A note on public good provision under polarization

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Abstract

This paper studies some theoretical properties of polarization by considering a population grouped in two significantly-sized groups regarding the preferences its member have about a single public good. The equilibrium amount of public good is that of the majoritarian group in such a way that it is far from the Pareto-efficient one. This feature allows us to characterize a deadweight loss function which depends on the inter-group heterogeneity and the relative size of each group. Our main conclusions are that while both the increase in the relative size of minoritarian group and its radicalization increase the deadweight loss, the radicalization of the majoritarian group can decline it whenever the change in the amount of public good in equilibrium is low enough.
1 Introduction

The allocation of public goods is one of the main topics of interest in public economics. As is well-known, the fact that the amount of public good is equal for a set of heterogeneous individuals rules out the working of the price mechanism. As an alternative to this impossibility, societies have naturally developed political processes to solve these public choice problems. Within these processes Bowen (1943) proposed the simple majority voting equilibrium as the most direct allocation mechanism that can be considered. The most common version of Bowen’s model considers a continuum of individuals whose heterogeneity is characterized by a unimodal distribution on their (quasi-linear) preferences for the public good, so that equilibrium corresponds to the median voter’s amount. Moreover, the skewness of that distribution emerges as a source of inefficiency, even when the public good is financed by uniform taxes, since the efficient allocation arises when the average willingness to pay for the public good equals its marginal cost. Therefore, this standard approach assumes that there exists a moderate or middle ground point of view in public opinion represented by the median voter’s outcome. Nevertheless, there is a broad set of public choice problems where public opinion confronts and, even goes into two extremes. From a formal point of view, this polarization is characterized by a bimodal distribution of a given set of characteristics or individual attributes, intuitively reflecting the presence of opposing principles. Esteban and Ray (1994) stated three basic features that a given distribution of attributes has to exhibit to consider polarization: (i) a high degree of homogeneity within each group; (ii) a high degree of heterogeneity across groups, and; (iii) a small number of significantly sized groups.

This paper explores some simple theoretical properties of polarization by considering a population grouped into two significantly-sized clusters or groups, regarding their quasi-linear preferences about a single public good, with significant intra-group homogeneity and inter-group heterogeneity. There is consensus about majority voting as the rule aggregating preferences and the public good is funded through uniform taxes. One of the groups is majoritarian in such a way that it rules the amount of public good. Of course this allocation is far from the Pareto-efficient one, which also considers the preferences of the minoritarian group, in such a way that the political equilibrium yields to a deadweight loss. Moreover, this loss depends on the inter-group heterogeneity and the relative size of each group such that we can posit the effects of an increase in the relative size of the minoritarian group and a radicalization of each group. Effects which indeed represent an increase in polarization according to Esteban and Ray (1994). Our main conclusions are that while both the increase in the relative size of the minoritarian group and its radicalization increase the deadweight loss, the radicalization of the majoritarian group can decrease it whenever the change in the amount of public good in equilibrium is low enough.

The structure of the paper is the usual one: a second section which describes the model, the Pareto-efficient allocation and the equilibrium, a third section which studies de effects on welfare and a final section with final remarks.
2 The model

There are two groups of individuals $l$ and $h$ belonging to a normalized population sized 1, where $\pi$ is the measure of group $l$ and $1 - \pi$ is that of group $h$. Within each group each individual is endowed with the same quantity $m_i$ of numerarie and shows identical quasi-linear preferences on the quantity $x_i$ of private good $X$ and the quantity $y$ of public good $Y$, so that

$$u_i(x_i, y) = x_i + v(\theta_i, y); \quad i = l, h.$$  

For sake of simplicity we assume that the functional form $v$ is equal for each group and which differs between them is only the parameter $\theta_i$, which shows the leaning of each group for the public good. We also assume that $\theta_h > \theta_l$.

First, second and cross partial derivatives of $v$ are denoted by subindexes and throught the paper we assume that $v_y > 0, v_yy < 0, v_{\theta_i} > 0$ and $v_{y\theta_i} > 0$. The first and second conditions are the usual ones which guaranties the existence and unicity of the equilibrium, and the third and four conditions says that, given a quantity of public good, an increase of its parameter of preference increases both the consumer surplus and the marginal willingness to pay for it. In other words, since $\theta_h > \theta_l$, $v(\theta_h, y) > v(\theta_l, y)$ and $v_y(\theta_h, y) > v_y(\theta_l, y) \forall y$. In addition, given the symmetry in the second cross derivative, $v_{y\theta_i} = v_{\theta_i y}$, the assumption on the positiveness of this second cross derivative entails that $v_{\theta_i}(\theta_i, y_1) > v_{\theta_i}(\theta_i, y_0) \forall y_1 > y_0$. Finally, it is assumed that the public good absorbs the quantity $C(y) = y$ of numerarie for its production and $0 < \pi < \frac{1}{2}$. That is, more than a half of the population has the leaning $\theta_h$ for the public good. In this trend, we will call $h$ as the majoritarian group meanwhile $l$ is called the minoritarian group.

2.1 Pareto-efficient allocation

Since the preferences of consumers are quasilinear let us consider a Benthamite social welfare function so that the Pareto-efficient allocation of public good is given by the solution of

$$\max_y M - y + \pi v(\theta_i, y) + (1 - \pi)v(\theta_h, y),$$

where $M = \pi m_h + (1 - \pi)m_l$. Let $y(\theta, \pi)$ be the per-capita efficient allocation of public good which fulfills the Pareto-efficient condition:

$$\pi v_y(\theta_1, y(\theta, \pi)) + (1 - \pi)v_y(\theta_h, y(\theta, \pi)) = 1. \quad (1)$$

This characterization of the per-capita efficient allocation of public good allow us to study their properties with respect the parameters $\pi$ and $\theta = (\theta_1, \theta_h)$.

Proposition 1 $\frac{\partial y(\theta, \pi)}{\partial \pi} < 0; \frac{\partial y(\theta, \pi)}{\partial \theta_i} > 0, i = l, h.$
Proof: Deriving the first order condition (1) with respect to $\pi$ we have

\[
v_y(\theta_i, y(\theta, \pi)) + \pi v_{yy}(\theta_i, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \pi} - v_y(\theta_h, y(\theta, \pi)) + (1 - \pi) v_{yy}(\theta_h, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \pi} = 0,
\]

and clearing we have

\[
\frac{\partial y(\theta, \pi)}{\partial \pi} = \frac{v_y(\theta_h, y(\theta, \pi)) - v_y(\theta_i, y(\theta, \pi))}{\pi v_{yy}(\theta_i, y(\theta, \pi)) + (1 - \pi) v_{yy}(\theta_h, y(\theta, \pi))} < 0.
\]

For the other partial derivative we derive the first order condition (1) with respect to $\theta_h$

\[
\pi v_{yy}(\theta_i, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \theta_h} + (1 - \pi) \left[ v_{yy}(\theta_h, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \theta_h} + v_y(\theta_h, y(\theta, \pi)) \right] = 0,
\]

and clearing we have

\[
\frac{\partial y(\theta, \pi)}{\partial \theta_h} = \frac{-(1 - \pi) v_y(\theta_h, y(\theta, \pi))}{\pi v_{yy}(\theta_i, y(\theta, \pi)) + (1 - \pi) v_{yy}(\theta_h, y(\theta, \pi))} > 0.
\]

For the partial derivative with respect $\theta_i$ is analogous to the former, obtaining

\[
\frac{\partial y(\theta, \pi)}{\partial \theta_i} = \frac{-\pi v_{\theta_i}(\theta_i, y(\theta, \pi))}{\pi v_{yy}(\theta_i, y(\theta, \pi)) + (1 - \pi) v_{yy}(\theta_h, y(\theta, \pi))} > 0. \quad \blacksquare
\]

Therefore, the per-capita efficient allocation of public good decreases whenever the relative size of minoritarian group increases. This is because the average marginal willingness to pay is shifted towards the marginal willingness to pay of the minoritarian group. On the other hand, given $\pi$, when the parameter of preference for the public good increases, whatever it may be, the average willingness to pay increases with the consequence of an increase in the per-capita efficient allocation of public good.

### 2.2 Majority voting equilibrium

We are to consider the simple majority mechanism where the public good is financed by a uniform lump-sum tax $T$ on each individual. In this trend, since $0 < \pi < \frac{1}{2}$, the equilibrium allocation of public good will be that the majoritarian group $h$, that is

\[
\max_{x_h, y} x_h + v(\theta_h, y) \text{ s. t. } m_h = x_h + T,
\]

where $\pi T + (1 - \pi) T = y$ or $T = y$, thus the above problem becomes

\[
\max_y m_h - y + v(\theta_h, y).
\]
Let \( y(\theta_h) \) be the solution of this problem which fulfills its first order condition

\[
v_y(\theta_h, y(\theta_h)) = 1. \tag{2}
\]

Since \( \theta_l < \theta_h \), it is fairly easy to see, equaling first order conditions (1) and (2) and rearranging terms, that \( y(\theta_h) > y(\theta, \pi) \) and that \( \lim_{\pi \to 0} y(\theta, \pi) = y(\theta_h) \).

On the other hand, we see, in a difference with the per-capita efficient allocation of public good, that the per-capita allocation of public good under the majority voting equilibrium only depends of the parameter of preference of the majoritarian group, this allow us to state the following Proposition:

**Proposition 2** \( \frac{dy(\theta_h)}{d\theta_h} > 0 \).

Proof: Deriving the first order condition (2) with respect to \( \theta_h \) we have

\[
v_{\theta h}(\theta_h, y(\theta_h)) + v_{yy}(\theta_h, y(\theta_h)) \frac{\partial y(\theta_h)}{\partial \theta_h} = 0,
\]

and clearing we have

\[
\frac{\partial y(\theta_h)}{\partial \theta_h} = -\frac{v_{\theta h}(\theta_h, y(\theta_h))}{v_{yy}(\theta_h, y(\theta_h))} > 0. \qed
\]

That is, the per-capita allocation of public good in equilibrium increases whenever the parameter of preference for the public good of the majoritarian group increases.

### 3 Deadweight loss and polarization

This section concerns with how an increase in polarization affects welfare. According to Esteban and Ray (1994), in our model, the polarization increases if:

(i) given \( \theta = (\theta_l, \theta_h) \), \( \pi \) (identification) increases and (ii) given \( \pi \), the difference \( \theta_h - \theta_l \) (alienation) increases.\(^1\) The first case says that when the proportion of individuals in the minoritarian group becomes more significant polarization increases. The second says that polarization increases as a consequence of the increase in the heterogeneity across groups. Provided that thorough the paper we assume that \( h \) is the majority group \((0 < \pi < \frac{1}{2})\) and that \( \theta_l < \theta_h \), let us introduce the concept of *radicalization* to study the increase in polarization due to an increase in alienation. Thus, an increase in \( \theta_h \) supposes a radicalization of group \( h \), while a decrease in \( \theta_l \) supposes a radicalization of group \( l \).

Let us define the deadweight loss function as

\[
L(\theta, \pi) = V(\theta, \pi) - V(\theta_h),
\]

\(^1\)The Esteban and Ray’s Polarization Measure applied to our two-point distribution case is given by \( P(\pi, \theta, \delta) = \pi(1 - \pi)(\pi^2 + (1 - \pi)^2)(\theta_h - \theta_l), \quad 1 < \delta < 2 \).
Where

\[ V(\theta, \pi) = M - y(\theta, \pi) + \pi v(\theta_1, y(\theta, \pi)) + (1 - \pi) v(\theta_h, y(\theta, \pi)) \]

\[ V(\theta) = M - y(\theta) + \pi v(\theta_1, y(\theta)) + (1 - \pi) v(\theta_h, y(\theta)), \]

that is, \( V(\theta, \pi) \) is the total surplus evaluated in the Pareto-efficient allocation and \( V(\theta) \) is the surplus evaluated in the majority voting equilibrium. Thus,

\[ L(\theta, \pi) = y(\theta_h) - y(\theta, \pi) + \pi [v(\theta_1, y(\theta, \pi)) - v(\theta_1, y(\theta))], \]

is the deadweight loss generated in majority voting equilibrium. Let us study how \( L(\theta, \pi) \) changes as a consequence of an increase in the size of the minoritarian group and the radicalization of both groups.

**Proposition 3** The deadweight loss generated by the majority voting equilibrium increases whenever the minoritarian group becomes more significant.

Proof: Deriving (3) with respect to \( \pi \),

\[
\frac{\partial L(\theta, \pi)}{\partial \pi} = -\frac{\partial y(\theta, \pi)}{\partial \pi} + v(\theta_1, y(\theta, \pi)) - v(\theta_1, y(\theta_h)) + \pi v_y(\theta_1, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \pi} \\
- \left[ v(\theta_h, y(\theta, \pi)) - v(\theta_h, y(\theta_h)) \right] + (1 - \pi) v_y(\theta_h, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \pi}.
\]

Taking into account the first order condition (1),

\[
\frac{\partial L(\theta, \pi)}{\partial \pi} = v(\theta_1, y(\theta, \pi)) - v(\theta_1, y(\theta_h)) - \left[ v(\theta_h, y(\theta, \pi)) - v(\theta_h, y(\theta_h)) \right],
\]

or

\[
\frac{\partial L(\theta, \pi)}{\partial \pi} = \int_{y(\theta, \pi)}^{y(\theta_h)} [v_y(\theta_h, y) - v_y(\theta_1, y)] dy > 0. \]

Therefore when the relative size of the minoritarian group increases the average willingness to pay for the public good shifts towards the willingness to pay of the minoritarian group so that, according with Proposition 1, the per-capita efficient allocation of public good decreases. Thus, the deadweight loss generated by this change in \( \pi \) is equivalent to the difference between the variation in consumer surplus of the majoritarian group less that of the minoritarian one, when the amount of public good changes from the efficient one to that of majority voting equilibrium.

**Proposition 4** The deadweight loss generated by the majority voting equilibrium increases whenever the minoritarian group radicalizes.
Proof: Deriving (3) with respect to $\theta_i$

$$\frac{\partial L(\theta, \pi)}{\partial \theta_i} = -\frac{\partial y(\theta, \pi)}{\partial \theta_i} + \pi \left[ v_y(\theta_i, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \theta_i} + ... \right] + v_{\theta_i}(\theta_i, y(\theta, \pi)) - v_{\theta_i}(\theta_i, y(\theta)) + (1 - \pi)v_y(\theta, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \theta_i}.$$  

Taking into account the first order condition (1),

$$\frac{\partial L(\theta, \pi)}{\partial \theta} = \pi \left[ v_{\theta_i}(\theta_i, y(\theta, \pi)) - v_{\theta_i}(\theta_i, y(\theta)) \right] < 0.$$  

The above Proposition says that deadweight loss changes in opposite way of changes in the parameter of preference for the public good of the minoritarian group. Since the radicalization of this group entails a decrease in its parameter the deadweight loss increases as a consequence of the radicalization of the minoritarian group. This radicalization entails a shift inward of both the willingness to pay curve of the minoritarian group and the average willingness to pay curve in such a way that, according with Proposition 1, the per-capita efficient allocation of public good decreases. However, as in the former case, the per-capita allocation of public good under majority voting equilibrium remains constant. Thus the increase in the deadweight loss comes exclusively from the worsening of the minoritarian group.

**Proposition 5** The deadweight loss generated by the majority voting equilibrium does not increase when the radicalization of the majoritarian group increases whenever

$$\frac{\partial y(\theta_h)}{\partial \theta_h} \leq \frac{(1 - \pi) \left[ v_{\theta_h}(\theta_h, y(\theta_h)) - v_{\theta_h}(\theta_h, y(\theta)) \right]}{\pi \left[ 1 - v_y(\theta_h, y(\theta)) \right]}.$$  

Proof: Deriving (3) with respect to $\theta_h$

$$\frac{\partial L(\theta, \pi)}{\partial \theta_h} = \frac{\partial y(\theta_h)}{\partial \theta_h} - \frac{\partial y(\theta, \pi)}{\partial \theta_h} + \pi \left[ v_y(\theta_h, y(\theta, \pi)) \frac{\partial y(\theta, \pi)}{\partial \theta_h} - v_y(\theta_i, y(\theta)) \frac{\partial y(\theta_i)}{\partial \theta_h} \right] ... + (1 - \pi) \left[ v_{\theta_h}(\theta_h, y(\theta, \pi)) \frac{\partial y(\theta_h)}{\partial \theta_h} - v_{\theta_h}(\theta_h, y(\theta)) \frac{\partial y(\theta_h)}{\partial \theta_h} + v_{\theta_h}(\theta_h, y(\theta)) - v_{\theta_h}(\theta_h, y(\theta)) \right].$$  

Taking into account the first order condition (1)

$$\frac{\partial L(\theta, \pi)}{\partial \theta_h} = \left[ 1 - \pi v_y(\theta_h, y(\theta_h)) - (1 - \pi)v_y(\theta_h, y(\theta)) \right] \frac{\partial y(\theta_h)}{\partial \theta_h} ... + (1 - \pi) \left[ v_{\theta_h}(\theta_h, y(\theta, \pi)) - v_{\theta_h}(\theta_h, y(\theta)) \right].$$
Taking into account the first order condition (2),
\[ \frac{\partial L(\theta, \pi)}{\partial \theta_h} = \pi [1 - v_y(\theta_l, y(\theta_h))] \frac{\partial y(\theta_h)}{\partial \theta_h} \cdots \]
\[ -(1 - \pi) [v_{\theta_h}(\theta_h, y(\theta_h)) - v_{\theta_h}(\theta_h, y(\theta, \pi))]. \]

In this case the radicalization implies an increase in the parameter of preference for the public good of the majoritarian group, entailing a shift outward of both its willingness to pay and the average willingness to pay curves and, thus, the increase in both the efficient and the equilibrium allocation of public good. A brief checking of Equation (4) allow us to split the total effect upon the deadweight loss in two parts. The first part of (4) shows how this radicalization makes worse off the minoritarian group: a higher amount of public good in equilibrium implies a fall in the willingness to pay of this group, and the deadweight loss arises as the difference between its valuation and the marginal cost times the change in the amount of public good in equilibrium. On the other hand, the second part is similar to that of Proposition 4 and shows how the radicalization of the majoritarian group improves itself: the increase in the preference for the public good of this group increases both its own and the average willingness to pay for it, with the consequence of a fall in the deadweight loss. The final effect depends on how the amount of public good in equilibrium changes when majoritarian group radicalizes, respecting a sized-weighted ratio which measures the increase in the majoritarian group surplus due to its radicalization over the difference between public good marginal cost and the willingness to pay for it of the minoritarian group.

4 Final remarks

This paper has addressed certain theoretical effects that an increase in the polarization about preferences on a single public good can have on welfare. In a very parsimonious model where such preferences are split into two groups and there is no conflict concerning the mechanism of allocation, the equilibrium allocation of the public good is that of the majoritarian group. This equilibrium generates a deadweight loss since it is far from the efficient allocation. In this standard framework we have attained a stark result, which contrasts with the usual point of view in the literature on the subject: it is possible that an increase in polarization will lead to a decline in deadweight loss. This happens whenever the increase in polarization comes from a radicalization of the majoritarian group. In this case both the efficient and the majority voting equilibrium are affected, in such a way that both minoritarian and majoritarian group surplus changes. The majoritarian group becomes better off as a consequence of its own radicalization, but the minoritarian group ends up worse off. This last effect depends on the change in the amount of public good in equilibrium. The total effect on the deadweight loss is compounded by the sum of both effects, and it is possible, under very reasonable conditions, that the first effect will dominate the second.
References