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Optimal tax policy in a small open economy with credit constraints.
Theoretical model plus calibration and conclusions for Poland.

Abstract

This paper presents an extension of the endogenous growth model by Turnovsky (2009). We allow for public deficit and 6 types of taxes including the inflation tax. By assumption interest rates on public and private debt are linear functions of the debt-to-GDP ratios. Two extreme situations are analyzed: the model of “decentralized economy” where economic agents do not take into account any externalities, and the model of “benevolent social planner”. We derive the rules of optimal fiscal policy, i.e. we show how the government can induce economic agents to internalize all externalities (this is known as the problem of replication of the first-best solution). Theoretical results are illustrated with an empirical analysis for Poland. We calculate the optimal values of several fiscal policy instruments and we show that these values depend on the rate of inflation.

Keywords: budget deficit, optimal fiscal policy, imperfect capital mobility, credit constraints


1. Introduction

Recently researchers put intense effort into extending closed-economy growth theory to incorporate at least some aspects of openness, including foreign trade and international capital flows. Nevertheless, there are few publications concerning fiscal policy in the open-economy Ramsey-type endogenous growth theory. Some of the earliest examples are Nielsen and Sorensen (1991), Rebelo (1992), Razin and Yuen (1994), (1996). More recent examples are Lee & Gordon (2005), Agenor (2007), and Dhont & Heylen (2009).

One of the latest examples is the monograph by Stephen Turnovsky (2009), who presents several models of optimal fiscal policy in a small open economy (SOE) under perfect or imperfect mobility of capital. He examines productive government expenditures and 3 types of taxes: on consumption, on production, and on foreign debt of the private sector. Important qualitative differences between closed economy and SOE are exposed by Turnovsky. For example, the capital income tax ceases to have any effect on the long-run growth rate of the economy. The equilibrium growth rate is independent of almost all fiscal instruments, including public expenditures. The only tool that has any influence is the tax rate on foreign interest income. However, the government debt and deficit by assumptions do not exist so that there is no public debt (neither domestic, nor foreign). In a striking contrast, the private sector can borrow from abroad.

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The assumption of permanently balanced government budget is in fact typical for vast majority of open economy growth models (though it is not in the closed economy growth theory). To the best of our knowledge, there are only a few exceptions, e.g. Greiner and Semmler (2000), Ghosh and Mourmouras (2004a,b), Futagami et.al. (2008). Ignoring budget deficit and public debt in the open-economy setting is – in our view – unjustified. Some other assumptions in Turnovsky’s (2009) models are also worth waiving, for example zero capital depreciation and the absence of public consumption in the social welfare (utility) function.

In this paper Turnovsky’s model is modified and generalized. The government can run deficit (or surplus) financed by public debt composed of domestic and foreign debt. We analyze 6 types of taxes: on wages, capital income, consumption, interest paid by the private sector to foreign lenders, interest on government bonds held by domestic investors, and inflation tax. Capital depreciates at an exogenous rate. We introduce public consumption as a substitute to private consumption in the intertemporal utility function. Finally, following Acemoglu (2008), we apply a slightly modified utility function with the rate of discount which depends on the rate of growth of population.

Similarly to Turnovsky (2009, p. 67), we assume, that the real interest rate on foreign debt (both private and public) is a linear function of the debt-to-GDP ratio. However, contrary to Turnovsky, we distinguish between real and nominal interest rates, which allows us to analyze the inflation tax.

There are 4 types of externalities in our model (all except for the last one are explicitly or implicitly present in the model of Turnovsky, 2009):

a) Individual producers do not realize positive externalities of investment in capital, related to learning-by-doing and spillover-effects.

b) They treat all prices (of output and factors of production) as exogenous values, on which their individual decisions have no noticeable effect. Meanwhile, their aggregated decisions do impact prices.

c) They neglect negative externalities associated with the aggregate level of private foreign debt, i.e. they assume that their individual decisions regarding borrowing from abroad have negligible impact on the interest rate. However, in reality their aggregated decisions do influence the risk premium, which translates into the cost of borrowing.

d) Similarly, they do not take into account negative externalities associated with an increasing ratio of public debt to GDP, i.e. they treat the interest rate on government bonds as an exogenous value, on which their individual decisions have no impact. Meanwhile, their aggregated consumption and investment decisions impact on the
revenues and expenditures of the public sector, and therefore public debt, which translates into the interest rate on government bonds.

Identically to Turnovsky, we analyze two distinct situations: the model of atomized representative agents (or “decentralized economy”) who do not take into account any externalities, and the model of the benevolent social planner (or “centrally planned” economy). Technically both models in this paper are relatively complex optimal control problems. Neither of them can be fully solved analytically, however many precise qualitative conclusions can be formulated.

The remainder of the paper is organized as follows. Section 2 presents basic assumptions and budget constraints of the private and public sector. In section 3 we solve the optimal control problem for the decentralized economy – we find the balanced growth path and analyze transitory dynamics plus we derive a mathematical formula for welfare obtained by consumers over the entire (infinite) time horizon. In section 4 we do the same from the point of view of the benevolent social planner, i.e. we derive the first-best solution. Section 5 shortly describes sensitivity of the two types of economies to fiscal policy. In section 6 we solve the problem of replication of the first-best equilibrium, i.e. we find the optimal fiscal policy which allows the decentralized economy to internalize all externalities and reach the first-best outcome. Section 7 contains the solution of the problem of partial replication (of the steady state only).

The second part of the paper is an empirical analysis for Poland. In section 8 we calibrate the model for Poland. Section 9 presents the base scenario which assumes that all parameters will preserve their current values long into the future. (There are, of course, two base scenarios: one for the decentralized economy, and one for the benevolent social planner.) Next we present the solution of the problem of replication for Polish economy. It requires strongly negative tax rate on capital income (in practice, subsidizing investment in productive capital), and in parallel high positive tax rate on interest paid by domestic borrowers to foreign lenders (to discourage foreign financing). (These conclusions are similar to Turnovsky’s.) Finally, in section 11 we investigate the relationships between selected parameters of fiscal policy and consumer welfare under the assumption that the economy is governed by the benevolent social planner. This allows us to search (in section 12) for the global optimum: such set of values of fiscal policy instruments that maximize welfare of the nation. Section 13 summarizes the main theoretical and empirical results.
2. Assumptions

2.1. The interest rates

Let $Z$ be the net foreign debt of the private sector. Similarly to Turnovsky (2009, page 67), we assume, that the real interest rate on private foreign debt is a linear function of the debt-to-GDP ratio$^2$:

$$ r_Z = r_Z(Z/Y) = r_Z(z/y) = \varepsilon_Z + p_Z(z/y). $$(1)

where $\varepsilon_Z$ is the base interest rate$^3$ to private sector, and $p_Z > 0$ is the risk-premium parameter. An analogous equation applies to the real interest rate on public debt:

$$ r_D = r_D(D/Y) = r_D(d/y) = \varepsilon_D + p_D(d/y). $$$(2)$

where $\varepsilon_D$ is the base interest rate$^4$ to public sector, and $p_D > 0$ is the risk-premium parameter for public sector. Nominal interest rates are given by: $r^N_Z = r_Z + \vartheta$, $r^N_D = r_D + \vartheta$, where $\vartheta$ is the domestic inflation rate. For simplicity, all assets and liabilities in the model are expressed in domestic currency, so we apply domestic inflation rate everywhere. To keep things simple, we assume that $\vartheta$ is a decision parameter (of the central bank), and it is constant over time.

2.2. Technology and the markets for factors of production.

The production of a representative ($i$-th) firm is described by the Cobb-Douglas production function with constant returns to scale:

$$ Y_i = F(K_i, L_i) = aK_i^\alpha (EL_i)^\beta \quad \text{with} \quad \alpha + \beta = 1, \alpha, \beta > 0, a > 0, $$$(3)$

where $K_i$ denotes the stock of physical capital, and $L_i$ represents raw labor in the $i$-th firm and $E$ is the labor-augmenting technology index. Constant returns to scale allow a straightforward aggregation. Let $N$ be the number of representative firms in the country. Then, the aggregate output of the country is equal to

$$ Y = NY_i = a(NK_i)^\alpha (ENL_i)^\beta = aK^\alpha (EL)^\beta, $$$(4)$

where $K$ denotes the aggregate stock of physical capital and $L$ is the supply of labor in the country. We assume that $L = L_0e^{\mu t}$, where $L_0 > 0$ denotes the initial stock of labor, and $t \geq 0$ is a continuous time index.

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$^2$Throughout the paper we apply the following convention: capital letters denote real values of variables in domestic currency (e.g. $Z$); superscript $N$ denotes its nominal value (e.g. $Z^N = P \cdot Z$); lowercase letters denote real values per capita, e.g. $z = Z / L$.

$^3$Statistically, it could be interpreted as the average cost of foreign borrowing to private sector in countries with net international investment position (NIIP) of the private sector close to zero; see section 8 (calibration).

$^4$It can be interpreted as the average cost of foreign borrowing to public sector in countries with public debt close to zero.
From a mathematical point of view the production function (3) is identical with (4), so the economy as a whole can be analyzed in such a way as if it were a single firm under perfect competition, whose production is described with the function (4).

We assume positive externalities related to learning-by-doing and spillover-effects\(^5\). These externalities are reflected in the labor-augmenting technology index \(E\), which is proportional to the capital per worker ratio, i.e.

\[
E = x \frac{K}{L}, \quad \text{with} \quad x = \text{const.} > 0.
\]  

(5)

Dividing both sides of (4) by \(L\) yields the per capita production function:

\[
y = \frac{Y}{L} = ak^\alpha (E)^\beta.
\]  

(6)

Having regard to (5), the function (6) can be written in a simpler form:

\[
y = ak^\alpha (E)^\beta = Ak^\alpha (k)^\beta = Ak
\]  

(7)

with \(A = ax^\beta = \text{const} > 0\). Similarly, the aggregate output function (4) can be written as

\[
Y = aK^\alpha (EL)^\beta = AK^\alpha (K)^\beta = AK.
\]  

(8)

Therefore, de facto, we use a production function of the AK type, very popular in the theory of endogenous growth. By assumption, firms are maximizing profits in the perfectly competitive markets, which implies that the marginal product of capital must be equal to the real rental rate, and simultaneously the marginal product of labor must be equal to the real wage rate, i.e.

\[
\forall t \quad MPK = \frac{\partial Y}{\partial K} = aK^{\alpha - 1} (EL)^\beta = \frac{\alpha Y}{k} = \alpha y = \alpha A = w_k,
\]  

(9)

\[
\forall t \quad MPL = \frac{\partial Y}{\partial L} = \beta aK^\alpha E(EL)^{\beta - 1} = \frac{\beta Y}{L} = \beta y = w.
\]  

(10)

The accumulation of capital is described in a standard way (per capita):

\[
\dot{k} = i - (n + \delta)k,
\]  

(11)

where \(\delta > 0\) is the depreciation rate. Investment requires the so-called adjustment cost, described by the following equation:

\[
C(I, K) = \chi \frac{I^2}{2K} \quad \text{with} \quad \chi > 0.
\]  

(12)

The concept of adjustment costs is attributed to Hayashi (1982). In order to achieve net investment equal to \(I\), one needs expenditures equal to

\(^{5}\) (These ideas were first introduced by Arrow (1962) and Lucas (1988); an overview of the literature related to these externalities is provided by Romer (1986), and Barro & Sala-i-Martin (2004).
\[
\Phi(I, K) = I + C(I, K) = I \left(1 + \frac{\chi I}{2 K}\right) \quad \text{with} \quad \chi > 0, \quad (13)
\]

or, in per capita terms,

\[
\phi(i, k) = \left(1 + \frac{\chi i}{2 k}\right). \quad (14)
\]

2.3. Consumer preferences

The level of welfare of a representative household at any given time \( t \) is described by the following instantaneous utility function:

\[
u(t) = \frac{1}{\gamma} \left(c_t \cdot g_{C_t}^\kappa\right)^\gamma \quad \text{with} \quad \gamma < 0, \quad \kappa > 0, \quad (15)\]

where \( c_t \) is personal (private) consumption per capita, and \( g_{C_t} \) denotes public consumption per capita at time \( t \). The parameter \( \kappa \) expresses the elasticity of substitution between both types of consumption. A fraction \( \gamma/(1-\gamma) \) is equal to the intertemporal elasticity of substitution. The assumption \( \gamma < 0 \) is justified by empirical research; see e.g. Turnovsky (2009), p. 177. Notice that these assumptions guarantee strict concavity of the utility function given by (15) with respect to both types of consumption.

The level of welfare resulting from the present and future consumption is described by the following intertemporal utility function:

\[
U = \int_0^\infty u(t) e^{-(\rho-n) t} dt = \int_0^\infty \frac{1}{\gamma} \left(c_t \cdot g_{C_t}^\kappa\right)^\gamma e^{-(\rho-n) t} dt, \quad \rho > 0. \quad (16)
\]

The parameter \( \rho > 0 \) is the subjective discount rate (of the future consumption). We must explain an unusual effective rate of discount equal to \( \rho - n \), which is rare in the literature. This is adopted from Acemoglu (2008), p. 310. It reflects the assumption that a household derives utility from both its own consumption, and consumption of its future members (children, grandchildren, etc.), the number of which is growing at the rate \( n \). Therefore, the higher the rate of population growth (\( n \)) in a country, the smaller the effective discount rate, because the number of children, grandchildren, etc., which will be consuming in the future is greater. Speaking a little more intuitively – the more children (per family), the more we value future consumption. In our opinion this is a realistic assumption – parents of 3 (as myself) or 4 kids plan their lifetime spending flow differently (leaving more for the future and drawing satisfaction from bequests that their children get) than parents of 1 child, not to mention a family without children or singles. In our view, this assumption brings the standard Ramsey-type theory somewhat closer to the overlapping generations model. We assume that \( \rho > n \).
Otherwise, the integral in (16) would not be convergent, and it would be impossible to solve any optimization problems involving it.

2.4. The public sector (the government)

The total nominal tax revenue of the government is

\[ T^N = \tau_L WL + \tau_K W_k K + \tau_C C^N + \tau_Z r_Z^N Z^N + \tau_D r_D^N D_D^N, \]  

(17)

where \( \tau_L, \tau_K, \tau_C, \tau_Z, \tau_D \) are the average tax rates on wages, capital income, consumption, interest paid by private sector to foreign lenders, and interest on government bonds held by domestic investors. Dividing both sides by the domestic price level \( P \) yields real tax revenue:

\[ T = \tau_L wL + \tau_K w_k K + \tau_C C + \tau_Z r_Z^N Z + \tau_D r_D^N D_D, \]  

(18)

Notice that there is an inflation tax: given the real interest rates, the higher the inflation rate, the higher is the real tax burden on interest income. The nominal deficit of the public sector is the difference between government spending and tax receipts, i.e. \( J^N = G^N + r_D^N D^N - T^N, \) which can be expressed in real terms:

\[ J = G + r_D^N D - T, \]  

(19)

where \( G \) is government spending in real terms, and \( D \) represents the total public debt. We assume that the budget deficit is a fixed percentage of the GDP, i.e. \( J = \xi Y, \) where \( \xi = const > 0 \) is a decision parameter. Hence the budgetary rule (19) can be written as:

\[ G = T - r_D^N D + \xi Y. \]  

(20)

The (nominal) deficit is financed by government bonds, which increases the nominal public debt according to the equation: \( \dot{D}^N = \xi Y^N. \) Certain part (\( \omega \)) of bonds is sold to foreign investors, and the rest to domestic creditors:

\[ \dot{D}_D^N = \omega \dot{D}^N = \omega \xi Y^N, \quad 0 \leq \omega \leq 1, \]  

\[ \dot{D}_F^N = (1 - \omega) \dot{D}^N = (1 - \omega) \xi Y^N, \]  

(21)

(22)

where \( D_D^N \) is the nominal stock of domestic debt of the government, and \( D_F^N \) is the nominal stock of foreign public debt. By definition: \( D^N = D_D^N + D_F^N. \) The real value of public debt is defined as \( D = D^N/P, \) which upon differentiation of both sides with respect to \( t \) yields\(^6\):

\[ \dot{D} = \frac{\dot{D}^N P - \dot{P} D^N}{P^2} = \frac{\dot{D}^N}{P} - \frac{\dot{P}}{P} D = \xi Y - \delta D, \]  

(23)

which can be decomposed into:

\(^6\) Dots denote time derivatives.
\[
\dot{D}_F = \omega \xi Y - \partial D_F ,
\]  
\[
\dot{D}_D = (1 - \omega) \xi Y - \partial D_D ,
\]  

(24) 

(25) 

Obviously, high inflation helps to reduce the real value of public debt. Government spending includes two components:

\[
G = G_C + G_T = \sigma_c C + G_T , \quad 0 < \sigma_c < 1 ,
\]  

(26) 

where \( G_C \) is the public consumption (by assumption proportional to the private consumption), and \( G_T \) represents the cash transfers to the private sector.

### 2.5. The private sector

The private sector receives income in the form of remuneration of labor and capital, the interest on domestic public debt, and cash transfers from the government. It must, however, pay interest to foreign creditors. The nominal disposable income of the private sector after taxes is defined as:

\[
Y_d = (1 - \tau_L)WL + (1 - \tau_K)W_K + (1 - \tau_D)r_D D_D^N - (1 + \tau_Z)r_Z^N Z^N + G_T .
\]  

(27) 

i.e., in real terms:

\[
Y_d = (1 - \tau_L)wL + (1 - \tau_K)w_K + (1 - \tau_D)r_D D_D - (1 + \tau_Z)r_Z^N Z + G_T .
\]  

(28) 

This income is spent on consumption and investment, as well as purchases of government bonds. Any difference is covered by borrowing from abroad. Therefore the (instantaneous) budget constraint in nominal terms is of the form:

\[
Y_d = C^N (1 + \tau_c) + P \cdot \Phi(I, K) + \dot{D}_D^N - \dot{Z}^N .
\]  

(29) 

It’s straightforward to demonstrate that its counterpart in real terms is:

\[
Y_d = C(1 + \tau_c) + \Phi(I, K) + \dot{D}_D - \dot{Z} + \partial(D_D - Z) .
\]  

(30) 

Substituting (13) together with (25) and rearranging yields:

\[
\dot{Z} = C(1 + \tau_c) + \dot{I} \left[ 1 + \frac{Z}{2} K \right] + (1 - \omega)\xi Y - Y_d - \partial Z .
\]  

(31) 

Using (28), we can transform this budget constraint per capita form:

\[
\dot{Z} = c(1 + \tau_c) + \dot{I} \left[ 1 + \frac{Z}{2} K \right] + \left[ (1 + \tau_Z)r_Z^N - n \right] \xi - (1 - \tau_L)w\dot{I} - \\
- (1 - \tau_K)w_K k - (1 - \tau_D)r_D^N d_D - g_T + (1 - \omega)\xi y - \partial \xi .
\]  

(32) 

It’s worth to emphasize that the representative agent treats all prices and fiscal variables as exogenous, because alone his influence on the market is negligible (just like a single firm under perfect competition). Hence, when making decisions that will be written down as an
optimal control problem, he pays attention to the budget constraint (32) treating \( w, w_k, g_T, g_c \) and \( d_D \) as constants.

To the contrary, the benevolent social planner has a different perspective – he has full information about the economy, including all externalities, aggregate effects, and fiscal rules. Therefore, even though his optimal control problem must formally incorporate the same budget constraint (32), it can be transformed to a simpler form if we use all information about the economy. From (20) and (26) it follows that \( g_T = t + \xi y - r^N_d d - g_c \). Substituting it into (32) yields:

\[
\dot{z} = c(1 + \tau_c) + \frac{\nu i}{2} \left( 1 + \tau_z \right) r^N_z - n \right] z \right) w_l \right) -
\left( (1 - 1) \right) w_k - \left( 1 - \tau_D \right) r^N_D d_D - t - \xi y + r^N_d d + g_c + (1 - \omega) \xi y - \xi z.
\]

From (18) it follows that \( t = \tau_k w + \tau_k w_k k + \tau_z r^N_z z + \tau_D r^N_D d_D + \tau_c c \), so that (33) reduces to:

\[
\dot{z} = c + \frac{\nu i}{2} \right) \right) + \left( r^N_z - n \right) \right) - w - w_k k + r^N_D (d - d_D) + g_c - \omega \xi y - \xi z.
\]

From equations (8), (9) and (10) we know that \( w + w_k k = (\alpha + \beta) y = y \), and of course \( d_d = d_f \). Therefore the budget constraint for the “central planner” (34) can be written as:

\[
\dot{z} = c + \frac{\nu i}{2} \right) + \left( r^N_z - n \right) \right) - (1 + \omega \xi) y + r^N_D d_f + g_c.
\]

3. The representative agent (the decentralized economy)

3.1. The optimal control problem and its solution

The private sector determines the flows of consumption and investment, so as to achieve the highest level of utility described by the function (16), with a budget constraint of (32). That decision problem boils down to the following optimal control problem:

\[
\max \int_0^\infty \left( c g^x \right) e^{-(\rho - \nu) t} dt, \]

\[
\dot{z} = c(1 + \tau_c) + \frac{\nu i}{2} \right) \right) + \left( r^N_z - n \right) \right) - (1 - \tau_L) w_l -
\left( (1 - 1) \right) w_k - \left( 1 - \tau_D \right) r^N_D d_D - g_T + (1 - \omega) \xi y - \xi z,
\]

\[
\dot{k} = i - (n + \delta) k.
\]
Control variables: \( c, i \). State variables: \( z, k \). The initial values of variables (endowments):
\[ z_0, \quad k_0 > 0, \quad d_0 \geq 0, \quad d_{F0} \geq 0, \quad d_{D0} \geq 0, \quad \text{with} \quad d_{F0} + d_{D0} = d_0. \]
The variables treated by an individual decision-maker as exogenous: \( w, w_k, g_T, g_C, d_F, d_D, r_Z, r_D \).

We must emphasize that the individual decision-maker treats the eight listed variables as exogenous, because alone he does not have any noticeable impact on them. However, the individual decisions summed together (aggregated) do affect these variables, which has been described above by the appropriate equations. It’s worth to provide a simple example of such a situation. A single firm considering how many people to hire assumes that changing the number of its employees has no impact on the labor market, and so assumes that the wage rate it must offer is independent of the number of its employees. But if all firms in the country increase or reduce employment at the same time, the labor market will be influenced, and the wage rate will rise. Similarly, an individual decision-maker assumes that alone he has no impact on the rental rate of capital, as well as the size of public consumption, the volume of cash transfers from the government, and the size of public debt (all of these in per capita terms). Nevertheless, aggregated decisions of all agents obviously do have influence on these variables, which has been described above with the relevant equations. Therefore, while formulating and solving the problem (36) we assume that the values of the above listed 7 variables are exogenous. But at the same time the solution must satisfy all equations describing the relationships between these variables and aggregated decisions. The solution must therefore satisfy all equations written down in section 2.

A mathematician would probably say that at first we solve the problem (36) taking 8 listed variables as exogenous (constants). In this way we obtain a bundle (a set) of potential solutions, among which we have to select these solutions that satisfy all the equations listed in section 2. This process of selecting a feasible solution(s) can be simplified in such a way that while solving the necessary and sufficient conditions for optimality we immediately take advantage of equations from section 2. This is exactly the way Turnovsky (2009) proceeds, and we will follow the same approach.

The current value hamiltonian is:

\[
H_c = \frac{1}{\gamma} \left[ c g_C^\gamma \right] + \lambda' \left[ c(1 + \tau_C) + i \left( 1 + \frac{x}{2} \right) - (1 + \tau_L)w \right] - (1 - \tau_K)w_k - (1 - \tau_D) \lambda^N_Z d_Z - g_T - (1 - \omega) \xi \lambda^Y + \lambda^2 \left[ \left( i - (n + \delta)k \right) \right].
\] (37)
Obviously the shadow price of debt is negative: \( \lambda'_d < 0 \). Hence, following Turnovsky (2009), we will replace it with \( \lambda'_d = -\lambda'_d \) which will allow us to use the ratio of shadow prices \( q = \frac{\lambda'_d}{\lambda'_d} = -\frac{\lambda'_d}{\lambda'_d} > 0 \) which can be interpreted as the market price of capital in relation to the market price of (private) foreign debt (bonds).

The optimal solution of the problem (36) must meet the following conditions (necessary and sufficient):

\[
\forall t \quad \frac{\partial H_c}{\partial c} = 0 ,
\]

(38a)

\[
\forall t \quad \frac{\partial H_c}{\partial i} = 0 ,
\]

(38b)

\[
\dot{\lambda}_1 = -\frac{\partial H_c}{\partial z} + \lambda'_d (\rho - n) ,
\]

(38c)

\[
\dot{\lambda}_2 = -\frac{\partial H_c}{\partial k} + \lambda'_d (\rho - n) ,
\]

(38d)

\[
\lim_{t \to \infty} e^{-(\rho - n) t} \lambda'_d(t) z(t) = 0 ,
\]

(38e)

\[
\lim_{t \to \infty} e^{-(\rho - n) t} \lambda'_d(t) k(t) = 0 .
\]

(38f)

The condition (38a) has the form:

\[
\lambda'_d (1 + \tau_c) = e^{-\gamma^{-1} g_C^0} ,
\]

(39)

which means that the shadow price of wealth (in the form of bonds), adjusted for the size of consumption tax must be (for each \( t \)) equal to the marginal utility of private consumption. By differentiating this equation with respect to \( t \), after transformation we get:

\[
\hat{\lambda}_d = (\gamma - 1) \hat{c} + \kappa \hat{g}_C .
\]

(40)

Note that from the equation (26) it follows that private and public consumption must grow at equal rates, say \( \psi \). Hence \( \hat{g}_C = \hat{c} = \psi \). The condition (38c) can be written as:

\[
\hat{\lambda}_d = \rho - (1 + \tau_z) r^C_z + \theta = \rho - (1 + \tau_z) r_z - \tau_z \theta .
\]

(41)

Substituting (41) into (40), and using \( \hat{g}_C = \hat{c} = \psi \), we can calculate the growth rate of consumption (both private and public) per capita:

\[
\psi = \frac{\dot{c}}{c} = \frac{\frac{(1 + \tau_z) r_z - \rho + \tau_z \theta}{1 - \gamma(1 + \kappa)}}{1 - \gamma(1 + \kappa)} = \frac{r_z - \rho + \tau_z r^C_z}{1 - \gamma(1 + \kappa)} .
\]

(42)
The growth rate of per capita consumption depends only on the parameters that describe the consumer preferences \((\gamma, \kappa \text{ and } \rho)\), on the real interest rate on external debt of the private sector \(r_Z\), on the tax rate on interest on the external debt of the private sector and on inflation. Importantly, the optimal trajectory of the real interest rate \(r_Z(t)\) is not necessarily constant over time. Thus the optimal trajectory of private consumption per capita can only be written in a general form:

\[
c(t) = c_0 \cdot e^{\int \nu(x)dx},
\]

whereas the trajectory of public consumption per capita has the form:

\[
g_c(t) = \sigma_c c(t).
\]

The condition (38b) can be written in the form:

\[
q = \frac{\lambda_2}{\lambda_1} = 1 + \frac{i}{k},
\]

The ratio of the shadow prices \(q = \lambda_2/\lambda_1\) can be broadly interpreted as the market price of capital in relation to the market price of foreign bonds. According to (45) it must be equal to the marginal cost of an additional unit of investment (inclusive of the adjustment cost). Dividing both sides of (11) by \(k\) and having regard to the (45) we get the growth rate of capital (as well as production) per capita:

\[
\hat{\varphi} = \hat{k} = \hat{\gamma} = \frac{q - 1}{\chi} - (n + \delta).
\]

This growth rate is not necessarily constant, because it is related to the trajectory \(q(t)\). Therefore the trajectory \(k(t)\) can be written only in a general form:

\[
k(t) = k_0 e^{\int \varphi(t)dt}.
\]

To determine the path of \(q(t)\), we need to use the last of the necessary conditions for optimality, i.e. (38d). Having regard to (41) and (45), and using equality \(w_k = \alpha A\), it can be written in the form of the following equation:

\[
\hat{\lambda}_2 = -\lambda_1 \left[ \frac{(q - 1)^2}{2\chi} + (1 - \tau_k)\alpha A - (1 - \omega)\xi A \right] + \lambda_2 (\rho + \delta).
\]
Dividing both sides of this equality by $\lambda$, and taking into account (41) and (45) together with (1), after minor manipulation we get:

$$\dot{q} = \left[ (1 + \tau_z)(\varepsilon_z + p_z z) + \tau_z \vartheta + \delta \right] q - (1 - \tau_k) \alpha A + (1 - \omega) \hat{\varphi} A - \frac{(q - 1)^2}{2 \chi}.$$ \hspace{1cm} (49)

where $z = z / y$.

To derive the steady-state it’s convenient to replace original (per capita) variables with their shares in the GDP\(^7\). Let us denote these shares in the GDP with an underline, e.g.:

$$\zeta = \frac{c}{y}, \quad \bar{z} = \frac{\bar{z}}{\bar{y}}, \quad d_D = \frac{d_D}{\bar{y}}, \quad d_F = \frac{d_F}{\bar{y}}, \quad \text{etc.} \hspace{1cm} (50)$$

Hereafter we will refer to these variables as to the **ratio of consumption, the ratio of private foreign debt**, etc. The growth rates of consumption-to-GDP and debt-to-GDP are equal to:

$$\dot{\zeta} = \dot{\bar{z}} = \bar{z} - \bar{y} = \varphi - \vartheta, \hspace{1cm} (51)$$

$$\dot{\bar{z}} = \bar{z} - \bar{y} = \bar{z} - \varphi. \hspace{1cm} (52)$$

Using (1), (42) and (46), from (51) we can obtain:

$$\dot{\zeta} = \left[ (1 + \tau_z) r_z + \tau_z \vartheta - \rho - \frac{q - 1}{\chi} + n + \delta \right] \zeta, \hspace{1cm} (53)$$

where $r_z = \varepsilon_z + p_z \bar{z}$.

The derivation of the equation of motion of $\bar{z}$ requires several substitutions. First, from (44) it follows that $g_c = \sigma_c c$. Substituting that into (35) yields:

$$\dot{\bar{z}} = (1 + \sigma_c) c + \left( 1 + \frac{\chi}{2 \kappa} \right) r_z - n \bar{z} + r_D^y d_F - (1 + \omega \bar{z}) y. \hspace{1cm} (54)$$

Dividing both sides by $\bar{z}$, and using (45), we obtain:

$$\dot{\bar{z}} = (1 + \sigma_c) \frac{c}{\bar{z}} + \frac{q^2 - 1}{2 A \chi} \frac{\bar{z}}{\bar{z}} + \left( r_z - n \bar{z} + r_D^y d_F - (1 + \omega \bar{z}) \right) \frac{1}{\bar{z}}. \hspace{1cm} (55)$$

From (52) it follows that: $\dot{\bar{z}} = \dot{\bar{z}} - \bar{z} \bar{z} = (\bar{z} - \varphi) \bar{z}$. Together with (55) and (46), it can be rearranged to:

$$\dot{\bar{z}} = (1 + \sigma_c) \frac{c}{\bar{z}} + \frac{q^2 - 1}{2 A \chi} - (1 + \omega \bar{z}) + \left( r_z - \frac{q - 1}{\chi} + \delta \right) \bar{z} + r_D^y d_F + \partial d_F, \hspace{1cm} (56)$$

where $r_z = \varepsilon_z + p_z \bar{z}$ and $r_D = \varepsilon_D + p_D d$. The equations (53), (56) and (49) determine the evolution of variables: $\zeta$, $\bar{z}$ and $q$. However, apart from these 3 variables, the right-hand

---

\(^7\) Turnovsky (2009) uses the same approach in chapter 4, only instead of shares in the GDP he uses ratios to capital. With the AK production function our approach is in fact identical.
sides of these equations contain 2 additional ones: \( d_D \) and \( d_E \). Therefore, in order to close this system of differential equations we need to append 2 additional equations describing the dynamics of \( d_D \) and \( d_E \). It follows from (24) and (25) that:

\[
\hat{d}_F = \left( \frac{D_F}{Y} \right) = \dot{D}_F - \dot{Y} = \frac{\omega \xi}{d_F} - \vartheta - n - \varphi .
\]

(57)

\[
\hat{d}_D = \left( \frac{D_D}{Y} \right) = \dot{D}_D - \dot{Y} = \frac{(1-\omega)\xi}{d_D} - \vartheta - n - \varphi .
\]

(58)

Since \( \dot{d}_F = \dot{d}_F \cdot d_F \) and \( \dot{d}_D = \dot{d}_D \cdot d_D \), we have:

\[
\dot{d}_F = (-\vartheta - \varphi - \vartheta) d_F + \omega \xi .
\]

(59)

\[
\dot{d}_D = (-\vartheta - \varphi - \vartheta) d_D + (1-\omega) \xi .
\]

(60)

Obviously, \( \dot{d} = \dot{d}_F + \dot{d}_D \). The equations (53), (56), (49), (59) and (60) constitute a nonlinear, autonomous system of differential equations of the following form:

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\bar{z}} \\
\dot{\bar{q}} \\
\dot{d}_F \\
\dot{d}_D
\end{bmatrix}
= \begin{bmatrix}
f^1(c, z, \bar{q}, d_F, d_D) \\
f^2(c, z, \bar{q}, d_F, d_D) \\
\bar{q} \\
f^3(c, z, \bar{q}, d_F, d_D) \\
f^4(c, z, \bar{q}, d_F, d_D)
\end{bmatrix} .
\]

(61)

3.2. The steady-state equilibrium (the balanced growth path)

The steady-state equilibrium is defined as such set of values: \( [\xi \  \bar{z} \  \bar{q} \  \hat{d}_F \  \hat{d}_D] \) which satisfies the condition: \( [\xi \  \bar{z} \  \bar{q} \  \hat{d}_F \  \hat{d}_D] = 0 \). In order to find the steady-state we must solve the system of 5 equations: \( f^i(c, z, q, d_F, d_D) = 0 \) \( (i = 1, \ldots, 5) \), i.e.

\[
\frac{(1+\tau_z)\bar{r}_z + \tau_z \vartheta - \rho}{1-\gamma(1+\kappa)} - \bar{\varphi} = 0 ,
\]

(62)

\[
(1+\sigma_c)\xi + \frac{\bar{q}^2-1}{2A\chi} - (1+\omega \xi) + (\bar{r}_z - n - \bar{\varphi})\xi + (\bar{r}_D + \vartheta)\xi = 0 ,
\]

(63)

\[
[1+\tau_z]\bar{r}_z + \tau_z \vartheta + \delta \bar{q} - (1-\tau_k)\alpha A + (1-\omega)\xi A - \frac{(\bar{q} - 1)^2}{2\chi} = 0 ,
\]

(64)

\[
(-n - \bar{\varphi} - \vartheta)\hat{d}_F + \omega \xi = 0 ,
\]

(65)

\[
(-n - \bar{\varphi} - \vartheta)\hat{d}_D + (1-\omega)\xi = 0 ,
\]

(66)
where \( \tilde{r}_z = \varepsilon_z + p_z \tilde{z} \) and \( \bar{\varphi} = \frac{q-1}{\chi} - (n + \delta) \). First notice that (62) immediately yields:

\[
\bar{\varphi} = \frac{(1 + \tau_Z) \tilde{r}_z + \tau_Z \vartheta - \rho}{1 - \gamma(1 + \kappa)}.
\]

which is the steady-state rate of growth of all per capita variables:

\[
\hat{y} = \hat{k} = \hat{c} = \hat{z} = \bar{\varphi}.
\]

From (64) it follows that

\[
(1 + \tau_Z) \tilde{r}_z + \tau_Z \vartheta = \frac{(1 - \tau_K) \alpha A - (1 - \omega) \xi A}{q} + \frac{(q - 1)^2}{2 \chi q} - \delta.
\]

Substituting this into (62), and rearranging yields a quadratic equation in \( \bar{q} \):

\[
a_1 \cdot \bar{q}^2 + a_2 \cdot \bar{q} + a_3 = 0,
\]

with

\[
a_1 = 1 - 2 \gamma (1 + \kappa) > 0,
\]

\[
a_2 = 2 \left[ \chi \rho + \gamma (1 + \kappa) - \chi n + \chi \gamma (1 + \kappa) (n + \delta) \right],
\]

\[
a_3 = -\left( 1 + 2 \chi A \left[ (1 - \tau_K) \alpha - (1 - \omega) \xi \right] \right).
\]

Notice that \( a_1 > 0 \), because \( \gamma < 0 \). Moreover, we will assume that:

\[
a_3 < 0.
\]

Otherwise the deficit would have to be extremely high – one the level that nowhere in the world does occur. (More formal justification for this assumption is presented in the appendix.) The signs of \( a_1 \) and \( a_3 \) imply that the equation (70) has 2 real roots, one positive, and one negative. The negative one is rejected for the sake of economic interpretation of \( q \). The only viable solution to the equation (70) is:

\[
\bar{q} = \frac{\sqrt{\Delta} - a_2}{2a_1} > 0, \quad \Delta = a_2^2 - 4a_1a_3 > 0.
\]

From (46) it follows that the steady-state rate of growth of the economy is:

\[
\bar{\varphi} = \frac{q - 1}{\chi} - (n + \delta).
\]

The remaining unknown steady-state values can be found easily. The equation (67) yields:

\[
\tilde{r}_z = \frac{1 - \gamma (1 + \kappa)}{1 + \tau_Z} \bar{\varphi} + \rho - \tau_Z \vartheta,
\]

From \( \tilde{r}_z = \varepsilon_z + p_z \tilde{z} \), and (65) with (66) we obtain the indicators of indebtedness:
\[ \bar{z} = \frac{\bar{r}_Z - \bar{e}_Z}{p_Z}. \]  
(78)

\[ \bar{a}_f = \frac{\omega \bar{\xi}}{n + \bar{\varphi} + \bar{\gamma}}, \]  
(79)

\[ \bar{a}_D = \frac{(1 - \omega)\bar{\xi}}{n + \bar{\varphi} + \bar{\gamma}}. \]  
(80)

The equations (79), (80) and (2), after some manipulation yield:

\[ \bar{r}_D = \bar{e}_D + p_D \frac{\bar{\xi}}{n + \bar{\varphi} + \bar{\gamma}}. \]  
(81)

Finally, from (63) we can derive the steady-state consumption-to-GDP ratio:

\[ \bar{c} = \frac{1 + \omega \bar{\xi} - \bar{q}^2 - 1}{2A\chi} - \frac{(\bar{r}_Z - n - \bar{\varphi})\bar{\xi} - (\bar{r}_D + \bar{\gamma})\bar{a}_f}{1 + \sigma_c}. \]  
(82)

In the appendix we prove that the transversality conditions (38ef) are satisfied if, and only if,

\[ (1 + \tau_Z)\bar{r}_Z^N > \bar{\varphi} + \bar{\gamma} + n, \]  
(83)

which means that the nominal rate of interest on the private sector external debt augmented by the special tax must be higher than the rate of growth of nominal GDP on the balanced growth path. Using (77) this condition can be rewritten as:

\[ \rho > n + \gamma(1 + \kappa)\bar{\varphi} \]  
(84)

which implies that the rate of discount must simply be sufficiently high.

### 3.3. The local stability of equilibrium

Equations (61) are nonlinear, hence we will only investigate the local stability of equilibrium using a standard method of linearization in the neighborhood of equilibrium by approximating non-linear functions \( f^i \) with Taylor series truncated to the linear part only:

\[ f^i(\varepsilon, \bar{z}, \bar{q}, \bar{a}_f, \bar{a}_D) \approx \left. \frac{\partial f^i}{\partial \varepsilon} \right|_{E} \cdot \varepsilon + \left. \frac{\partial f^i}{\partial \bar{z}} \right|_{E} \cdot \bar{z} + \left. \frac{\partial f^i}{\partial \bar{q}} \right|_{E} \cdot \bar{q} + \left. \frac{\partial f^i}{\partial \bar{a}_f} \right|_{E} \cdot \bar{a}_f + \left. \frac{\partial f^i}{\partial \bar{a}_D} \right|_{E} \cdot \bar{a}_D. \]  

\((i = 1, \ldots, 5)\) where “ waved” variables are deviations from the steady-state, i.e. \( \bar{z} = z - z_e, \quad \bar{q} = q - q_e, \quad \bar{a}_f = a_f - a_{fe}, \quad \bar{a}_D = a_D - a_{de}. \) The linear approximation of (61) around the equilibrium has the following form:
It follows that:

\[ \frac{d}{dt} z(0) = M z(0), \]

with the matrix of values of partial derivatives (Jacobian) calculated in the equilibrium:

\[
M = \begin{bmatrix}
0 & \frac{(1 + \tau_x) p x \bar{c}}{1 - \gamma(1 + \kappa)} & -\frac{\bar{c}}{\chi} & 0 & 0 \\
1 + \sigma_c & \bar{r}_x + p_i \bar{z} - n - \bar{\theta} & \frac{-\bar{q} - A \bar{z}}{\chi} & \varepsilon_d + p_D \bar{c}(1 + \omega) + \vartheta \n + \bar{\theta} + \vartheta \n + \bar{\theta} + \vartheta \n + \bar{\theta} + \vartheta \\
0 & 0 & -\frac{\omega \bar{c}}{\chi(n + \bar{\theta} + \vartheta)} & -n - \bar{\theta} - \vartheta & 0 \\
0 & 0 & \frac{-\omega \bar{c}}{\chi(n + \bar{\theta} + \vartheta)} & \frac{\chi(n + \bar{\theta} + \vartheta)}{\chi(n + \bar{\theta} + \vartheta)} & 0 - n - \bar{\theta} - \vartheta
\end{bmatrix}. \tag{86}

The general solution of the linear system of equations (85) can be written as:

\[
\begin{bmatrix}
\dot{c} \\
\dot{z} \\
\dot{q} \\
\dot{d}_e \\
\dot{d}_d
\end{bmatrix} = \begin{bmatrix}
\ddot{c} \\
\ddot{z} \\
\ddot{q} \\
\ddot{d}_e \\
\ddot{d}_d
\end{bmatrix} + \sum_{i=1}^S s_i e^{r_i t} v^i, \tag{87}
\]

where \( r_i \) are the eigenvalues of the matrix \( M \), \( v^i \) are its eigenvectors, and \( s_i \) are unknown constants dependent on the starting point (endowments). The stability of equilibrium depends on the signs of eigenvalues of the matrix \( M \). The product of these eigenvalues is equal to \( \det M \), whereas the sum is equal to \( \text{tr} M \). It follows that:

\[
\det M = -(n + \bar{\theta} + \vartheta)^2 (1 + \tau_x)(1 + \sigma_c) p x \bar{c} \frac{\bar{q}}{\chi} + \frac{(1 + \tau_x) \bar{r}_x - n - \bar{\theta} + \tau_x \vartheta}{1 - \gamma(1 + \kappa)}, \tag{88}
\]

\[
\text{tr} M = \bar{r}_x + p_i \bar{z} + (1 + \tau_x) \bar{r}_x - 4(n + \bar{\theta}) + \vartheta(\tau_x - 2). \tag{89}
\]

The transversality condition (83) implies that \( \det M < 0 \). However, with all assumptions taken so far, it’s not possible to determine the sign of \( \text{tr} M \). From \( \det M < 0 \) we conclude that \( M \) has an odd number of negative eigenvalues, i.e. 1, 3 or 5 such values. The first of these possibilities is rejected, because from equations (59) and (60) it follows that two variables \( d_e \) and \( d_d \) are globally stable. On the other hand, 5 negative eigenvalues would imply local

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9 Strictly speaking, negative real parts, because some eigenvalues may be complex (conjugate) numbers.
stability of all 5 variables in the system of equations (85) which is virtually impossible in the light properties of Turnovsky (2009, chapter 4) model, as well as in the light of our own analyses. Therefore the only viable possibility seems to be 3 negative eigenvalues, which we henceforth assume\(^{10}\). Strictly speaking, we assume 3 eigenvalues with negative real parts, out of which one is certainly real, and 2 may be complex, conjugate numbers. Hence, the equilibrium has the form of the stable saddle path. Because there are 2 positive eigenvalues, 2 out of 5 variables must “jump” to accommodate any shock instantly, and the remaining 3 variables evolve continuously over time. For obvious reasons, the “jump” variables are consumption \(c\) and \(q\), whereas all three debt indicators must (by definition) be continuous.

Let us denote positive eigenvalues of the matrix \(M\) as \(r_i\) and \(r_5\). Obviously \(s_4 = s_5 = 0\), and the solution (87) boils down to:

\[
\begin{bmatrix}
\bar{c} \\
\bar{z} \\
q \\
d_F \\
d_D
\end{bmatrix}
= \begin{bmatrix}
\bar{c} \\
\bar{z} \\
\bar{q} \\
d_F \\
d_D
\end{bmatrix} + \sum_{i=1}^{3} s_i e^{r_i t} \mathbf{v}^i ,
\]

where \(r_i\) \((i = 1, ..., 3)\) are negative eigenvalues, \(\mathbf{v}^i\) are corresponding eigenvectors. The unknown constants \(s_i\) can be found by plugging the initial values of debt indicators into (90), which results in the following system of 3 equations in 3 unknowns:

\[
\begin{align*}
\bar{z}_0 &= \bar{z} + \sum_{i=1}^{3} s_i v_2^i , \\
\bar{d}_F &= \bar{d}_F + \sum_{i=1}^{3} s_i v_4^i , \\
\bar{d}_D &= \bar{d}_D + \sum_{i=1}^{3} s_i v_5^i .
\end{align*}
\]

Knowing the eigenvalues, eigenvectors and constants \(s_i\) from the remaining two equations of the system (90) we can find initial values of \(c\) and \(q\). We obtain:

\[
\begin{align*}
\bar{c}_0 &= \bar{c} + \sum_{i=1}^{3} s_i v_1^i , \\
\bar{q}_0 &= \bar{q} + \sum_{i=1}^{3} s_i v_3^i .
\end{align*}
\]

\(^{10}\) We have confirmed this in numerous simulations.
Finally, notice that even though we have an analytical formula for the matrix $M$ in (86), it’s not possible to derive an analytical solution, because the eigenvalues of $M$ are the roots of the polynomial of the 5th degree, and so analytical formulas for them do not exist (one of fundamental theorems in algebra). Because of this fact, the only viable way to analyze transitional dynamics is by numerical methods.

3.4. Welfare in the decentralized economy

The level of welfare in the economy is described by the integral (16). Using (44) we can express its value for the obtained optimal solution as:

$$\Omega = \frac{1}{\gamma} \sigma_c \int_{0}^{\infty} (c_i)_{t+1} e^{-(\rho - \alpha) \gamma} dt.$$  \hspace{1cm} (94)

Substituting the trajectory of private consumption (43) we get:

$$\Omega = \frac{1}{\gamma} \sigma_c \int_{0}^{\infty} e^{\gamma (1 + \kappa)} \int_{0}^{\infty} e^{-(\rho - \alpha) \gamma} dt.$$  \hspace{1cm} (95)

The exact value of this integral cannot be established due to non-linearity of the model. More precisely, as we have seen, there is no way to derive the exact formula of the trajectory $z(t)$ which in turn determines the trajectory $r_Z(t)$, which finally determines the trajectory $\psi(t)$. Therefore, in order to estimate the value of the integral $\Omega$, we use a linear approximation of the model. From (90) we know that (around equilibrium):

$$z(t) \approx \bar{z} + \sum_{i=1}^{3} s_i e^{\epsilon_i \gamma} v_i.$$  \hspace{1cm} (96)

Substituting this to the formula (1) we get a linear approximation of the trajectory $r_Z(t)$:

$$r_Z(t) \approx \epsilon_Z + p_Z \bar{z} + p_Z \sum_{i=1}^{3} s_i e^{\epsilon_i \gamma} v_i = \bar{r}_Z + p_Z \sum_{i=1}^{3} s_i e^{\epsilon_i \gamma} v_i.$$  \hspace{1cm} (97)

Substituting (96) to (42) we get a linear approximation of the trajectory $\psi(t)$:

$$\psi(t) \approx \frac{(1 + \tau_Z)}{1 - \gamma (1 + \kappa)} \left[ \bar{r}_Z + p_Z \sum_{i=1}^{3} s_i e^{\epsilon_i \gamma} v_i \right] - \frac{\rho - \tau_Z \bar{\theta}}{1 - \gamma (1 + \kappa)}.$$  \hspace{1cm} (98)

Therefore:

$$\int_{0}^{\infty} \psi(s)ds \approx \int_{0}^{\infty} \left[ \frac{(1 + \tau_Z)}{1 - \gamma (1 + \kappa)} \left[ \bar{r}_Z + p_Z \sum_{i=1}^{3} s_i e^{\epsilon_i \gamma} v_i \right] - \frac{\rho - \tau_Z \bar{\theta}}{1 - \gamma (1 + \kappa)} \right] ds,$$  \hspace{1cm} (99)

which can be transformed to:
\[ \int_0^t \psi(s)ds \approx \frac{(1 + \tau_Z) \tau Z}{1 - \gamma(1 + \kappa)} + \frac{(1 + \tau_Z)}{1 - \gamma(1 + \kappa)} p_Z \sum_{i=1}^3 s_i v_i \int_0^t e^{r^i t}ds - \frac{(\rho - \tau_Z q) t}{1 - \gamma(1 + \kappa)}. \]  
\[ (99) \]

Notice that \( \int_0^t e^{r^i t}ds = \frac{1}{r_i} e^{r^i t} \bigg|_0^t = \frac{1}{r_i} (e^{r^i t} - 1) \). Hence:

\[ \int_0^t \psi(s)ds \approx \left[ \frac{(1 + \tau_Z) \tau Z - \rho + \tau_Z q} {1 - \gamma(1 + \kappa)} \right] + \frac{(1 + \tau_Z)}{1 - \gamma(1 + \kappa)} p_Z \sum_{i=1}^3 s_i v_i \left( e^{r^i t} - 1 \right). \]  
\[ (100) \]

Finally, substituting (100) to (95) we obtain the formula:

\[ \Omega \approx \frac{1}{\gamma} \sigma C e^0 \int_0^{\gamma(1+\kappa)} e^{\left( \left[ \frac{(1+\tau_Z) \tau Z - \rho + \tau_Z q} {1 - \gamma(1 + \kappa)} \right] + \frac{(1+\tau_Z)}{1 - \gamma(1 + \kappa)} p_Z \sum_{i=1}^3 s_i v_i \left( e^{r^i t} - 1 \right) \right) \gamma(1+\kappa)} dt. \]  
\[ (101) \]

The integral in the above formula is convergent which follows immediately from the transversality condition (83) together with the fact that \( \text{Re}(r_i) < 0 \) \( (i = 1, \ldots, 3) \) and \( \gamma < 0 \). Unfortunately, as we have demonstrated above, analytical formulas for \( r_i \) do not exist. It follows that the examination of the impact of each parameter on the equilibrium and achieved welfare is only possible by numerical methods. We will take care of this in the empirical part of the paper, after calibration of the model for the Polish economy.

4. The benevolent social planner (the “centrally planned” economy)

Recall that the representative agent treats all prices and fiscal variables as exogenous, because alone his influence on the market is negligible. Hence his optimal control problem is subject to the budget constraint (32) with \( w, w_K, s_T, g_C \) and \( d_D \) as exogenous. He also neglects all externalities related to investment and borrowing.

On the other hand, the benevolent social planner knows everything about the economy, including all externalities, aggregate effects, and fiscal rules. It’s worth to realize how many. First, he comprehends positive externalities of investment in productive capital, described by the assumption (5). Second, he realizes and takes into account the aggregate effects of individual decisions – in particular the influence of changes in individual demand for labor and capital on the wage rate and rental rate of capital, as described by equations (9) and (10). Third, he knows all rules of fiscal policy governing the deficit and its financing. Fourth, he can freely decide about the size of public consumption, which will therefore become an additional control variable. Last but not least, the social planner is aware of negative
externalities of running into debt, i.e. the relationships between interest rates and debt indicators described by equations (1) and (2).

The budget constraint of the social planner can be obtained from the representative agent constraint (32) by taking into account all information about the economy, i.e. equations (9) and (10), and fiscal rules (17) – (26), however \( g_c \) is now a control variable. In this way, we get the budget constraint (35).

Note that, in accordance with the equation (1) the interest rate on public debt is a function of the total public debt equal to \( d = d_D + d_F \). Therefore, the optimal control problem needs 2 additional state variables \( d_D \) and \( d_F \) together with appropriate equations of motion, which can be derived easily from (59) and (60), and have the following form:

\[
\begin{align*}
\dot{d}_F &= \omega \xi y - (n + \vartheta)d_F, \\
\dot{d}_D &= (1 - \omega) \xi y - (n + \vartheta)d_D.
\end{align*}
\]

### 4.1. The optimal control problem and its solution

The optimal control problem of the benevolent social planner is:

\[
\max \int_0^\infty \left( 1 \gamma g_c^x \right) e^{-(\rho-n)\tau} dt,
\]

\[
\begin{align*}
\dot{z} &= c + \left( 1 + \frac{\lambda_1}{2} \right) + (r_D - n)z - \left( 1 + \omega \xi \right) y + (r_D + \vartheta)d_F + g_c, \\
\dot{k} &= i - (n + \delta)k, \\
\dot{d}_F &= \omega \xi y - (n + \vartheta)d_F, \\
\dot{d}_D &= (1 - \omega) \xi y - (n + \vartheta)d_D.
\end{align*}
\]

Control variables: \( c, i, g_c \). State variables: \( z, k, d_F, d_D \). The initial values of variables (endowments): \( z_0, k_0 \geq 0, d_0 \geq 0, d_{F0} \geq 0, d_{D0} \geq 0 \), with \( d_{F0} + d_{D0} = d_0 \). Decision parameters of the government and the central bank: \( \tau_K, \tau_Z, \tau_D, \tau_c, \tau_L, \xi, \omega, \vartheta \). We will solve this problem using the same method as for the representative agent. The current value hamiltonian is:

\[
H_c = \frac{1}{\gamma} \left( 1 \gamma g_c^x \right) + \lambda_1 \left[ c + \left( 1 + \frac{\lambda_1}{2} \right) + (r_D - n)z - \left( 1 + \omega \xi \right) y + (r_D + \vartheta)d_F + g_c \right] +
\]

\[
+ \lambda_2 \left[ i - (n + \delta)k \right] + \lambda_3 \left[ \omega \xi y - (n + \vartheta)d_F \right] + \lambda_4 \left[ (1 - \omega) \xi y - (n + \vartheta)d_D \right].
\]
The optimal solution of (105) must satisfy the following conditions:

\[ \forall t \quad \frac{\partial H_c}{\partial c} = 0, \]  
\[ \forall t \quad \frac{\partial H_c}{\partial i} = 0, \]  
\[ \forall t \quad \frac{\partial H_c}{\partial g_c} = 0, \]  
\[ \lambda_1' = -\frac{\partial H_c}{\partial z} + \lambda_1' (\rho - n), \]  
\[ \lambda_2' = -\frac{\partial H_c}{\partial k} + \lambda_2' (\rho - n), \]  
\[ \lambda_3' = -\frac{\partial H_c}{\partial d_f} + \lambda_3' (\rho - n), \]  
\[ \lambda_4' = -\frac{\partial H_c}{\partial d_D} + \lambda_4' (\rho - n), \]  
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_1'(t) z(t) = 0, \]  
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_2'(t) k(t) = 0, \]  
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_3'(t) d_f(t) = 0, \]  
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_4'(t) d_D(t) = 0. \]

The condition (106a) can be written as:

\[ \lambda_1 = c^{-1} g_c^\psi, \]  
which is a counterpart of (39) and has similar interpretation. The important difference, however, is the absence of the consumption tax. The condition (106c) is:

\[ \lambda_1 = \kappa c^\psi g_c^{\psi - 1}, \]

which means that the shadow price of wealth (in the form of bonds) must be equal to the marginal utility of public consumption. Equating the right-hand sides of (107) and (108) immediately yields:

\[ g_c = \kappa c, \]

Which implies that the two kinds of consumption must grow at the same pace:

\[ \hat{g}_c = \hat{c} = \psi. \]
Recall that an identical rule was true for the decentralized economy, but it resulted from the assumption (26), whereas now it follows from the necessary conditions of optimality. By differentiating equation (107) with respect to time $t$, after transformation we get:

$$\hat{\lambda}_1 = (\gamma - 1)\dot{c} + \kappa \hat{g}_c.$$  \hfill (111)

The condition (106d) can be written as:

$$\hat{\lambda}_i = \rho - r_Z - p_Z \bar{z},$$  \hfill (112)

which differs from its counterpart (41). Substituting (112) into (111), and using $\hat{g}_c = \dot{c} = \psi$, we get the growth rate of per capita consumption:

$$\psi = \frac{\dot{c}}{c} = \frac{r_Z + p_Z \bar{z} - \rho}{1 - \gamma(1 + \kappa)}.$$  \hfill (113)

Notice that this formula is significantly different from its counterpart in the decentralized economy. Unlike in the decentralized economy, the growth rate of consumption does not depend on the tax rate on interest on the external debt of the private sector, but it depends on the external debt-to-GDP ratio of the private sector $\bar{z}$ and the risk premium $p_Z$. So far we don’t know whether in the optimal solution the interest rate $r_Z$ and the debt ratio $\bar{z}$ are constant over time. Thus the optimal trajectory of private consumption per capita can only be written in the same general form as before:

$$c(t) = c_0 \cdot e^{\int_0^t \psi(s)ds}. \hfill (114)$$

The condition (106b) can be written as:

$$q = \frac{\bar{z}}{\hat{\lambda}_1} = 1 + \chi \frac{i}{k},$$  \hfill (115)

identically as before. Its interpretation is identical as in the decentralized economy. Thus the growth rate of capital and the trajectory of capital are also described by identical formulas (46) and (47).

Now consider two additional necessary conditions: (106f) and (106g). The former can be written as

$$\dot{\lambda}_3 = \lambda_1 (p_D d_F + r_D + \mathcal{G}) + \lambda_3 (\rho + \mathcal{G}).$$  \hfill (116)

Dividing both sides by $\lambda_3$ and using (112) yields:

$$\frac{\dot{\lambda}_3}{\lambda_3} - \frac{\dot{\lambda}_1}{\lambda_1} = \frac{\lambda_1}{\lambda_3} (p_D d_F + r_D + \mathcal{G}) + r_Z + p_Z \bar{z} + \mathcal{G}.$$  \hfill (117)

It’s convenient to introduce another ratio of shadow prices, analogous to $q$:
\[ u = \frac{\lambda_1}{\lambda_4}, \]  

Using this we can transform the equation (117) to the following form:

\[ \dot{u} = r_D + p_D \frac{d}{d_F} + \mathcal{G} + (r_z + p_z \frac{z}{z} + \mathcal{G})u. \]  

(119)

The phase diagram of this differential equation is presented on fig. 1. A positive slope of the function \( \dot{u}(u) \) comes from the fact that \( r_z + p_z \frac{z}{z} + \mathcal{G} > 0 \). On the other hand, the positive value at \( u = 0 \) follows from the fact that \( r_D + p_D \frac{d}{d_F} + \mathcal{G} > 0 \).

\[ \begin{array}{c}
\dot{u} \\
\hline
r_D + p_D \frac{d}{d_F} + \mathcal{G} \\
\hline
u_0 \quad 0 \quad u
\end{array} \]

Fig. 1. The phase diagram of the equation (119)

Obviously it is an unstable saddle point. Thus the optimal solution must satisfy the condition: \( \forall t \quad \dot{u} = 0 \), i.e.

\[ \lambda_3 = -\lambda_1 \cdot \frac{r_D + p_D \frac{d}{d_F} + \mathcal{G}}{r_z + p_z \frac{z}{z} + \mathcal{G}}. \]  

(120)

An identical analysis of the condition (106e) leads to a similar conclusion:

\[ \lambda_4 = -\lambda_1 \cdot \frac{p_D \frac{d}{d_F}}{r_z + p_z \frac{z}{z} + \mathcal{G}}. \]  

(121)

To determine the trajectory \( q(t) \), we will use the last necessary condition for optimality, i.e. (106e), which can be written as:

\[ \dot{\lambda}_2 = -\left[ \lambda_1 \left( -\frac{X}{2 \kappa^2} \right) + \frac{\partial \lambda_3}{\partial \kappa} \frac{z}{z} - (1 + \omega \xi)A + \frac{\partial r_D}{\partial \kappa} \frac{d}{d_F} \right] + \lambda_3 (n + \omega \xi A - \lambda_4 (1 - \omega \xi A) + \lambda_2 (\rho - n). \]

(122)

Since \( y = Ak \), it follows from equations (1) and (2) that:

\[ \frac{\partial r_z}{\partial \kappa} = -\frac{p_z \frac{z}{z}}{Ak^2}. \]  

(123)
\[
\frac{\partial r_D}{\partial k} = -\frac{p_D d}{Ak^2}.
\]

Substituting these formulae into (122) and using (115) together with \( \lambda_i = -\lambda_i' \) yields:

\[
\dot{\lambda}_2 = -\lambda_1 \left[ \frac{(q-1)^2}{2\chi} + (1 + \omega\xi)A + Ap_Z \frac{z^2}{y^2} + Ap_D \frac{d \cdot d_F}{y^2} \right] + \lambda_2 (\rho + \delta) - \lambda_3 \omega\xi A - \lambda_4 (1 - \omega)\xi A.
\]

Finally, dividing both sides by \( \dot{\lambda}_2 \) and using (112), (120) and (121) after transformation we get:

\[
\dot{q} = \left[ A + \omega\xi A + \frac{(q-1)^2}{2\chi} + Ap_Z \frac{z^2}{y^2} + Ap_D \frac{d \cdot d_F}{y^2} \right] + (r_z + p_z z + \delta) q + \omega A \frac{\omega r_D + p_d d_F + \omega \partial}{r_z + p_z z + \delta},
\]

where \( d = d_F + d_D \). This is the social planner’s counterpart of the equation (49), though far more complex – it’s possible to demonstrate by substitutions that instead of a quadratic function in \( q \) this time we have a polynomial of the 5th order in \( q \).

Obviously, along the balanced growth path all debt-to-GDP ratios \((z, d_F, d_D)\) must be constant – otherwise, we would have \( \dot{q} \neq 0 \), and therefore \( \varphi \neq const \). Now we will prove more – namely, that a stationary state will be achieved when all variables in relation to production reach constant values. As before, we will use the ratios \( \zeta \) and \( \bar{z} \). Recall that the growth rates of these ratios are expressed by equations (51) and (52). Having regard to (1), (113) and (115), equation (51) can be written in the form:

\[
\dot{\zeta} = \left[ \frac{r_z + p_z \bar{z} - \rho}{1 - \gamma(1 + \kappa)} - \frac{q-1}{\chi} + n + \delta \right] \cdot \zeta,
\]

where \( r_z = \epsilon_z + p_z \bar{z} \). Derivation of dynamics equations of the debt ratio \( \bar{z} \) requires a few transformations. First, dividing both sides of the equation (35) by \( z \), and taking into account (109) and (115) we get:

\[
\ddot{z} = (1 + \kappa) \frac{\zeta}{z} + \frac{q^2 - 1}{2A\chi} \cdot \frac{1}{z} + [r_z - n] + r_D^\chi \frac{d_F}{z} - \frac{1}{z} \cdot (1 + \omega\xi).
\]

From (52) it follows that: \( \dot{\bar{z}} = \dot{\bar{z}} \cdot z = (\dot{\bar{z}} - \varphi)\bar{z} \). Together with (128) and (115), it can be rearranged to:

\[
\dot{\bar{z}} = (1 + \kappa) \zeta + \frac{q^2 - 1}{2A\chi} - (1 + \omega\xi) + \left( r_z - \frac{q-1}{\chi} + \delta \right) \bar{z} + (r_D + \partial) d_F.
\]
where $r_z = \varepsilon_z + p_Z z$, $r_D = \varepsilon_D + p_D d$. Just like in the decentralized economy, the equations (127), (129) and (126) determine the evolution of variables: $c$, $z$ and $q$. In order to close this system of differential equations we need to append 2 equations describing the dynamics of $d_D$ and $d_F$ given by (59) and (60). In that way we have obtained a nonlinear, autonomous system of 5 differential equations in 5 variables of the form (61).

4.2. Steady-state equilibrium (the balanced growth path)

In order to find the steady state we solve the system of 5 equations: $f^i(c, z, q, d_F, d_D) = 0$ ($i = 1, \ldots, 5$), which this time has the following form:

$$\bar{r}_z + p_Z \bar{z} - \rho \frac{1}{1 - \gamma(1 + \kappa)} - \bar{\phi} = 0,$$

$$\bar{r}_D + p_D \bar{d} - \rho \frac{1}{1 - \gamma(1 + \kappa)} - \bar{\phi} = 0,$$

$$(1 + \kappa)\bar{c} + \frac{\bar{q}^2 - 1}{2A\chi} - (1 + \omega\xi) + (\bar{r}_z - n - \bar{\phi})\bar{z} + \bar{r}_D = 0,$$

$$(\bar{r}_z + p_Z \bar{z} + \delta)\bar{q} - A(1 + \omega\xi) + \frac{(\bar{q} - 1)^2}{2\chi} - Ap_Z \bar{z}^2 - Ap_D \bar{d}_D \bar{d} + \xi A \frac{\omega \bar{d}_D + p_D \bar{d} + \omega \bar{\phi}}{\bar{r}_z + p_Z \bar{z} + \bar{\phi}} = 0,$$

$$(-n - \bar{\phi} - \delta)\bar{d}_F + \omega \xi = 0,$$

$$(-n - \bar{\phi} - \delta)\bar{d}_D + (1 - \omega)\xi = 0,$$

where $r_z = \varepsilon_z + p_Z z$, $r_D = \varepsilon_D + p_D d$, $\bar{d} = \bar{d}_F + \bar{d}_D$, $\bar{\phi} = \frac{\bar{q} - 1}{\chi} - (n + \delta)$. First notice that (130) immediately yields:\n
$$\bar{\phi}^* = \frac{\bar{r}_z + p_Z \bar{z} - \rho}{1 - \gamma(1 + \kappa)},$$

which is the steady-state rate of growth of all per capita variables:

$$\hat{\gamma} = \hat{k} = \hat{c} = \hat{z} = \hat{\phi}^*.\n
From (130) it follows that

$$\bar{r}_z + p_Z \bar{z} = \rho + \bar{\phi} A_i.$$

where $A_i = 1 - \gamma(1 + \kappa)$. Using (1) we get:

$$\bar{r}_z = \frac{\rho + \varepsilon_z + \bar{\phi} A_i}{2}.\n
^{11}$ The optimal solution obtained by the benevolent social planner will be denoted with stars, to distinguish it from the second best solution obtained by representative agents in the decentralized economy.
\[
\tilde{z} = \frac{\rho - \varepsilon_z + \bar{\rho}A_1}{2p_Z}.
\]

In addition, it follows from (2) that: \( r_D = \varepsilon_D + p_D \tilde{d} \). Using (137) – (139), the equality (132) can be written in the form:

\[
(\rho + \bar{\rho}A_1 + \delta) \cdot \bar{q} - A(1 + \omega \xi) - \frac{(\bar{q} - 1)^2}{2\chi} - \frac{A}{4p_Z} (\rho + \bar{\rho}A_1 - \varepsilon_Z)^2 - Ap_{D} \frac{\omega \xi^2}{(n + \bar{\rho} + \vartheta)^2} + \xi A \frac{\omega (\varepsilon_D + p_D \bar{d})}{n + \bar{\rho} + \vartheta} + \omega \vartheta = 0.
\]

From equations (133) and (134), we get stationary values of government debt ratios which are expressed by identical formulae as in the decentralized economy:

\[
\tilde{d}_F^* = \frac{\omega \xi}{n + \bar{\rho} + \vartheta},
\]

(141)

\[
\tilde{d}_D^* = \frac{(1 - \omega) \xi}{n + \bar{\rho} + \vartheta},
\]

(142)

where \( \tilde{d}^* = \tilde{d}_F^* + \tilde{d}_D^* \). Substituting these formulae into (140) yields:

\[
(\rho + \bar{\rho}A_1 + \delta) \cdot \bar{q} - A(1 + \omega \xi) - \frac{(\bar{q} - 1)^2}{2\chi} - \frac{A}{4p_Z} (\rho + \bar{\rho}A_1 - \varepsilon_Z)^2 - Ap_{D} \frac{\omega \xi^2}{(n + \bar{\rho} + \vartheta)^2} + \xi A \frac{\omega (\varepsilon_D + p_D \bar{d})}{n + \bar{\rho} + \vartheta} + \omega \vartheta = 0.
\]

It follows from (115) that:

\[
\bar{q} = 1 + \chi (\bar{\rho} + n + \delta).
\]

(144)

Replacing \( \bar{q} \) in the equation (143) with this formula will yield an equation with one unknown, \( \bar{\rho} \):

\[
(\rho + \bar{\rho}A_1 + \delta) \cdot [1 + \chi (\bar{\rho} + n + \delta)] - A(1 + \omega \xi) - \frac{\chi (\bar{\rho} + n + \delta)^2}{2} - \frac{A}{4p_Z} (\rho + \bar{\rho}A_1 - \varepsilon_Z)^2 - Ap_{D} \frac{\omega \xi^2}{(n + \bar{\rho} + \vartheta)^2} + \xi A \frac{\omega (\varepsilon_D + p_D \bar{d})}{n + \bar{\rho} + \vartheta} + \omega \vartheta = 0.
\]

(145)

Multiplying both sides by the expression \( 4p_Z (\rho + \bar{\rho}A_1 + \vartheta)(n + \bar{\rho} + \vartheta)^2 \), after rearrangement we get a polynomial equation of the fifth degree:

\[
w_5 \bar{\rho}^5 + w_4 \bar{\rho}^4 + w_3 \bar{\rho}^3 + w_2 \bar{\rho}^2 + w_1 \bar{\rho} + w_0 = 0,
\]

(146)

where the coefficients are very complicated functions of parameters\textsuperscript{12}:

\textsuperscript{12} Calculated with Mathematica 9.0.
\[ w_0 = 2p_x \left[ 2A_x z^2 - (n + \vartheta - \rho) x \xi \omega \right] - \\
\left\{ 2(2n + \vartheta - \rho) \left[ 2n + \vartheta - \rho + n^2 \vartheta - 2n \vartheta + 2n \vartheta \right] + \\
2A(n + \vartheta)(\vartheta - \rho - \epsilon_z) \right\} \] (147)

\[ w_1 = 2p_x \left[ 2A_x z^2 - (n + \vartheta) \left[ \vartheta (n + \vartheta) \vartheta - 2n \vartheta + 2n \vartheta \right] + \\
2A(n + \vartheta)(\vartheta - \rho - \epsilon_z) \right] \] (148)

\[ w_2 = 2p_x \left[ (n + \vartheta) \left[ \vartheta (n + \vartheta) \vartheta - 2n \vartheta + 2n \vartheta \right] + \\
2A(n + \vartheta)(\vartheta - \rho - \epsilon_z) \right] \] (149)

\[ w_3 = -A(n + \vartheta) A_x^2 - 4(2n + \vartheta - \rho) x \xi \omega \right] - \\
2A(n + \vartheta)(\vartheta - \rho - \epsilon_z) \right\} \] (150)

\[ w_4 = -A(n + \vartheta) A_x^2 - 2(n + \vartheta) x \xi \omega \right] + \\
2A(n + \vartheta)(\vartheta - \rho - \epsilon_z) \right\} \] (151)

\[ w_5 = -A \left( A A_x z^2 - 2x \xi \omega \right) \] (152)

The polynomial equation of the 5th degree can have up to 5 real roots. Therefore the model has a potential problem of nonuniqueness and nonexistence of a balanced growth equilibrium. Due to the complexity of the equation (146), there is no way to eliminate this problem by any simple assumptions. The only way to cast some light on this issue is by numerical methods. We have carried out numerous (virtually thousands) simulations for this model calibrated for Polish economic data, varying decision parameters, as well as many exogenous parameters. In all cases, without any exception, the equation (146) turned out to have 1 positive real root, 2 negative real roots (not acceptable, because it would mean the economy shrinks along the balanced growth path, eventually disappearing), and 2 complex conjugate roots (not feasible). Therefore, henceforth we will assume that the equation (146) has a unique viable (positive) solution \( \vartheta^* \) which is the balanced growth rate (the BGR). Knowing \( \vartheta^* \) we can calculate all other steady-state values. First, out of the equation (144), we get \( \vartheta^* \), from equations (138) and (139) we calculate \( \epsilon^*_x \) and \( \epsilon^*_d \), from (79) and (80) we obtain \( \vartheta^*_f \) and \( \vartheta^*_n \), from (81) we calculate \( \epsilon^*_d \). Finally from (131) we obtain the steady-state ratio of private consumption to GDP:
1 + \omega \xi - \bar{q}^* \frac{1}{2 \chi} - \left( \bar{F}_z^* - n - \bar{q}^* \right) \bar{z}_{t-1}^* - (\bar{F}_D^* + \bar{q}) \bar{d}_F^* \\
\bar{z}^* = \frac{1 + \omega \xi - \bar{q}^* \frac{1}{2 \chi} - \left( \bar{F}_z^* - n - \bar{q}^* \right) \bar{z}_{t-1}^* - (\bar{F}_D^* + \bar{q}) \bar{d}_F^*}{1 + \kappa}.

(153)

In the appendix we prove that the transversality conditions (106h–k) are satisfied if, and only if:

\rho > n + \gamma (1 + \kappa) \bar{q}^* ,

(154)

which is identical with (84) for the decentralized economy.

4.3. The local stability of equilibrium

We will investigate the stability of the system (61) just like we did for the decentralized economy. Linearization in the neighborhood of equilibrium yields approximations of the functions \( f^i \):

\[ f^i(c, \bar{z}, \bar{q}, d_F, d_D) \approx f^i(c^*, \bar{z}^*, \bar{q}^*, d_F^*, d_D^*) + \]

\[ + \frac{\partial f^i}{\partial c} \bar{c} + \frac{\partial f^i}{\partial \bar{z}} \bar{z} + \frac{\partial f^i}{\partial \bar{q}} \bar{q} + \frac{\partial f^i}{\partial d_F} \bar{d}_F + \frac{\partial f^i}{\partial d_D} \bar{d}_D \]

\((i = 1, ..., 5)\) where “waved” variables are deviations from the steady-state, i.e. \( \bar{c} = c - c^* \), \( \bar{z} = z - z^* \), \( \bar{q} = q - q^* \), \( \bar{d}_F = d_F - d_F^* \), \( \bar{d}_D = d_D - d_D^* \). The linear approximation of (61) around the equilibrium has the form (85) with the matrix of values of partial derivatives (Jacobian) calculated in the equilibrium\(^{13}\):

\(^{13}\) Formally, the steady-state values in (155) should be labeled with stars, but to make it more “readable” we skipped stars.
The general solution of the linear system of equations (85) can be written as:

\[
M^{*T} = \begin{bmatrix}
0 & 1+\kappa & 0 & 0 & 0 \\
\frac{2p_x \tilde{c}}{A_1} & \rho + \tilde{\rho} A_1 - n - \tilde{\rho} & 2p_x (\tilde{q} - \tilde{A}_2) - \frac{\omega D}{\rho + \tilde{\rho} A_1 + \vartheta} & \frac{\omega D}{\rho + \tilde{\rho} A_1 + \vartheta} & 0 & 0 \\
-\frac{\tilde{c}}{\chi} & \frac{\tilde{q} - \tilde{A}_2}{A\chi} & \rho + \tilde{\rho} A_1 - n - \tilde{\rho} & -\frac{\tilde{d}_f}{\chi} & -\frac{\tilde{d}_d}{\chi} & 0 \\
0 & p_\beta (2\tilde{d}_f + \tilde{d}_d) + \varepsilon_d + \omega \vartheta & \frac{-Ap_\beta}{\rho} & \frac{-Ap_\beta}{\rho + \tilde{\rho} A_1 + \vartheta} & \frac{-2\tilde{d}_f + \tilde{d}_d}{\rho + \tilde{\rho} A_1 + \vartheta} & -n - \tilde{\vartheta} - \vartheta & 0 \\
0 & \rho_\beta \tilde{d}_f + \omega \vartheta & \frac{-Ap_\beta}{\rho} & \frac{-Ap_\beta}{\rho + \tilde{\rho} A_1 + \vartheta} & \frac{-\tilde{d}_f}{\rho + \tilde{\rho} A_1 + \vartheta} & -n - \tilde{\vartheta} - \vartheta & 0
\end{bmatrix}
\]

(155)

The general solution of the linear system of equations (85) can be written as:\[14:\]

\[
\begin{bmatrix}
\tilde{c}^* \\
\tilde{z}^* \\
q^* \\
\tilde{d}_f^* \\
\tilde{d}_d^*
\end{bmatrix}
= \begin{bmatrix}
\tilde{c}^* \\
\tilde{z}^* \\
\tilde{q}^* \\
\tilde{d}_f^* \\
\tilde{d}_d^*
\end{bmatrix}
+ \sum_{i=1}^{S} s_i e^{ri} v^i,
\]

(156)

where \(r_i\) are the eigenvalues of the matrix \(M^*\), \(v^i\) are its eigenvectors, and \(s_i\) are unknown constants dependent on the starting point (endowments). It follows that:

\[
\det M^* = \frac{-(1+\kappa)(n + \tilde{\rho}^* A_1 + \vartheta)\tilde{c}^*}{2p_x \chi A_1 (\rho + \tilde{\rho}^* A_1 + \vartheta)(\rho + \tilde{\rho}^* A_1 + \vartheta)^2} \cdot [expr].
\]

(157)

\[
\text{tr} M^* = 2\left[\rho + \tilde{\rho}^* A_1 - 2(n + \tilde{\rho}^*) - \vartheta\right].
\]

(158)

where \(expr\) is such a complex expression, that it is impossible to determine its sign\[15\]. But even the sign of \(\text{tr} M^*\) cannot be predetermined without additional assumptions. It’s clear, therefore, that it’s possible to analyze the stability of equilibrium only by numerical methods, after calibrating the model.

To be able to continue our discussion, let’s assume now that the matrix \(M^*\) has the same properties as in the decentralized economy, i.e. it has 3 eigenvalues with negative real parts. In this case the equilibrium has the form of the **stable saddle path**, and the solution (156) can be written as:


\[15\] The appendix contains the full formula for \(expr\).
where \( r_i \) (\( i = 1, \ldots, 3 \)) are negative eigenvalues, \( \mathbf{v}' \) are corresponding eigenvectors. The unknown constants \( s_i \) can be found by plugging the initial values of debt indicators into (159), which results in the following system of 3 equations in 3 unknowns:

\[
\begin{align*}
\mathbf{z}_0 &= \mathbf{z}^* + \sum_{i=1}^{3} s_i v_{2i}', \\
\mathbf{d}_F &= \mathbf{d}_F^* + \sum_{i=1}^{3} s_i v_{4i}', \\
\mathbf{d}_D &= \mathbf{d}_D^* + \sum_{i=1}^{3} s_i v_{5i}'.
\end{align*}
\]  

(160)

Knowing the eigenvalues, eigenvectors and constants \( s_i \) from the remaining two equations of the system (159) we can find initial values of \( c \) and \( q \). We obtain:

\[
\begin{align*}
c_0 &= \mathbf{c}^* + \sum_{i=1}^{3} s_i v_{1i}', \\
q_0 &= \mathbf{q}^* + \sum_{i=1}^{3} s_i v_{3i}'.
\end{align*}
\]  

(161) (162)

4.4. Welfare provided by the benevolent social planner

Using (109) we can express the level of welfare in the economy as:

\[
\Omega = \frac{1}{\gamma} \kappa^{\nu \gamma} \int_{0}^{\infty} (c_t)^{(1+\kappa)} e^{-(\rho-n)\nu} dt, \quad \rho > n.
\]  

(163)

Substituting the trajectory of private consumption given by (114) yields:

\[
\Omega = \frac{1}{\gamma} \kappa^{\nu \gamma} c_0^{(1+\kappa)} \int_{0}^{\infty} \left( \int_{0}^{t} \nu s(t) dt \right)^{(1+\kappa)} e^{-(\rho-n)\nu} dt.
\]  

(164)

In order to evaluate the value of this integral we use the linear approximation of the model. It follows from (159) that around the steady state:

\[
\mathbf{z}(t) \approx \mathbf{z}^* + \sum_{i=1}^{3} s_i e^{\nu t} v_{2i}'.
\]  

(165)
Substituting (165) to the formula (1) we get a linear approximation of the trajectory \( r_Z^*(t) \):

\[
r_Z^*(t) = e_Z + p_Z z^*(t) \approx e_Z + p_Z z^* + p_Z \sum_{i=1}^{3} s_i e^{r_i} v_i = r_Z^* + p_Z \sum_{i=1}^{3} s_i e^{r_i} v_i.
\]

(166)

Substituting (165) and (166) to (113) we get an approximation of \( \psi^*(t) \):

\[
\psi^*(t) = \frac{e_Z + 2 p_Z z^*(t) - \rho}{1 - \gamma(1 + \kappa)} - \frac{1}{1 - \gamma(1 + \kappa)} \left[ r_Z^* + p_Z z^* + 2 p_Z \sum_{i=1}^{3} s_i e^{r_i} v_i \right] - \frac{\rho}{1 - \gamma(1 + \kappa)}.
\]

(167)

Using these formulae to calculate \( \int_0^t \psi^*(s)ds \) yields:

\[
\int_0^t \psi^*(s)ds = \int_0^t \left( \frac{1}{1 - \gamma(1 + \kappa)} \left[ r_Z^* + p_Z z^* + 2 p_Z \sum_{i=1}^{3} s_i e^{r_i} v_i \right] - \frac{\rho}{1 - \gamma(1 + \kappa)} \right) ds,
\]

(168)

which can be written as:

\[
\int_0^t \psi^*(s)ds = \frac{\left( r_Z^* + p_Z z^* \right) t}{1 - \gamma(1 + \kappa)} + \frac{2 p_Z}{1 - \gamma(1 + \kappa)} \sum_{i=1}^{3} s_i v_i e^{r_i} ds - \frac{\rho t}{1 - \gamma(1 + \kappa)}.
\]

(169)

It’s straightforward to verify that:

\[
\int_0^t e^{r_i} ds = \frac{1}{r_i} \left[ e^{r_i} \right]_0^t = \frac{1}{r_i} (e^{r_i} - 1).
\]

(170)

Using (170), the equation (168) can be written in the form:

\[
\int_0^t \psi^*(s)ds = \frac{\left( r_Z^* + p_Z z^* - \rho \right) t}{1 - \gamma(1 + \kappa)} + \frac{2 p_Z}{1 - \gamma(1 + \kappa)} \sum_{i=1}^{3} s_i v_i e^{r_i} - \frac{\rho t}{1 - \gamma(1 + \kappa)}.
\]

(171)

Finally, substituting (171) to (164) we get the (approximate) formula for welfare in the economy controlled by the benevolent social planner:

\[
\Omega^* = \frac{1}{\gamma} \kappa^\gamma c_0 (r_{1+\kappa}) \int_0^t e^{\left[ \frac{(r_Z^* + p_Z z^* - \rho)}{1 - \gamma(1 + \kappa)} + \frac{2 p_Z}{1 - \gamma(1 + \kappa)} \right] \gamma(1 + \kappa)} e^{-(\rho-n)\gamma} dt.
\]

(172)

The integral in this formula converges, which follows from the transversality conditions (154) and the fact that \( Re(r_i) < 0 \) \((i = 1, 2, 3)\) and \( \gamma < 0 \).
5. Sensitivity of the two types of economies to fiscal policy

Careful analysis of the derived formulae provides some interesting conclusions.

5.1. Income and consumption taxes
All income and consumption tax rates are fully neutral for the benevolent social planner, i.e. they influence neither the steady state, nor the trajectories converging to the steady state. (It follows directly from a simple observation that all five tax rates $r_i$ do not show up in any of the formulae describing the balanced growth path and linear approximation of the trajectories converging to this path.) This leads to an important conclusion: the level of prosperity achieved by the social planner $\Omega^*$ does not depend on any of the tax rates. The taxation of labor, capital, consumption, foreign bonds, and domestic bonds is neutral for the “centrally planned” economy – has no effect on the level of welfare reached by consumers in the economy governed by the social planner.

In the decentralized economy three tax rates remain fully neutral, just like in the “centrally planned” economy: taxes on labor, consumption and domestic bonds. However, two other tax rates are no longer neutral – taxes on capital income and interest on private foreign debt do effect the economy. The derivatives of the steady state with respect to these tax rates and their signs are reported in table 1. An important conclusion is that welfare in the decentralized economy may depend on these two tax rates, but investigating this relationship is only possible by numerical methods.

5.2. Inflation
The inflation tax is not neutral for both economies in the long run. Specifically, in the decentralized economy it does not influence the BGR – in that sense it is neutral for the “real” economy. Nevertheless, inflation impacts the level of welfare measured by the value of the intertemporal utility function. On the other hand, for the benevolent social planner inflation is not neutral – it impacts all real variables including the BGR. The difficulty is that without numerical calculations it’s impossible to say anything not only about the magnitude of these relationships, but also about the sign.
Table 1. How the steady state in the decentralized economy is influenced by taxation of capital income and interest on private foreign debt

<table>
<thead>
<tr>
<th></th>
<th>( \frac{\partial(.)}{\partial \tau_k} )</th>
<th>sign</th>
<th>( \frac{\partial(.)}{\partial \tau_z} )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\bar{q}} )</td>
<td>( -\frac{2\chi A \alpha}{\sqrt{\Delta}} )</td>
<td>–</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{\bar{l}} )</td>
<td>( -\frac{2\alpha}{\sqrt{\Delta}} )</td>
<td>–</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{\phi} )</td>
<td>( -\frac{2\alpha A}{\sqrt{\Delta}} )</td>
<td>–</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{r}_Z )</td>
<td>( -\frac{A_i}{1+\tau_z} \frac{2\alpha A}{\sqrt{\Delta}} )</td>
<td>–</td>
<td>( -\frac{\bar{r}_Z + \bar{\phi}}{1+\tau_z} )</td>
<td>–</td>
</tr>
<tr>
<td>( \bar{z} )</td>
<td>( -\frac{A_i}{p_z(1+\tau_z)} \frac{2\alpha A}{\sqrt{\Delta}} )</td>
<td>–</td>
<td>( -\frac{\bar{r}_Z + \bar{\phi}}{p_z(1+\tau_z)} )</td>
<td>–</td>
</tr>
<tr>
<td>( \bar{r}_D )</td>
<td>( \frac{2\alpha A \bar{y}_D}{(n+\bar{\phi}+\bar{\phi})^2 \sqrt{\Delta}} )</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{d}_F )</td>
<td>( \frac{2\alpha A \bar{z}_D}{(n+\bar{\phi}+\bar{\phi})^2 \sqrt{\Delta}} )</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{d}_D )</td>
<td>( \frac{2\alpha A \bar{z}(1-\omega)}{(n+\bar{\phi}+\bar{\phi})^2 \sqrt{\Delta}} )</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{z} )</td>
<td>quite complex formula; could be positive, negative or zero.</td>
<td>?</td>
<td>( \frac{\bar{r}_Z [p_z \bar{z} + \bar{r}_Z - n - \bar{\phi}]}{p_z (1+\tau_z)(1+\sigma_c)} )</td>
<td>?</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>analytical formula does not exist</td>
<td>?</td>
<td>analytical formula does not exist</td>
<td>?</td>
</tr>
</tbody>
</table>

Source: own calculations

5.3. Budget deficit and its financing
All remaining parameters of fiscal policy also influence both solutions, so that trajectories of many variables and welfare depend on the share of public consumption in the GDP, the size of public deficit and the structure of public debt (the share of foreign creditors). Consequently, all these parameters do impact welfare in both economies. However, since an analytical solution to the model of the benevolent social planner does not exist, an analysis of the relations between these parameters and the BGR and welfare requires numerical methods.
6. Replication – optimal fiscal policy

The optimal control problem of the benevolent social planner incorporates all externalities and rules governing the economy, which are not necessarily taken into account by the representative agent. This implies that the decentralized economy is not able to achieve greater welfare than the social planner, i.e. $\Omega \leq \Omega^*$. However, the social planner may induce individual economic agents to internalize all externalities by proper adjustment of fiscal policy. This is known in the literature as the problem of replication of the “centrally planned” economy, or the replication of the first-best solution.

**Definition 1.** The parameters of fiscal policy (in our model, tax rates and the size of public consumption) are optimal, if the decentralized economy reaches the same welfare as the benevolent social planner, that is, if $\Omega = \Omega^*$.

The following proposition seems to be so obvious, that many authors do not even mention it. Nevertheless we present it together with the proof, because the devil is in the detail: the proof hinges on the uniqueness of the balanced growth path, which is not necessarily guaranteed in many models presented by some authors in the literature.

**Proposition 1.** The parameters of fiscal policy (in our model, tax rates and the size of public consumption) are optimal, if and only if they are replicating the “centrally planned” economy, that is, the trajectories of all variables in the decentralized economy are identical to those in the “centrally planned” economy.

**Proof.** First we prove (by contradiction) the following implication: If $\Omega = \Omega^*$, then the decentralized economy fully replicates the “centrally planned” economy, that is, the trajectories of all variables in the decentralized economy are identical to those in the “centrally planned” economy. Assume for a moment that $\Omega = \Omega^*$, and at the same time (at least) one of the trajectories in the decentralized economy differs from its counterpart in the “centrally planned” economy. This would imply that the optimal control problem of the benevolent social planner has more than one solution (at least two distinct solutions).
However, it’s impossible, because we have demonstrated that it has exactly one (unique) solution\textsuperscript{16}.

The implication in the opposite direction is obvious: if parameters of fiscal policy replicate the “centrally planned” economy, then the decentralized economy reaches the same level of welfare as the social planner, i.e. \( \Omega = \Omega^* \).

\[ \star \]

It follows from proposition 1 that the problem of finding the optimal parameters of fiscal policy is tantamount with the problem of finding parameters that replicate the “centrally planned” economy. The following proposition provides the solution of this problem for our model.

**Proposition 2** (proof in the appendix). If the parameters of fiscal policy are optimal, then for every \( t \geq 0 \) the following conditions hold:

\[
\sigma_C^{opt} = \kappa, \quad (173)
\]

\[
\tau_Z^{opt}(t) = \frac{r_Z^*(t) - \epsilon_Z}{r_Z(t) + \partial}, \quad (174)
\]

and the tax rate on capital must appropriately evolve over time. However, it is impossible to derive an analytical formula for the optimal trajectory \( \tau_{K}^{opt}(t) \) – not only for the whole trajectory, but even for a single value of this parameter in a selected moment of time \( t \).

Individual (momentary) values of the trajectory \( \tau_{K}^{opt}(t) \) can only be calculated numerically, for a chosen set of values of all model parameters.

Note that other tax rates (\( \tau_D, \tau_C, \tau_L \)) do not affect replication, because (as we showed in section 5) they are neutral for both types of the economy. It is interesting and less obvious, however, that three other parameters of fiscal policy (\( \xi, \omega, \partial \)) also do not affect replication, which of course does not necessarily mean that they do not affect the level of welfare in the “centrally planned” economy. (We will investigate this issue in sections 11 and 12.)

Proposition 2 is a prescription for the internalization of all external effects that occur in a decentralized market economy, which we listed in Introduction. However, from a practical

\textsuperscript{16} Strictly speaking, we have eliminated the problem of nonuniqueness by assuming that the equation (146) has a unique positive solution \( \overline{\Theta}^* \). This assumption is strongly supported empirically: we have carried out numerous simulations varying widely all parameters, and not even once have we encountered the problem of nonuniqueness.
point of view, an implementation of this recipe is difficult to imagine, because it requires that
the two tax rates must be continuously (in practice, very often, perhaps once per year)
adjusted over time. For this reason, it’s worth to define and solve a simpler problem of partial
replication (of the steady state only).

7. Partial replication

Definition 2. We say that the decentralized economy replicates the steady state of the
“centrally planned” economy if these economies have identical steady states, i.e. $\bar{c} = \bar{c}^*$, $\bar{z} = \bar{z}^*$, etc.

Obviously, partial replication means that all trajectories in the decentralized economy
converge to the same balanced growth path as in the first-best equilibrium, but they do not necessarily coincide with the trajectories of the social planner. For example

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} c^*(t) = \bar{c}^*, \quad (175)$$

but at least for some values of $t$ we may have:

$$c(t) \neq c^*(t). \quad (176)$$

Directly from the definition 1 and 2 it follows that (full) replication implies partial
replication, but not otherwise. The next proposition provides the solution of the problem of
partial replication for our model.

Proposition 3. If the decentralized economy replicates the steady state for the “centrally
planned” economy, then the following (necessary) conditions hold together:

$$\sigma_c = \sigma_c^{rep} = \kappa, \quad (177)$$

$$\tau_Z = \tau_Z^{rep} = \frac{\bar{p}_Z^* - c_Z^*}{\bar{r}_Z^* + \gamma} \quad (178)$$

and the tax rate on capital $\tau_K$ is constant over time and equal to $\tau_K^{rep}$, the value of which can only be calculated numerically. (An explicit formula for this value does not exist.)

Proof. It follows directly from the formulae describing the steady states of both models that
the replication of the steady state occurs if and only if the following conditions are satisfied
(necessary and sufficient):
Let us look at each one of these conditions. From the formula (1) it follows that if the condition (179) holds, then \( \bar{r}_Z = \bar{r}_Z^* \) and vice versa. Put simply, the condition (179) is equivalent to \( \bar{r}_Z = \bar{r}_Z^* \). This fact together with the formulae for the rates of growth of consumption in both types of economies implies that if (179) holds, then \( \bar{\psi} = \bar{\psi}^* \). Previously we have shown that \( \psi(t) = \psi^*(t) \) if, and only if, \( \tau_Z(t) = \tau_Z^{opt}(t) \). It implies that

\[ \bar{\psi} = \bar{\psi}^* \iff \tau_Z = \lim_{t \to \infty} \tau_Z^{opt}(t). \] (182)

It follows that if (179) holds, then the tax rate \( \tau_Z \) is equal to:

\[ \tau^{rep}_Z = \frac{\bar{r}_Z^* - \bar{c}_Z}{\bar{r}_Z^* + \delta}, \] (183)

which is the second necessary condition for partial replication.

The first necessary condition can be derived from the condition (180). Recall that \( \bar{q}^* = 1 + \chi(\bar{\phi}^* + n + \delta) \) and \( \bar{q} = 1 + \chi(\bar{\phi} + n + \delta) \). It follows that the condition (181) is equivalent to the equality: \( \bar{q} = \bar{q}^* \). On the other hand, if \( \bar{\phi} = \bar{\phi}^* \), then \( \bar{r}_D = \bar{r}_D^* \) and \( \bar{d}_D = \bar{d}_D^* \). Taking all this into account it's easy to see that if (181) holds, then the condition (180) is satisfied if and only if

\[ \sigma_C = \sigma_C^{rep} = \kappa, \] (184)

which is the first necessary condition for partial replication.\(^\text{17}\)

Finally, we need to analyze the last necessary and sufficient condition of partial replication (181). Recall that the social planner’s rate of balanced growth in \( \bar{\phi}^* \) is a real, positive solution of a polynomial equation of the fifth degree, and so there is no analytical formula for \( \bar{\phi}^* \). On the other hand, the balanced growth rate in the decentralized economy \( \bar{\phi} \) is given by the simple formula, which through appropriate substitutions can be written in the form:

\(^{17}\) Notice that these two conditions coincide with the conditions of full replication (173) and (174) – technically, they could be derived from these two conditions by assuming that the “centrally planned” economy is on the balanced growth path from the very beginning (\( t = 0 \)).
where $\Delta$ is a linear function of $\tau_K$ described by equations (75) together with (71) – (73). Therefore, the rate of taxation $\tau_K$ which assures the equality $\bar{\varphi} = \bar{\varphi}^*$ can only be found numerically (there is no analytical formula for this rate). In order to do this, one needs to set all values of parameters, and then find such value of $\tau_K$ for which $\bar{\varphi}$ calculated in accordance with the equation (185) is at the same time an appropriate solution of the equation (146), i.e. it satisfies the equation (146) and is equal to $\bar{\varphi}^*$.

**Technical note.** The easiest procedure to calculate the replicating rate $\tau_K^{rep}$ is this: derive the function $\bar{\varphi}(\tau_K)$ by plugging equations (75) together with (71) – (73) into (185). Then, substitute the function $\bar{\varphi}(\tau_K)$ into the equation (146) and solve it numerically for $\tau_K$.

### 8. Calibration for Poland

There is not enough space for detailed exposition of calibration. Therefore, in table 2 we only present the **base set of parameters and initial values (endowments)** with a very concise explanation of sources and/or methods of calibration. Generally speaking, calibration was based on macroeconomic statistics for the period 2000-2013 published by the Eurostat, the National Bank of Poland, the Central Statistical Office of Poland, and the Kiel Institute for the World Economy, and existing research regarding OECD countries.
Table 2. The summary of calibration

<table>
<thead>
<tr>
<th>Parameters and endowments</th>
<th>Sources and methods of calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Technology</strong></td>
<td></td>
</tr>
<tr>
<td>( \alpha = \frac{2}{3}, \beta = \frac{1}{3} )</td>
<td>Review of empirical literature. Sources: Mankiw, Romer, Weil (1992), Bernanke, Gurkaynak (2002), Balisteri et al. (2002), Konishi, Nishiyama (2002), Willman (2002), Turnovsky (2009), Growiec (2012). Note: in our model capital ( K ) must be interpreted widely, as an aggregate of physical capital and human capital. See Turnovsky (2009) or Rebelo (1991).</td>
</tr>
<tr>
<td>( A = \frac{1}{3} )</td>
<td>OECD statistics and the database published by the Kiel Institute for the World Economy</td>
</tr>
<tr>
<td>( \delta = 4% )</td>
<td>Eurostat statistics and review of empirical literature. Sources: Easterly, Rebelo (1993), Turnovsky (2009), Cichy (2008), Manuelli, Seshadri (2005), Arrazola, de Hevia (2004). Note: Since capital ( K ) is interpreted widely, as the sum of physical capital and human capital, this depreciation rate is the average of the two depreciation rates: of human capital (about 1.5%) and physical capital (about 6.5%).</td>
</tr>
<tr>
<td>( \kappa = 0.27 )</td>
<td>Review of empirical and theoretical literature. Sources: Turnovsky (1999) and (2004), Park, Philippopoulos (2004), Dhont, Heylen (2009)</td>
</tr>
<tr>
<td>( \rho = 0.04 )</td>
<td>Metanalysis by Nijkamp, Percoco (2006) of 42 previous analyses; European Commission (2002)</td>
</tr>
<tr>
<td>( \gamma = -1 )</td>
<td>Huge metanalysis by Havranek et al. (2013) of 169 previous analyses</td>
</tr>
<tr>
<td><strong>Population</strong></td>
<td></td>
</tr>
<tr>
<td>( n = 0% )</td>
<td>Demographic forecasts for Poland published by the Central Statistical Office of Poland</td>
</tr>
<tr>
<td><strong>Fiscal policy</strong></td>
<td></td>
</tr>
<tr>
<td>( \sigma_c = 28.7% )</td>
<td>According to Eurostat, during the period 2000-2013 public consumption share of GDP in Poland amounted to an average of 18.1%, while private consumption was the 62.9% of GDP. Hence ( \sigma_c = 18.1% / 62.9% = 28.7% ).</td>
</tr>
<tr>
<td>( \xi = 4.8% )</td>
<td>The average deficit of the public sector in Poland during the period 2000-2012 (according to Eurostat methodology).</td>
</tr>
<tr>
<td>( \omega = 0.4 )</td>
<td>The average share of foreign debt in public debt in Poland during the period 2000-2012</td>
</tr>
<tr>
<td>( \tau_k = 21.2% )</td>
<td>Eurostat statistics: the average taxation rates (implicit tax rates) in Poland in the period 2000-2010 (the latest available data).</td>
</tr>
<tr>
<td>( \tau_L = 32.8% )</td>
<td>We have tried to calculate this rate based on the balance of payments for Poland, but the results turned out to vary wildly year-to-year. In the end we assumed the average value between ( \tau_k ) and ( \tau_D ).</td>
</tr>
<tr>
<td>( \tau_D = 19% )</td>
<td></td>
</tr>
<tr>
<td>( \tau_c = 19.4% )</td>
<td></td>
</tr>
<tr>
<td>( \tau_Z = 20% )</td>
<td></td>
</tr>
<tr>
<td>( \theta = 3.0% )</td>
<td>The average inflation in Poland in the period 2001-2012 measured by the consumer price index (HICP, according to Eurostat)</td>
</tr>
</tbody>
</table>
The interest rates: base interest rates and risk premiums

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_D = 1.85% )</td>
<td>Calibrated so as to be consistent with the (real) average 10-year Treasury bond yields in Poland in 2001-2012 (equal to 3.29%), the average public debt-to-GDP ratio (48.1%) and ( p_D = 0.03 ), i.e. calculated from the equation: ( 3.29% = \varepsilon_D + 0.03 \cdot 48.1% )</td>
</tr>
<tr>
<td>( \varepsilon_Z = 3.37% )</td>
<td>First, the average real cost of foreign borrowing to private sector in Poland was calculated based on the balance of payments statistics. In the period 2000-2012 it was on average 5.99%. Second, it was “disaggregated” into the base interest rate and the risk premium so as to turn out identical proportions as in the public sector, i.e. by assuming that ( \varepsilon_Z / \varepsilon_D = p_Z / p_D = 5.99 / 3.29 ).</td>
</tr>
<tr>
<td>( p_Z = 0.05 )</td>
<td></td>
</tr>
</tbody>
</table>

The initial values (endowments)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{z}_0 = 59.5% )</td>
<td>Statistical data for Poland published by the National Bank of Poland (NBP): net international investment position (NIIP) of the private sector and the public sector in 2012.</td>
</tr>
<tr>
<td>( d_{F0} = 29.2% )</td>
<td>The difference between the public debt (source: the NBP) and the NIIP of the public sector in 2012.</td>
</tr>
<tr>
<td>( d_{P0} = 26.4% )</td>
<td>The initial stock of capital per capita is set arbitrarily (as the numeraire); 300 is convenient because it yields ( y_0 = 100 ), so that values of other variables at ( t = 0 ) represent not only levels, but also ratios to GDP.</td>
</tr>
<tr>
<td>( k_0 = 300 )</td>
<td></td>
</tr>
</tbody>
</table>

9. The base scenario

9.1. The steady state (the balanced growth path)

Table 5.7 contains the calculation results obtained with the base set of parameters and initial values. The table presents only the steady state. The transitory dynamics, that is trajectories of selected variables are presented in the appendix (section A6).

Table 3. The balanced growth path in the base scenario.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Decentralized economy</th>
<th>Benevolent social planner</th>
</tr>
</thead>
<tbody>
<tr>
<td>The balanced growth rate (BGR)</td>
<td>( \bar{\varphi} )</td>
<td>1.274%</td>
</tr>
<tr>
<td>Private consumption (% of GDP)</td>
<td>( \bar{c} )</td>
<td>58.66%</td>
</tr>
<tr>
<td>Initial private consumption (% of GDP)</td>
<td>( \bar{c}_0 )</td>
<td>56.99%</td>
</tr>
<tr>
<td>Investment (% of GDP)</td>
<td>( \bar{i} )</td>
<td>15.82%</td>
</tr>
<tr>
<td>The real interest rate on private foreign debt</td>
<td>( \bar{r}_Z )</td>
<td>5.24%</td>
</tr>
<tr>
<td>Private foreign debt (% of GDP)</td>
<td>( \bar{z} )</td>
<td>37.46%</td>
</tr>
<tr>
<td>The real interest rate on public debt</td>
<td>( \bar{r}_D )</td>
<td>5.22%</td>
</tr>
</tbody>
</table>
You might want to compare the decentralized economy with the *first-best* equilibrium. The most important difference is the size of investment and consumption. The social planner spends for investment much more (27%) than the representative agent in the decentralized economy (16%). Thanks to this significant rate of capital formation the social planner is able to achieve and maintain a very high growth rate of GDP of 4.9% per year. For comparison, in the decentralized economy firms invest less than 16% of GDP, and so on the balanced growth path the GDP grows at less than 1.3%.

Interestingly, in the “centrally planned” economy the private sector borrows from abroad much more than in the decentralized economy. This is due to the fact that the social planner takes into account positive externalities of capital accumulation, which increases significantly the sense (the profitability) of borrowing abroad in order to invest. This is partly offset by another externality – this time negative – the social planner knows that borrowing more from abroad raises the cost of borrowing (the interest rate). Nevertheless, the net effect is positive, resulting in a higher foreign debt and higher interest rate in the “centrally planned” economy.

Conversely, the public sector in the “centrally planned” economy borrows less (61% of GDP) than in the decentralized economy (112%). This is mainly due to the significant difference in the rate of GDP growth, but also in part due to the fact that the social planner takes into account the negative effects of the external indebtedness of the public sector (an increase in the cost of servicing public debt), which clearly translates into a lower tendency for the government to run into debt.

Admittedly, the share of private consumption in GDP along the balanced growth path is lower in the “centrally planned” economy, nevertheless, thanks to the much faster growth of GDP, the social planner provides much higher welfare (utility of the flow of consumption): $\Omega^* > \Omega$. Figure 2 presents the trajectories of consumption per capita in both types of economies. After a brief period of sacrifices (about 6 years), during which the per capita

<table>
<thead>
<tr>
<th></th>
<th>$\bar{d}_F$</th>
<th>$\bar{d}_F^*$</th>
<th>$\bar{d}_D$</th>
<th>$\bar{d}_D^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foreign debt of the</td>
<td>44.9%</td>
<td>24.3%</td>
<td>44.9%</td>
<td>24.3%</td>
</tr>
<tr>
<td>government (% of GDP)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Domestic debt of the</td>
<td>67.4%</td>
<td>36.4%</td>
<td>67.4%</td>
<td>36.4%</td>
</tr>
<tr>
<td>government (% of GDP)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total public debt</td>
<td>112.3%</td>
<td>60.7%</td>
<td>112.3%</td>
<td>60.7%</td>
</tr>
<tr>
<td>(% of GDP)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare (utility)</td>
<td>$\Omega$</td>
<td>$\Omega^*$</td>
<td>$\Omega$</td>
<td>$\Omega^*$</td>
</tr>
<tr>
<td>Lost consumption</td>
<td>24.16%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>indicator (LCI)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: own calculations.
consumption in the “centrally planned” economy is lower than in the decentralized economy, there begins an infinitely long period of prosperity with higher consumption.

Fig. 2. Trajectories of consumption per capita in the base scenario.

For mathematical reasons, the values of utility functions $\Omega^*$ and $\Omega$ cannot be easily compared, for example, by calculating the difference or quotient. Therefore, in order to obtain an intuitive view of the welfare difference between the two types of economies, let us define the lost consumption indicator (LCI) that states by what percentage higher both types of consumption would have to be to obtain the same welfare (utility) as in the first-best equilibrium. It’s easy to show that in our model the LCI can be calculated as follows:

$$LCI = \left( \frac{\Omega^*}{\Omega} \right)^{\frac{1}{1+\alpha}} - 1.$$  \hspace{1cm} (186)

In the base scenario, LCI is equal to about 24%. Therefore, the consumption flow in the decentralized economy would have to be increased by approximately 24% (in every moment of the infinite time horizon) to obtain the first-best level of utility. This number can be roughly interpreted as a welfare cost of all externalities that economic agents fail to internalize.

9.2. A few words of caution

The results presented in table 3 deviate from the actual statistics reported for the period 2000-2013, on the basis of which the model has been calibrated. It does not mean, however, that the calibration was wrong: if you compare these results with statistical data regarding, for example, the rate of growth of GDP, the rate of investment, private consumption, etc., it is clear that the baseline scenario for the decentralized economy differs from actual data in minus, whereas the “centrally planned” economy differs in plus. For example, in the period...
2000-2013 Polish GDP grew on average at a rate of 3.7%, while in the base scenario we have 1.3% for the decentralized economy and 4.9% for social planner. This may mean that, in fact, economic agents in Poland have been internalizing a substantial part of externalities “by themselves”. The comparison of GDP growth rates suggests that they might have internalized as much as 2/3 of externalities, because we have: \[
\frac{3.7 - 1.3}{4.9 - 1.3} = \frac{2.4}{3.6} = \frac{2}{3}.
\] Therefore, when interpreting the results of all our simulations, we have to keep in mind that they represent extreme cases: the decentralized economy where agents take into account 0% of the external effects and the “centrally planned” economy where they take into account 100% of the external effects. Probably the truth (the real world) lies somewhere between these extremes, and it is plausible that the reality is much closer to the benevolent social planner than to the decentralized economy.

For example, above we have estimated the welfare cost of externalities at 24% (the LCI in the base scenario). If we take these words of caution seriously, we shall say, that in the real world this welfare cost is only about 8%.

### 10. Replication of the balanced-growth path (partial replication)

Now we determine the values of the three parameters of fiscal policy that allow replication of the first-best steady state in the baseline scenario. Values of two parameters can be calculated directly from (177) and (178). However, the value of the rate of taxation of income from capital \( \tau_K^{rep} \) must be calculated numerically, in accordance with the procedure described in section 4.4. We get the following values:

\[
\sigma_C^{rep} = \kappa = 0.27 ,
\]

\[
\tau_Z^{rep} = \frac{\bar{r}_Z - \delta_Z}{\bar{r}_Z + \delta} = 48.01\% ,
\]

\[
\tau_K^{rep} = -66.89\%.
\]

Partial replication requires strongly negative tax rate on capital income (in practice, subsidizing investment in productive capital), and in parallel high positive tax rate on interest paid by domestic borrowers to foreign lenders (to discourage foreign financing). These conclusions are similar to Turnovsky’s. Table 4 presents the steady-state values of selected variables in two scenarios: the first-best steady state in the baseline scenario (obviously the same as in table 3), and the steady state in the decentralized economy replicating the first-best steady state.
### Table 4. The replication of the first-best steady state.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Benevolent social planner in the base scenario</th>
<th>Decentralized economy replicating the first-best steady state</th>
</tr>
</thead>
<tbody>
<tr>
<td>The balanced growth rate (BGR)</td>
<td>( \bar{\phi} * ) 4.904%</td>
<td>( \bar{\phi} ) 4.904%</td>
</tr>
<tr>
<td>Private consumption (% of GDP)</td>
<td>( \bar{c} * ) 41.74%</td>
<td>( \bar{c} ) 41.74%</td>
</tr>
<tr>
<td>Initial private consumption (% of GDP)</td>
<td>( \bar{c}_0 * ) 51.10%</td>
<td>( \bar{c}_0 ) 58.59%</td>
</tr>
<tr>
<td>Investment (% of GDP)</td>
<td>( \bar{r} * ) 26.71%</td>
<td>( \bar{l} ) 26.71%</td>
</tr>
<tr>
<td>The real interest rate on private foreign debt</td>
<td>( \bar{r}_z * ) 9.25%</td>
<td>( \bar{r}_z ) 9.25%</td>
</tr>
<tr>
<td>Private foreign debt (% of GDP)</td>
<td>( \bar{z} * ) 117.63%</td>
<td>( \bar{z} ) 117.63%</td>
</tr>
<tr>
<td>The real interest rate on public debt</td>
<td>( \bar{r}_D * ) 3.67%</td>
<td>( \bar{r}_D ) 3.67%</td>
</tr>
<tr>
<td>Foreign debt of the government (% of GDP)</td>
<td>( \bar{d}_F * ) 24.3%</td>
<td>( \bar{d}_F ) 24.3%</td>
</tr>
<tr>
<td>Domestic debt of the government (% of GDP)</td>
<td>( \bar{d}_D * ) 36.4%</td>
<td>( \bar{d}_D ) 36.4%</td>
</tr>
<tr>
<td>Total public debt (in % of GDP)</td>
<td>( \bar{d} * ) 60.7%</td>
<td>( \bar{d} ) 60.7%</td>
</tr>
<tr>
<td>Welfare (utility)</td>
<td>( \Omega * ) (-0.1090)</td>
<td>( \Omega ) (-0.1102)</td>
</tr>
<tr>
<td>Lost consumption indicator (LCI)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Source: own calculations.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Recall (we wrote about this in section 7) that the replication of the steady state only (partial replication) is not synonymous with full replication of entire trajectories, corresponding to the “centrally planned” economy. Therefore the parameter values (187)–(189) only bring decentralized economy closer to the first-best equilibrium, but not necessarily exactly to the first-best solution. However, notice that according to our calculations partial replication reduces the LCI from 24.2% (base scenario) to just 0.8%. Full replication would, of course reduce it to zero, but it’s probably not worth the effort. Recall that full replication would require an application of tax rates \( \tau_{z opt}(t) \) and \( \tau_{K opt}(t) \) that (continuously) change over time. (Moreover, calculating the trajectory \( \tau_{K opt}(t) \) is a complex numerical problem described at section 6). Even we could determine both trajectories, it’s hard to imagine putting them in life. Updating tax rates even once a year would be cumbersome and expensive (e.g. the menu costs). It is not impossible that the total economic costs would exceed these 0.84% of lost welfare.

To summarize, partial replication seems to be almost equivalent to the full replication, and therefore we will not go beyond that.
11. Numerical sensitivity analysis

In this section we investigate the relationships between selected parameters of fiscal policy and the balanced growth paths in the two types of economy. Every time just one selected parameter is modified, whereas all remaining parameters and the initial state are taken from the base set (table 2).

11.1. Budget deficit

First, we have investigated various levels of budget deficit $\xi$: from 0% to 5.5% of GDP. Figures 3 – 12 present the steady-state values of important variables in both types of economy as functions of the level of budget deficit. The solid line represents the “centrally planned” economy, and the dashed line – the decentralized economy.

![Fig. 3. Investment (% of GDP)](image1)

![Fig. 4. The growth rate of GDP per capita](image2)

![Fig. 5. Private foreign debt (% of GDP)](image3)

![Fig. 6. The interest rate on private foreign debt.](image4)
First, let's look at the decentralized economy. The lower the deficit rate $\xi$ is, the higher the rate of investment and the rate of growth of GDP along the balanced growth path. In the extreme case of zero budget deficit the GDP grows at 1.48% and the public debt is reduced to zero. On the other hand, the lower the budget deficit, the higher is the level of foreign debt of the private sector. In the extreme case of $\xi = 0$ private foreign debt is reduced to 45% of...
GDP (compared to current level of almost 60%). From the point of view of welfare, the best situation would be the total elimination of the budget deficit.

In the economy governed by the benevolent social planner we observe a totally different situation. Here the rate of investment and the GDP rate of growth are positively dependent on the deficit-to-GDP ratio, though this relationship is very mild. An increase of the deficit to 5.5 percent of GDP, leads to the public debt stabilizing at around 69% of the GDP which corresponds to the real interest rate on public debt equal to 3.9%. The external debt of the private sector rises and stabilizes at a very high level of almost 120% of GDP, and the real interest rate on private debt rises to 9.3%. Despite the high growth rate of GDP, from the point of view of welfare such a situation is far from optimal. Utility measured by the value of $\Omega^*$ reaches a maximum for $\xi = 3.98\%$, which means the deficit-to-GDP level of almost 1 percentage point lower than in the base period 2000–2012.

Supposedly, these conclusions heavily depend on the risk premium parameter $p_D$ associated with foreign debt of the public sector. Intuitively, the higher the risk premium is, the more dangerous (and costly) to the economy it gets to borrow from abroad, so the optimal budget deficit should be the lower, the higher the risk premium $p_D$. Indeed, this intuition is confirmed by calculations we made for the following values of $p_D$: 0.03 (baseline scenario), 0.06, 0.1 and 0.2. Figure 13 presents the relationship between first-best level of welfare $\Omega^*$ achieved by the central planner and the value of the parameter $\xi$ (the ratio of public deficit to GDP) calculated for these four values of the risk premium $p_D$. The upper line corresponds to the base scenario ($p_D = 0.03$), and lower lines correspond to higher values of the risk premium. Two important conclusions follow. First, the higher the risk premium, the lower the maximum welfare the economy is able to achieve for any chosen level of public deficit (except for the extreme case of zero deficit). Second, the optimal debt-to-GDP ratio ($\xi$) decreases with the increase in the risk premium. In the base scenario, in which $p_D = 0.03$, the optimal value of $\xi$ is 3.98%. With twice the risk premium (equal to 0.06) the optimal value of $\xi$ is only 2.2%. If the risk premium is equal to 0.1 or more, the optimal deficit is zero, that is balanced budget (on average, in the long term).
11.2. The share of foreign lenders in the public debt

If we consider the share of foreign creditors in the public debt ($\omega$) to be a decision parameter of the government (or at least a parameter under the influence of the government), we may be interested in the consequences of different choices. The figures 14 – 23 illustrate the steady state in both types of economy as functions of the level of $\omega$ ranging from 0 to 55%. As before, solid line represents the “centrally planned” economy, whereas the dashed line – the decentralized economy.

Fig. 13. The relationship $\Omega^*(\zeta)$ for four different values of the risk premium $p_D$

Source: own calculations.

Fig. 14. Investment (% of GDP)
Fig. 15. The growth rate of GDP per capita
Fig. 16. Private foreign debt (% of GDP)

Fig. 17. The interest rate on private foreign debt.

Fig. 18. Public debt (% of GDP)

Fig. 19. The interest rate on public debt.

Fig. 20. Foreign public debt (% of GDP)

Fig. 21. Domestic public debt (% of GDP)
Fig. 22. Private consumption (% of GDP)

Fig. 23. Welfare (utility)

The higher the value of \( \omega \), the higher the balanced growth rate in both types of economies. However, it does not necessarily mean a higher welfare. In particular, in the “centrally planned” economy welfare measured by \( \Omega^* \) begins to drop rapidly when \( \omega \) increases above 50%. The benevolent social planner delivers maximum welfare for \( \omega = 33.4\% \). The situation is different in the decentralized economy: the higher the value of \( \omega \), the higher is the level of utility \( \Omega \).

Interestingly, we observe a negative relationship between foreign lenders share in public debt and the steady-state public debt-to-GDP ratio. This relationship is clear in the decentralized economy (fig. 18), but very mild in the “centrally planned” economy. The foreign debt of the private sector also depends on \( \omega \), though in the opposite direction: the larger the percentage of public debt in the hands of foreign creditors, the higher the steady-state value of private-foreign-debt-to-GDP.

We may suppose that the optimal value of \( \omega \) heavily depends on the risk premium parameter \( p_D \) associated with foreign debt of the public sector. As mentioned in the previous section: the higher the risk premium is, the more dangerous (and costly) to the economy it is to borrow from abroad, so – intuitively – the optimal value of \( \omega \) should be falling with an increase in the risk premium \( p_D \). This intuition is confirmed by our calculations for the following values of \( p_D \): 0.03 (baseline scenario), 0.06, 0.1 and 0.2. Figure 24 presents the relationship between first-best level of welfare \( \Omega^* \) achieved by the central planner and the value of the parameter \( \omega \) (the share of foreign creditors in public debt) calculated for these four values of the parameter \( p_D \) (all other parameters remain the same as in the base scenario). The upper line corresponds to the base scenario (\( p_D = 0.03 \)), and other lines
correspond to higher values of the risk premium. Two conclusions follow. First, the higher the risk premium is, the lower the maximum welfare the economy is able to achieve for any chosen level of $\omega$. Put simply, an increase in the risk premium is always undesirable, because it reduces the maximum obtainable welfare. Second, the optimal value of $\omega$ decreases with the increase in the risk premium. In the base scenario, in which $p_D = 0.03$, the optimal value of $\omega$ is 33.4%. With twice the risk premium (equal to 0.06) the optimal value of $\omega$ is only 17.1%, whereas for the risk premium equal to 0.1 it is a mere 4.3%. For sufficiently high risk premium an entire government debt should be the internal debt.

Fig. 24. The relationship $\Omega^*(\omega)$ for four different values of the risk premium $p_D$

Source: own calculations.

11.3. The rate of inflation

Analogous simulations were done for a wide range of possible values of inflation, from 0% up to 10% (annually), assuming that all other parameters are the same as in the base scenario. The results are presented on figures 25 – 34. As before, the solid lines correspond to the benevolent social planner, and the dashed lines – the decentralized economy.

Fig. 25. Investment (% of GDP)  Fig. 26. The growth rate of GDP per capita
Fig. 27. Private foreign debt (% of GDP)

Fig. 28. The interest rate on private foreign debt.

Fig. 29. Public debt (% of GDP)

Fig. 30. The interest rate on public debt.

Fig. 31. Foreign public debt (% of GDP)

Fig. 32. Domestic public debt (% of GDP)
The level of inflation negatively influences the steady-state GDP growth rate, but only in the “centrally planned” economy. The sensitivity to inflation rate, however, is very small – the reduction of inflation from 10% to 0% (annually) increases long-term growth by a mere 0.07 p.p. (from 4.87% to 4.94%). Inflation has much stronger impact on the government's debt indicators, reflecting the importance of inflation tax. At zero inflation, public debt in the “centrally planned” economy stabilizes at 97% of GDP; at inflation equal to 3% – at the rate of 60%; at 5% inflation – at 48%. Naturally, this is mirrored in the real interest rate on public debt, which decreases with an increase in inflation. Remember, that does not necessarily mean the decline in nominal interest rates. For example, with zero inflation the steady-state value of the nominal interest rate on public debt is 4.8%, whereas for 5% inflation it is 8.3%. In the decentralized economy these relationships are the same as for direction, but much stronger.

Inflation also affects the foreign debt of the private sector, which decreases with an increase in inflation. This relationship is very mild in the “centrally planned” economy, but quite strong in the decentralized economy. Similarly, there exists a positive relationship between inflation and the steady-state value of private consumption-to-GDP ratio. Again, this is clearly visible in the decentralized economy, but almost unnoticeable under the governance of the benevolent social planner.

All these relationships are in a complex way translating into consumer welfare. In the decentralized economy the optimal inflation is zero, whereas for the social planner optimal inflation is equal to 2.43%. The impact of inflation on welfare in the “centrally planned” economy, however, is small.
12. The quest for the global optimum

At this point we leave the base scenario and we'll try to answer a more general question. We will search for such values of fiscal policy parameters that maximize welfare of the society measured by $\Omega^*$. It follows from the properties of the model that the procedure of searching for optimal parameters can be divided into two stages: first, we will try to find the optimal parameter values for the “centrally planned” economy. Second, we will solve the problem of (partial) replication finding the appropriate values of the two tax rates $\tau_K$ and $\tau_Z$.

To have a fixed point of reference, we will compare the results obtained in any new scenario with the first-best solution in the base scenario. We will use the indicator of lost consumption (LCI), however, in the opposite direction: we will calculate by what percentage should the consumption flow be increased in the first-best base scenario in order to deliver the same level of welfare as in the analyzed (new) scenario. To emphasize the difference, we call this measure the gained consumption indicator (GCI). Formally, the GCI can be defined as follows.

**Definition 3.** Let $c(t)$ be the trajectory of consumption per capita in some scenario, and $\Omega$ is the value of the corresponding welfare (utility). Let $c^B(t)$ be the trajectory of consumption per capita in the “centrally planned” economy in the base scenario, whereas $\Omega^B = -0.1090433$ is the value to the corresponding welfare (utility). The gained consumption indicator (GCI) is such a number $x$, for which the equality $\Omega = \Omega^B$ is satisfied, where $\Omega$ is the value of welfare (utility) corresponding to the following trajectory of consumption: $c(t) = (1 + x) \cdot c^B(t)$.

It’s straightforward to show that the GCI in our model can be calculated as follows:

$$GCI = \left( \frac{\Omega}{\Omega^B} \right)^{\frac{1}{\gamma(1+\kappa)}} - 1.$$  \hspace{1cm} (190)

**Stage 1. The optimal fiscal policy in the “centrally planned” economy.**

At this stage, looking at the economy from the position of the benevolent social planner, we are searching for the optimal values of all parameters under the control of the government (and the central bank), namely: $\sigma_c, \xi, \omega, \theta$. First, notice that utility maximization requires the equality: $\sigma_c = \kappa$, and this condition is independent of the values of other parameters.
Secondly, taking into account the results of the analyses presented in section 11 we conclude that maximizing welfare in the “centrally planned” economy requires that:

a) the budget deficit $\xi$ was equal to about 3.98% (cf. section 11.1),

b) the share of foreign lenders in the public debt $\omega$ was equal to about 33.44% (cf. section 11.2).

c) inflation rate was equal to about 2.4% (cf. section 11.3).

Of course, this is not global optimum yet – these are only the results of partial optimization: searching for the optimal value of a single parameter, keeping all others same as in the base scenario. In order to find the global optimum (optimal values of these three parameters at the same time) we need to design and numerically solve a complex mathematical problem. In the simplest possible form it could be an algorithm solving one optimal control problem of the social planner for each point of a three-dimensional grid (with some chosen accuracy), because we have three parameters. In what follows we present the results of a slightly simplified version of this procedure. We have applied a two-dimensional grid, obtained by presetting inflation, and then applying an algorithm for a grid of two parameters: $\xi$ and $\omega$. Paradoxically, this simplified procedure stays closer to reality than the more general one. It imitates the fact that the independent central bank chooses his (long-term) inflation target, and the government has to adjust its fiscal policy.

Let us consider four different values of inflation: $\vartheta = 1.5\%$, $\vartheta = 2.0\%$, $\vartheta = 2.5\%$, which is exactly equal to current inflation target of the National Bank of Poland, and $\vartheta = 3.0\%$, which corresponds to the actual average level of inflation in Poland in the years 2000-2012. At each of these levels of inflation we have been numerically searching for such values of $\xi$ and $\omega$, that maximize welfare $\Omega^*$ in the model of the “centrally planned” economy. The table 5 presents the results. Interestingly, for all levels of inflation we have obtained interior solutions (rather than corner solutions) in all cases, except for the case of the lowest inflation, where the optimal value of $\omega$ is above 100%, so it is a corner solution (more precisely, an edge solution). The results are presented with an accuracy of 1 p.p. in the case of $\omega$, and 0.01 p.p. in the case of $\xi$. 

[Table 5 is not shown in the image]
Table 5. The optimal deficit-to-GDP ratio and the share of foreign lenders in the public debt at various inflation targets for the benevolent social planner.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Inflation target (%)</th>
<th>$\xi$</th>
<th>$\omega$</th>
<th>The maximum value of welfare</th>
<th>Gained consumption indicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.5</td>
<td>2.31</td>
<td>100</td>
<td>-0.1080743</td>
<td>0.477%</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>2.51</td>
<td>88</td>
<td>-0.1084092</td>
<td>0.311%</td>
</tr>
<tr>
<td>C</td>
<td>2.5</td>
<td>2.97</td>
<td>67</td>
<td>-0.1086487</td>
<td>0.193%</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>3.43</td>
<td>54</td>
<td>-0.1088144</td>
<td>0.112%</td>
</tr>
</tbody>
</table>

Source: own calculations.

For a more complete picture, the following figure presents the relationship $\Omega^*(\xi)$ in four scenarios from the table above. In each case both inflation and $\omega$ have the values indicated in table 5.

Fig. 35. Welfare in the “centrally planned” economy as a function of budget deficit (as % of GDP) at four different levels of inflation

Several conclusions follow from these calculations. First of all, the inflation target should be as low as possible, because it allows the social planner to assure the highest level of welfare to the society.

If the central bank pursues the inflation target $\vartheta = 3\%$ (which corresponds to the average inflation in Poland in the past decade) welfare $\Omega^*$ reaches a maximum, when the budget deficit is equal to 3.4% of GDP, while foreign investors hold about 54% of public debt. However, compared to the baseline scenario, the gain is very small – the GCI is only 0.1%.

The lower the inflation target is, the lower should be the budget deficit, and at the same time, the larger the share of foreign lenders should be in public debt. At the lowest level of
inflation \( \theta = 1.5\% \) the optimal deficit amounts to 2.3\% of GDP and the optimum foreign lenders share in public debt is 100\% (a corner solution). This choice allows the central planner to gain nearly 0.5\% of the consumption compared to the baseline scenario. We call this variant **scenario A**.

It is worth noting that the values of GCI are relatively small, which suggests that the size of the budget deficit and the structure of its financing, as well as inflation are far less important than the choice of tax rates for the replication of the “centrally planned” economy. We turn to this issue in the next stage.

**Stage 2. Partial replication of the first-best solution in scenario A**

Just like we did earlier, the two tax rates, \( \tau_K \) and \( \tau_Z \), must be adjusted in such a way that the decentralized economy replicates the first-best steady state in scenario A. According to the proposition 3, we compute the rate \( \tau_Z^{rep} \) directly from the formula (178), whereas the value of \( \tau_K^{rep} \) is calculated numerically, in accordance with the procedure described in section 4.4. We get:

\[
\tau_Z^{rep} = \frac{\bar{r}_Z - \bar{E}_Z}{\bar{r}_Z + \theta} = 54.84\% ,
\]

\[
\tau_K^{rep} = -63.40\% .
\]

Table 6 presents the steady states in two scenarios – the “centrally planned” economy in the scenario A and the decentralized economy replicating the steady state of scenario A.
Table 6. Partial replication of the scenario A.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Scenario A (first-best steady state)</th>
<th>Decentralized economy replicating scenario A</th>
</tr>
</thead>
<tbody>
<tr>
<td>The balanced growth rate (BGR)</td>
<td>$\bar{\varphi}^*$ 4.93%</td>
<td>$\bar{\varphi}$ 4.93%</td>
</tr>
<tr>
<td>Private consumption (% of GDP)</td>
<td>$\overline{c}^*$ 41.90%</td>
<td>$\overline{c}$ 41.90%</td>
</tr>
<tr>
<td>Initial private consumption (% of GDP)</td>
<td>$\xi_0^*$ 50.96%</td>
<td>$\xi_0$ 49.10%</td>
</tr>
<tr>
<td>Investment (% of GDP)</td>
<td>$\overline{t}^*$ 26.80%</td>
<td>$\overline{t}$ 26.80%</td>
</tr>
<tr>
<td>The real interest rate on private foreign debt</td>
<td>$\overline{r}_z^*$ 9.28%</td>
<td>$\overline{r}_z$ 9.28%</td>
</tr>
<tr>
<td>Private foreign debt (% of GDP)</td>
<td>$\overline{z}^*$ 118.3%</td>
<td>$\overline{z}$ 118.3%</td>
</tr>
<tr>
<td>The real interest rate on public debt</td>
<td>$\overline{r}_D^*$ 2.93%</td>
<td>$\overline{r}_D$ 2.93%</td>
</tr>
<tr>
<td>Foreign debt of the government (% of GDP)</td>
<td>$\overline{d}_F^*$ 35.9%</td>
<td>$\overline{d}_F$ 35.9%</td>
</tr>
<tr>
<td>Domestic debt of the government (% of GDP)</td>
<td>$\overline{d}_D^*$ 0%</td>
<td>$\overline{d}_D$ 0%</td>
</tr>
<tr>
<td>Total public debt (in % of GDP)</td>
<td>$\overline{d}^*$ 35.9%</td>
<td>$\overline{d}$ 35.9%</td>
</tr>
<tr>
<td>Welfare (utility)</td>
<td>$\Omega^*$ -0.1081</td>
<td>$\Omega$ -0.1083</td>
</tr>
</tbody>
</table>

Source: own calculations.

Recall that the replication of the steady state only (partial replication) is not synonymous with full replication of entire trajectories converging to the balanced growth path. Therefore the parameter values (191)–(192) only bring decentralized economy closer to the first-best trajectories, but not necessarily exactly to the first-best solution. However, partial replication reduces the LCI from 20.0% (the decentralized economy in scenario A\textsuperscript{18}) to just 0.79%. Full replication would reduce it to zero, but – as we argued in section 10 – it’s probably not worth the effort.

‘Almost optimal’ tax rates

The replication of the first-best steady state in scenario A requires substantial subsidies to capital (investment) coupled with very high positive tax rate on interest paid by private sector to foreign lenders (to discourage foreign financing). This kind of policy might be impossible to implement in practice, for many reasons\textsuperscript{19}: political, ethical, moral, etc. Therefore let us shortly discuss slightly more realistic scenarios. Assume that the government may consider

\textsuperscript{18} Detailed results are not presented at this paper.

\textsuperscript{19} This could be an example of the trade-off between efficiency and equity.
lowering the tax rate on capital income to 10% or zero, or – at best – introduce small subsidies amounting to 10% or 20%. Assuming that public consumption is at the optimal level, i.e. \( \sigma_c = \sigma_c^{opt} = \kappa = 0.27 \), for each of these hypothetical levels of tax rate on capital income \( \tau_k \) we have calculated the optimal (welfare maximizing) tax rate on interest paid by private sector to foreign lenders. The results are presented in the next table.

Table 7. ‘Almost optimal’ tax rates compared to scenario A

<table>
<thead>
<tr>
<th>assumed tax rate ( \tau_k )</th>
<th>optimal tax rate ( \tau_z )</th>
<th>LCI in comparison to scenario A</th>
<th>balanced growth rate (( \bar{\rho} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20%</td>
<td>13.4%</td>
<td>2.4%</td>
<td>3.29%</td>
</tr>
<tr>
<td>-10%</td>
<td>3.1%</td>
<td>4.7%</td>
<td>2.88%</td>
</tr>
<tr>
<td>0</td>
<td>-8.2%</td>
<td>7.7%</td>
<td>2.45%</td>
</tr>
<tr>
<td>10%</td>
<td>-20.9%</td>
<td>11.6%</td>
<td>2.00%</td>
</tr>
<tr>
<td>20%</td>
<td>-40.0%</td>
<td>16.6%</td>
<td>1.54%</td>
</tr>
</tbody>
</table>

Source: own calculations.

If we leave the capital tax rate at its current level (\( \tau_k = 20\% \)), then from the point of view of welfare it would be optimal to set the tax rate on interest paid by private sector to foreign lenders at the level of \( \tau_z = -40\% \). This means that the government should encourage the private sector to borrow capital abroad by subsidizing an inflow of foreign capital to Poland (in the form of direct and portfolio investment, and possibly also other types of capital inflow). The value of \( \tau_z = -40\% \) means that the compensation of foreign investors (e.g. profits transferred abroad, interest on bonds, loans, etc.) not only should not be taxed, but the government should increase these payments by an extra bonus of about 40%. This situation is, however, unfavorable from the point of view of welfare, because in comparison to the solution obtained by the benevolent social planner in scenario A the country is losing as much as 16.6% of consumption (LCI), and the growth rate of GDP is only 1.54%.

Without doubt, it would be far better to reverse the situation: set \( \tau_k = -20\% \), that is to encourage the private sector to invest by subsidizing income from capital, and at the same time discourage borrowing abroad by setting \( \tau_z = 13.4\% \). In that case the LCI in comparison to scenario A is only 2.4%, and the rate of GDP growth in the steady state is 3.29%.
13. Summary

13.1. Summary of theoretical results
Over time both types of economies are converging to their (separate) balanced growth paths with different rates of growth. In the decentralized economy the balanced growth rate (the BGR) is given by a simple mathematical formula. To the contrary, in the model of the benevolent social planner an analytical formula for the BGR does not exist – it can only be calculated numerically by finding a positive real root of the 5\textsuperscript{th} order polynomial equation.

Several conclusions regarding fiscal policy follow:

1. All income and consumption taxes are neutral for the benevolent social planner (they do not influence the solution of the model), but not for the decentralized economy.

2. The inflation tax is not neutral for the economy in the long run. Specifically, in the decentralized economy it does not influence the BGR – in that sense it is neutral for the “real” economy. However, it impacts the level of welfare measured by the value of the intertemporal utility function. On the other hand, for the benevolent social planner inflation is not neutral – it impacts all real variables including the BGR. The difficulty is that without numerical calculations it’s impossible to say anything not only about the magnitude of these relationships, but also about the sign.

3. All remaining parameters of fiscal policy also influence both solutions, so that trajectories of many variables and welfare depend on the share of public consumption in the GDP, the size of public deficit and the structure of public debt (the share of foreign creditors). However, since an analytical solution to the model of the benevolent social planner does not exist, an analysis of the relations between these parameters and the BGR and welfare requires numerical methods.

4. The social planner may induce individual economic agents to internalize all externalities by proper adjustment of fiscal policy. This is known in literature as the problem of replication (of the “centrally planned economy”, or of the first-best solution). In our model replication requires 3 instruments of fiscal policy: the share of public consumption in the GDP, the tax rate on interest paid by the private sector to foreign lenders, and the tax rate on capital income. The optimal value of the first of these parameters is constant over time, the optimal value of the second changes over time (we derive an analytical formula of the appropriate trajectory). The optimal value of the third changes over time and additional difficulty is the fact that there is no analytical formula for the optimal value of this parameter – not only for the whole trajectory (as a function of time), but even for the single value of this...
parameter in a selected moment of time. Therefore, from a practical point of view the replication is a pretty difficult task.

5. Though full replication of the trajectories generated by the social planner is a complex numerical problem, it’s possible to solve analytically a simplified problem of partial replication, i.e. the replication of the steady state only, so that over time the decentralized economy converges to the same balanced growth path as the social planner. The necessary and sufficient condition for this to happen is that the three above-mentioned parameters of fiscal policy be at certain optimal levels (constant over time). We derive analytical formulas for the first two of them and we show that the optimal value of the third (the tax rate on capital income) is a unique, feasible solution of the 5th order polynomial equation.

13.2. Summary of empirical results
We arrived at 3 main empirical conclusions for Poland.

1. The lower the inflation rate, the higher is welfare (as measured by the value of the intertemporal utility function), and so the inflation target of the National Bank of Poland (NBP) should be as low as possible.

2. The optimal (welfare maximizing) values of parameters of fiscal policy are dependent on inflation. We considered several scenarios corresponding to several levels of inflation. In the case of 3% inflation target (annually, which corresponds to the actual average inflation in Poland in the past decade) welfare reaches a maximum, when the budget deficit is 3.4% of GDP, while foreign investors hold about 54% of the public debt (we treat this share as an instrument of fiscal policy, assuming that the government can somehow control it). The lower the inflation is, the lower is the optimal level of the budget deficit, and at the same time, the larger the optimal share of foreign lenders in public debt. At the lowest considered level of inflation equal to 1.5% the optimal deficit amounts to 2.3% of GDP and the optimum share of foreign lenders in public debt is 100%.

3. If the NBP successfully targets 1.5% inflation, then the optimal tax rate on capital is –63.4%, whereas the optimal tax rate on interest on the external debt of the private sector is almost 55%. These values allow partial replication of the long-run balanced growth path (as defined above).

The results of our empirical analysis should be treated with caution for at least two reasons. First, although most parameters have been calibrated based on statistical data about the Polish economy, several important parameters do not have their counterparts in the
available data. Therefore, they have been calibrated on the basis of the average values observed in other OECD countries or on the basis of the so-called consensus – values that are widely accepted in the literature.

Secondly, we found that the obtained baseline scenario deviates from the actual statistics recorded in Poland in the period 2000-2013, on the basis of which the model was calibrated (see section 9.2). In particular, the baseline scenario for the decentralized economy differs from actual data in minus, whereas the “centrally planned” economy differs in plus. This may mean that, in reality, economic agents internalize a large part of externalities. By comparing the actual growth rate of GDP with values obtained in the baseline scenario, we found that it is likely that they internalize as much as 2/3 of the external effects. Therefore, if we would like to determine a scenario reflecting the actual economic situation in Poland (that could be even viewed as a forecast), we should take some kind of “weighted average” of the base scenarios for the two types of the economy. In other words, a realistic, reliable scenario (forecast) probably lies somewhere between the base scenario for the decentralized economy and the base scenario describing the “centrally planned” economy. It is highly likely that the actual economic processes correspond more to the “centrally planned” economy rather than to the decentralized one.

Because of these two objections, all results presented above should be considered with caution. This also applies to section 12 where we tried to determine the optimal values of fiscal policy parameters. With the emphasis we stress that values obtained in stage 1 deserve much more confidence than those obtained in stage 2. In stage 1 the optimal size of the public consumption, the budget deficit-to-GDP ratio and the share of foreign lenders in the public debt have been established on the basis of an analysis of the “centrally planned” economy only. So the optimal values obtained in stage 1 do not depend on what percentage of externalities economic agent internalize. As long as we neglect the first objection mentioned above, these optimal values can be regarded as the true first-best recipe for the fiscal policy.

The results of stage 2 are far more problematic and less trustworthy, because the obtained replicating values of the two tax rates crucially depend on what part of externalities agents internalize “by themselves”. Our results of stage 2 hinge on the assumption that they internalize nothing. However, if it is true that economic agents in Poland internalize a significant part of these effects – replicating tax rates are quite different, much less radical. Subsidies to capital income need not be so high, and, on the other hand, the taxation of interest income on private foreign debt can be much lower.
Appendix

A1. Rationale for the assumption (74)

Notice that $a_i < 0$, if:

$$(1 - \tau_k)\alpha > (1 - \omega)\xi. \quad (a.0)$$

(This condition is sufficient, but not necessary.) Let us estimate the value of the left-hand side, based on realistic empirical data. The rate of capital tax anywhere in the world is less than 50% (in many countries, much lower), while the share of capital in output is estimated at around $1/3$. Hence $(1 - \tau_k)\alpha > 1/6$. Therefore, the inequality (a.0) could be violated, only if $(1 - \omega)\xi > 1/6$, which means that the budget deficit (in % of GDP) multiplied by the share of foreign creditors in the public debt $(1 - \omega)$ exceeds $1/6$. This is not possible in the light of real-world data. Even in an extreme case of $(1 - \omega) = 1$, it would require the budget deficit to be above 16.7% of the GDP. In the long-run such level of deficit is never observed. To summarize, from an empirical point of view, it’s obvious that the condition (a.0) is satisfied, so that $a_i < 0$.

Note: In our model capital $K$ is defined broadly and includes both physical capital and human capital. (See Table 2). Therefore, in our calibration we apply $\alpha = 2/3$. Notice that with this value of $\alpha$ our justification for assumption (74) still holds, and is even stronger.

A2. Proof that transversality conditions (38ef) are satisfied if, and only if,

$$(1 + \tau_Z)R_Z^N > \bar{\theta} + \mathcal{G} + n, \quad (83)$$

1. Let’s start with the condition (38f).

$$\lim_{t \to \infty} e^{-(\rho - n)H} \lambda_2(t)k(t) = 0. \quad (38f)$$

Substituting $\dot{\lambda}_2(t) = q(t) \cdot \dot{\lambda}_1(t)$ this condition can be written in the equivalent form:

$$\lim_{t \to \infty} e^{-(\rho - n)H} \lambda_1(t)q(t)k(t) = 0. \quad (a.1)$$

From (46) and (47) it follows that the trajectory of capital is of the following form:

$$k(t) = k_0 e^{\int_0^t \left( \frac{q(s) - n - \delta}{Z} \right) ds} = k_0 e^{\int_0^t \left( \frac{q(s) - 1}{Z} \right) ds} \cdot e^{-(n + \delta)H}. \quad (a.2)$$

Meanwhile, (41) implies that

$$\dot{\lambda}_1 \lambda_1 = \rho - (1 + \tau_Z)R_Z(z) - \tau_Z \mathcal{G}, \quad (a.3)$$

where $R_Z(z) = \mathcal{E}_Z + p_Z z$. Hence the trajectory $\lambda_1(t)$ has the following form:
\[ \lambda_i(t) = \lambda_i(0) e^{(\rho - \tau \varrho) t} e^{-t(1 + \tau \varrho)} \int_0^t (r_x(z(s)) ds). \]  

(a.4)

Having regard to (a.2) and (a.4) condition (a.1) can be written in the form:

\[ \lambda_i(0) k_0 \cdot \lim_{t \to \infty} \left\{ q(t) e^{-(\delta + \tau \varrho) t} \cdot e^{-t(1 + \tau \varrho)} \int_0^t \left[ r_x(z(s)) ds \right] e^{\int_0^t \left[ \frac{q(s)}{x} \right] ds} \right\} = 0 , \]

which is equivalent to

\[ \lambda_i(0) k_0 \cdot \lim_{t \to \infty} \left\{ q(t) e^{-t(1 + \tau \varrho)} \cdot e^{-t(1 + \tau \varrho)} \int_0^t \left[ r_x(z(s)) ds \right] e^{\int_0^t \left[ \frac{q(s)}{x} \right] ds} \right\} = 0 . \]

(a.6)

In order to examine this condition we need to know the trajectories of variables \( z(s) \) and \( q(s) \). Because the model is non-linear, we will use approximate trajectories obtained by solving a linearized model. From equations (87) we know that

\[ z(t) = \bar{z} + s_1 e^{\varrho^i} v_2^1 + s_2 e^{\varrho^2} v_2^2 + s_3 e^{\varrho^2} v_2^3 = \bar{z} + \sum_{i=1}^{3} s_i v_2^i e^{\varrho^i} , \]

\[ q(t) = \bar{q} + s_1 e^{\varrho^i} v_3^1 + s_2 e^{\varrho^2} v_3^2 + s_3 e^{\varrho^2} v_3^3 = \bar{q} + \sum_{i=1}^{3} s_i v_3^i e^{\varrho^i} . \]

(a.7)

(a.8)

We are interested in stable equilibria only, and so we assume that:

\[ \forall i \quad (r_i \geq 0 \implies s_i = 0). \]

(a.9)

(All nonnegative eigenvalues of the matrix \( M \), must have zero constants \( s_i \).) Using (a.7) we obtain:

\[ \int_0^t z(s) ds = \bar{Z} - \sum_{i=1}^{3} s_i v_2^i \frac{r_i}{r_1} + \sum_{i=1}^{3} s_i v_2^i e^{\varrho^i} . \]

(a.10)

which implies that

\[ e^{-(1 + \tau \varrho) p_1} \int_0^t z(s) ds = e^{-(1 + \tau \varrho) p_1} \bar{z} - \sum_{i=1}^{3} s_i v_2^i \frac{r_i}{r_1} e^{\varrho^i} . \]

(a.11)

Similarly, on the basis of (a.8), we compute:

\[ \int_0^t q(s) ds = \bar{q} - \sum_{i=1}^{3} s_i v_3^i \frac{r_i}{r_1} + \sum_{i=1}^{3} s_i v_3^i e^{\varrho^i} . \]

(a.12)

which implies that

\[ e^{t(1 + \tau \varrho)} \int_0^t q(s) ds = e^{t(1 + \tau \varrho)} \bar{q} - \sum_{i=1}^{3} s_i v_3^i \frac{r_i}{r_1} e^{\varrho^i} . \]

(a.13)

Using (a.11) and (a.12) we can rewrite the condition (a.6) as:
The assumption (a.9) implies that \( \lim_{t \to \infty} e^{-(1+\tau_Z) \frac{\Sigma}{\chi}} = 1 \) and \( \lim_{t \to \infty} e^{\frac{\Sigma}{\chi^2} v' t} = 1 \). Therefore, the condition (a.14) is satisfied if, and only if:

\[
\lim_{t \to \infty} \left\{ q(t) e^{\left( \frac{\Sigma}{\chi} (1+\tau_Z) + \Sigma + \frac{1}{\chi} p_Z (1+\tau_Z) z - \frac{z}{\chi} \right)} \right\} = 0. \tag{a.15}
\]

Using (a.8) and the equality \( \bar{r}_Z = \epsilon_Z + p_Z \bar{z} \) we can rewrite this condition in the form:

\[
\bar{q} \cdot \lim_{t \to \infty} e^{\frac{\Sigma}{\chi} (1+\tau_Z) - \delta - \tau_Z \theta} + \lim_{t \to \infty} \left\{ \sum_{i=1}^{3} s_i v_i' e^{\varepsilon t} \cdot e^{\left( \frac{\Sigma}{\chi} (1+\tau_Z) - \delta - \tau_Z \theta \right)} \right\} = 0. \tag{a.16}
\]

It follows from (a.9) that \( \lim_{t \to \infty} \sum_{i=1}^{3} s_i v_i' e^{\varepsilon t} = 0 \) and \( e^{\left( \frac{\Sigma}{\chi} (1+\tau_Z) - \delta - \tau_Z \theta \right)} > 0 \) (positive and finite number), and so the second part of the sum in (a.16) is zero. Therefore, for (a.16) to hold it must be that \( \lim_{t \to \infty} e^{\frac{\Sigma}{\chi} (1+\tau_Z) - \delta - \tau_Z \theta} = 0 \). This equality holds if, and only if,

\[
\frac{\bar{q} - 1}{\chi} - \bar{r}_Z (1+\tau_Z) - \delta - \tau_Z \theta < 0, \tag{a.17}
\]

which can be written in a more convenient form:

\[
(1+\tau_Z) \bar{r}_Z^N > \bar{\theta} + \bar{\theta} + n. \tag{a.17}
\]

2. Now let’s tackle the second transversality condition, that is (38e):

\[
\lim_{t \to \infty} e^{-(\rho-n)\varepsilon} \lambda_i'(t) z(t) = 0. \tag{38e}
\]

Since \( \lambda'_i(t) = -\lambda_i(t) \) and \( z(t) = z(t) \cdot y(t) = z(t) \cdot Ak(t) \), this condition is equivalent to:

\[
\lim_{t \to \infty} e^{-(\rho-n)\varepsilon} \lambda_i(t) z(t) \cdot Ak(t) = 0. \tag{a.18}
\]

The equation (78) implies that \( \lim_{t \to \infty} z(t) = \bar{z} \neq +\infty \), so that (a.18) can be rewritten as:

\[
A \bar{z} \lim_{t \to \infty} e^{-(\rho-n)\varepsilon} \lambda_i(t) \cdot k(t) = 0. \tag{a.19}
\]

Notice that this condition is equivalent to (38f), as long as we neglect the special case of \( \bar{z} = 0 \). It follows that the necessary and sufficient condition for both transversality conditions to hold is (a.17).
A2. Proof that transversality conditions (106h–k) are satisfied if, and only if,

\[ \rho > n + \gamma (1 + \kappa) \bar{\theta}^*, \]  

(154)

Let’s start with the condition (106i).

\[ \lim_{t \to \infty} e^{-(\rho - n) t} \lambda_2(t) k^*(t) = 0. \]  

(106i)

Substituting \( \lambda_2(t) = q^*(t) \cdot \lambda_1(t) \) this condition can be written in the equivalent form:

\[ \lim_{t \to \infty} e^{-(\rho - n) t} \lambda_1(t) q^*(t) k^*(t) = 0. \]  

(a.20)

Recall that both formulas (46) and (47) hold for the „centrally planned” economy. It follows that the trajectory of capital is of the form:

\[ k^*(t) = k_0 e^{\int_0^t \left( \frac{q(s) - 1}{z} \right) ds} = k_0 e^{\int_0^t \left( \frac{q(s) - 1}{z} \right) ds} \cdot e^{-(\rho + \kappa) t}. \]  

(a.21)

Meanwhile, (112) implies that

\[ \frac{\dot{\lambda}_1}{\lambda_1} = \rho - r_Z(z^*) - p_Z z^*(t), \]  

(a.22)

where \( r_Z(z) = \varepsilon_Z + p_Z z^*(t) \). Therefore,

\[ \frac{\dot{\lambda}_1}{\lambda_1} = \rho - \varepsilon_Z - 2p_Z z^*(t). \]  

(a.23)

Hence the trajectory \( \lambda_1(t) \) has the following form:

\[ \lambda_1(t) = \lambda_1(0) e^{(\rho - \varepsilon_Z) t} e^{\int_0^t \left( -2p_Z \int_0^s (z^*) ds \right) ds}. \]  

(a.24)

Having regard to (a.21) and (a. 24) condition (a.20) can be written in the form:

\[ \lambda_1(0) k_0 \lim_{t \to \infty} \left\{ q^*(t) e^{-(\varepsilon_Z + \kappa) t} e^{-2p_Z \int_0^t (z^*) ds} e^{\int_0^t \left( \frac{q(s) - 1}{z} \right) ds} \right\} = 0, \]  

(155)

which is equivalent to

\[ \lambda_1(0) k_0 \lim_{t \to \infty} \left\{ q^*(t) e^{-\left( \varepsilon_Z + \frac{1}{\kappa} \right) t} e^{-2p_Z \int_0^t (z^*) ds} e^{\int_0^t \left( \frac{q(s) - 1}{z} \right) ds} \right\} = 0. \]  

(a.25)

In order to examine this condition we need to know the trajectories of variables \( z^*(t) \) and \( q^*(t) \). Because the model is non-linear, we will use approximate trajectories obtained by solving a linearized model. From equations (156) we know that

\[ z^*(t) = z^* + s_1 e^{\alpha t} v_1^1 + s_2 e^{\alpha t} v_2^2 + s_3 e^{\alpha t} v_3^3 = z^* + \sum_{i=1}^3 s_i v_i e^{\alpha t}, \]  

(a.27)
\[ q^*(t) = \sqrt{q^*} + s_1 e^{t\sigma} v_1^1 + s_2 e^{t\sigma} v_2^2 + s_3 e^{t\sigma} v_3^3 = \sqrt{q^*} + \sum_{i=1}^{3} s_i v_i^i e^{t\sigma}. \] (a.28)

We are interested in stable equilibria only, and so we assume that:
\[ \forall i \quad (r_i \geq 0 \Rightarrow s_i = 0). \] (a.29)

(All nonnegative eigenvalues of the matrix \( \mathbf{M} \), must have zero constants \( s_i \)). Using (a.27) we obtain:
\[ \int_0^t z^*(s)ds = z^* t - \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} + \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}. \] (a.30)

which implies that
\[ e^{-2p_1 \int_0^t z^*(s)ds} = e^{-2p_1 \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i}} e^{-2p_1 \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}}. \] (a.31)

Analogously, on the basis of (a.28), we compute:
\[ \int_0^t q^*(s)ds = \sqrt{q^*} t - \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} + \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}, \] (a.32)

which implies that
\[ e^{\int_0^t q^*(s)ds} = e^{\frac{1}{x} \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i}} e^{-\frac{1}{x} \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}}. \] (a.33)

Using (a.31) and (a.32) we can rewrite the condition (a.26) as:
\[ \lambda_1(0) k_0 e^{2p_1 \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}} \lim_{t \to \infty} \left\{ q^*(t)e^{\left( \epsilon_{2} + \frac{1}{x} \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma} \right)} \cdot e^{-2p_1 \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}} \right\} = 0. \] (a.34)

The assumption (a.29) implies that \( \lim_{t \to \infty} e^{-2p_1 \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}} = 1 \) and \( \lim_{t \to \infty} e^{-\frac{1}{x} \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma}} = 1. \) Therefore, the condition (a.34) is satisfied if, and only if:
\[ \lim_{t \to \infty} \left\{ q^*(t)e^{\left( \epsilon_{2} + \frac{1}{x} \sum_{i=1}^{3} \frac{S_i v_i^i}{r_i} e^{t\sigma} \right)} \right\} = 0. \] (a.35)

Using (a.28) together with the equality \( z^*_x = \epsilon_x + p_1 z^* \) we can rewrite this condition as:
\[ \sqrt{q} \cdot \lim_{t \to \infty} e^{\left( \frac{S_1 - p_1 z^*}{x} \right)} + \lim_{t \to \infty} \left\{ \sum_{i=1}^{3} \frac{s_i v_i^i}{r_i} e^{t\sigma} \cdot e^{\left( \frac{S_1 - p_1 z^*}{x} \right)} \right\} = 0. \] (a.36)

It follows from (a.29) that \( \lim_{t \to \infty} \sum_{i=1}^{3} s_i v_i^i e^{t\sigma} = 0 \) and \( e^{\left( \frac{S_1 - p_1 z^*}{x} \right)} > 0 \) (positive and finite number), and so the second part of the sum in (a.36) is zero. Therefore, for (a.36) to hold it
must be:  \[ \lim_{t \to \infty} e^{\left( \frac{\overline{q} - r^*_z - p_z \overline{z} - \delta}{\chi} \right) t} = 0. \]
This equality holds if, and only if, 
\[ \left( \frac{\overline{q} - 1}{\chi} - r^*_z - p_z \overline{z} - \delta \right) < 0, \]
which can be written in a more convenient form:
\[ r^*_z + p_z \overline{z} > \frac{\overline{q} - 1}{\chi} - \delta = \overline{\sigma} + n. \]

It’s straightforward to show that it is equivalent to:
\[ \rho > n + \gamma(1 + \kappa)\overline{\sigma}. \quad (a.37) \]

2. Now let’s tackle the second transversality condition, that is (106h):
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_1(t) \overline{z}^*(t) = 0. \quad (106h) \]
Since \( \lambda_1'(t) = -\lambda_1(t) \) and \( \overline{z}^*(t) = \overline{z}^*(t) \cdot y^*(t) = \overline{z}^*(t) \cdot Ak^*(t) \), this condition is equivalent to:
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_1(t) \overline{z}^*(t) \cdot Ak^*(t) = 0. \quad (a.38) \]
The equation (139) implies that \( \lim_{t \to \infty} \overline{z}^*(t) = \overline{z}^* \neq \pm \infty \), and so (a.38) can be rewritten as:
\[ A_{\overline{z}} \overline{z}^* \cdot \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_1(t) \cdot k^*(t) = 0. \quad (a.39) \]
Notice that this condition is equivalent to (106i), as long as we neglect the special case of \( \overline{z}^* = 0 \).

3. Now let’s tackle the third transversality condition, that is (106j):
\[ \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_3(t) d_F^*(t) = 0. \quad (106j) \]
Since \( d_F^*(t) = d_F^*(t) \cdot y^*(t) \) and \( y^*(t) = Ak^*(t) \), this condition is equivalent to:
\[ A \cdot \lim_{t \to \infty} e^{-(\rho - n)t} \lambda_3(t) d_F^*(t) k^*(t) = 0. \quad (a.40) \]
From (120) it follows that \( \forall t \)
\[ \lambda_3(t) = -\lambda_1(t) \cdot \frac{r^*_d(t) + p_d d_F^*(t) + \theta}{r^*_z(t) + p_z \overline{z}^*(t) + \theta}. \quad (a.41) \]
Using the equalities \( r^*_d(t) = \varepsilon_d + p_d d_F^*(t) \) and \( r^*_z(t) = \varepsilon_z + p_z \overline{z}^*(t) \), we can rewrite (a.41) as
\[ \lambda_3(t) = -\lambda_1(t) \cdot \frac{\varepsilon_d + p_d d_F^*(t) + 2p_d d_F^*(t) + \theta}{\varepsilon_z + 2p_z \overline{z}^*(t) + \theta}. \quad (a.42) \]
Having regard to (a.42), condition (a.40) can be converted to the form:
\[ A \cdot \lim_{t \to +\infty} e^{-(\rho-n)t} \lambda_i(t) \frac{\varepsilon_D + p_D \cdot d_D^*(t) + 2 p_D \cdot d_F^*(t) + \vartheta \cdot k^*(t)}{\varepsilon_Z + 2 p_Z \cdot z^*(t) + \vartheta} d_F^*(t) = 0. \] (a.43)

The equation (139) implies that \( \lim_{t \to +\infty} z^*(t) = Z^* \neq +\infty \). Similarly, (141) implies that \( \lim_{t \to +\infty} d_F^*(t) = \overline{d}_F^* \neq +\infty \). Since by assumption all these trajectories of public and private debt are positive, their limits cannot be equal to \(-\infty\).

These four facts imply that \( \lim_{t \to +\infty} \frac{\varepsilon_D + p_D \cdot d_D^*(t) + 2 p_D \cdot d_F^*(t) + \vartheta \cdot d_F^*(t)}{\varepsilon_Z + 2 p_Z \cdot z^*(t) + \vartheta} = \pm \infty \). (this limit is finite). For this reason, the condition (a.43) can be written in the equivalent form:

\[ A \cdot \frac{\varepsilon_D + p_D \cdot \overline{d}_D^*(t) + 2 p_D \cdot \overline{d}_F^*(t) + \vartheta \cdot \overline{d}_F^*(t)}{\varepsilon_Z + 2 p_Z \cdot \overline{z}^*(t) + \vartheta} \cdot \lim_{t \to +\infty} e^{-(\rho-n)t} \lambda_i(t) \cdot k^*(t) = 0. \] (a.44)

Notice that this condition is equivalent to (106i).

4. Analogously, we can show that the fourth transversality condition (106k) is also equivalent to (106i).

5. It follows that the necessary and sufficient condition for all four transversality conditions to hold is (a.37).
A4. The formula for $expr$:

\[
expr = 4\bar{\phi}^3(n + \theta + \bar{\theta})(\bar{q} - A\bar{z} + \chi \bar{\theta})A_1^2 p_Z^2 - \frac{4\rho^2 p_Z^2(n + \theta)(n - \rho)(\theta + \rho)\chi + (2n + \theta - \rho)(\theta + \rho)\chi \bar{\theta} + (\theta + \rho)\chi \bar{\theta}^2 + A\rho \left[\omega \bar{d}_F \left(\xi(1 + A + A\omega) - (\theta + \rho)\bar{d}_F \right) - \frac{A\xi}{1 + \omega} - (1 + 3\omega)\bar{d}_F \left(\bar{d}_F - 2(\theta + \rho)\omega \frac{\bar{z}}{\bar{p}} \right) \right] - 4\bar{\phi}^2 A_1^2 p_Z^2 - (\theta + 3\rho)\bar{q}(n + \theta + \bar{\theta}) + A(\theta + 3\rho)\bar{z}(n + \xi + \bar{\theta}) + \bar{\phi} \left[\xi(n + \theta)(n - \theta - 4\rho) \chi + \chi \bar{\theta}(2n - 4\rho + \bar{\theta}) - A\rho \left[\omega \bar{d}_D^2 + (1 + 3\omega)\bar{d}_D \bar{d}_F + 2\omega \frac{\bar{z}}{\bar{p}} \right] \right]
\]

\[
A_1 \left[8\rho(2n + \theta + \bar{\theta} + 3n - 2\rho^2)\chi \bar{p}^2 p_Z^2 + 4\rho(2n + \theta + 3\rho)\chi \bar{p}^3 p_Z^2 + \bar{\phi} \left[-4\rho \bar{p}_Z^2 - (n + \theta)(2n + \theta + 3\rho - 3\rho^2) \chi + \rho(\theta + \rho)\bar{q} - A\rho(\theta + \rho)\bar{z} + A\rho \left[\omega \bar{d}_D \left(-2\xi(1 + A + A\omega) + (\theta + 3\rho)\bar{d}_D \right) + \left(-2\xi(1 + \omega) + (\theta + 3\rho)(1 + 3\omega)\bar{d}_D \bar{d}_F + 2\frac{\bar{z}}{\bar{p}} \right) \right] + A\xi(\theta + \rho) \left[\bar{d}_F + \omega(\theta + \bar{\theta}) \right] + \left(\theta + 3\rho\right) \left[\frac{1}{\theta + 3\rho} \xi \bar{p}_Z^2 + \bar{\phi} \left[\xi(n + \theta)(n + 3\rho) \chi + \chi \bar{\theta}(2n + 3\rho + \bar{\theta}) - A\rho \left[\omega \bar{d}_D^2 + (1 + 3\omega)\bar{d}_D \bar{d}_F + 2\omega \frac{\bar{z}}{\bar{p}} \right] \right] \right] \right]
\]

A5. Proof of proposition 2.

Proposition 1 implies that the rates of taxes and public consumption are optimal if, and only if, for each $t \geq 0$ the following equalities hold:

\[
z(t) = z^*(t), \quad \xi(t) = \xi^*(t), \quad \xi(t) = \xi^*(t), \quad q(t) = q^*(t),
\]

\[
d_F(t) = d_F^*(t), \quad d_D(t) = d_D^*(t), \quad g_C(t) = g_C^*(t),
\]

because these 6 conditions directly implicate the equality (identity) of all other trajectories. Namely, (4.7) together with (46) which holds for both types of economies imply that
\[ \varphi(t) = \varphi^*(t). \] (a.51)

Similarly, equations (a.51) and (47) which is true for both types of economies imply that
\[ k(t) = k^*(t) \] (a.52)
and
\[ y(t) = y^*(t). \] (a.53)

From (a.45) and (a.53) it follows that
\[ c(t) = c^*(t). \] (a.54)

Equations (a.46) and (1) implicate the following:
\[ r_Z(t) = r_Z^*(t). \] (a.55)

On the other hand, (a.48), (a.49), (a.53) together with identities \( d_F(t) = d_F^*(t) \cdot y(t) \) and \( d_D(t) = d_D^*(t) \cdot y(t) \) for both types of economy imply that
\[ d_F(t) = d_F^*(t), \] (a.56)
\[ d_D(t) = d_D^*(t). \] (a.57)
which in turn implies that
\[ d(t) = d^*(t). \] (a.58)

From (a.46), (a.53) and identity \( z(t) = z^*(t) \cdot y(t) \) for both economies it follows that
\[ z(t) = z^*(t). \] (a.59)

Finally, (a.48), (a.49) and (2) imply that
\[ r_D(t) = r_D^*(t). \] (a.60)

If all these equations are satisfied, the decentralized economy replicates the “centrally planned” economy. What we have shown above is that it will suffice to ensure that the six conditions (a.45)–(a.50) are fulfilled, because they constitute the necessary and sufficient conditions. Now, let us examine what particular values of fiscal policy parameters will ensure that.

First, note that because (a.50) implies (a.54), the following condition must hold:
\[ \sigma_C^{opt} = \kappa. \] (173)

Second, note that using formulae for the trajectories of consumption (43) and (114), we can rewrite (a.53) in the equivalent form:
\[ c(t) = c_0 \cdot e^0 = c_0^* \cdot e^0 = c^*(t). \] (a.61)
This equality must hold for any moment \( t \). Substituting \( t = 0 \) we get:
\[ c_0^* = c_0. \]  

The equality (a.61) holds if, and only if, (a.62) is true and
\[ \psi(t) = \psi^*(t). \]  

Substituting the formulae (42) and (113) we can write this equality as:
\[ \frac{(1 + \tau_Z)r_Z(t) - \rho + \tau_Z\theta}{1 - \gamma(1 + \kappa)} = \frac{r_Z^*(t) + p_Zz^*(t) - \rho}{1 - \gamma(1 + \kappa)}. \]

Using (1) we can rewrite it as:
\[ \frac{(1 + \tau_Z)r_Z(t) - \rho + \tau_Z\theta}{1 - \gamma(1 + \kappa)} = \frac{\varepsilon_Z + 2p_Zz^*(t) - \rho}{1 - \gamma(1 + \kappa)}. \]

This condition holds if, and only if,
\[ (1 + \tau_Z)r_Z(t) + \tau_Z\theta = \varepsilon_Z + 2p_Zz^*(t). \]

Using (1) together with (a.59) it can be transformed to:
\[ (1 + \tau_Z)[\varepsilon_Z + p_Zz^*(t)] + \tau_Z\theta = \varepsilon_Z + 2p_Zz^*(t), \]

which can be reduced to the form:
\[ \tau^\text{opt}_Z(t) = \frac{r_Z^*(t) - \varepsilon_Z}{r_Z^*(t) + \theta}. \]

Now, we will show that the rate of tax on capital income must vary in time, but it is impossible to provide an explicit analytical formula for its trajectory \( \tau^\text{opt}_Z(t) \). Using the (approximate) formulae for the trajectories \( q(t) \) and \( q^*(t) \), we can write the condition (a.47) as follows:
\[ \tilde{q} + \sum_{i=1}^{3} s_i e^{r_i'} v_i = \tilde{q}^* + \sum_{i=1}^{3} s_i^* e^{r_i'} w_i. \]

It must hold for any time \( t \). Let us put \( t \to \infty \). It yields:
\[ \tilde{q} = \tilde{q}^*. \]

(Obviously, the equality of trajectories \( q(t) \) and \( q^*(t) \) implies the equality of their limit values). From (a.68) and (a.69) it follows that the replication requires the fulfillment of the equality:
\[ \sum_{i=1}^{3} s_i e^{r_i'} v_i = \sum_{i=1}^{3} s_i^* e^{r_i'} w_i. \]

Notice that the right-hand side of this equation is independent of all tax rates, because the matrix \( \mathbf{M} \) given by (155) is independent from them. Therefore the right-hand side of (a.70) can be considered as a given number, while the left side is dependent on five fiscal policy
parameters, i.e. \( \tau_K, \tau_Z, \sigma_c, \omega, \xi \). The parameters \( \tau_Z \) and \( \sigma_c \) are already fixed by conditions (173) and (174). The parameters \( \omega \) and \( \xi \) are also fixed by conditions (a.56) and (a.57) – this equalities hold if, and only if, \( \omega \) and \( \xi \) are exactly the same as assumed by the social planner in the “centrally planned” economy. Therefore, the only parameter that is free to manipulate is \( \tau_K \). It follows from this that in order to ensure the condition (a.70) the parameter \( \tau_K \) must be properly adjusted for each moment of time \( t \geq 0 \). Notice that on the left-hand side of (a.70) we have the (negative) eigenvalues of the matrix \( M \), which cannot be described by analytical formulae, because they are the roots of the polynomial of the 5th degree. Therefore, it is not possible to solve the equation (a.70), even for a given moment of time (some chosen \( t \)). For this reason it is impossible to derive an analytical formula for the optimal trajectory \( \tau_K^{opt}(t) \) – not only for the whole trajectory, but even for a single value of \( \tau_K \) in a selected moment of time \( t \). Individual (momentary) values of the trajectory \( \tau_K^{opt}(t) \) can only be calculated numerically, for a chosen set of values of all model parameters.

A6. Transitory dynamics in the baseline scenario.

The solid line represents the “centrally planned” economy, and the dashed line – the decentralized economy in the base scenario. Trajectories were calculated on the basis of the linear approximation of the model. For each type of the economy first the steady state was calculated, then the matrices \( M \) and \( M^* \), their eigenvalues and eigenvectors, and finally trajectories in accordance with formulae presented in sections 3.3 and 4.3.
Fig. 38. The growth rate of GDP per capita

Fig. 39. The rate of growth of private consumption per capita

Fig. 40. GDP per capita

Fig. 41. Consumption per capita

Fig. 42. Private foreign debt (% of GDP)

Fig. 43. The interest rate on private foreign debt.
Fig. 44. Public debt (% of GDP)  
Fig. 45. The interest rate on public debt.  
Fig. 46. Foreign public debt (% of GDP)  
Fig. 47. Domestic public debt (% of GDP)  

References


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