Financial asset bubble with heterogeneous agents and endogenous borrowing constraints

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Abstract

This paper studies the root of financial asset bubble in an infinite horizon general equilibrium model with heterogeneous agents and borrowing constraints. We say that there is a bubble at equilibrium if the price of the financial asset is greater than its fundamental value. First, we found that bubble can occur only if there exists an agent and an infinite sequence of dates such that borrowing constraints of this agent are binding at each such date. Second, we prove that there is a bubble if and only if interest rates are low, which means that the sum (over time) of interest rates (in term of financial asset) is finite. Last, we give a condition on exogenous variables, under which a financial asset bubble occurs at equilibrium.

Keywords: Financial asset bubble, intertemporal equilibrium, infinite horizon, borrowing constraints.

JEL Classifications: C62, D5, D91, G10

1 Introduction

This paper is to study fundamental questions about rational bubbles: What is an asset price bubble? What is the root of bubbles? We also give an answer for the following debate:

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"However, despite the widespread belief in the existence of bubbles in the real world, it is difficult to construct model economies in which bubbles exist in equilibrium."

Kocherlakota (2008)

To do this, we construct an infinite horizon general equilibrium model with heterogeneous agents and endogenous borrowing constraints. There is a single consumption good and a financial asset. On the one hand, the financial asset will give dividends in term of consumption good. On the other hand, agents can resell it. This structure is similar to the one in Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000).

There is an endogenous borrowing constraint when agents want to borrow: at each date, each agent can borrow an amount but the delivery of this amount at next date cannot be greater than a fraction of the endowment of this agent. Because of the borrowing constraints, the financial market is dynamically incomplete.

Before studying the bubbles, we have to prove the existence of equilibrium. We do so by proving the existence of equilibrium for each truncated economy, and then pass to the limit. The existence of equilibrium for each truncated economy is proved by two steps: (i) consider bounded truncated economy and prove that there exists equilibrium for each bound, (ii) let bound tend to infinity to obtain an equilibrium for truncated economy. Our proof is crucial because we do not require any condition on endowments of agents as in Levine (1989), Levine and Zame (1996), Magill and Quinzii (1994), Araujo, Pascoa, Torres-Martinez (2002). Instead, endowments of agents may be zero in our paper. We overcome this difficulty by two steps: (1) we prove that there exists an equilibrium for the economy in which every agent has $\epsilon > 0$ units of endowment; as a result we obtain a sequence of equilibria parameterized by $\epsilon$, (2) let $\epsilon$ tend to zero, this sequence has a limit; we prove that such limit is an equilibrium by using the positivity of financial dividend (since the financial dividend is in term of consumption good and can be consumed).

Second, we move to study bubble of financial asset. We say that a financial asset bubble occurs at an equilibrium (for short, bubble) if the price of the financial asset is greater than its fundamental value. Some significant papers studied rational asset bubbles. A well known result is that if the present value of aggregate endowments is finite, there is no bubble (Santos and Woodford (1997), Huang and Werner (2000)). However, the present value of aggregate endowments is endogenously determined. Why the present value of aggregate endowments is finite?

Montrucchio (2004) gave conditions on endogenous variables, under which there is a bubble. Unfortunately, they did not explain the nature of these conditions. Although there are some examples of bubbles (Kocherlakota (1992), Huang and Werner (2000)), no one gives conditions of exogenous variables under which there is a bubble at equilibrium. Our paper will fill these gaps.

We begin by pointing out that, at equilibrium, individual transversality condition of each agent is satisfied but real transversality condition of each agent may be not held. If the real transversality condition of each agent is satisfied, there is no bubble.
We also find that if a bubble appear, there exists an agent $i$ and an infinite sequence of date $(t_n)_{n=0}^\infty$ such that borrowing constraint of this agent is binding at each date $t_n$. This finding complements the one in Kocherlakota (1992) where he wanted to claim that borrowing constraint is binding infinitely often. However, he only proved that the inferior limit of difference between asset amount of each agent and exogenous borrowing bound equals zero.

We then introduce new concepts: low interest rates and high interest rates. An equilibrium is said to have low interest rates if the sum (over time) of interest rates (in term of financial asset) is finite, otherwise we say interest rates are high. A novel result is that interest rates are low if and only if bubbles exist. Our definition of low interest rates is different from the one in Alvarez and Jermann (2000) where implied interest rates are called to be high if the present value of aggregate endowments is finite. We proved that if equilibrium is high implied interest rates, it will be high interest rates. The inverse sense is not true. In our bubble example, the present value of aggregate endowments is finite and bubble may exist.

Our last contribution is to give a condition of exogenous variables, under which bubble occurs at equilibrium. The intuition of our condition is the following: if there exists an agent whose highest subjective interest rate is less than the interest rate of the economy, this agent accepts to buy financial asset with a price which is greater than its fundamental value. Consequently, there is a bubble.

**Related literature:** In a rational expectation model without endowment, Tirole (1982) proved that there is no financial asset bubble. His result can be viewed as a particular case of our model.

One can also define bubble for a long-lived asset which does not give dividends. For such an asset, its fundamental value is zero. The standard definition of bubble is the following: We say that there is a bubble if the market price of this asset is strictly positive. There is a large litterature on this kind of bubbles. Some of them are Tirole (1985), Ventura (2012), Farhi and Tirole (2012).

A survey on bubble in models as asymmetric information, overlapping generation, heterogeneous-beliefs can be found in Brunnermeier and Oehmke (2012).

Our paper is also related to physical capital bubble introduced by Becker, Bosi, Le Van and Seegmuller (2014) in a standard Ramsey model with heterogeneous agents and stationary concave technology. The physical capital is viewed as a long-lived asset whose price (in term of consumption goood) is 1 and capital returns can be viewed as dividends. However, these dividends are endogenous. They then defined the fundamental value of the physical capital and that bubble exists if the market price of the physical capital is greater than its fundamental value. In Becker, Bosi, Le Van and Seegmuller (2014), they proved that physical capital bubble is ruled out. Bosi, Le Van and Pham (2014) allowed non-stationary technologies and prove that physical capital bubble exists if and only if the sum (over time) of capital return is finite.

The remainder of the paper is organized as follows. Section 2 describes the model. In section 3, the existence of equilibrium is proved. Section 4 studies the root of financial asset bubble. Conclusion will be presented in Section 5. Technical details are gathered in Appendix.
2 Model

We consider a standard exchange economy in an infinite horizon model. At each date, agents are endowed with an amount of consumption good. \( p_t \) is the price of consumption good at date \( t \).

Financial market: there is one long-lived financial asset. At date \( t \), if agent \( i \) buys \( a_{i,t} \geq 0 \) units of financial asset with price \( q_t \), at the next date (date \( t+1 \)), this agent will receive \( \xi_{t+1} \) units of consumption good as dividend and she will be able to sell \( a_{i,t} \) units of financial asset with price \( q_{t+1} \).

Each household \( i \) takes sequences of prices \( (p, q) = (p_t, q_t)_{t=0}^{\infty} \) as given and maximizes her utility:

\[
(P_i(p, q)) : \max_{(c_{i,t}, a_{i,t})_{t=0}^{\infty}} \left[ \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \right] \tag{1}
\]

subject to

\[
p_t c_{i,t} + q_t a_{i,t} \leq p_t e_{i,t} + (q_t + p_t \xi_{t}) a_{i,t-1} \tag{2}
\]

\[
-(p_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} \leq f_i p_{t+1} e_{i,t+1} \tag{3}
\]

where \( \beta_i \) is the discount factor of agent \( i \) and \( u_i \) is the utility function of agent \( i \). Borrowing constraint (3) means that the payment of agent \( i \) cannot exceed a fraction of her endowments. If \( f_i = 0 \), agent \( i \) cannot borrow. Our setup is different from the one in Kocherlakota (1992) where he considers that borrowing constraints are exogenous, which is given by \( a_{i,t} \geq a^* \) where \( a^* \) do not depend on time. Borrowing constraints in our framework allow us to bound the trade volume of asset by an exogenous bound. Indeed, assume that prices are strictly positive, we observe that (3) implies that

\[
-a_{i,t} \leq \frac{f_i p_{t+1} e_{i,t+1}}{q_{t+1} + p_{t+1} \xi_{t+1}} \leq \frac{f i e_{i,t+1}}{\xi_{t+1}}. \tag{4}
\]

It means that agent \( i \) cannot borrow more than \( \frac{f i e_{i,t+1}}{\xi_{t+1}} \) which is exogenous but varies in time \( t \). Although \( \frac{f i e_{i,t+1}}{\xi_{t+1}} \) is exogenous, it may tend to infinity.

**Definition 1.** A sequence of prices and quantities \( (\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{a}_{i,t})_{i=1}^{m})_{t=0}^{+\infty} \) is an equilibrium of the economy \( E = \left( (u_i, \beta_i, a_{i,-1}, f_i)_{i=1}^{m} \right) \) if

(i) Price positivity: \( \bar{p}_t, \bar{q}_t > 0 \) for \( t \geq 0 \).

(ii) Market clearing: at each \( t \geq 0 \),

\[
\text{Consumption good : } \sum_{i=1}^{m} \bar{c}_{i,t} = \sum_{i=1}^{m} e_{i,t} + \xi_t, \tag{5}
\]

\[
\text{Financial asset : } \sum_{i=1}^{m} \bar{a}_{i,t} = 1. \tag{6}
\]
(iii) Optimal consumption plans: for each \(i\),

\[
(\bar{c}_{i,t}, \bar{a}_{i,t})_{t=0}^{\infty}
\]

is a solution of the problem \((P_i(p, \bar{q}))\).

2.1 The existence of equilibrium

We need some standard assumptions.

**Assumption (H1):** The utility function \(u_i : \mathbb{R}_+ \rightarrow \mathbb{R}\) is \(C^0\), strictly increasing, concave and \(u_i(0) = 0\), \(u'(0) = +\infty\).

**Assumption (H2):** \(e_{i,t} \geq 0\) such that for every \(t \geq 0\) and \(i = 1, \ldots, m\).

**Assumption (H3):** For each \(i = 1, \ldots, m\), \(a_{i,-1} \geq 0\) and \(\sum_{i=1}^{m} a_{i,-1} = 1\).

**Assumption (H4):** \(\xi_t > 0\) for every \(t\).

**Assumption (H5):** For each \(i\), utility of agent \(i\) is finite

\[
\sum_{t=0}^{\infty} \beta^t_i u_i(W_t) < \infty,
\]

where \(W_t := \sum_{i=1}^{m} e_{i,t} + \xi_t\).

**Theorem 1.** Assume that Assumptions (H1) – (H5) are satisfied, there exists an equilibrium in the infinite-horizon economy if \((e_{i,0}, a_{i,-1}) \neq (0, 0)\) for any \(i\).

We see that even if agents do not hold endowment, i.e., \(e_{i,t} = 0\), there may be an equilibrium. This is the case where \(a_{i,-1} > 0\) for every \(i\). This result also gives a foundation in order to study financial bubble in Tirole (1982) where he assumes that households do not hold endowments.

We prove Theorem 1 by two main steps: (1) we prove the existence of equilibrium for each \(T\)–truncated economy, we have a sequence of equilibria which depend on \(T\); (2) we prove that this sequence has a limit (for the product topology) which is an equilibrium for the infinite horizon economy.

The detailed proof is presented in Appendices 5 and 6.

2.2 Individual behavior

We consider an equilibrium \((p, \bar{q}, (c_i, a_i)_{i=1}^{m})\). Let \(\lambda_{i,t}\) denote the multiplier associated with budget constraints of agent \(i\) at period \(t\), and the multiplier of borrowing constraint is denoted by \(\mu_{i,t} (\mu_{i,t} \geq 0)\). We have

\[
\beta^t_i u'_i(c_{i,t}) = p_t \lambda_{i,t} \quad (8)
\]

\[
\lambda_{i,t} q_t = (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + p_{t+1} \xi_{t+1}) \quad (9)
\]

\[
\mu_{i,t+1} (q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} + f_t p_{t+1} e_{i,t+1} = 0. \quad (10)
\]
For each $i$, we define $S_{i,0} = 1$, $S_{i,t} := \frac{\beta_i u_i'(c_{i,t})}{u_i'(c_{i,0})}$ is the agent $i$’s discount factor from initial period to period $t$.

We now define the discount factor of the economy. Since $\sum_{i=1}^{m} a_{i,t} = 1$, there exists $i(t)$ such that $a_{i(t),t} > 0$, hence $\mu_{i(t),t} = 0$. Therefore, we get

$$
\frac{q_t}{q_{t+1} + p_{t+1} \xi_{t+1}} = \frac{\lambda_{i,t+1}}{\lambda_{i,t}} = \max_{i \in \{1, \ldots, m\}} \frac{\bar{\mu}_{i,t+1}}{\bar{\mu}_{i,t}} = \gamma_{t+1} \frac{p_t}{p_{t+1}}.
$$

(11)

where $\gamma_{t+1} := \max_{i \in \{1, \ldots, m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$. For each $t \geq 0$ we have

$$
\frac{q_t}{p_t} = \gamma_{t+1} (\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}).
$$

(12)

We define $Q_0 := 1$, and for each $t \geq 1$, $Q_t := \prod_{s=1}^{t} \gamma_s$ is the discount factor of the economy from initial period to period $t$. Since $Q_{t+1} = \gamma_{t+1} Q_t$, we have

$$
Q_t \frac{q_t}{p_t} = Q_{t+1} (\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}).
$$

(13)

Remark 1. It is clear that $Q_t \geq S_{i,t}$ for every $t$, i.e., the market discount factor is greater individual discount factors. If borrowing constraints of agent $i$ are not binding from date $t_0$, we have $\frac{Q_t}{Q_{t_0}} = \frac{S_{i,t}}{S_{i,t_0}}$ for every $t \geq t_0$.

Remark 2. We can see that borrowing constraint $(q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} \geq -f_i p_{t+1} \epsilon_{i,t+1} \geq 0$ is equivalent to $Q_{t+1} (q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} \geq -f_i Q_{t+1} p_{t+1} \epsilon_{i,t+1}$. According to (13), this can be rewritten as

$$
Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} \epsilon_{i,t+1} \geq 0
$$

This means that borrowing value of agent $i$ does not exceed a fraction value of its endowments.

We state the a fundamental result showing the information of borrowing constraints.

Lemma 1. (Transversality conditions)

We have

$$
\lim_{t \to \infty} S_{i,t} \frac{q_t}{p_t} a_{i,t} = 0
$$

$$
(\frac{q_0}{p_0} + \xi_0) a_{i,-1} + \sum_{t=0}^{\infty} S_{i,t} \epsilon_{i,t} + \sum_{t=1}^{\infty} f_i \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} \epsilon_{i,t} = \sum_{t=0}^{\infty} S_{i,t} c_{i,t} < \infty.
$$

(15)

Proof. We first prove that

$$
\lim sup_{t \to \infty} S_{i,t} \frac{q_t}{p_t} a_{i,t} \leq 0.
$$

(16)
We say that the path \( a_i := (a_{i,0}, a_{i,1}, \ldots, a_{i,t-1}, a_{i,t}, a_{i,t+1}, \ldots) \) is feasible if, for any \( t \),
\[
q_i a_{i,t} \leq p_t e_{i,t} + (q_t + p_t \xi_t) a_{i,t-1}
\]
\[
-(q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} \leq f_i p_t e_{i,t+1}.
\]

We claim that: if \( a_i \) is feasible, then for \( \lambda \in [0, 1] \) the path
\[
a_i(\lambda, t) := (a_{i,0}, a_{i,1}, \ldots, a_{i,t-1}, \lambda a_{i,t}, \lambda a_{i,t+1}, \ldots)
\]
is feasible too. We prove our claim in three steps.

Step 1: We prove that \( f_i e_{i,t} + \lambda(\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \geq 0 \). This is true if \( a_{i,t-1} \geq 0 \). If \( a_{i,t-1} < 0 \), we have
\[
\lambda a_{i,t-1} \geq a_{i,t-1}.
\]
Therefore, \( f_i e_{i,t} + \lambda(\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \geq f_i e_{i,t} + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \geq 0 \).

Step 2: We prove that \( e_{i,t} + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} - \lambda \frac{q_i}{p_t} a_{i,t} \geq 0 \). Indeed, we always have
\[
e_{i,t} + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \geq f_i e_{i,t} + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \geq 0.
\]

If \( a_{i,t} > 0 \), we have \( \lambda \frac{q_i}{p_t} a_{i,t} \leq \frac{q_i}{p_t} a_{i,t} \leq e_t + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \).

If \( a_{i,t} \leq 0 \), we have \( \lambda \frac{q_i}{p_t} a_{i,t} \leq 0 \leq e_t + (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \).

Step 3: We prove that \( e_{i,t} + \lambda(\frac{q_i}{p_t} + \xi_t)a_{i,t-1} - \lambda \frac{q_i}{p_t} a_{i,t} \geq 0 \). This is equivalent to \( e_{i,t} \geq \lambda(\frac{q_i}{p_t} a_{i,t} - (\frac{q_i}{p_t} + \xi_t)a_{i,t-1}) \). This inequality is true if \( \frac{q_i}{p_t} a_{i,t} - (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} < 0 \).

When \( \frac{q_i}{p_t} a_{i,t} - (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} > 0 \), we have
\[
\lambda(\frac{q_i}{p_t} a_{i,t} - (\frac{q_i}{p_t} + \xi_t)a_{i,t-1}) \leq \frac{q_i}{p_t} a_{i,t} - (\frac{q_i}{p_t} + \xi_t)a_{i,t-1} \leq e_{i,t}.
\]

To prove (16), we borrow the argument in the proof of Theorem 2.1 in Kamihi-gashi (2002). Our proof is presented in the following.

Given \( a_i \) be feasible, we denote \( c_i(a_i) \) as follows
\[
c_{i,s} := e_{i,s} + (\frac{q_s}{p_s} + \xi_s)a_{i,s-1} - \frac{q_s}{p_s} a_{i,s}.
\]
We then denote
\[
u_i(c_i(a_i)) := \sum_{s=0}^{+\infty} \beta^s_t u_i(c_{i,s})
\]
By the optimality of \( (c_i, a_i) \) we have \( u_i(c_i(a_i(\lambda, t))) - u_i(c_i(a_i))) \leq 0 \) which is equivalent to
\[
\beta^s_t \left(u_i(e_{i,t} + (\frac{q_t}{p_t} + \xi_t)a_{i,t-1} - \lambda \frac{q_t}{p_t} a_{i,t}) - u_i(e_{i,t} + (\frac{q_t}{p_t} + \xi_t)a_{i,t-1} - \frac{q_t}{p_t} a_{i,t}) \right)
\]
\[
\leq \sum_{s=t+1}^{+\infty} \beta^s_t \left(u_i(e_{i,s} + (\frac{q_s}{p_s} + \xi_s)a_{i,s-1} - \frac{q_s}{p_s} a_{i,s}) - u_i(e_{i,s} + (\frac{q_s}{p_s} + \xi_s)\lambda a_{i,s-1} - \lambda \frac{q_s}{p_s} a_{i,s}) \right).
\]
As a consequence, we get
\[
\beta_i^t u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \lambda \frac{q_t}{p_t} a_{i, t}) - u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \frac{q_t}{p_t} a_{i, t})
\]
\[
\leq \sum_{s=t+1}^{\infty} \beta_i^s u_i(e_{i, s} + \left( \frac{q_s}{p_s} + \xi_s \right) a_{i, s-1} - \frac{q_s}{p_s} a_{i, s}) - u_i(e_{i, s} + \left( \frac{q_s}{p_s} + \xi_s \right) \lambda a_{i, s-1} - \lambda \frac{q_s}{p_s} a_{i, s})
\]
\[
1 - \lambda
\]
(24)

For each \( s \), the function
\[
f_s(\lambda) := u_i(e_{i, s} + \left( \frac{q_s}{p_s} + \xi_s \right) \lambda a_{i, s-1} - \lambda \frac{q_s}{p_s} a_{i, s})
\]
is concave on \((0, 1)\). Therefore, we have
\[
\frac{f_s(1) - f_s(\lambda)}{1 - \lambda} \leq f_s(1) - f_s(0)
\]
(25)
which implies that
\[
\frac{u_i(e_{i, s} + \left( \frac{q_s}{p_s} + \xi_s \right) a_{i, s-1} - \frac{q_s}{p_s} a_{i, s}) - u_i(e_{i, s} + \left( \frac{q_s}{p_s} + \xi_s \right) \lambda a_{i, s-1} - \lambda \frac{q_s}{p_s} a_{i, s})}{1 - \lambda}
\]
\[
\leq u_i(c_{i, s}) - u_i(e_{i, s}).
\]
(27)

Combining with (24), we get
\[
\beta_i^t \leq \sum_{s=t+1}^{\infty} \beta_i^s u_i(c_{i, s}) - u_i(e_{i, s})
\]
(28)

We also have
\[
\lim_{\lambda \to 1} \frac{u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \lambda \frac{q_t}{p_t} a_{i, t}) - u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \frac{q_t}{p_t} a_{i, t})}{1 - \lambda}
\]
\[
= -\lim_{\lambda \to 1} \frac{u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \frac{q_t}{p_t} a_{i, t}) - u_i(e_{i, t} + \left( \frac{q_t}{p_t} + \xi_t \right) a_{i, t-1} - \frac{q_t}{p_t} \lambda a_{i, t})}{1 - \lambda}
\]
\[
= -u_i'(c_{i, t}) \left[ - \frac{q_t}{p_t} a_{i, t} \right] = u_i'(c_{i, t}) \frac{q_t}{p_t} a_{i, t}.
\]
(31)
Let \( \lambda \) tend to 1 in (28), we have
\[
\beta_i^t u_i'(c_{i, t}) \frac{q_t}{p_t} a_{i, t} \leq \sum_{s=t+1}^{\infty} \beta_i^s u_i(c_{i, s}).
\]
(32)
Let $t$ tend to infinity, we obtain

$$\limsup_{t \to \infty} S_{i,t} \frac{q_t}{p_t} a_{i,t} \leq 0.$$  \hfill (33)

We now prove (15). At equilibrium, the utility of agent $i$ is finite. Hence, we have

$$\infty > \sum_{t=0}^{\infty} \beta_t^i u_i(c_{i,t}) \geq \sum_{t=0}^{\infty} \beta_t^i u'_i(c_{i,t}) c_{i,t}.$$  

As a consequence, there exists $\sum_{t=0}^{\infty} S_{i,t} c_{i,t} < \infty$. Therefore, we obtain (15).

FOCs imply that

$$\lambda_{i,t} q_t a_{i,t} = (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t}$$

$$= \lambda_{i,t+1}(q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} - \mu_{i,t+1} f p_{t+1} e_{i,t+1}.$$  

Thus we get

$$S_{i,t} \frac{q_t}{p_t} a_{i,t} = S_{i,t+1} \left(\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}\right) a_{i,t} - f_i \frac{\mu_{i,t+1}}{\lambda_{i,t+1}} e_{i,t+1}.$$  \hfill (34)

We rewrite the budget constraint of agent $i$ at date $t$ as follows

$$S_{i,t} c_{i,t} + S_{i,t} \frac{q_t}{p_t} a_{i,t} = S_{i,t} c_{i,t} + S_{i,t} \left(\frac{q_t}{p_t} + \xi_t\right) a_{i,t-1}.$$  

By taking the sum of budget constraints from $t = 0$ until $T$ and using (34), we get that

$$\left(\frac{q_0}{p_0} + \xi_0\right) a_{i,-1} + \sum_{t=0}^{T} S_{i,t} e_{i,t} + \sum_{t=1}^{T} f_i \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} e_{i,t} = \sum_{t=0}^{T} S_{i,t} c_{i,t} + S_{i,t} \frac{q_t}{p_t} a_{i,t}.$$  \hfill (35)

Since $\sum_{t=0}^{\infty} S_{i,t} c_{i,t} < \infty$ and condition (14) holds, there exists

$$\left(\frac{q_0}{p_0} + \xi_0\right) a_{i,-1} + \sum_{t=0}^{\infty} S_{i,t} e_{i,t} + \sum_{t=1}^{\infty} f_i \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} e_{i,t} < \infty.$$  \hfill (36)

We thus have $\lim_{t \to \infty} S_{i,t} e_{i,t} = \lim_{t \to \infty} f_i \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} e_{i,t} = 0$ and there exists $\lim_{t \to \infty} S_{i,t} \frac{q_t}{p_t} a_{i,t}$.

By using (34), we rewrite borrowing constraints $(q_{t+1} + p_{t+1} \xi_{t+1}) a_{i,t} + f_t p_{t+1} e_{i,t+1} \geq 0$ as follows

$$f_t S_{i,t+1} e_{i,t+1} + S_{i,t} \frac{q_t}{p_t} a_{i,t} + f_t \frac{\mu_{i,t+1}}{\lambda_{i,t+1}} S_{i,t+1} e_{i,t+1} \geq 0.$$  \hfill (37)

Since $\lim_{t \to \infty} S_{i,t} e_{i,t} = \lim_{t \to \infty} f_t \frac{\mu_{i,t}}{\lambda_{i,t}} S_{i,t} e_{i,t} = 0$, we obtain that $\lim_{t \to \infty} S_{i,t} \frac{q_t}{p_t} a_{i,t} \geq 0$. Combining with (16), we get (14).
Proposition 1. (Fluctuation of borrowing constraints)

1. For each \( i \), there are only 2 cases

   (a) there does not exist \( \lim_{t \to \infty} \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) \).

   (b) \( \lim_{t \to \infty} \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) = 0 \).

2. We have, for each \( i \),

\[
\liminf_{t \to \infty} \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) = 0 \tag{38}
\]

Proof. Assume that there exists \( \lim_{t \to \infty} Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} =: Q_i \).

If \( Q_i > 0 \), there exists \( t_0 \) such that \( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} > 0 \) for each \( t \geq t_0 \).

However, this condition is equivalent to \( Q_{t+1} \left( \frac{q_{t+1}}{p_{t+1}} + \xi_{t+1} \right) a_{i,t+1} + f_i Q_{t+1} e_{i,t+1} > 0 \).

It means that the borrowing constraints of agent \( i \) are not binding from date \( t_0 \).

This implies that \( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} = 0 \) for every \( t \geq t_0 \). According to Lemma 1, we get \( \lim_{t \to \infty} Q_t \frac{q_t}{p_t} a_{i,t} = 0 \), and \( \lim_{t \to \infty} Q_{t+1} e_{i,t+1} = 0 \), contradiction!

(38) is proved by using the same argument.

\[
\blacksquare
\]

3 Financial asset bubble

According (12), we have \( \frac{q_t}{p_t} = \gamma_{t+1} \left( \frac{q_{t+1}}{p_{t+1}} + \xi_{t+1} \right) \). Therefore, for each \( t \geq 1 \), we have

\[
\frac{q_0}{p_0} = \gamma_1 \left( \frac{q_1}{p_1} + \xi_1 \right) = Q_1 \xi_1 = \gamma_1 q_1 + \gamma_1 \gamma_2 \left( \frac{q_2}{p_2} + \xi_2 \right) = Q_1 \xi_1 + Q_2 \xi_2 + Q_2 \frac{q_2}{p_2} = \ldots = \sum_{s=1}^{t} Q_s \xi_s + Q_t \frac{q_t}{p_t}
\]

Interpretation: In this model, financial asset is a long-lived asset whose price at date 0 is \( q_0 \).

1. At date 1, one unit (from date 0) of this asset will give back 1 units of the same asset and \( \xi_1 \) units of consumption good as its dividend.

2. At date 2, one units of long lived asset will give one unit of the same asset and \( \xi_2 \) units of consumption good ...

This leads us to have the following concept.

**Definition 2.** The fundamental value of financial asset

\[
FV_0 := \sum_{t=1}^{+\infty} Q_t \xi_t
\]
Denote \( b_0 := \lim_{t \to +\infty} Q_t \frac{q_t}{p_t} \), \( b_0 \) is called financial asset bubble. We have

\[
q_0 = b_0 + FV_0.
\]

It means that the price of the financial asset equals its fundamental value plus its bubble.

**Definition 3.** We say there is a bubble on financial asset if the price of financial asset is greater than its fundamental value: \( q_0 > FV_0 \).

### 3.1 Bubbles and borrowing constraints

On the relationship between bubble and borrowing constraints, Kocherlakota (1992) suggests that borrowing constraints are binding infinitely often if bubbles exists.\(^1\) However, what he proved was that \( \lim \inf_{t \to \infty} (a_{i,t} - a^*) = 0 \). We now prove that borrowing constraint is binding.

**Proposition 2.** *(Borrowing constraint is binding at infinitely many date)*

If bubble occurs, there exists \( i \) and an infinite sequence \( (t_n)_{n \geq 1} \) such that borrowing constraint of agent \( i \) is binding at each date \( t_n \).

**Proof.** Assume that for each \( i \), there exists \( t_i \geq 0 \) such that borrowing constraints of agent \( i \) are not binding from \( t_i \). We define \( t_0 = \max_{i=1,\ldots,m} t_i \). Hence, borrowing constraints of all agents are not binding from date \( t_0 \). By using the same argument in the proof of Proposition 1, we have \( \lim_{t \to \infty} Q_t \frac{q_t}{p_t} a_{i,t} = 0 \). As a consequence, we have

\[
\lim_{t \to \infty} Q_t \frac{q_t}{p_t} = 0.
\]

This result says that, if there is a bubble, there exists an agent whose borrowing constraints are binding at infinitely many dates. Our finding complements the one in Kocherlakota (1992).

The following result (as the one in Kocherlakota (1992)) shows that at bubble equilibrium, there is a fluctuation in financial asset volume of some agent.

**Proposition 3.** If bubble occurs, there exists \( i \) such that the sequence \( (a_{i,t}) \) has no limit.

**Proof.** If the sequence \( (a_{i,t}) \) converges for every \( i \), there exists \( i \) such that \( \lim_{t \to \infty} a_{i,t} > 0 \). Borrowing constraint of this agent are not binding from some date \( t_i \). Therefore, by using the same argument as in the proof of Proposition 1, we have \( \lim_{t \to \infty} Q_t \frac{q_t}{p_t} a_{i,t} = 0 \).

Hence, \( \lim_{t \to \infty} Q_t \frac{q_t}{p_t} = 0 \).

\(^1\)Recall that in Kocherlakota (1992), borrowing constraint is \( a_{i,t} \geq a^* \).
3.2 Bubbles and low interest rates

We firstly define what does low interest rates mean. We recall budget constraint of agent $i$ at date $t-1$ and $t$.

\[
\begin{align*}
p_{t-1}c_{i,t-1} + q_{t-1}a_{i,t-1} & \leq p_{t-1}c_{i,t-1} + (q_{t-1} + p_{t-1}\xi_{t-1})a_{i,t-2} \\
p_{t}c_{i,t} + q_{t}a_{i,t} & \leq p_{t}c_{i,t} + q_{t}(1 + \frac{p_{t}\xi_{t}}{q_{t}})a_{i,t-1}.
\end{align*}
\]

One can interpret that if agent $i$ buys $a_{i,t-1}$ units of financial asset at date $t-1$ with price $q_{t-1}$, she will receive $(1 + \frac{p_{t}\xi_{t}}{q_{t}})a_{i,t-1}$ units of financial asset with price $q_{t}$ at date $t$. Therefore, $\frac{p_{t}\xi_{t}}{q_{t}}$ can be viewed as the interest rate of the financial asset at date $t$.

**Definition 4.** We say that interest rates are low at equilibrium if

\[
\sum_{t=1}^{\infty} \frac{p_{t}\xi_{t}}{q_{t}} < \infty.
\] (41)

Otherwise, we say that interest rates are high.

**Remark 3.** In Alvarez and Jermann (2000), they define high implied interest rates as a situation in which the present value of aggregate endowments is finite, i.e.,

\[
\sum_{t=0}^{\infty} Q_{t}e_{t} < \infty,
\]

where $e_{t} := \sum_{i=1}^{m} e_{i,t}$. We will compare these two concepts (high interest rates and high implied interest rates) at the end of this subsection.

We now present relationship between financial bubble and low interest rates.

**Proposition 4.** There is a bubble if and only if interest rates are low.

**Proof.** According to (13), we imply that

\[
\frac{q_{0}}{p_{0}} = Q_{T} \frac{q_{T}}{p_{T}} \prod_{t=1}^{T} (1 + \frac{p_{t}\xi_{t}}{q_{t}}).
\] (42)

Since $q_{0} > 0$, we see that $\lim_{t \to +\infty} Q_{t}q_{t} > 0$ if and only if

\[
\lim_{t \to \infty} \prod_{t=1}^{T} (1 + \frac{\xi_{t}}{q_{t}}) < \infty.
\]

It is easy to prove that this condition is equivalent to (41). □

We give some consequences of condition (41).
Corollary 1. (i) Assume that \( \sum_{t=1}^{\infty} \xi_t < \infty \). If \( \frac{q_t}{p_t} \) is bounded away from zero, there is a bubble.

(ii) Assume that \( \sum_{t=1}^{\infty} \xi_t = \infty \). If there is a bubble, we have \( \lim_{t \to +\infty} \frac{q_t}{p_t} = \infty \).

(iii) Assume that there exists \( x, X \in (0, \infty) \) such that \( x \leq \frac{\xi_t}{\xi_{t+1}} \leq X \). If bubble occurs; the real return \( \frac{\frac{q_{t+1}}{p_{t+1}} + \xi_{t+1}}{\frac{q_t}{p_t}} \) is bounded from below.

Proof. Point (i) and (ii) are clear. Let us prove point (iii). Since bubble occurs, condition 41 is held, thus there exists \( M \in (0, \infty) \) such that

\[
\frac{\frac{p_{t+1} \xi_{t+1}}{q_{t+1} p_t}}{\frac{q_t}{p_t}} \leq M.
\]

By combining with the fact that rate of growth of financial dividend is bounded, we implies that rate of growth of financial asset prices \( \frac{q_t}{p_t} \) is bounded from below. Consequently, the real return is bounded from below.

Proof. Use Proposition 1 and Remark 2.

We now study the relationship between bubble and present values of agents. Recall that the present value of agent \( i \) is given by \( \sum_{t=0}^{\infty} Q_t e_{i,t} \).

**Proposition 5.** If a bubble occurs, there exists \( i \) such that \( \sum_{t=1}^{+\infty} Q_t e_{i,t} = \infty \), and then

\[
\limsup_{t \to \infty} \frac{Q_t}{S_{i,t}} = +\infty.
\]

Proof. Suppose that a bubble occurs. Assume that for every \( i \), \( \sum_{t=1}^{+\infty} Q_t e_{i,t} < \infty \). Thus, \( \lim_{t \to \infty} Q_t e_{i,t} = 0 \). We observe that

\[
\begin{align*}
\left( \frac{q_0}{p_0} + \xi_0 \right) a_{i,-1} + \sum_{t=0}^{T} Q_t e_{i,t} + f_i Q_{t+1} e_{i,t+1} &= \sum_{t=0}^{T} Q_t c_{i,t} + \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) \\
&\geq \sum_{t=0}^{T} Q_t c_{i,t}.
\end{align*}
\]

(43) (44)

Therefore, there exists \( \sum_{t=1}^{+\infty} Q_t c_{i,t} \). Consequently, there exists \( \lim_{t \to \infty} \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) \).

According Proposition 1, we have \( \lim_{t \to \infty} \left( Q_t \frac{q_t}{p_t} a_{i,t} + f_i Q_{t+1} e_{i,t+1} \right) = 0 \) which implies that \( \lim_{t \to \infty} \frac{Q_t}{p_t} = 0 \). \( \square \)
**Interpretation:** We define the gross interest rate \((1 + R_t)\) of the economy and gross interest rate \((1 + R_{i,t})\) of agent \(i\) at date \(t\) as follows

\[
\frac{1}{1 + R_t} = \frac{q_t - 1}{q_t + \xi_t} = \gamma_t = \max_{i \in \{1, \ldots, m\}} \frac{\beta_i u'_i(c_{i,t})}{u'_i(c_{i,t-1})} \quad (45)
\]

\[
\frac{1}{1 + R_{i,t}} = \frac{\beta_i u'_i(c_{i,t})}{u'_i(c_{i,t-1})} \quad (46)
\]

Proposition 5 indicates that if bubble occurs, there exists an agent \(i\) and an infinite sequence date \((t_n)\) such that

\[
\frac{Q_{t_n}}{S_{i,t_n}} = (1 + R_{i,1}) \ldots (1 + R_{i,t_n}) \quad (1 + R_t) \ldots (1 + R_{t_n})
\]

(47)

tends to infinity. It means that the individual interest rate is greater than the interest rate of the economy. We will come back to this point in next section.

We point out some consequences of Proposition 5.

**Corollary 2.** If there exists \(\alpha > 0\) such that \(\xi_t \geq \alpha \sum_{i=1}^{m} e_{i,t}\), there is no bubble.

Therefore, interest rates are high

*Proof.* Note that \(\sum_{t=1}^{\infty} Q_t \xi_t \leq \frac{p_0}{p_0} < \infty\), we implies that \(\sum_{t=1}^{\infty} Q_t e_{i,t} < +\infty\) for every \(i\). According to Proposition 5, there is no bubble. \(\Box\)

Corollary allows us to point out some interesting results:

1. If we consider a situation where endowments are uniformly bounded, bubbles appear only if dividends of asset tend to zero.

2. If we take \(\xi_t = \xi > 0\) for every \(t\), bubbles appear only if the aggregate endowment tends to infinity.

3. Assume that \(e_{i,t} = 0\) for every \(t\) and every \(i\). There is no bubble at equilibrium. This result is in line with Tirole (1982) where he proved that bubble does not exist in a fully dynamic rational expectation equilibrium model without endowments. However, he did not prove the existence of equilibrium.

**Corollary 3.** (Santos and Woodford (1997), Huang and Werner (2000))

Assume that \(\sum_{t=0}^{\infty} Q_t e_t < \infty\). There is no sequential price bubble.

*Proof.* This is a direct consequence of Proposition 5. \(\Box\)

In next Section, we prove that high implied interest rates is only an sufficient condition for no bubble. We end this subsection by making clear the difference between high interest rates and high implied interest rates.
**Proposition 6.** At equilibrium, if \( \sum_{t=0}^{\infty} Q_t e_t < \infty \), we have \( \sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} = \infty \). It means that

**Proof.** According to Corollary 3, if the present value of aggregate endowment is finite, there is no bubble. As a consequence of Proposition 4, there is no bubble if and only if \( \sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} = \infty \). Therefore, we obtain the result. □

This result shows that if an equilibrium has high implied interest rates, it has high interest rates.

### 3.3 A bubbles example

We consider an economy with two infinitely-lived agents \((i = 1, 2)\) characterized by the same CRRA instantaneous utility function

\[
 u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \theta > 0,
\]

having common discount factor \( \beta \in (0, 1) \) and initial asset holdings \( a_{i,-1} = 1/2 \) for \( i = 1, 2 \). The asset’s dividend process \((\xi_t)_{t \in T}\) satisfies: \( \xi_0 > 1 \) and \( \sum_{t=1}^{\infty} \xi_t \leq 1 \). Let \( f_i = 0 \) for every \( i \).

Denote by \( \gamma \) the following constant

\[
 \gamma = \frac{1}{\beta^{1/\theta}}
\]

and consider the sequence \((b_t)_{t \geq 0}\) defined as follows: \( b_0 = 1, b_1 = 1 - \xi_1, \ldots, b_t = 1 - \sum_{s=1}^{t} \xi_t, \ldots \).

The endowments are specified in Table 1. Observe that for both agents we have \((\omega^t_i)_{t \in T} \in \text{int}(\ell^t_{\infty})\).

<table>
<thead>
<tr>
<th>Table 1: Endowments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Endowments for agent 1 ((\tau \geq 1))</strong></td>
</tr>
<tr>
<td>( e_{1,0} = 1 )</td>
</tr>
<tr>
<td>( e_{1,1} + b_0 = \beta^{1/\theta}(e_{1,0} + \xi_0) \times 1/2 )</td>
</tr>
<tr>
<td>( e_{1,2} - b_2 = \beta^{2/\theta}(e_{1,0} + \xi_0) \times 1/2 + \gamma )</td>
</tr>
<tr>
<td>( e_{1,2r+1} + b_{2r} = \beta^{1/\theta}(e_{1,2r} - b_{2r}) )</td>
</tr>
<tr>
<td>( e_{1,2r+2} - b_{2r+2} = \beta^{1/\theta}(e_{1,2r+1} + b_{2r}) + \gamma )</td>
</tr>
</tbody>
</table>

The consumption and long-lived asset allocations are specified in Table 2.

**Equilibrium prices:** We set \( p_t = 1 \) for every \( t \) and we chose the price sequence \( q_t \) to be equal to the process \( b \), i.e., \( q_t = b_t \) for all \( t \).

**Lagrange multipliers:** We define \( \lambda_{i,t} = \beta^t u'_i(c_{i,t}) \), and then \((\mu_{i,t})_{t \in T}\) are specified in Table 3.
Table 2: Equilibrium allocations

<table>
<thead>
<tr>
<th>Allocations for agent 1</th>
<th>Allocations for agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{1,t} = 1, a_{1,t+1} = 0$</td>
<td>$a_{2,t} = 0, a_{2,t+1} = 1$</td>
</tr>
<tr>
<td>$c_{1,0} = (e_{1,0} + \xi_0) \times 1/2$</td>
<td>$c_{2,0} = e_{2,0} + (\xi_0 + q_0) \times 1/2$</td>
</tr>
<tr>
<td>$c_{1,1} = e_{1,1} + q_0$</td>
<td>$c_{2,1} = e_{2,1} - q_1$</td>
</tr>
<tr>
<td>$c_{1,2t} = e_{1,2t} - q_{2t}$</td>
<td>$c_{2,2t} = e_{2,2t} + q_{2t-1}$</td>
</tr>
<tr>
<td>$c_{1,2t+1} = e_{1,2t+1} + q_{2t}$</td>
<td>$c_{2,2t+1} = e_{2,2t+1} - q_{2t+1}$</td>
</tr>
</tbody>
</table>

Table 3: Lagrange multipliers ($\mu_i^t$)_{t \in T}

<table>
<thead>
<tr>
<th>Multipliers for agent 1</th>
<th>Multipliers for agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0^1 = 0$</td>
<td>$\mu_0^2 = \lambda_0^2 - \lambda_1^2$</td>
</tr>
<tr>
<td>$\mu_1^1 = \lambda_1^1 - \lambda_2^1$</td>
<td>$\mu_1^2 = 0$</td>
</tr>
<tr>
<td>$\mu_2^1 = 0$</td>
<td>$\mu_2^2 = \lambda_2^2 - \lambda_2^1$</td>
</tr>
<tr>
<td>$\mu_{2t+1}^1 = \lambda_{2t+1}^1 - \lambda_{2t+2}^1$</td>
<td>$\mu_{2t+1}^2 = 0$</td>
</tr>
</tbody>
</table>

To see that these prices and allocations are an equilibrium, we first note that markets clear at every date, the budget and short-sales constraints are all satisfied. Moreover, the first order conditions as well as the Transversality Condition

$$\lim_{t \to \infty} \beta^t u'(c_t) x_t = 0$$

for each agent $i$ is also satisfied.

Observe that, for each $t \geq 0$,

$$Q_{t+1} = \frac{q_t}{q_{t+1} + \xi_{t+1}} Q_t = Q_t, \quad (48)$$

It means that $Q_t = 1$ for every $t$. Therefore, we obtain $\lim_{t \to \infty} Q_t = 1 - \sum_{t=1}^{\infty} \xi_t$.

If $\sum_{t=1}^{\infty} \xi_t < 1$ then there is a sequential price bubble.

If $\sum_{t=1}^{\infty} \xi_t = 1$ then bubbles are excluded.

Remark 4. In our example, our computation shows that bubbles exists if and only if $\sum_{t=1}^{\infty} \xi_t < 1$. This condition is exactly the low interest rates condition (41) that is

$$\sum_{t=1}^{\infty} \frac{p_t \xi_t}{q_t} < \infty.$$  

Indeed, we have $q_t = 1 - \sum_{s=1}^{t} \xi_s$, and then

$$\sum_{t=1}^{\infty} p_t \xi_t = \sum_{t=1}^{\infty} \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s} \quad (49)$$

We observe that $\sum_{t=1}^{\infty} \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s} < \infty$ if and only if $\lim_{T \to \infty} \prod_{t=1}^{T} (1 + \frac{\xi_t}{1 - \sum_{s=1}^{t} \xi_s}) < \infty$. 

16
We see that

\[
\prod_{t=1}^{T} \left( 1 + \frac{\xi_t}{1 - \sum_{s=1}^{t-1} \xi_s} \right) = \prod_{t=1}^{T} \frac{1}{1 - \sum_{s=1}^{t-1} \xi_s} = \frac{1}{1 - \sum_{s=1}^{T} \xi_s}
\]  

(50)

As a consequence, \(\sum_{t=1}^{\infty} p_t \xi_t < \infty\) is equivalent to \(\sum_{t=1}^{\infty} \xi_t < 1\).

Remark 5. In our example, in both cases (bubble or no-bubble), the present value of aggregate endowments is finite. Indeed, \(e_{1,2\tau} \geq \gamma\) for every \(\tau\). As a consequence, 

\[
\sum_{t=0}^{\infty} Q_t e_t = \sum_{t=0}^{\infty} e_t \geq \sum_{\tau=0}^{\infty} e_{1,2\tau} = \infty.
\]

Therefore, high implied interest rates is only an sufficient condition for no bubble.

3.4 An exogenous sufficient condition for bubble

We have so far given necessary conditions or some sufficient (on endogenous variables) of bubble. Although there are some examples of bubble (Kocherlakota (1992), Huang and Werner (2000)), no one gives conditions of exogenous variables under which there is a bubble at equilibrium.

Our novel contribution is to give a sufficient condition (on exogenous parameters) under which a financial asset bubble occurs.

For simplicity, we assume that \(f_i = 0\) for every \(i\). Let us begin by the following result.

Lemma 2. At each date \(t\), there exists \(i\) such that \(a_{i,t} \geq a_{i,t+1}\) and \(a_{i,t} > 0\).

Proof. Define \(i_0\) such that 

\[
a_{i_0,t} - a_{i_0,t+1} = \max_i \{a_{i,t} - a_{i,t+1}\}.
\]

Then \(a_{i_0,t} - a_{i_0,t+1} \geq 0\).

Case 1: \(a_{i_0,t} - a_{i_0,t+1} > 0\) then \(a_{i_0,t} > a_{i_0,t+1} \geq 0\).

Case 2: \(a_{i_0,t} - a_{i_0,t+1} = 0\) then \(a_{i,t} - a_{i,t+1} \leq 0\) for every \(i\). Since \(\sum_i (a_{i,t} - a_{i,t+1}) = 0\), we imply that \(a_{i,t} - a_{i,t+1} = 0\) for every \(i\). Choose \(i_1\) such that \(a_{i_1,t} > 0\), we have \(a_{i_1,t} = a_{i_1,t+1}\) and \(a_{i_1,t} > 0\). \(\square\)

We now bound the size of the discount rate \(\gamma_t\) by exogenous bounds.

Lemma 3. (Size of discount rate \(\gamma_t\))

We have 

\[
A_t < \gamma_t < D_t,
\]

where \(D_t := \max_{i \in \{1, \ldots, m\}} \frac{\beta u'_i(e_{i,t})}{u'_i(W_{i,t-1})}\), \(A_t := \min_{i \in \{1, \ldots, m\}} \frac{\beta u'_i(W_{i,t})}{u'_i(W_{i-1,t})}\), and \(W_t := \sum_{i=1}^{m} e_{i,t} + \xi_t\).
Note that $A_t, D_t$ are exogenous.

**Proof.** Recall that $\gamma_t := \max_{i \in \{1, \ldots, m\}} \frac{\beta_i u'_i(c_{i,t})}{u'_i(c_{i,t-1})}$. By using Lemma 2, there exists $i$ such that $a_{i,t-1} \geq a_{i,t}$ and $a_{i,t-1} > 0$. Since $a_{i,t-1} > 0$, we get $\mu_{i,t-1} = 0$, and then

$$\gamma_t = \frac{\beta_i u'_i(c_{i,t})}{u'_i(c_{i,t-1})}.$$  \hfill (52)

On the one hand, we have $c_{i,t-1} < W_{t-1}$, so $u'_i(c_{i,t-1}) > u'_i(W_{t-1})$. On the other hand, we have

$$c_{i,t} + q_t a_{i,t} = e_{i,t} + (q_t + \xi_t) a_{i,t-1} \geq e_{i,t} + (q_t + \xi_t) a_t,$$  \hfill (53)

hence $c_{i,t} \geq e_{i,t}$. Therefore, we get that

$$\gamma_t = \frac{\beta_i u'_i(c_{i,t})}{u'_i(c_{i,t-1})} \leq \frac{\beta_i u'_i(c_{i,t})}{u'_i(W_{t-1})} \leq D_t$$  \hfill (54)

It is easy to see that there exists $j$ such that $c_{j,t-1} \geq \frac{W_{t-1}}{m}$, so

$$\gamma_t \geq \frac{\beta_j u'_j(c_{j,t})}{u'_j(c_{j,t-1})} \geq \frac{\beta_j u'_j(W_t)}{u'_j(\frac{W_{t-1}}{m})} \geq A_t.$$  \hfill (55)

**Interest rates:** Before giving a sufficient condition for financial asset bubble, let us study the gross interest rate. We define $r_t$ by $\frac{1}{1+r_t} = D_t$. It is easy to see that

$$r_t < R_t = \min_i \{R_{i,t}\}.$$

We recall that $FV_t$ is the fundamental value of financial asset at date $t$. We have

$$q_0 = b_0 + \sum_{t=1}^{+\infty} Q_t \xi_t = b_0 + FV_0$$  \hfill (56)

$$q_1 = b_1 + \sum_{t=1}^{+\infty} Q_t^1 \xi_t = b_1 + FV_1,$$  \hfill (57)

where $Q_t^1 := \frac{Q_t}{\gamma_1}$, and $b_1 = \frac{b_0}{\gamma_1}$.

**Lemma 4.** We have

$$\frac{q_1 + \xi_1}{q_0} = \frac{FV_1 + \xi_1}{FV_0}.$$  \hfill (58)

**Proof.** We have $\frac{q_0}{q_1 + \xi_1} = \frac{b_0 + FV_0}{b_1 + FV_1 + \xi_1}$. Note that $b_0 = b_1 \frac{q_0}{q_1 + \xi_1}$, and then we get (58).
According to Condition (58), the real return of an asset can be computed by using its prevent value: it equals the ratio between the sum its dividend and its fundamental value at date 1 and its fundamental value at date 0.

We now state our main result in this subsection.

**Theorem 2. (An exogenous sufficient condition for financial asset bubble)**

We normalize by setting \( p_t = 1 \) for every \( t \).

There is a financial asset bubble at equilibrium if the following conditions hold:

(i) \( B := \sum_{t=1}^{\infty} B_t \xi_t < \infty \), where \( B_t := \prod_{k=1}^{t} D_k \).

(ii) There exists \( i \) such that

\[
\frac{\beta_i u'_i(W_1)}{u'_i(e_{i,0} + \xi_0 a_{i,-1} - B(1 - a_{i,-1}))} \geq \frac{B}{A + \xi_i}. \tag{59}
\]

where \( A := \sum_{t=2}^{\infty} (\prod_{s=2}^{t} A_s) \xi_s \).

Note that these conditions is satisfied if \( \xi_1, \xi_2, \ldots, \) are small.

**Proof.** First, according to Lemma 3, we have

\[
q_0 = b_0 + \sum_{t=1}^{+\infty} Q_t \xi_t < b_0 + \sum_{t=1}^{+\infty} B_t \xi_t = b_0 + B \tag{60}
\]

\[
q_1 \geq \sum_{t=2}^{\infty} (\prod_{s=2}^{t} \gamma_s) \xi_s > \sum_{t=2}^{\infty} (\prod_{s=2}^{t} A_s) \xi_s = A. \tag{61}
\]

Condition (59) implies that

\[
\frac{\beta_i u'_i(W_1)}{u'_i(e_{i,0} + \xi_0 a_{i,-1} - B(1 - a_{i,-1}))} \geq \frac{B}{q_1 + \xi_1} = \frac{q_0 - b_0}{q_1 + \xi_1}. \tag{62}
\]

We also have

\[
e_{i,0} + \xi_0 a_{i,-1} - B(1 - a_{i,-1}) < e_{i,0} + \xi_0 a_{i,-1} - (q_0 - b_0)(1 - a_{i,-1})
= e_{i,0} + (q_0 + \xi_0) a_{i,-1} - q_0 + b_0(1 - a_{i,-1})
\leq e_{i,0} + (q_0 + \xi_0) a_{i,-1} - q_0 a_{i,0} + b_0(1 - a_{i,-1})
= e_{i,0} + b_0(1 - a_{i,-1}). \tag{63}
\]

Hence,

\[
\frac{\beta_i u'_i(c_{i,1})}{u'_i(c_{i,0})} \leq \max_{j \in \{1, \ldots, m\}} \frac{\beta_j u'_j(c_{j,1})}{u'_j(c_{j,0})} = \frac{q_0}{q_1 + \xi_1} < \frac{b_0}{q_1 + \xi_1} + \frac{\beta_i u'_i(c_{i,1})}{u'_i(c_{i,0} + b_0(1 - a_{i,-1}))}.
\]

Since the function \( f(x) = \frac{x}{q_1 + \xi_1} + \frac{\beta_i u'_i(c_{i,1})}{u'_i(c_{i,0} + x(1 - a_{i,-1}))} \) is increasing in \( x \), we obtain \( b_0 > 0 \). \( \Box \)
**Interpretation:** $B$ is an upper bound of the fundamental value $FV_0$ of financial asset at initial date 0. $A$ is a lower bound of the fundamental value $FV_1$ of financial asset at date 1. We define $\bar{r}_1$ by \[
\frac{1}{1 + \bar{r}_1} = \frac{B}{A + \xi_1}.
\] We see that $\bar{r}_1$ is a lower bound of the interest rate $R_1$. Indeed, \[
1 + \bar{r}_1 = \frac{B}{A} + \xi_1.
\] (64) So, we get $\bar{r}_1 < \bar{R}_1$.

Define $\bar{r}_{i,1}$ by \[
\frac{1}{1 + \bar{r}_{i,1}} = \frac{\beta_i u'_i(W_i)}{u'_i(e_{i,0} + \xi_0 - B(1 - a_{i,-1}))}.
\] (65) For any equilibrium without bubble, by using (63) we see that, for each $i$, \[
\frac{\beta_i u'_i(W_i)}{u'_i(e_{i,0} + \xi_0 - B(1 - a_{i,-1}))} \leq \frac{\beta_i u'_i(c_1)}{u'_i(c_{i,0})}.
\]

Therefore, $\bar{r}_{i,1}$ is an upper bound of the agent $i$'s subjective interest rate for no-bubble equilibria.

We now observe that Condition (59) is equivalent to \[
1 + \bar{r}_1 \geq 1 + \bar{r}_{i,1}.
\] (66) This implies that the agent $i$'s subjective interest rate $\bar{r}_{i,1}$ is less than the interest rate of the economy $R_1$. Therefore, agent $i$ accepts to buy financial asset with a price which is greater than the fundamental value. Consequently, there is a bubble.

**3.4.1 When $f^i \in (0, 1)$**

**QUESTION:** HOW TO BOUND from below $c_{i,t}$?

**Remark 6.** We observe that, for any $i$, \[
-a_{i,t-1} \leq \frac{f_i p_t e_{i,t}}{q_t + p_t \xi_t} < \min\left(\frac{f_i e_{i,t}}{\xi_t}, \frac{f_i p_t e_{i,t}}{q_t}\right)
\] \[
(67)
\]
hence $\xi_t a_{i,t-1} + f^i e_{i,t} > 0$ and $q_t a_{i,t-1} + f^i p_t e_{i,t} > 0$ for any $i$.

**Remark 7.** Let $\mathcal{A} := \{i : (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} > 0\}$.

Since $(q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} \geq 0$ for any $i \in I$, we have \[
\sum_{i \in \mathcal{A}} (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} = \sum_{i \in I} (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t}
\]
\[
= q_t + p_t \xi_t + \sum_{i \in \mathcal{A}} f^i e_{i,t}.
\] (68) \[
(69)
\]

**QUESTION:** Does there exist some $i \in \mathcal{A}$ satisfying the following condition? \[
(q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} - q_t a_{i,t} > 0
\] (70)
Lemma 5. Given an equilibrium. At each date $t$, there exists an agent $i$ such that

\[
(q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} - q_t a_{i,t} > 0 \quad (71)
\]

\[
(q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} > 0. \quad (72)
\]

Proof. Let a date $t$. Define $i_t$ such that

\[
(q_t + p_t \xi_t) a_{i_t,t-1} - q_t a_{i_t,t-1} = \max_{i \in I} \{(q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_i - q_t a_{i,t-1}\}.
\]

It is easy to see that

\[
(q_t + p_t \xi_t) a_{i_t,t-1} + f^i p_t e_{i_t,t} - q_t a_{i_t,t-1} > 0.
\]

We now see that, for any $i$,

\[
-a_{i,t-1} \leq \frac{f^i p_t e_{i,t}}{q_t + p_t \xi_t} < \frac{f^i e_{i,t}}{\xi_t}
\]

hence $p_t a_{i,t-1} + f^i p_t e_{i,t} > 0$ for any $i$.

Choose $j$ such that $a_{j,t-1} \geq a_{j,t}$ then we have

\[
(q_t + p_t \xi_t) a_{i_t,t-1} + f^i p_t e_{i_t,t} - q_t a_{i_t,t} = \max_{i \in I} \{(q_t + p_t \xi_t) a_{i,t-1} + f^i e_i - q_t a_{i,t}\} \quad (74)
\]

\[
\geq p_t \xi_t a_{j,t-1} + f^i p_t e_{j,t} > 0. \quad (75)
\]

It is ok.

In this case, with $i_t$ chosen above, we have

\[
p_t c_{i,t} = (q_t + p_t \xi_t) a_{i_t,t-1} + p_t e_{i,t} - q_t a_{i,t} = (1 - f^i) p_t e_{i,t} + (q_t + p_t \xi_t) a_{i,t-1} + f^i p_t e_{i,t} - q_t a_{i,t} \quad (76)
\]

\[
>(1 - f^i) e_{i,t} p_t. \quad (77)
\]

Remark 8. By using Lemma 5 and the same argument in Lemma 3, we can give an upper bound of the discount factor $\gamma_t$ as follows

\[
\gamma_t \leq \frac{\beta u_i((1 - f^i) e_{i,t})}{u'_i(W_{t-1})}. \quad (78)
\]

4 Conclusion

We considered an infinite horizon general equilibrium asset pricing model with heterogeneous agents and endogenous borrowing constraints. We proved the existence of equilibrium in this model without any condition about endowments.

At equilibrium, if the market price of the financial asset is greater than its fundamental value, we say that there is a bubble. Borrowing constraints play an important role on bubble: a bubble can occur only if there exists an agent whose borrowing constraints are binding in infinitely many times. We prove that the existence of a bubble is equivalent to low interest rates. We also give an sufficient condition (on exogenous variables) for the existence of bubble: the highest subjective interest rate of agent is smaller than the interest rate of the economy.
5 Appendix 1: Existence of equilibrium for truncated economy

For each $T \geq 0$, we define $T-$ truncated economy $E^T$ as $E$ but there are no activities from period $T + 1$, i.e., $c_{i,t} = a_{i,t-1} = 0$ for every $i = 1, \ldots, m$, $t \geq T + 1$.

5.1 Existence of equilibrium for bounded economy

We define the bounded economy $E^T_b$ as $E^T$ but all variables (consumption demand, asset investment) are bounded. \[ C_i := [0, Bc]^{T+1}, \quad Bc > 1 + \max_{t \leq T} W_t \]
\[ A_i := [-Ba, Ba]^T, \quad Ba > 1 + B, \]
where $B$ is satisfied $B > \max\{ \max_{t \leq T} \frac{1+W_t}{\xi_t}, 1 + m \max_{t \leq T} \frac{1+W_t}{\xi_t} \}$.

Denote $\Delta := \{(z_0 = (p, q) : 0 \leq p_t, q_t \leq 1, p_t + q_t = 1 \quad \forall t = 0, \ldots, T)\}$.

For each $\epsilon > 0$ such that $2m\epsilon < 1$, we define $\epsilon-$economy $E^{T,\epsilon}_b$ by adding $\epsilon$ units of each asset (consumption good and financial asset) at date 0 for each agent in the bounded economy. More presise, the feasible set of agent $i$ is given by \[ C^{T,\epsilon}_i(p, q) := \{(c_{i,t}, a_{i,t})_{t=0}^T \in \mathbb{R}^{T+1} \times \mathbb{R}^{T+1} : (a) \quad a_{i,T} = 0, \]
\[ (b) \quad p_0c_{i,0} + q_0a_{i,0} \leq p_0(e_{i,0} + \epsilon) + (q_0 + \bar{p}_0\xi_0)(a_{i,t-1} + \epsilon) \]
\[ (c) \quad \text{for each } 1 \leq t \leq T : \]
\[ 0 \leq (q_t + p_t\xi_t)a_{i,t-1} + f_i p_t(e_{i,t} + \epsilon) \]
\[ p_t c_{i,t} + q_t a_{i,t} \leq p_t(e_{i,t} + \epsilon) + (q_t + p_t\xi_t)a_{i,t-1} \} \].

Definition 5. A sequence of prices and quantities \( (\bar{p}_t, \bar{q}_t, \bar{r}_t, (\bar{c}_{i,t}, \bar{a}_{i,t}, \bar{k}_{i,t})_{i=1}^m, \bar{K}_t)_{t=0}^T \) is an equilibrium of the economy $E^{T,\epsilon}_b$ if the following conditions are satisfied

(i) Price are strictly positive, i.e., $\bar{p}_t, \bar{q}_t > 0$ for $t \geq 0$.

(ii) All markets clear:

Consumption good
\[ \sum_{i=1}^m \bar{c}_{i,0} = \sum_{i=1}^m \bar{e}_{i,0} + 2m\epsilon + \xi_0 \]
\[ \sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m \bar{e}_{i,t} + m\epsilon + \xi_t \]

Financial asset
\[ \sum_{i=1}^m \bar{a}_{i,0} = \sum_{i=1}^m (\bar{a}_{i,-1} + \epsilon), \quad \sum_{i=1}^m \bar{a}_{i,t+1} = \sum_{i=1}^m \bar{a}_{i,t}, \forall t \geq 0. \]
(iii) Optimal consumption plans: for each $i$, $(\bar{c}_{i,t}, \bar{a}_{i,t})_{t=0}^{\infty}$ is a solution of the maximization problem of agent $i$ with the feasible set $C_i^{T^*}(p,q)$.

The first step is to prove the existence of equilibrium for each $\epsilon-$ economy when $\epsilon$ is small. We then take a sequence $(\epsilon_n)$ converging to zero. When $n$ tends to infinity, the sequence of equilibria depending on $\epsilon_n$ has a limit who will be proved to be an equilibrium for bounded economy $E_0^T$.

5.1.1 Existence of equilibrium for $\epsilon$-economy

We also define $B_i^{T^*}(p,q, \cdot)$ as follows.

$$B_i^{T^*}(p,q) := \{(c_{i,t}, a_{i,t})_{t=0}^{T} \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : \text{(a) } a_{i,T} = 0,$$

$$(b) \quad p_0 c_{i,0} + q_0 a_{i,0} < p_0(e_{i,0} + \epsilon) + (q_0 + p_0\xi_0)(a_{i,-1} + \epsilon)$$

$$(c) \text{ for each } 1 \leq t \leq T :$$

$$0 < (q_t + p_t\xi_t)a_{i,t-1} + f_i(e_{i,t} + \epsilon)$$

$$p_t c_{i,t} + q_t a_{i,t} < p_t(e_{i,t} + \epsilon) + (q_t + p_t\xi_t)a_{i,t-1} \}.$$

We write $C_i^{T}(p,q), B_i^{T}(p,q)$ instead of $C_i^{T,0}(p,q), B_i^{T,0}(p,q)$.

Lemma 6. $B_i^{T^*}(p,q, \cdot) \neq \emptyset$ and $B_i^{T^*}(p,q) = C_i^{T^*}(p,q)$.

Proof. We write

$$B_i^{T^*}(p,q) := \{(c_{i,t}, a_{i,t})_{t=0}^{T} \in \mathbb{R}_+^{T+1} \times \mathbb{R}_+^{T+1} : a_{i,T} = 0,$$

$$0 < p_0(e_{i,0} + \epsilon + \xi_0 a_{i,t-1} - c_{i,0}) + q_0(a_{i,-1} + \epsilon - a_{i,0})$$

and for each $1 \leq t \leq T :$

$$0 < q_t a_{i,t-1} + p_t(\xi_t a_{i,t-1} + f_i(e_{i,t} + \epsilon))$$

$$0 < p_t(e_{i,t} + \epsilon + \xi_t a_{i,t-1} - c_{i,t}) + q_t(a_{i,t-1} - a_{i,t}).$$

Since $e_{i,0} + \epsilon + \xi_0 a_{i,t-1} > 0$ and $a_{i,t-1} + \epsilon > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $a_{i,0} \in (0, B_a)$ such that

$$0 < p_0(e_{i,0} + \epsilon + \xi_0 a_{i,t-1} - c_{i,0}) + q_0(a_{i,-1} + \epsilon - a_{i,0}).$$

By induction, we see that $B_i(p,q, \cdot)$ is not empty. \qed

Lemma 7. $B_i(p,q)$ is lower semi-continuous correspondence on $\Delta$. And $C_i(p,q)$ is upper semi-continuous on $\Delta$ with compact convex values.

Proof. Clearly, since $B_i(p,q)$ is empty and has open graph. \qed

We define $\Phi := \Delta \times \prod_{i=1}^{m} (C_i \times A_i)$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m}$ where $z_0 := (p,q,r), z_i := (c_i, a_i, k_i)$ for each $i = 1, \ldots, m$. 

\begin{align*}
\text{(iii) Optimal consumption plans: for each } i, (\bar{c}_{i,t}, \bar{a}_{i,t})_{t=0}^{\infty} \text{ is a solution of the maximization problem of agent } i \text{ with the feasible set } C_i^{T^*}(p,q). 
\end{align*}
We now define correspondences. First, we define $\varphi_0$ (for additional agent 0)

$$\varphi_0 : \prod_{i=1}^m (C_i \times A_i) \rightarrow 2^\Delta$$

$$\varphi_0((z_i)_{i=1}^m) := \arg \max_{(p,q) \in \Delta} \left\{ p_0 \left( \sum_{i=1}^m (c_{i,0} - e_{i,0}) - 2m\epsilon - \xi_0 \right) + q_0 \sum_{i=1}^m (a_{i,0} - a_{i,-1} - \epsilon) 
+ \sum_{t=1}^T p_t \left( \sum_{i=1}^m (c_{i,t} - e_{i,t}) - m\epsilon - \xi_t \right) + \sum_{t=1}^{T-1} q_t \sum_{i=1}^m (a_{i,t} - a_{i,t-1}) \right\}.$$ 

For each $i = 1, \ldots, m$, we define

$$\varphi_i : \Delta \times \rightarrow 2^{C_i \times A_i},$$

$$\varphi_i(p, q) := \arg \max_{(c_i, a_i) \in C_i} \left\{ \sum_{t=0}^T \beta_t u_t(c_{i,t}) \right\}.$$ 

**Lemma 8.** The correspondence $\varphi_i$ is lower semi-continuous and non-empty, concex, compact valued for each $i = 0, 1, \ldots, m + 1$.

**Proof.** This is a direct consequence of the Maximum Theorem. 

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{q}) \in \varphi_0((\bar{c}_i, \bar{a}_i)_{i=1}^m) \quad (79)$$

$$(\bar{c}_i, \bar{a}_i) \in \varphi_i((\bar{p}, \bar{q})) \quad (80)$$

Denote

$$\bar{X}_0 := \sum_{i=1}^m (c_{i,0} - e_{i,0}) - 2m\epsilon - \xi_0 \quad (81)$$

$$\bar{X}_t := \sum_{i=1}^m (c_{i,t} - e_{i,t}) - m\epsilon - \xi_t, \quad t \geq 1 \quad (82)$$

$$\bar{Z}_0 = \sum_{i=1}^m (\bar{a}_{i,0} - \epsilon - \bar{a}_{i,-1}), \quad \bar{Z}_t = \sum_{i=1}^m (\bar{a}_{i,t} - \bar{a}_{i,t-1}), \quad t \geq 1 \quad (83)$$

For every $(p, q) \in \Delta$, we have

$$\sum_{t=0}^T (p_t - \bar{p}_t) \bar{X}_t + \sum_{t=0}^{T-1} (q_t - \bar{q}_t) \bar{Z}_t \leq 0 \quad (84)$$

By summing the budget constraints, we get that: for each $t$

$$\bar{p}_t \bar{X}_t + \bar{q}_t \bar{Z}_t \leq 0 \quad (85)$$

Hence, we have: for every $(p, q) \in \Delta$

$$p_t \bar{X}_t + q_t \bar{Z}_t \leq \bar{p}_t \bar{X}_t + \bar{q}_t \bar{Z}_t \leq 0 \quad (86)$$
Therefore, we have $\bar{X}_t, \bar{Z}_t \leq 0$, which implies that
\[
\sum_{i=1}^{m} \bar{c}_{i,0} \leq \sum_{i=1}^{m} \bar{c}_{i,t} + 2m\epsilon + \xi_0 \tag{87}
\]
\[
\sum_{i=1}^{m} \bar{c}_{i,t} \leq \sum_{i=1}^{m} \bar{c}_{i,t} + m\epsilon + \xi_t, \quad t \geq 1 \tag{88}
\]
\[
\sum_{i=1}^{m} \bar{a}_{i,0} \leq \sum_{i=1}^{m} (\bar{a}_{i,-1} + \epsilon), \quad \sum_{i=1}^{m} \bar{a}_{i,t} \leq \sum_{i=1}^{m} \bar{a}_{i,t-1}, \quad t \geq 1. \tag{89}
\]

**Lemma 9.** $\bar{p}_t > 0$ and $\bar{q}_t \geq 0$ for $t = 0, \ldots, T$. Moreover, $\bar{q}_t > 0$ if $\xi_{t+1} > 0$.

**Proof.** If $\bar{p}_t = 0$, the optimality implies that $\bar{c}_{i,t} = B_c > 1 + W_t$. Therefore, we get $\bar{c}_{i,t} > \sum_{i=1}^{m} e_{i,t} + 2m\epsilon + \xi_t)$, contradiction. Hence, $\bar{p}_t > 0$

We now assume that $\xi_{t+1} > 0$. If $\bar{q}_t = 0$, we have $\bar{a}_{i,t} = B_a$ for each $i$. Thus, $\sum_{i=1}^{m} \bar{a}_{i,t} \geq mB_a > 1 + B_a$. However, we have $\sum_{i=1}^{m} \bar{a}_{i,t} \leq \sum_{i=1}^{m} \bar{a}_{i,-1} + m\epsilon = 1 + m\epsilon < 1 + B_a$, contradiction! \hfill \qed

**Lemma 10.** $\bar{X}_t = \bar{Z}_t = 0$.

**Proof.** Since consumption good prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing budget constraints at date $t$ we have which implies that

$$\bar{p}_t \bar{X}_t + \bar{q}_t \bar{Z}_t = 0.$$ \tag{90}

By combining with the fact that $\bar{X}_t, \bar{Z}_t \leq 0$, we obtain $\bar{X}_t = \bar{Z}_t = 0$. \hfill \qed

The optimality of $(c_i, a_i)$ is from (80).

### 5.1.2 When $\epsilon$ tends to zero

We have so far proved that for each $\epsilon_n = 1/n > 0$, where $n$ is integer number and high enough, there exists an equilibrium, say

$$equi(n) := \left(\bar{p}_t(n), \bar{q}_t(n), (\bar{c}_{i,t}(n), \bar{a}_{i,t}(n))_{i=1}^{m}\right)^T_{t=0},$$

for the economy $\mathcal{E}^{T,\epsilon_n}_b$.

By using borrowing constraint, we get that: for every $n$,

$$-\bar{a}_{i,t}(n) \leq \frac{f_t\bar{p}_{t+1}(n)(e_{i,t+1} + \epsilon_n)}{\bar{q}_{t+1}(n) + \xi_{t+1}\bar{p}_{t+1}(n)} \leq \frac{f_t(e_{i,t+1} + \epsilon_n)}{\xi_{t+1}} \leq \frac{1 + W_{t+1}}{\xi_{t+1}}.$$
By combining with $\sum_{i=1}^{m} a_{i,t}(n) = 1$, we see that $a_{i,t}(n)$ is uniformly bounded when $n$ tends to infinity. Moreover, we have $\bar{p}_t(n) + \bar{q}_t(n) = 1$. Therefore, we can assume that
\[
(p(n), q(n), (\bar{c}_i(n), \bar{a}_i^T(n)))_{i=1}^{m} \xrightarrow{n \to \infty} (\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^{m}).
\]

**Markets clearing conditions:** By taking limit of market clearing conditions for economy $E_b^{T,\epsilon}$, we obtain market clearing conditions for the economy $E_b^T$.

**Lemma 11.** $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$ if $(e_{i,0}, a_{i,-1}) \neq (0,0)$.

**Proof.** Recall that
\[
B_i^{T,\epsilon}(\bar{p}, \bar{q}) := \{(c_{i,t}, a_{i,t})_{t=0}^{T+1} \in \mathbb{R}^{T+1}_+ \times \mathbb{R}^{T+1}_+ : a_{i,T} = 0,
0 < \bar{p}_0(e_{i,0} + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} - a_{i,0})
\text{ and for each } 1 \leq t \leq T :
0 < \bar{q}_i a_{i,t-1} + \bar{p}_t (\xi_1 a_{i,t-1} + f_i e_{i,t})
0 < \bar{p}_i (e_{i,t} + \xi_0 a_{i,t-1} - c_{i,t}) + \bar{q}_t (a_{i,t-1} - a_{i,t}).
\]

If $(e_{i,0}, a_{i,-1}) \neq (0,0)$, we have $e_{i,0} + \xi_0 a_{i,t-1} > 0$. By combining with $\bar{p}_t + \bar{q}_t = 1$, we can choose $c_{i,0} \in (0, B_c)$ and $a_{i,0} \in (0, B_a)$ such that
\[
0 < \bar{p}_0(e_{i,0} + \xi_0 a_{i,t-1} - c_{i,0}) + \bar{q}_0(a_{i,-1} - a_{i,0})
0 < \bar{q}_1 a_{i,0} + \bar{p}_1 (\xi_1 a_{i,0} + f_i e_{i,1}).
\]

\[\square\]

**Lemma 12.** We have $\bar{p}_t, \bar{q}_t > 0$.

**Proof.** Since $\sum_{i=1}^{m} a_{i,-1} = 1 > 0$, there exists an agent $i$ such that $a_{i,-1} > 0$. According Lemma 11, we have $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$. We are going to prove that the optimality of allocation $(\bar{c}_i, \bar{a}_i)$.

Let $(c_i, a_i)$ be an feasible allocation of the maximization problem of agent $i$ with the feasible set $C_i^T(\bar{p}, \bar{q})$. We have to prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

Since $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$, there exists sequences $(h)_{h \geq 0}$ and $(c_i^h, a_i^h) \in B_i^T(\bar{p}, \bar{q})$ such that $(c_i^h, a_i^h)$ converges to $(c_i, a_i)$. We have
\[
\bar{p}_t c_{i,t}^h + \bar{q}_t a_{i,t}^h < \bar{p}_t e_{i,t} + (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h
0 < (\bar{q}_t + \bar{p}_t \xi_t) a_{i,t-1}^h + f_i \bar{p}_t e_{i,t}.
\]

\[2\text{In fact, since prices and allocations are bounded, there exists a subsequence } (n_1, n_2, \ldots) \text{ such that } equi(n) \text{ converges. However, without loss of generality, we can assume that } equi(n) \text{ converges.}\]
Fixe $h$. Let $n_0$ ($n_0$ depends on $h$) be high enough such that for every $n \geq n_0$, $(c^h_i, a^h_i) \in C^{T,1/n}_i(\bar{p}(n), \bar{q}(n))$. Therefore, we have $\sum_{t=0}^{\infty} \beta^t_i u_t(c^h_i) \leq \sum_{t=0}^{\infty} \beta^t_i u_t(\bar{c}_{i,t}(n))$.

Let $n$ tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta^t_i u_t(c^h_i) \leq \sum_{t=0}^{\infty} \beta^t_i u_t(\bar{c}_{i,t})$.

Let $h$ tend to infinity, we have $\sum_{t=0}^{\infty} \beta^t_i u_t(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta^t_i u_t(\bar{c}_{i,t})$. It means that we have just proved the optimality of $(\bar{c}_i, \bar{a}_i, \bar{k}_i)$.

We now prove $\bar{p}_t > 0$ for every $t$. Indeed, otherwise we have $c_{i,t} = B_c > 1 + W_t$, contradiction. $\bar{q}_t > 0$ is from the positivity of financial dividend.

**Lemma 13.** For each $i$, $(\bar{c}_i, \bar{a}_i)$ is optimal.

**Proof.** Consider agent $i$. Since $(c_{i,0}, a_{i,-1}) \neq (0, 0)$, we have $B^T_i(\bar{p}, \bar{q}) \neq \emptyset$. By using the same argument as in Lemma 12, we obtain the optimality of $(\bar{c}_i, \bar{a}_i)$.

## 5.2 Existence of equilibrium for unbounded economy

We claim that an equilibrium of $E^T_i$ is also an equilibrium for $E^T$. Indeed, let $(\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{a}_{i,t})_{t=1}^{m})_T$ be an equilibrium of $E^T_i$. Note that $a_{i,T} = 0$ for every $i = 1, \ldots, m$. It is easy to see that prices are strictly positive and all markets clear. We will prove the optimality of allocation.

Let $z_i := (c_{i,t}, a_{i,t})_T$ be a feasible plan of household $i$. Assume that $\sum_{t=0}^{T} \beta^t_i u_t(c_{i,t}) > \sum_{t=0}^{T} \beta^t_i u_t(\bar{c}_{i,t})$. For each $\gamma \in (0,1)$, we define $z_i(\gamma) := \gamma z_i + (1-\gamma)\bar{z}_i$. By definition of $E^T_i$, we can choose $\gamma$ sufficiently close to 0 such that $z_i(\gamma) \in C_i \times A_i$. It is clear that $z_i(\gamma)$ is satisfied budget constraints.

By the concavity of the utility function, we have

$$\sum_{t=0}^{T} \beta^t_i u_t(c_{i,t}(\gamma)) \geq \gamma \sum_{t=0}^{T} \beta^t_i u_t(c_{i,t}) + (1-\gamma) \sum_{t=0}^{T} \beta^t_i u_t(\bar{c}_{i,t})$$

Contradiction to the optimality of $\bar{z}_i$.

## 6 Appendix 2: Existence of equilibrium in the infinite horizon economy

We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $E^T$. We denote by $(\bar{p}^T, \bar{q}^T, (\bar{c}^T_i, \bar{a}^T_i)_{i=1}^{n})$ an equilibrium of $T-$ truncated economy $E^T$. 

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We can normalize prices by setting $\bar{p}_t^T + \bar{q}_t^T = 1$ for every $t \leq T$.
It is easy to see that
\[
0 < c^T_{i,t} < D_t \quad \text{and} \quad -\bar{a}^T_{i,t} \leq \frac{W_{t+1}}{\xi_{t+1}} \quad \forall i, \text{ and } \sum_{i=1}^{m} \bar{a}^T_{i,t} = 1.
\]

Therefore, endogenous variables are bounded for the product topology. Therefore, we can assume that
\[
(\bar{p}^T, \bar{q}^T, (c^T_{i,t}, \bar{a}^T_{i,t})_{i=1}^{m}) \xrightarrow{T \to \infty} (\bar{p}, \bar{q}, (\bar{c}_i, \bar{a}_i)_{i=1}^{m}) \quad \text{(for the product topology ).}
\]

We prove that this limit is an equilibrium for the economy $E$.

It is easy to see that all markets clear.

**Lemma 14.** We have $\bar{p}_t > 0$ for each $t \geq 0$.

**Proof.** There exists $i$ such that $a_{i,-1} > 0$. By using the same argument in Lemma 11, we see that $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$.

Let $(c_i, a_i)$ be an feasible allocation of the problem $P_i(\bar{p}, \bar{q})$. We have to prove that
\[
\sum_{t=0}^{\infty} \beta^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta^t u_i(\bar{c}_{i,t}).
\]

Note that, without loss of generality, we can only consider feasible allocations such that $\bar{p}_T c_i + \bar{q}_T a_i \geq 0$. We define $(c'_{i,t}, a'_{i,t})_{t=0}^{T}$ as follows:
\[
c'_{i,t} = a_{i,t}, \text{ if } t \leq T - 1, = 0 \text{ if } t 
\]
\[
c'_{i,t} = c_{i,t}, \text{ if } t \leq T - 1; \bar{p}_T c'_{i,t} = \bar{p}_T c_i + \bar{q}_T a_i; \; c_{i,t} = 0 \text{ if } t > T.
\]

We see that $(c'_{i,t}, a'_{i,t})_{t=0}^{T}$ belongs to $C_i^T(\bar{p}, \bar{q})$. Since $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$, there exists a sequence $(c^n_{i,t}, a^n_{i,t})_{t=0}^{\infty}$ in $B_i^T(\bar{p}, \bar{q})$ with $a^n_{i,T} = 0$, and this sequence converges to $(c'_{i,t}, a'_{i,t})_{t=0}^{T}$ when $n$ tends to infinity. We have
\[
\bar{p}_t c^n_{i,t} + \bar{q}_t a^n_{i,t} < \bar{p}_t c_{i,t} + (\bar{q}_t + \bar{p}_t \xi_t) a^n_{i,t-1}.
\]

We can chose $s_0$ high enough and $s_0 > T$ such that: for every $s \geq s_0$, we have
\[
\bar{p}_t c^n_{i,t} + \bar{q}_t a^n_{i,t} < \bar{p}_t c_{i,t} + (\bar{q}_t + \bar{p}_t \xi_t) a^n_{i,t-1}.
\]

It means that $(c^n_{i,t}, a^n_{i,t})_{t=0}^{T} \in C_i^T(\bar{p}^s, \bar{q}^s)$. Therefore, we get
\[
\sum_{t=0}^{T} \beta^t u_i(c^n_{i,t}) \leq \sum_{t=0}^{s} \beta^t u_i(\bar{c}_{i,t}).
\]

Let $s$ tend to infinity, we obtain
\[
\sum_{t=0}^{T} \beta^t u_i(c^n_{i,t}) \leq \sum_{t=0}^{\infty} \beta^t u_i(\bar{c}_{i,t}).
\]
Let $n$ tends to infinity, we have $\sum_{t=0}^{T} \beta_t^i u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_t^i u_i(\bar{c}_{i,t})$ for every $T$. As a consequence, we have: for every $T$

$$\sum_{t=0}^{T-1} \beta_t^i u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_t^i u_i(\bar{c}_{i,t}).$$

Let $T$ tend to infinity, we obtain $\sum_{t=0}^{\infty} \beta_t^i u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_t^i u_i(\bar{c}_{i,t})$.

Therefore, we have proved that the optimality of $(\bar{c}_i, \bar{a}_i)$.

Prices $\bar{p}_t, \bar{q}_t$ is strictly positive since the utility function of agent $i$ is strictly increasing and $\xi_t > 0$ for every $t$.

\[\square\]

**Lemma 15.** For each $i$, $(\bar{c}_i, \bar{a}_i)$ is optimal.

**Proof.** Since $\bar{p}_t, \bar{q}_t$ and $(e_{i,0}, a_{i,-1}) \neq (0, 0)$, we get that $B_i^T(\bar{p}, \bar{q}) \neq \emptyset$. By using the same argument in Lemma 14, we can prove the optimality of $(\bar{c}_i, \bar{a}_i)$.

\[\square\]

**References**


