the adulthood period in generation $t$ choose the saving $s_t$, the number of children $n_t$ and the per-child private education investment $e_t$ to maximize their lifetime utility subject to the budget constraints (2), (3) and the technology of human capital accumulation (4). To find the solution to this individual’s optimization problem, I set up a Kuhn-Tucker problem as follows

$$
\mathcal{L} \equiv \log c_t + p \log d_{t+1} + \gamma \log n_t B(E_t + \psi e_t)\eta h_t
$$

$$
+ \lambda_1 \left\{ (1 - \tau)w_t h_t - (1 - \tau)\phi w_t h_t n_t - w_t h_t n_t e_t - c_t - \frac{d_{t+1}}{1 + r_{t+1}} \right\} + \lambda_2 e_t,
$$

where $\lambda_1$ and $\lambda_2$ are shadow prices which are associated with the lifetime budget and

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6This form of lifetime uncertainly is also adopted in Yakita (2001), Pecchenino and Pollard (2002).

7This human capital accumulation function is similar to the settings used in Eckstein and Zilcha (1994), Kaganovich and Zilcha (1999) and Azarnert (2010).
private educational investment respectively. Kuhn-Tucker conditions to this problem are

\[
\frac{1}{c_t} = \lambda_1, \\
\frac{p}{d_{t+1}} = \frac{\lambda_1}{1 + r_{t+1}}, \\
\frac{\gamma}{n_t} = \lambda_1 \{(1 - \tau)\phi w_t h_t + w_t h_t e_t\}, \\
\frac{\gamma \eta \psi}{E_t + \psi e_t} = \lambda_1 w_t h_t n_t + \lambda_2, \\
(1 - \tau)w_t h_t = c_t + \frac{d_{t+1}}{1 + r_{t+1}} + w_t h_t n_t e_t + (1 - \tau)\phi w_t h_t n_t, \\
\lambda_2 e_t = 0.
\]

From these conditions, the saving of generation \( t \) is

\[
s_t = \left( \frac{p}{1 + p + \gamma} \right) (1 - \tau)w_t h_t, \tag{5}
\]

There exist two cases of household’s private education investment: the one is an inner solution \( e > 0 \) and the other is corner solution \( e = 0 \).

\( \lambda_2 = 0 \): An optimal inner solutions for the per-child private education investment and the optimal number of children at time \( t \) is given by

\[
e_t = \frac{\psi \eta \phi (1 - \tau) - E_t}{\psi (1 - \eta)}, \tag{6}
\]

\[
n_t = \frac{\hat{\gamma} (1 - \tau)}{\psi \phi (1 - \tau) - E_t} \equiv n_t^{ME}, \tag{7}
\]

where \( \hat{\gamma} \equiv \frac{\gamma \phi (1 - \eta)}{1 + p + \gamma} \). I name this case as "the mixed education case" and place a superscript "ME" to the upper right of a letter.

\( \lambda_2 > 0 \): In contrast, the optimal corner solutions for the per-child private education investment and the optimal number of children at time \( t \) is

\[
e_t = 0, \tag{8}
\]

\[
n_t = \frac{\gamma}{\phi (1 + p + \gamma)} \equiv n_t^{PE}, \tag{9}
\]

I name this case as "the complete public education case" and place a superscript "PE" to the upper right of a letter. Note that the fertility rate in two cases are negatively relate to the life expectancy: \( \frac{\partial n^{ME}}{\partial p} < 0, \frac{\partial n^{PE}}{\partial p} < 0 \).
2.2 The production technology

There is a unique consumption goods in this economy. Production of this consumption
goods is carried out by a single representative firm. Production technology at time $t$ is as follows

$$Y_t = K_t^\alpha H_t^{1-\alpha},$$

(10)

where $K_t$ is the aggregate physical capital and $H_t$ is the aggregate labor at time $t$ and a
parameter $\alpha \in (0, 1)$ is the share of aggregate physical capital in production. Here, the
physical capital depletes completely after one period use in production.

The production sector chooses the optimal level of the aggregate physical capital de-
mand $K_t$ and the effective labor demand $H_t$ to maximize profits $Y_t - w_t H_t - r_t K_t$. From
the first order conditions, the wage rate and the rental price are derived as follows

$$w_t = (1 - \alpha) \left(\frac{K_t}{H_t}\right)^\alpha,$$

(11)

$$1 + r_t = \alpha \left(\frac{K_t}{H_t}\right)^{\alpha-1}.$$

(12)

2.3 Markets

The equilibrium condition in the physical capital market is

$$K_t = s_{t-1} P_{t-1},$$

(13)

The initial old-age individuals are endowed with the initial capital stock $K_0$ which is in-
stalled in the initial period $t = 0$.

The production sector employs workers in the production labor market for producing
the consumption goods.

$$H_t = P_t (1 - \phi n_t) h_t,$$

(14)

The LHS of (14) is the effective labor demand and the RHS of it means the effective labor
supply, that is to say, the population size $P_t$ times per worker labor hours $(1 - \phi n_t)$ and
worker’s labor efficiency $h_t$.

2.4 The government

In this economy, the public and private education investment is measured in units of
teachers whose average human capital is $h_t$ employed by the government and house-
holds. I abstract that both individuals and the government can employ any number of
teachers for the educational investment in period \( t \). The public education expenditure in period \( t \) is financed by the labor income tax rate collected in the period \( t \)

\[
E_t w_t h_t P_t n_t = \tau H_t w_t. \tag{15}
\]

From Eq.\((14)\) and \((15)\), the public educational investment is rewritten as

\[
E_t = \frac{\tau (1 - \phi n_t)}{n_t}. \tag{16}
\]

In the following discussions, the government determines and commits the sequences of education tax rate \( \tau \) before individuals solve their problem. And I assume that the labor income tax rate is assumed to be constant across time.

Now that the competitive equilibrium in this economy is defined as follows:

**Definition 1. Competitive Equilibrium:**

Given the initial stock of aggregate physical capital \( K_0 \), the initial stock of human capital per capita \( h_0 \), the initial population size \( P_0 \), an equilibrium consists of sequences of household’s decision rules \( \{c_t, d_{t+1}, s_t, n_t, e_t\}_{t=0}^{\infty} \), factor prices \( \{r_{t+1}, w_t\}_{t=0}^{\infty} \), the public education investments \( \{E_t\}_{t=0}^{\infty} \) and aggregate valuables \( \{K_{t+1}, H_t\}_{t=0}^{\infty} \) such that:

1. the households’ decision rules \( \{c_t, d_{t+1}, s_t, n_t, e_t\}_{t=0}^{\infty} \) maximizes their lifetime utility function \( (1) \) subject to the budget constraints \( (2), (3) \) and the human capital accumulation function \( (4) \);
2. the factor prices \( \{r_{t+1}, w_t\}_{t=0}^{\infty} \) are such that markets clear, i.e., \((11)\) and \((12)\) hold;
3. the aggregate valuables \( \{K_{t+1}, L_t\}_{t=0}^{\infty} \) hold the market equilibrium conditions \( (13) \) and \((14)\);
4. the per-child public education investments \( \{E_t\}_{t=0}^{\infty} \) hold the government budget constraint \((15)\).

3 The fertility rate and the long-run economic growth rate

In this section, I derive the fertility rate, the per-child private education investment and the per capita output growth rate in the balanced growth path (henceforth called "BGP").

3.1 The fertility rate

Using Eq.\((7)\) and \((16)\), the fertility rate in the ME case is

\[
n^{*\text{ME}} = \frac{\gamma + (1 - \gamma) \tau}{\phi \{\psi + (1 - \psi) \tau\}}. \tag{17}
\]
By substituting Eq. (17) to (16), the per-child public education investment in the ME case is rewritten by
\[
E^{*\text{ME}} = \frac{\tau \phi (1 - \tau)(\psi - \gamma)}{\gamma + (1 - \gamma)\tau}. \tag{18}
\]

In the PE case, the fertility rate (9) does not depend on the public education policy. Thus, by using Eq. (9) and (16), the per-child public education investment in the PE case is
\[
E^{*\text{PE}} = \frac{\tau \phi (1 + p)}{\gamma}. \tag{19}
\]

From Eq. (19), I can get the following lemma.

**Lemma 1.**
\[
\tilde{\tau} = -\gamma + \sqrt{\gamma} \quad \frac{1}{2(1 - \gamma)}.
\]

A labor income tax rate has an inverted-U shaped relationship to the per-child public education investment in the ME case. When the labor income tax rate is sufficiently low \(0 < \tau < \tilde{\tau}\), an increase in labor income tax increases the per-child public education investment: \(\frac{\partial E^{*\text{ME}}}{\partial \tau} > 0\). On the contrary, when the labor income tax rate is sufficiently high \(\tilde{\tau} < \tau < 1\), an increase in labor income tax decreases the per-child public education investment: \(\frac{\partial E^{*\text{ME}}}{\partial \tau} < 0\).

**Proof.** See Appendix. \(\square\)

A rise in the labor income tax rate increases the per-child public education investment per child directly. At the same time, a rise in the labor income tax also increases the number of children, thus indirectly decreases the per-child public education investment. When the labor income tax rate is high, the direct positive effect can be dominated by the indirect negative effect.

### 3.2 The per-child private education investment

From Eq. (6) and (18), I can rewrite the equilibrium per-child private education investment in the ME case as follows
\[
e^* = \frac{\psi \eta \phi (1 - \tau)}{\psi (1 - \eta)} - \frac{E^{*\text{ME}}}{\psi (1 - \eta)}. \tag{20}
\]

I can get the condition that households invest in their children’s education privately as follows
Lemma 2.

\[
\tau = \frac{\gamma \phi \eta}{1 + p + \gamma \phi \eta}.
\]

When the labor income tax rate is sufficiently low \((\tau < \bar{\tau})\), households spend some part of income to invest in their children’s education. On the contrary, when the labor income tax rate is sufficiently high \((\bar{\tau} \leq \tau)\), households do not invest in their children’s education at all.

**Proof.** See Appendix.

This relationship between the labor income tax rate and the private education expenditure is illustrated in Figure 2. When the labor income tax rate is sufficiently low, a rise of labor income tax rate have two negative effects on household’s private education investment in the interval \((0, \bar{\tau})\). First, a rise of labor income tax rate decreases individual’s disposable income and thus the private education investment directly. Second, the private education expenditure crowds out the public education investment because the public and private education investment is completely substitutive. When the labor income tax rate becomes higher than the threshold income rate \(\bar{\tau}\), the public education investment becomes equal to zero.
3.3 The BGP economic growth rate

Now using Eq.(4), (18), (19) and (20), I derive the human capital accumulation rate in two education cases:

\[
\frac{h_{t+1}}{h_t} = \begin{cases} 
B \left\{ \left( \frac{\eta}{1-\eta} \right) \left( E^{*ME} + \psi e^* \right) \right\}^{\eta} & (0 \leq \tau < \bar{\tau}), \\
B \left( E^{*PE} \right)^{\eta} & (\bar{\tau} \leq \tau \leq 1), 
\end{cases} 
\]  

(21)

In addition, by using Eq.(17) and (9), the fertility rate is rewritten by

\[
n^* = \frac{P_{t+1}}{P_t} = \begin{cases} 
\frac{\hat{\gamma} + (1-\hat{\gamma}) \tau}{\phi(\psi + (1-\psi)\tau)} n^{*ME} & (0 \leq \tau < \bar{\tau}), \\
\frac{\gamma}{\phi(1+\rho+\gamma)} n^{*PE} & (\bar{\tau} \leq \tau \leq 1).
\end{cases} 
\]  

(22)

From Eq.(10), the per-capita output growth rate at time \( t \) is

\[
1 + g_t = \frac{Y_{t+1}}{Y_t} \frac{P_t}{P_{t+1}}, \\
= \frac{h_{t+1}}{h_t} \left( \frac{k_{t+1}}{k_t} \right)^{1/\alpha}, 
\]  

(23)

where \( k = K/L \) is a physical capital and effective labor ratio. Then, by using Eq.(5), (11), (13) and (14), I derive the dynamics of the physical capital and effective labor ratio:

\[
k_{t+1} = \left\{ \left( \frac{p}{1+\rho+\gamma} \right) \frac{(1-\tau)(1-\alpha)}{(1-\phi n^*)(h_{t+1}/h_t)} \right\} k_t^{\alpha}. 
\]  

(24)

The dynamics (24) always converges to the BGP capital/effective labor ratio \( k^* = k_t = k_{t+1} \) monotonously because the human capital accumulation rate (21) and the fertility rate (22) in the competitive equilibrium are constant across time and the physical capital share parameter is assumed to be \( 0 < \alpha < 1 \):

\[
k^* = \left\{ \left( \frac{p}{1+\rho+\gamma} \right) \frac{(1-\tau)(1-\alpha)}{(1-\phi n^*)n^*(h_{t+1}/h_t)} \right\}^{1/\alpha}. 
\]  

(25)

From Eq.(22), (23) and (24), the per-capita BGP economic growth rate equals to the human capital accumulation rate (21):

\[
1 + g^* = \begin{cases} 
B \left\{ \left( \frac{\eta}{1-\eta} \right) \left( E^{*ME} + \psi e^* \right) \right\}^{\eta} & (0 \leq \tau < \bar{\tau}), \\
B \left( E^{*PE} \right)^{\eta} & (\bar{\tau} \leq \tau \leq 1), 
\end{cases} 
\]  

(26)
Figure 3: The effects of increasing the labor income tax rate to finance the public education investment on the fertility rate and the BGP economic growth rate.

4 The policy experiments

In this section, I derive the effects of increasing the labor income tax rate to finance the public education investment on the fertility rate and the BGP economic growth rate in two education cases. Then, I compare the BGP growth rate in the ME case and the PE case in order to derive a growth-maximizing public education policy.

4.1 The relationship between the public education investment and the BGP economic growth rate

At first, for the simplicity, I assume that a rise in the labor income tax rate increases the per-child public education investment in the interval $[0, \tau]$. From Lemma 1, I put the following assumption:

Assumption 1.

$$\overline{\tau} \leq \overline{\tau}.$$ 

Now, I can derive the relationship between the labor income tax rate and the fertility rate in the BGP:

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8 Considering the case in which a rise in the labor income tax rate decreases the public education investment in the interval $[0, \tau]$ does not influence on the main result.
Lemma 3. In the ME case \( (0 \leq \tau < \overline{\tau}) \), a rise in the labor income tax rate to finance the public education investment increases the fertility rate: \( \frac{\partial n_{ME}}{\partial \tau} > 0 \). On the contrary, in the PE case \( (\overline{\tau} \leq \tau < 1) \), a rise in the labor income tax rate to finance the public education investment does not influence on the fertility rate at all: \( \frac{\partial n_{PE}}{\partial \tau} = 0 \).

Proof. See Appendix.

This lemma is illustrated in the Figure 3. In the ME case, a rise in the labor income tax rate to finance the public education investment has three effects on the fertility rate in the BGP. First, a rise in labor income tax rate decreases the household’s disposable income and, therefore, decreases the number of children. Second, a rise in per-child public education investment reduces household’s cost of educating their children because the public education investment substitute for the private education investment completely. Third, a rise in the labor income tax rate also decreases a household’s opportunity cost to raise children. A rise in the labor income tax rate increases the fertility rate in the interval \( 0 < \tau < \overline{\tau} \) because the second positive effect and the third positive effect dominates the first negative effect.

In the PE case, a rise in the per-child public education investment does not influence on fertility at all. This is because the public education investment does not effect the private education investment in the PE case \( (e = 0) \) and furthermore the first negative effect is offset by the second positive effect completely.

Next, by differentiating Eq.\( (27) \) with respect to \( \tau \), I derive the relationship between the labor income tax rate and the BGP growth rate as follows:

\[
\frac{\partial (1 + g^{*\text{ME}})}{\partial \tau} = B\eta \left( \frac{\eta}{1 - \eta} \right) \eta \left( E^{*\text{ME}} + \psi e \right)^{\eta - 1} \left( \frac{\partial E^{*\text{ME}}}{\partial \tau} \right)_{>0} + \psi \frac{\partial e^{*}}{\partial \tau} \left( \frac{\partial E^{*\text{PE}}}{\partial \tau} \right)_{>0} < 0,
\]

\[
\frac{\partial (1 + g^{*\text{PE}})}{\partial \tau} = B\eta \left( E^{*\text{PE}} \right)^{\eta - 1} \left( \frac{\partial E^{*\text{PE}}}{\partial \tau} \right)_{>0} > 0.
\]

Therefore, the effect of public education investment on the BGP economic growth rate is represented by the following proposition:

Proposition 1. In the ME case \( (0 \leq \tau < \overline{\tau}) \), a rise in the labor income tax rate to finance the public education investment decreases the BGP growth rate: \( \frac{\partial (1 + g^{*\text{ME}})}{\partial \tau} < 0 \). On the contrary, in the PE case \( (\overline{\tau} < \tau \leq 1) \), a rise in the labor income tax rate to finance the public education investment increases the BGP growth rate: \( \frac{\partial (1 + g^{*\text{PE}})}{\partial \tau} > 0 \).

Proof. See Appendix.
This proposition is illustrated in Figure 3. In the ME case, a rise in the labor income tax rate to finance the public education investment influences on the BGP economic growth rate through two conflict effects. First, directly, a rise in labor income tax rate to finance the public education investment increases the per-child public education investment \( E^{*\text{ME}} \). Second, a rise in the labor income tax rate crowd out the per-child private education investment \( e^{*} \) (See Lemma 2). A rise in the labor income tax rate decreases the economic growth rate in the interval \( 0 < \tau < \bar{\tau} \) because the second negative effect dominates the first positive effect.

Here, it is important that increasing the public education investment influences on not only the return of the children’s quality but also the return of children’s quantity. First, a rise in the labor income tax rate decreases the private education investment because the public education investment is completely substitutive to the private education investment. This decrease in the private education investment simultaneously reduces the cost of rearing a child and, therefore, motivates the incentive to have more number of children (See Lemma 3). Having more number of children increases the cost of privately educating children and additively decreases the per-child private education investment. This multiplier effect of increasing the public education expenditure through the quality and quantity trade-off significantly decreases household’s per-child private education investment. As a result, a rise in the labor income tax rate decreases the per-child total educational investment \( (E + e) \) and impedes the BGP growth rate.

In the PE case, this negative multiplier effect of public education investment does not work because households do not face the quality and quantity trade-off. Thus, increasing the public education expenditure directly increases the BGP economic growth rate.

4.2 The mixed education vs. the complete public education

By differentiating Eq.(21) with respect to \( \tau \), I get the following proposition.

Proposition 2.

\[
\bar{p} = \frac{(1 + \psi\gamma)\eta - 1}{1 - \eta}.
\]

(1) When the life expectancy is sufficiently low \( (0 < p < \bar{p}) \), the economic growth rate in the ME case is higher than that in the PE case in the interval \((\bar{\tau} < \tau \leq 1)\) and the economic growth rate is maximized when \( \tau = 0 \).

(2) When the life expectancy is sufficiently high \( (\bar{p} < p < 1) \), the economic growth rate in the PE case is higher than that in the ME case \((0 \leq \tau < \bar{\tau})\) and the economic growth rate is maximized when \( \tau = 1 \).

\(\square\) Proof. See Appendix.
This proposition is illustrated in Figure 4. When the life expectancy at the beginning of the old-age period is sufficiently low, households want to have more number of children because they cannot consume their saving in the old-age period with high probability. As the large number of children lowers the per-child public education investment, the BGP economic growth rate in the PE case is low relative to the growth rate in the ME case. As a result, if the life expectancy is low, the BGP economic growth rate is maximized in the ME case.

On the contrary, if the life expectancy is sufficiently high, households have small number of children and the per-child public education investment is relatively large. In this case, the maximum economic growth rate is in the PE case.

This result gives a implication about the problem with the population aging. For example, some East-Asian countries such as Japan and Korea face the low birth rate and the high longevity, in which households bear a major part of children’s education. In these mixed education countries, shifting the mixed education case to the complete public education case can promote the BGP economic growth rate and resolve the population aging problem by supplying the sufficient public educational investment to households in kind.

5 Concluding remarks

A large number of endogenous growth literature concludes that the public education expenditure promotes human capital accumulation and consequently promotes the long-
run economic growth. But, the relationship between the size of public education expenditure and the economic growth rate is unclear in the empirical literature. This paper explain why the public education expenditure have the mixed effect on the long-run economic growth from the new view point.

I build the closed-economy three-period-lived overlapping generations model, incorporating both the trade-off between the quality and quantity of children. Considering the public education policy, households decides not only their consumption schedule but also the number of children and the level of private education respectively. There exist two education cases depending on the public education policy: One is the mixed education case and the other is the complete public education case.

This paper find that the increase in the public education investment has the V-shaped effect. A rise in the labor income tax rate to finance the public education expenditure impedes the BGP economic growth in the mixed education case and promotes the BGP economic growth in the complete public education case.

In addition, this paper find that the growth maximizing educational policy depends on the life expectancy of individuals. The life-expectancy influences on individual’s incentive to have children therefore per child public education investment. If the life expectancy is sufficiently high, the government can maximize the long-run economic growth by increasing the public education expenditure sufficiently. This result has the implication about the public education policy in countries that face the problem with the declining birth rate and long longevity.

It must be considered that this research only discuss about the effects of public education policy on the long-run economic growth rate with a simple theoretical model. Further researches to examine how the public education expenditure influences on welfare in the economy are necessary.

Appendix

Proof of Lemma 1

From Eq.(19), the per-child public education investment equals to 0 when $\tau = 0, 1$. And, by differentiating Eq.(19), I derive

$$\frac{\partial E^{PE}}{\partial \tau} = \frac{\phi (\psi - \hat{\gamma}) \left\{ - (1 - \hat{\gamma}) \tau^2 - 2 \hat{\gamma} \tau + \hat{\gamma} \right\}}{\left\{ \hat{\gamma} + (1 - \hat{\gamma}) \tau \right\}^2}.$$

Therefore, the per-child public education investment has the inverted-U shaped relationship to the labor income tax rate in the interval $[0, 1]$. In addition, we can confirm that the per-child public education investment has the maximum value when $\tau = \tilde{\tau}$ by using the quadratic formula. \qed
Proof of Lemma 2

By simplifying Eq.(20), I can rewrite the per-child private education investment as follows:

\[ e^* = \left( \frac{1}{1 + p + \gamma} \right) \left\{ X\tau^2 + Y\tau + Z \right\}. \]

The denominator of above equation is positive in the interval \([0, 1]\). By putting \( e^* = 0 \), I have

\[ X\tau^2 + Y\tau + Z = 0, \]

where \( X \equiv 1 + p + \gamma \psi \eta, \ Y \equiv -1 - p - 2\gamma \psi \eta, \ Z \equiv \psi \eta \gamma \). This is a quadratic equation of \( \tau \). The discriminant of this equation is written by:

\[ D_1 = Y^2 - 4XZ, \]

\[ = (1 + p)^2 > 0. \]

Thus, \( \tau \) has different two real number values in the interval \([0, 1]\). By using the quadratic formula, these values are:

\[ \tau = \frac{-Y - \sqrt{D_1}}{2X}, \frac{-Y + \sqrt{D_1}}{2X}. \]

I can illustrate the negative relationship between the labor income tax rate and the per-child private education investment as Figure 2. Here I define \( \bar{\tau} = \frac{-\gamma \psi \eta}{1 + \beta + \gamma \psi \eta}. \) The per-child private education investment has positive values in the interval \( 0 \leq \tau < \bar{\tau} \) and become equal to zero in the interval \( \bar{\tau} \leq \tau \leq 1. \)

Proof of Lemma 3

I differentiate Eq.(22) with respect to \( \tau \) in accordance with two education cases. First, I differentiate \( n^{* ME} \) with respect to \( \tau \):

\[ \frac{\partial n^{* ME}}{\partial \tau} = -\frac{\hat{\gamma}}{\phi \{ \psi + (1 - \psi)\tau \}} + \frac{1}{\phi \{ \psi + (1 - \psi)\tau \}} + \frac{(\psi - 1) \{ \psi + (1 - \psi)\tau \}^2}{\phi \{ \psi + (1 - \psi)\tau \}^2}, \]

\[ = \frac{\psi - \hat{\gamma}}{\phi \{ \psi + (1 - \psi)\tau \}^2} > 0. \]

where \( \psi \geq 1 > \hat{\gamma} \). Second, \( n^{* PE} \) does not depend on labor income tax \( \tau \): \( \frac{\partial n^{* PE}}{\partial \tau} = 0. \)
Proof of Proposition 1

By simplifying Eq. (26), I can rewrite the BGP growth rate as follows:

\[ 1 + g^* = \begin{cases} 
B \left[ \left( \frac{\eta}{1 - \eta} \right) \left\{ \frac{\phi \hat{\gamma} (1 - \tau) (\psi + (1 - \psi) \tau)}{\hat{\gamma} + (1 - \hat{\gamma}) \tau} \right\} \right]^{\eta} = 1 + g^{ME} (0 < \tau < \tau), \\
B \left\{ \frac{(1+p)\phi \tau}{\gamma} \right\}^{\eta} = 1 + g^{PE} (\tau \leq \tau < 1), 
\end{cases} \]

First, the derivative of \( 1 + g^{ME} \) with respect to \( \tau \) is:

\[ \frac{\partial (1 + g^{ME})}{\partial \tau} = -B \eta \left\{ \left( \frac{\eta}{1 - \eta} \right) \phi \hat{\gamma} \right\}^{\eta} \left[ (1 - \tau) \left\{ \phi + (1 - \phi) \tau \right\} \right]^{\eta-1} \]

\[ \times \left\{ \{ \psi + (1 - \psi) \tau \} \{ \hat{\gamma} + (1 - \hat{\gamma}) \tau \} \tau + (1 - \tau) (\psi - \hat{\gamma}) \right\} < 0, \]

where \( \psi > \hat{\gamma} \). Second, the derivative of \( 1 + g^{PE} \) with respect to \( \tau \) is:

\[ \frac{\partial (1 + g^{PE})}{\partial \tau} = B \eta \left\{ \left( \frac{1 + p}{\gamma} \right) \phi \right\}^{\eta} \tau^{\eta-1} > 0. \]

That is to say, a rise in the labor income tax rate impedes the BGP growth rate in the ME case and promotes the BGP growth rate in the PE case.

\[ \square \]

Proof of Proposition 2

Notice that the BGP economic growth rate is a monotone increasing function of \( \tau \) in the interval \([0, \tau]\) and a monotone increasing function in the interval \([\tau, 1]\) (See Proposition 1). Thus, the economic growth rate in BGP is maximized in either ME case with \( \tau = 0 \) or PE case with \( \tau = 1 \).

\[ 1 + g^{ME}_{\tau=0} = 1 + g^{ME}_{\tau=1}, \quad (27) \]

where

\[ 1 + g^{ME}_{\tau=0} = B \left\{ \left( \frac{\eta}{1 - \eta} \right) \phi \psi \right\}^{\eta}, \quad 1 + g^{PE}_{\tau=1} = B \left\{ \left( \frac{1 + p}{\gamma} \right) \phi \right\}^{\eta}. \]

By simplifying Eq.(27), a threshold life expectancy \( \overline{p} \) is derived as follows:

\[ p = \frac{(1 + \psi \gamma) \eta - 1}{1 - \eta} \equiv \overline{p}. \]

By using \( \overline{p} \), I can two cases about the growth maximizing education policy: the one is the case in which the life expectancy is sufficiently low and the other is the case in which the life expectancy is sufficiently high.
(1) \(0 < p < \bar{p}\): In this case, the BGP economic growth rate is maximized when \(\tau = 0\). Moreover, the BGP economic growth rate in the ME case is uniquely coincides with that in the PE case with \(\tau = 1\) in the interval \([0, \bar{\tau}]\). I define this labor income tax rate as \(\hat{\tau}\).

\[
1 + g^{*\text{ME}}|_{\tau=\hat{\tau}} = 1 + g^{*\text{PE}}|_{\tau=1},
\]

\[
\Leftrightarrow B \left[ \left( \frac{\eta}{1-\eta} \right) \left( \frac{\hat{\gamma}(1-\hat{\tau})\{\psi + (1-\psi)\hat{\tau}\}}{\hat{\gamma} + (1-\hat{\gamma})\hat{\tau}} \right) \right]^\eta = B \left\{ \frac{(1+p)\phi}{\gamma} \right\}^\eta,
\]

\[
\Leftrightarrow 0 = \Delta \hat{\tau}^2 + \Theta \hat{\tau} + \Xi \tag{28}
\]

where

\[
\Delta \equiv \left( \frac{\eta}{1-\eta} \right) (p-1)\hat{\gamma}, \quad \Theta \equiv \left( \frac{\eta}{1-\eta} \right) (2\psi-1)\hat{\gamma} - \left( \frac{1+p}{\gamma} \right) (1-\hat{\gamma}),
\]

\[
\Xi \equiv \left( \frac{\eta}{1-\eta} \right) \psi\hat{\gamma} + \left( \frac{1+p}{\gamma} \right) \hat{\gamma}.
\]

Because the discriminant of Eq.(28) is \(D_2 = \Theta^2 - 4\Delta\Xi > 0\), \(\hat{\tau}\) has two different real values: \(\hat{\tau} = \frac{-\Theta - \sqrt{D_2}}{4\Delta} < 0\), \(\hat{\tau} = \frac{-\Theta + \sqrt{D_2}}{4\Delta} > 0\). Here, I am looking forward the value in the interval \([0, \tau]\), thus \(\hat{\tau} = \frac{-\Theta + \sqrt{D_2}}{4\Delta}\). Thus, it is derived that the BGP economic growth rate in the ME case is higher than that in the PE case in the interval \(0 < \tau < \hat{\tau}\).

(2) \(\bar{p} < p < 1\): In this case, the BGP economic growth rate is maximized in the PE case with \(\tau = 1\). Moreover, the BGP economic growth rate in the PE case uniquely coincides with that in the ME case with \(\tau = 0\) in the interval \([0, \bar{\tau}]\). I define this labor income tax rate as \(\hat{\tau}\).

\[
1 + g^{*\text{ME}}|_{\tau=0} = 1 + g^{*\text{PE}}|_{\tau=\hat{\tau}},
\]

\[
\Leftrightarrow B \left\{ \left( \frac{\eta}{1-\eta} \right) \phi\hat{\gamma} \right\}^\eta = B \left\{ \frac{(1+p)\phi\hat{\tau}}{\gamma} \right\}^\eta,
\]

\[
\Leftrightarrow \hat{\tau} = \left( \frac{\eta}{1-\eta} \right) \left( \frac{\gamma}{1+p} \right) \phi.
\]

The BGP economic growth rate in the PE case is higher than that in the ME case in the interval \(\hat{\tau} < \tau \leq 1\).

\[\square\]

**References**


_ (2014) ‘Education at a glance 2014: OECD indicators.’ *OECD Indicators*


