Subgame Perfect Equilibrium of Ascending
Combinatorial Auctions*

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April 7, 2015

Abstract

This paper considers a class of combinatorial auctions with ascending prices, which includes the Vickrey-Clarke-Groves mechanism and core-selecting auctions. In every ascending auction, the Vickrey-target strategy, i.e., bidding up to the Vickrey price based on provisional valuations, constitutes a subgame perfect equilibrium when bidders are single-minded. This equilibrium outcome exists in the bidder-optimal core with respect to true valuations. However, the equilibrium outcome is unfair in the sense that winners with low valuations tend to earn high profits. This non-monotonic payoff can lead to inefficiency in the case of general valuations.

Keywords: combinatorial auction, ascending auction, Vickrey auction, core, single-minded bidders

JEL classification: D44, D47

*This paper was formerly entitled “The Vickrey-target strategy and core in ascending combinatorial auctions.” This research was supported by a Grant-in-Aid for Research Activity Start-up (KAKENHI 23830039) from the Japan Society for the Promotion of Sciences (JSPS).
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1 Introduction

Multi-object auctions have been examined for decades from both theoretical and practical perspectives. From the theoretical perspective, the Vickrey-Clarke-Groves mechanism (the Vickrey auction) is known to be a unique efficient mechanism that is incentive compatible in a dominant strategy and individually rational (Green and Laffont 1977; Holmstrom 1979). However, it has received criticism both in theory and practice, and the Vickrey auction is rarely used barring some Internet advertisement auctions.\(^1\) Instead, core-selecting auctions have recently been proposed and applied for spectrum license auctions in many countries, although they are not incentive compatible. Research on incentives and equilibria in such auctions is still progressing, and most existing studies focus on incentives in sealed-bid core-selecting auctions (Erdil and Klemperer 2010; Day and Milgrom 2008, DM hereafter).\(^2\)

From the viewpoint of practical auction design, on the other hand, ascending-price auctions are often preferred to sealed-bid auctions. Core-selecting auctions adopted for spectrum license allocations are a type of ascending auction.\(^3\) Numerous studies investigate and propose various combinatorial auction designs with ascending-price formats.\(^4\) Of these, auctions examined by Parkes and Ungar (2000), Ausubel and Milgrom (2002), and de Vries et al. (2007) are not ascending Vickrey auctions for general valuations, but rather ascending core-selecting auctions. They would be desirable in the sense that they have an ascending-price format and the core-selecting property. However, we need to consider the incentive problem. DM show that core-selecting auctions achieve an outcome in the core in a Nash equilibrium under complete information. A natural question then arises: what is the subgame perfect

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\(^1\)See Ausubel and Milgrom (2006) for the disadvantages of the Vickrey auction.

\(^2\)Sano (2012) is an exception, analyzing the perfect Bayesian equilibrium in an ascending core-selecting auction in a simple environment.

\(^3\)See Cramton (2013) for recent applications of core-selecting auctions to spectrum license auctions.

\(^4\)See Parkes (2006) for a review of several designs and the advantages of ascending auctions over sealed-bid auctions.
equilibrium (SPE) of an ascending auction? The strategic equivalence between an ascending and the “second-price”-type auctions often fails in multi-object auctions. Strategic bidding behavior differs between sealed-bid and ascending-price auctions. Is there an SPE that achieves an outcome in the core in ascending core-selecting auctions?

This paper answers this question and shows that under complete information, every ascending core-selecting auction and even other non-core-selecting auctions have an SPE that achieves an outcome in the core when bidders are single-minded. However, when bidders are not single-minded, there exists a situation such that an ascending core-selecting auction cannot achieve the efficient outcome in any SPE. A bidder is said to be single-minded when he is interested only in a particular bundle and places bids only for that bundle. The single-minded environment is restricted but is known to hold many features and incentive problems in multi-object auctions. Many studies focus on single-minded bidders (Lehmann et al. 2002; Mu’alem and Nisan 2008; Sano 2011; Milgrom and Segal 2013).

We consider a general form of ascending combinatorial auctions. We allow an arbitrary ascending price scheme and possible final discounts; that is, payments may differ from the terminal prices. Our formulation approximates the ascending auction used for recent spectrum license allocations, i.e., the so-called combinatorial clock auction.

We show that a particular strategy, which we call the Vickrey-target strategy, is an SPE in any specification of the auction rule. In the Vickrey-target strategy, a bidder aims to bid up to a type of Vickrey price, which is calculated by using the true and revealed values in each round. The equilibrium outcome is in the core with respect to true values. In addition, the specified equilibrium outcome is unique in a certain case. This result contrasts with the fact that there are possibly many Nash equilibria in sealed-bid auctions. An ascending-price format and subgame perfection work as equilibrium selection.

Although the specified equilibrium outcome is in the core, it can be “unfair” in
the sense that the higher a winner’s value, the lower his payoff. This non-monotonic payoff leads to the possibility that the Vickrey-target strategy may not be an SPE in a general multi-minded environment. Consequently, an ascending core-selecting auction may achieve an inefficient outcome.

The intuition behind these results is as follows. In an ascending auction, bidders decide whether or not to continue bidding in each round. Note that there exists a best core outcome for each bidder in which he obtains the Vickrey payoff. If a bidder stops bidding at his Vickrey price and the auction finally selects the efficient allocation, he wins the goods with at most the Vickrey price, which is the best outcome in the core for him. The Vickrey payments of winners with lower valuations are generally lower than those of high-value winners. Hence, prices first reach the Vickrey prices of the low-value winners, and low-value winners achieve their most preferred outcomes. When a winner stops bidding, the remaining bidders need to raise their bids even further to win. High-value bidders tend to pay dearly and earn little net payoff.

The contribution of this paper is that it analyzes an SPE for dynamic combinatorial auctions. An ascending combinatorial auction has a unique subgame perfect outcome in the bidder-optimal core with respect to true values. This contrasts with the multiple equilibria of sealed-bid combinatorial auctions (Bernheim and Whinston 1986; DM; Sano 2013). As Milgrom (2007, p. 952) discusses, the preceding analyses are not satisfactory even if we accept the assumption of complete information because there are many plausible equilibria. Our result can be interpreted as an equilibrium selection, and it indicates the most plausible outcome. However, these positive findings hold only for the single-minded environment, and inefficiency can arise in the equilibrium in a general multi-minded environment.

1.1 Related Literature

Various ascending-price auctions have been proposed by Parkes and Ungar (2000), Ausubel and Milgrom (2002), Ausubel (2006), and de Vries et al. (2007). All these auctions terminate with the Vickrey outcome and are incentive compatible for sub-


The single-minded environment is introduced by Lehmann et al. (2002) and examined by Mu’alem and Nisan (2008) and Babaioff and Blumrosen (2008) among others. These studies consider strategy-proof and approximately efficient mechanisms from the perspective of computational feasibility. Milgrom and Segal (2013) have recently considered ascending auctions like ours. Motivated by recent spectrum license reallocation in the U.S., they consider approximate efficiency and characterize strategy-proof ascending auctions that generalizes the concept of the deferred acceptance algorithm in the two-sided matching theory. We consider (fully) efficient auctions regarding reported values and incentive problem, which is the main difference from these studies.

For equilibrium analysis of multi-object auctions, most studies limit their attention to the complete information case. Bernheim and Whinston (1986) consider the pay-as-bid combinatorial auction and demonstrate that many full-information Nash equilibria can possibly exist in the core. Ausubel and Milgrom (2002) consider another mechanism, the ascending proxy auction, and obtain a similar result. DM and Sano (2013) generalize their results to all sealed-bid core-selecting auctions and

\footnote{Specifically, Milgrom and Segal (2013) formulate a model of reverse auctions and descending prices.}
Several studies have analyzed ascending-price non-package auctions and equilibria with low prices. Ausubel and Schwartz (1999) and Grimm et al. (2003) examine the SPE in a multi-unit uniform-price auction with complete information. They show the existence of a low-price SPE in which bidders initially split goods to avoid competition.

The remainder of this paper proceeds as follows. Section 2 formulates the model and the auction. Section 3 demonstrates that the Vickrey-target strategies constitute an equilibrium and lead to an outcome in the bidder-optimal core. We then examine the uniqueness of this equilibrium. Section 4 considers the case of general valuations and indicates that inefficiency arises in the SPE with a multi-minded bidder. Section 5 concludes the paper.

2 The Model

A seller allocates multiple indivisible goods, and $K$ denotes the set of goods. Let $N \equiv \{0, 1, 2, \ldots, n\}$ be the set of all players. $I = \{1, \ldots, n\}$ is the set of all bidders and 0 denotes the seller. Let $X_i \subseteq 2^K$ be the set of admissible bundles for bidder $i$. For each $i \in I$, a null bundle is denoted by $\emptyset_i$ (instead of $\emptyset$), and $x_i \in X_i$. $X \subseteq X_1 \times \cdots \times X_n$ denotes the set of feasible allocations. All bidders have quasi-linear utilities. Suppose that valuations for bundles of goods are integer-valued. Let $v_i : X_i \to \mathbb{Z}_+$ be a bidder $i$’s valuation function. Suppose that each $v_i$ is monotone and $v_i(x_i) = 0$ for all $i \in I$. Bidder $i$ earns a payoff $\pi_i = v_i(x_i) - p_i$ where $x_i \in X_i$ denotes the bundle allocated to $i$ and $p_i$ is the monetary transfer to the seller. The seller’s payoff is the revenue from the auction: $\pi_0 = \sum_{i \in I} p_i$.

Most analyses in this paper assume bidders are single-minded; i.e., each bidder is interested in a particular bundle of goods. Formally, for each $i \in I$, there exists a
non-null bundle \( y_i \in X_i \) and

\[
v_i(x_i) = \begin{cases} 
    v_i & \text{if } y_i \subseteq x_i \\
    0 & \text{otherwise}
\end{cases}
\]  

(1)

For simplicity, we assume for each \( i \in I \), \( X_i = \{x_i, y_i\} \): i.e., bidders’ interests are commonly known to all players. In addition, we assume that valuations are also common knowledge among all bidders, as in Bernheim and Whinston (1986) and DM.

Let \( X^*(v) \subseteq X \) be the set of efficient allocations with respect to the profile of valuation functions \( v = (v_i)_{i \in I} \):

\[
X^*(v) \equiv \arg \max_{x \in X} \sum_{i \in I} v_i(x_i).
\]

Just for simplicity, the efficient allocation with respect to the true values is uniquely determined by \( X^*(v) = \{x^*\} \). Given \( v \), a coalition value of a set of players \( J \subseteq N \) is the maximum total value that can be generated by \( J \). The coalition value function \( V \) is defined by

\[
V(J, v) = \begin{cases} 
    \max_{x \in X} \sum_{i \in J} v_i(x_i) & \text{if } 0 \in J \\
    0 & \text{if } 0 \notin J
\end{cases},
\]

where \( J \subseteq N \) and \( v_0(\cdot) \equiv 0 \). We sometimes use the notation \( V(\cdot) \) instead of \( V(\cdot, v) \) when no confusion will arise. Given the valuation profile \( v \), payoff profile \( \pi \in \mathbb{R}^{n+1} \) is feasible if \( \sum_{i \in N} \pi_i \leq V(N) \). Payoff profile \( \pi \) is individually rational if \( \pi \geq 0 \). A payoff profile is in the core if it is efficient, individually rational, and not blocked by any coalition. Given \( v \), the core of the auction game is

\[
\text{Core}(N, V) = \{ \pi \geq 0 | \sum_{i \in N} \pi_i = V(N) \text{ and } (\forall J \subseteq N) \sum_{i \in J} \pi_i \geq V(J) \}.
\]

A payoff profile \( \pi \in \text{Core}(N, V) \) is bidder-optimal if there is no \( \pi' \in \text{Core}(N, V) \setminus \{\pi\} \) such that \( \pi'_i \geq \pi_i \) for all \( i \in I \).

\footnote{This is often said to be the known single-minded model.}

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2.1 Ascending Auctions

An ascending clock auction is formulated as a discrete time game. In each round \( t \),
the auctioneer offers a price to each bidder, denoted by \( p_t^i(y_i) \in \mathbb{R}_+ \), or simply \( p_t^i \). A
price vector at round \( t \) is denoted by \( p_t = (p_t^i)_{i \in I} \in \mathbb{R}_+^n \). Each bidder’s action at \( t \) is
denoted by \( a_t^i \in \{C, S\} \). Action \( C \) indicates “continuing,” and implies that a bidder’s
value is greater than \( p_t^i \). Action \( S \) indicates “stopping,” and implies that a bidder’s
value is equal to or less than \( p_t^i \). The auctioneer gradually increases the prices for
continuing bidders. Bidder \( i \) is said to be active at \( t \) if for all \( \tau \leq t \), \( a_\tau^i = C \). Let
\( I_t \subseteq I \) be the set of all active bidders at \( t \).

An ascending auction is defined as follows. We do not fully specify the auction
rule. Our formulation below includes most of the auctions considered in preceding
studies.

1. The auctioneer initializes the price vector as \( p^1 = (0, \ldots, 0) \).

2. At each round \( t = 1, 2, \ldots \), each bidder chooses \( a_t^i \in \{C, S\} \). The auctioneer
chooses a set of active bidders \( J_t \subseteq I_t \). If \( i \in J_t \), then \( p_{t+1}^i = p_t^i + 1 \). Otherwise,
let \( p_{t+1}^i = p_t^i \).

3. The process terminates at \( T \leq \bar{T} \) where \( I_T = \emptyset \). The auctioneer selects an
allocation \( x \in X \) and determines bidders’ payments \( p \in \mathbb{R}_+^n \).

Let \( (\{J_t\}_t, g, (p_t)_{i \in I}) \) be an ascending auction mechanism. \( J^t : H^t \rightarrow 2^I \) specifies the
set of bidders who face price increase at \( t \), where \( H^t \) denotes the set of history up to \( t \).
The auction terminates at \( T \) when \( J^T(h^T) = \emptyset \). A final allocation \( g(h) \in X \) and the
payments \( (p_i(h))_{i \in I} \in \mathbb{R}_+^n \), where \( h \in H \) denotes a history throughout the ascending
auction. We call bidder \( i \) a winner if \( g_i(h) = y_i \). Conversely, bidder \( i \) is a loser if
\( g_i(h) = x_i \). The bidders’ payments do not have to equal the terminal prices. Ascend-
ing auctions generally terminate strictly before \( \bar{T} \). However, we assume that \( T = \bar{T} \)
without loss of generality by adding artificial rounds following round \( T \) and ignoring

\(^7\text{Mishra and Parkes (2007) require that auction outcome should be determined only from the}
information at the terminal round.\)
All information after $T$.

Different auctions have different termination conditions, and our formulation is able to deal with a range of termination conditions.

Once a bidder chooses $S$, then he cannot raise his bid any further. This reflects the bidding restriction imposed in real applications known as the “Activity Rule.”

We define the revealed value (and the bid) as $\hat{v}_i \equiv p_i^T$ for each $i \in I$. Efficiency and individual rationality in an ascending auction are defined with respect to $\hat{v}$ similar to those in sealed-bid auctions. An ascending auction is efficient if for all $h$, $g(h) \in X^*(\hat{v})$. An ascending auction is individually rational if $p_i(h) \leq \hat{v}_i$ for every winner and $p_i(h) = 0$ for every loser. An ascending auction is monotone if for each $i \in I$, $p_i(h)$ is non-decreasing in $\hat{v}_i$ given $\hat{v}_{-i}$.

Let $\hat{V}(. \equiv V(. , \hat{v})$, which is the coalition value function with respect to $\hat{v}$. A revealed payoff $\hat{\pi}$ is the payoff profile calculated from the auction outcome: $\hat{\pi}_i \equiv \hat{v}_i 1_{(g_i(h)=y_i)} - p_i(h)$ for each bidder and $\hat{\pi}_0 = \pi_0 = \sum p_i(h)$ for the seller. Various auctions are defined as follows:

**Definition 1** An ascending auction mechanism is an ascending Vickrey auction if it is efficient and the payments are defined by

$$p_i^V(h) = \hat{V}(N_{-i}) - \sum_{j \neq i} \hat{v}_j 1_{(g_j(h)=y_j)}.$$  

In addition, $\hat{\pi}_i$ denotes bidder $i$'s Vickrey payoff

$$\hat{\pi}_i \equiv V(N, \hat{v}) - V(N_{-i}, \hat{v}).$$

**Definition 2 (DM)** An ascending auction mechanism is core-selecting if $\forall h$, $\hat{\pi} \in Core(N, \hat{V})$.

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8Even if an auction outcome is determined at $T$, the price clock for each remaining active bidder increases and the auction continues until all the bidders choose $S$. The auction outcome is not affected by the information in these artificial rounds after $T$. We can assume $T = \bar{T}$ because the outcome is determined by all the history $h$ and we allow final discounts.

9Namely, the termination condition for finding an efficient allocation is different from the condition for finding the Vickrey outcome. See Mishra and Parkes (2007) for the detail.
**Definition 3 (Sano, 2013)** An ascending auction mechanism is *Vickrey-reserve* if it is efficient, individually rational, and \( p(h) \geq p^V(h) \) for all \( h \).

We consider an arbitrary ascending Vickrey-reserve auction that is monotone. Ausubel and Milgrom (2002) and Bikhchandani and Ostrov (2002) show that every core-selecting auction is Vickrey-reserve. In addition, Vickrey-reserve auctions are equivalent to core-selecting auctions if goods are substitutes. However, goods complementarities exist in the single-minded environment, so that a Vickrey-reserve auction need not be core-selecting.

### 2.2 Examples of Ascending Auctions

An ascending auction is specified by determining \( J^t \) in each round and the proper termination condition. Several specifications are proposed by preceding studies.

#### 2.2.1 The iBundle Auction

Parkes and Ungar (2000) and Ausubel and Milgrom (2002) consider that an auctioneer selects a tentative allocation in each round. A tentative allocation \( x(t) \) is selected by

\[
x(t) \in \arg \max_{x \in X} \sum_{i \in I} p^i_t(x_i).
\]

Then, \( J^t \) is defined as the set of tentative losers at \( t \); that is, \( J^t = \{i \in I^t | x_i(t) = x \} \).

The auction terminates at \( T \leq \bar{T} \) if \( J^T = \emptyset \). The tentative allocation \( x(T) \) at the terminal round is the final allocation, and the winners pays \( p^T_i \). This auction is known to be core-selecting.

#### 2.2.2 The Primal-Dual Auction

Another specification is proposed by de Vries et al. (2007) from the linear programming approach. They determine \( J^t \) as the set of minimally undersupplied bidders,
which is a generalization of overdemanded goods in Demange et al. (1986). The auction terminates when price increases stop, and winners pay the terminal prices.

### 2.2.3 The iBEA Auction

The iBEA auction is discussed by Mishra and Parkes (2007). $J^t$ is determined in a similar manner as in the iBundle auction: the auctioneer selects a tentative allocation and $J^t$ is set as tentative losers. However, the termination condition differs from that in the iBundle auction. Even if no tentative losers exist and the efficient allocation is identified, prices for remaining bidders increase until the determination of the efficient allocations for the marginal economies without each bidder. After the auction’s termination, each winner’s payment is determined by

$$p_t = p_t^T - \sum_{j \in W} p_j^T + \sum_{j \in W - i} p_j^T,$$

where $W$ denotes the set of winners and $W - i$ denotes the set of winners in the marginal economy without $i$.

### 2.2.4 The Combinatorial Clock Auction

Our model also approximates the combinatorial clock auction, which has been used for spectrum license auctions in many countries (Porter et al. 2003; Cramton 2013). The combinatorial clock auction originally employs a linear price vector rather than the personalized price vector. Let $q \in \mathbb{R}_+^K$ be a price vector for goods. In the combinatorial clock auction, the initial price vector is set as $q^1 = (0, \ldots, 0)$, and in each round $t$, bidders report their demands at current prices. The price $q^k$ of good $k$ is gradually raised whenever the aggregate demand for $k$ is greater than 1. We can convert the combinatorial clock auction to fit our model by letting $p_t^i = \sum_{k \in y_i} q^k_k$. When the bundle price $p_t^i$ jumps up in a round, we divide the jump into several rounds.\(^{10}\)

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\(^{10}\) In the combinatorial clock auction, a supplementary sealed-bid auction follows after the clock auction, so the auction rule is not completely covered by our model.
2.3 Strategy and Equilibrium

We assume complete information and that each bidder observes the price vector and the other bidders’ actions in each round. Just for simplification, we assume that bidders make choices sequentially. Assumption 1 is dispensable for our main result, Theorems 1 and 2.

**Assumption 1 (Sequential Decisions)** Each round has $n$ subrounds. At sub-round $i$ of round $t$, bidder $i$ makes a choice. Each bidder observes all actions of the other bidders before making his decision.

An ascending auction is now a perfect information game. We relabel auction rounds by each bidder’s decision nodes (subrounds). We refer to subround $i$ of round $s$ as round $t (= n(s - 1) + i)$.

Bidder $i$’s (pure) strategy $\sigma_i$ is a mapping from his decision nodes to his action set $\{C, S\}$. $\Sigma_i$ denotes the set of strategies for $i$. Let $\Sigma \equiv \Sigma_1 \times \cdots \times \Sigma_n$ be the set of strategy profiles. The equilibrium concept is a pure strategy subgame perfect equilibrium (SPE).

Let $v_{ti}$ be the **provisional value at** $t$, which is the possible value given the actions up to $t$: for each $i$

$$v_{ti} \equiv \begin{cases} \max\{v_i, p_{ti} + 1\} & \text{if } i \text{ is active at } t \\ p_{ti} & \text{otherwise} \end{cases}.$$  

The provisional value function is given by $v_{ti}(y_i) = v_{ti}$ and $v_{ti}(-) = 0$. The provisional value is equal to $i$’s true value whenever he is active and never bids over the true value. Once a bidder stops bidding, the provisional value is equal to the current price. Given $v' = (v'_i)_{i \in I}$, let $V'(\cdot) \equiv V(\cdot, v')$ for simplicity. In addition, let $\bar{\pi}_i^t$ be bidder $i$’s Vickrey payoff with respect to $v'$: $\bar{\pi}_i^t = V(\cdot) - V(N \setminus i)$. Let $X^t \equiv X^*(v')$ be the set of the efficient allocations with respect to $v'$, and let $X^t_i \equiv \{x_i \in X_i | x \in X^t\}$.

We impose two additional assumptions. One concerns ties. To simplify the analysis and clarify the results, we consider the following tie-breaking rule.
**Assumption 2** For each \( x \in X^*(\hat{v}) \), define \( t(x) \equiv \min\{t | (\forall s \geq t) \ x \in X^*\} \). Then, 
\( g(h) \in \arg\min_{x \in X^*(\hat{v})} t(x) \).

In auction models with complete information, ties are likely to occur, and an equilibrium may fail to exist with random tie-breaking when strategy space is continuous. Hence, ties are traditionally broken in a manner that depends on bidder values and not only on their bids. For example, in a first-price auction of a single object, the highest two bidders submit the same bid in a Nash equilibrium, which is the value of the second-highest bidder. In the analysis, we often assume that the bidder with the higher value is chosen in tie-breaking. This practice is acceptable because the selected outcome can be viewed as the limit of an equilibrium in an auction in which bidding is discrete with a small increment of \( \epsilon > 0 \). Since our model is discrete, we are able to avoid ties by slightly coordinating equilibrium strategies. However, we follow this practice to clarify the results and identify a striking property related to bidder-optimality of the core.

Another assumption deals with choice indifference.

**Assumption 3** Every bidder strictly prefers winning with a payment of \( v_i \) to losing.

An alternative assumption would be that bidders have a small additional (but negligible) utility for winning goods. We do not need Assumption 3 to show the existence of a proposed equilibrium (Theorem 1). However, it is crucial for the uniqueness of the equilibrium outcome. This can be justified when we additionally require trembling-hand perfection for the equilibrium concept.

### 3 Main Results

An SPE is derived in the standard manner of the backward induction. The key observation is that the efficient allocation according to \( v^t \), \( x \in X^t \) is a prediction of the final allocation in any SPE. Bidders never choose an action that changes \( X^t \) if they are restricted to bidding no more than true values.
**Proposition 1** Suppose Assumptions 1, 2, and 3. Further suppose that each bidder never bids over the true value. Then, any SPE of an ascending Vickrey-reserve auction satisfies $X^{t-1} \subseteq X^t$ for all $t$, both on and off equilibrium paths.

**Proof.** See Appendix B.

Proposition 1 implies $X^*(v) = X^0 \subseteq X^T = X^*(\hat{v})$ in equilibrium. Hence, any SPE is efficient as long as no one overbids.\(^{11}\) Suppose that bidder $i$ wins in the efficient allocation. By Proposition 1, given $\epsilon_{-i}$, it is optimal for $i$ to stop bidding at the minimum price $p_i^t$ such that $x^* \in X^*(p_i^t, \epsilon_{-i})$, since $i$’s payment never exceeds $p_i^t$ by individual rationality. This price is the Vickrey payment. Thus, it is an SPE for each bidder to stop at the Vickrey payment with respect to $v^t$. Formally, we define Vickrey-target strategy as follows.

**Definition 4** A strategy $\sigma_i^* \in \Sigma_i$ is said to be a **Vickrey-target strategy** if $\forall t$ and $\forall p_i^t$,

$$a_i^t = \begin{cases} C & \text{if } p_i^t < v_i - \bar{\pi}^{t-1}_i \\ S & \text{if } p_i^t = v_i - \bar{\pi}^{t-1}_i \end{cases}.$$  

Actions at rounds such that $p_i^t > v_i - \bar{\pi}^{t-1}_i$ are not specified. The optimal actions in such cases depend on the detail of an auction mechanism. However, the profile of any Vickrey-target strategies uniquely specifies a corresponding outcome because $p_i^t > v_i - \bar{\pi}^{t-1}_i$ arise off the equilibrium path only. The following theorems are the main result of this paper.

**Theorem 1** Suppose Assumptions 1 and 2. In every monotone ascending Vickrey-reserve auction, a profile of Vickrey-target strategies is an SPE.

**Proof.** See Appendix B.

**Theorem 2** The SPE outcome $\pi^*$ associated with Theorem 1 is in the bidder-optimal core with respect to the true values.

\(^{11}\)The efficient allocation with respect to true values is always chosen in the case of ties by Assumption 2.
Proof. See Appendix B.

Because of the assumption of complete information, the equilibrium possesses perfect foresight. In initial bidding rounds, every bidder is active and $\bar{\pi}^t = \bar{\pi}$. Hence, bidders first seek to stop bidding at their Vickrey payments. Once a bidder stops bidding, he can no longer renew his bid and his value is revealed. Each bidder revises his Vickrey payoff, considering the price of the stopping bidder as his true valuation. This revision weakly decreases active bidders’ Vickrey payoffs. The remaining bidders aim for the revised Vickrey prices. Losing bidders expect zero payoff $\bar{\pi}_i = 0$ and behave in an apparently truthful manner.

Note that Theorem 1 holds for any auction specification. In any ascending auction, an outcome in the true core is achievable in an equilibrium. Theorem 1 indicates an equivalence in equilibrium strategies of ascending Vickrey-reserve auctions. This is similar to the results of DM and Sano (2013), which show that a particular strategy profile is a Nash equilibrium of every sealed-bid auction. However, the equilibrium outcome $\pi^*$ can differ by ascending price rules.

Remark 1 Assumption 1 is not crucial. We obtain Theorem 1 without Assumption 1 by slightly modifying the strategy as follows. If two or more bidders (e.g., $\{i, j, \ldots \} \equiv M$) simultaneously face their stopping prices at $t$ with $\bar{\pi}_i^{t-1}$, $\bar{\pi}_j^{t-1}$, $\ldots$, then we find a maximal set $M^* \subseteq M$ that satisfies the following: (a) each $i \in M^*$ chooses $S$ at $t$, (b) each $i \in M \setminus M^*$ chooses $C$ at $t$, and (c) $X^{t-1} \subseteq X^t$.

3.1 Uniqueness

In typical ascending auctions such as Parkes and Ungar (2000), Ausubel and Milgrom (2002), winners pay the terminal prices. In such auctions, bidders never raise prices higher than the amounts necessary and sufficient to win. Therefore, the SPE outcome $\pi^*$ basically seems a unique SPE outcome.

The following theorem shows that the SPE outcome by Theorem 1 is a unique SPE outcome in the iBundle auction.
Theorem 3 Suppose Assumption 1, 2, and 3. If every losing bidder follows $\sigma_i^*, \pi^*$ is a unique SPE outcome in the iBundle auction.

Proof. See Appendix B.

For the uniqueness of SPE outcome, we need to take care of several possibility for multiple equilibria by indifferences. First, because any SPE is perfect foresight, losing bidders find it optimal to stop bidding at any round in an auction as long as they lose. Since any action is indifferent for losers in equilibrium, we focus on SPE such that losing bidders take the Vickrey-target strategy; i.e., they bid up until their true values. It would be natural and similar to those in several preceding studies (Ausubel and Milgrom 2002; DM).\footnote{This can be justified by trembling-hand perfection. If a loser stops at a price under his true value, he loses a chance to win with a small probability. Conversely, if he bids over the true values, he may suffer a loss.}

Second, when a winning bidder $i$ expects that his price will never increase further in equilibrium, it is indifferent for $i$ to choose $C$ or $S$. In such a case, the $i$’s action does not affect $i$’s payoff but it can affect the other bidders’ equilibrium payoffs through the influence on the price path of the others. Therefore, the uniqueness of SPE outcome is not obvious.

Theorem 3 contrasts with the case of sealed-bid auctions. When the Vickrey outcome is in the core, it is known to be a unique bidder-optimal outcome. Hence, the SPE outcome coincides with the Vickrey outcome. However, the Vickrey outcome is typically out of the core, and there are many bidder-optimal core outcomes. As shown by DM and Sano (2013), every bidder-optimal core outcome is achieved by a Nash equilibrium in a sealed-bid auction. Theorem 3 implies that the ascending price and subgame perfection work as equilibrium selection.

3.2 An Example

We view these results using a simple example of two goods and three bidders.
Example 1 There are two goods \{A, B\} and three bidders \{1, 2, 3\}. Suppose that bidder 1 wants good A, whereas bidder 2 wants B. Bidder 3 wants the bundle AB. Bidder 1’s value for A is 7, and bidder 2’s value for B is 8. Bidder 3’s value for AB is 10. In an efficient allocation, bidders 1 and 2 get A and B, respectively. The core of the auction game is described in terms of bidders’ payments as \(p_1(A) \leq 7\), \(p_2(B) \leq 8\) and \(p_1(A) + p_2(B) \geq 10\). In the bidder-optimal core, \(p_1(A) + p_2(B) = 10\).

Consider the iBundle auction of Parkes and Ungar (2000) and Ausubel and Milgrom (2002). In each round, the seller selects the tentative winning bids. \(J^t\) is the set of active tentative losers, and the auction terminates when \(J^T = \emptyset\).

At round 1, each bidder places bids of 1 for his interest: \(p_1^1(A) = p_2^1(B) = p_3^1(AB) = 1\). Thereafter, bidders 1 and 2 are tentative winners, and hence, bidder 3 raises the bid at round 2: \(p_2^2(AB) = 2\). If bidder 3 becomes the tentative winner at round 2, bidders 1 and 2 raise their bids to \(p_1^3(A) = p_2^3(B) = 2\) at round 3, and so on. If all bidders behave truthfully, then the auction terminates with \(p_1^T(A) = p_2^T(B) = 5\) and \(p_3^T(AB) = 10\). Bidders 1 and 2 win goods A and B, respectively, with the payment of 5.

In the SPE, bidders 1 and 2 plan to first stop at \(p_1^t(A) = 2\) and \(p_2^t(B) = 3\), respectively. Since \(p_1^t\) and \(p_2^t\) simultaneously increase in initial rounds, bidder 1 first stops at \(p_1^t = 2\) on the equilibrium path. Thereafter, bidder 2 updates the Vickrey payment from 3 to 8 and raises the price until \(p_2^t = 8\). Thus, bidders 1 and 2 win with the respective payments \(p_1 = 2\) and \(p_2 = 8\). This outcome is in the bidder-optimal core.

4 Multi-minded Bidders

We consider the general valuation case. Unfortunately, the logic in the previous section is not directly applied to a general “multi-minded” environment. We show that a straightforward extension of the Vickrey-target strategy is not optimal and that the iBundle auction may not achieve efficient allocation in the equilibrium. The
key observation for this negative result is non-monotonic payoff in the equilibrium in the single-minded model. In this section, we focus on the iBundle auction by Parkes and Ungar (2000) and Ausubel and Milgrom (2002).

4.1 Non-monotonic Payoff

When a winner has a relatively low valuation, he tends to earn a large profit in the SPE. Since a low-value bidder is likely to see his price reach his Vickrey price early on, he can enjoy the maximum payoff in the core. Conversely, a bidder with a relatively high value obtains a small payoff in the equilibrium due to the large amount of payment.

Example 1 (continued)  Recall the iBundle auction with two goods and three bidders. When the bidders’ actual values are \((v_1, v_2, v_3) = (7, 8, 10)\), bidders 1 and 2 respectively pay 2 and 8, and thus the equilibrium payoff allocation is \((\pi_0, \pi_1, \pi_2, \pi_3) = (10, 5, 0, 0)\). Note that bidder 1, who has a lower value than bidder 2, earns all the gains, whereas bidder 2 earns zero net payoff. Suppose that bidder 1’s value for A is \(v_1' = 9\), with everything else remaining the same. Then, in the equilibrium, bidder 2 stops at the price \(p_2^* = 1\) and bidder 1 bids up to 9 to win. The equilibrium payoff allocation is now \((10, 0, 7, 0)\). The equilibrium payoff of bidder 1 decreases as his valuation increases.

As this example shows, the SPE outcome is located on the corner of the bidder-optimal core. Moreover, the winner with a low value earns the Vickrey payoff, while the high-value winner earns 0.

This situation is similar to a free-rider problem. Suppose a private provision game of a public good whose marginal values differ among agents. In a Nash equilibrium, only the agent with the highest marginal value provides the good, while the others do not provide it at all.\(^{13}\) In our auction situation, bidders with low values free-ride on other bidders with higher values. Indeed, the incentive problem in core-

\(^{13}\)See Mas-Colell et al. (1995) for the free-rider problem in this public goods game.
selecting auctions is considered a kind of free-rider problem (Milgrom, 2000) called the “threshold problem.” In a sealed-bid format, the threshold problem is often interpreted as a type of coordination failure by bidders (Bykowsky et al., 2000). However, in an ascending-price format, it seems more appropriate to interpret the incentive problem as a free-rider problem.

4.2 Inefficiency in SPE

Payoff non-monotonicity makes the straightforward extension of the Vickrey-target strategy unprofitable for a multi-minded bidder and results in inefficiency even with complete information. The following example shows that the Vickrey-target strategy is not an SPE.

**Example 2** Suppose that there are three goods \{A, B, C\} and three bidders. Bidders 2 and 3 are single-minded. Their interests and value are given by \(v_2(B) = 7\) and \(v_3(ABC) = 10\). Bidder 1 is multi-minded and \(X_1 = \{x_1, A, C\}\): bidder 1 is interested in goods A and C, but not in the bundle. His valuation function is such that \(v_1(A) = 8\) and \(v_1(C) = 6\). In the efficient allocation, bidders 1 and 2 win the single goods A and B, respectively. Good C is unsold.

Suppose that the iBundle auction is conducted as in Example 1. For multi-minded environment, bidders face prices for each bundle of goods and respond with a demand correspondence in each round. Since bidders 2 and 3 are single-minded, the same argument applies for both of them, and the Vickrey-target strategy is defined. However, in contrast to the prior argument, bidder 1 faces a price vector \((p^1_1(A), p^1_1(C))\), which is initialized by \(p^1_1(A) = p^1_1(C) = 0\). Bidder 1 reports his demand correspondence under current prices. Throughout the auction, bidders are restricted by the Activity Rule, which requires them to report consistent demands based on revealed preferences. We can also define the Vickrey-target strategy for bidder 1.

\[\text{For a formal description of the ascending auction for general valuations, see Appendix A.}\]
The Vickrey payments for bidders 1 and 2 are $p_1^V = 3$ and $p_2^V = 2$, respectively. Hence, when every bidder follows the Vickrey-target strategy, bidder 2 stops earlier than bidder 1 as in Example 1. The associated terminal prices are $p_1(A) = 8$, $p_2(B) = 2$, (and $p_1(C) = 6$). Bidder 1’s corresponding payoff is 0.

Now, consider that bidder 1 behaves as if he is single-minded and interested only in $C$ with $v_1(C) = 6$. Under this specification, we have the SPE of the Vickrey-target strategies, in which bidder 1 wins good $C$ paying 3. Bidder 2 will need to pay 7. This is in fact a unique SPE outcome, which is verified as follows.\(^{15}\) The Activity Rule in Appendix A plays an important role in supporting this equilibrium. When bidder 1 initially makes a bid only for good $C$ under $(p_1(A), p_1(C)) = (0, 0)$, it implies that $\hat{v}_1(A) < \hat{v}_1(C)$. Since the Activity Rule restricts bidder 1 to report consistent demands, bidder 1 cannot raise $A$’s price to be greater than that of $C$. Hence, bidder 1 can never win good $A$ but only $C$. This makes the problem equivalent to the case where bidder 1 is a single-minded bidder interested in $C$. Hence, by Theorem 3, it is a unique SPE outcome in the subgame after the initial round that bidder 1 wins $C$ with a payment of 3 as long as bidder 3 follows the Vickrey-target strategy.

When bidder 1 reports demand for $A$ at the initial round, it implies that $\hat{v}_1(A) \geq \hat{v}_1(C)$. Since there is no chance for bidder 1 to win $C$, bidder 1 wins $A$ at price 8 in the subgame perfect outcome.\(^{16}\) Hence, it is not optimal for bidder 1 to report demand for $A$. Thus, when bidder 3 follows the Vickrey-target strategy, there exists a unique SPE outcome in which bidder 1 wins good $C$, and the efficient allocation is therefore not achieved.

**Proposition 2** The profile of Vickrey-target strategies is not an SPE with multi-minded bidders in general. In Example 2 and the iBundle auction, no efficient SPE exists as long as bidder 3 follows the Vickrey-target strategy.

This inefficiency stems from payoff non-monotonicity. Suppose that a bidder obtains a bundle of goods whose value is sufficiently large in an efficient allocation.

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\(^{15}\)The uniqueness is shown given that losing bidder 3 follows the Vickrey-target strategy.

\(^{16}\)When $\hat{v}_1(A) = \hat{v}_1(C)$, good $A$ is assumed to be allocated to bidder 1.
However, the true value is so large that other bidders stop earlier and he may end up paying too much. If he focuses on another good whose value is not so high, he may win it with a lower price.

**Remark 2** Inefficiency can exist in an SPE for another reason. Suppose that there are two goods and two bidders. Both bidders are interested in each individual goods as well as the package of both of them. In such a situation, bidders may coordinate bids and split up goods with low prices in an equilibrium even if the efficient outcome would result in one bidder winning both goods. This can be interpreted as implicit collusion. Such an equilibrium is considered by Ausubel and Schwartz (1999) and Grimm et al. (2003) in an ascending auction without package bidding.

5 Conclusion

We formulate a general class of ascending-price auctions with single-minded bidders. The Vickrey-target strategy is an SPE of every ascending Vickrey-reserve auction. The equilibrium outcome lies in the bidder-optimal core and is unique in a particular auction. In any ascending auction, bidders have an identical equilibrium strategy that leads to an outcome in the core. These results are positive findings as sealed-bid Vickrey-reserve auctions may have multiple Nash equilibria. Ascending price and subgame perfection can work as equilibrium selection.

The equilibrium outcome, however, can be “unfair” in the sense that bidders with lower values tend to obtain higher payoffs. This situation is similar to a standard free-rider problem. Non-monotonic payoff also provides inefficiency in the case of general valuations. The process of deriving an SPE under general valuations is open for future work.

A Ascending Auctions for General Valuations

For the general valuation case, we consider non-linear and non-anonymous prices $p \in \mathbb{R}^{\sum |X_i|}$. An element in $p$ is denoted by $p_i(x_i)$, which indicates the price of
bundle $x_i$ for $i$. For null bundles, let $p_i(x_i) \equiv 0$ for all $i$.

For a price vector $p$, let $D_i(p)$ be $i$’s (true) demand correspondence:

$$D_i(p) \equiv \arg \max_{x_i \in X_i} v_i(x_i) - p_i(x_i).$$

In each round of an ascending auction, each bidder reports his demand correspondence, which is denoted by $\hat{D}_i(p^t)$. Bidder $i$ is active at $t$ if for all $\tau \leq t$, $x_i \notin \hat{D}_i(p^\tau)$.

Following Mishra and Parkes (2007), an ascending clock auction for general valuations is defined as follows.

1. The auctioneer initializes the price vector as $p^1 = (0, \ldots, 0)$.

2. At each round $t = 1, 2, \ldots$, each bidder reports his demand correspondence $\hat{D}_i(p^t)$. The auctioneer chooses a set of active bidders $J^t \subseteq I^t$. If $i \in J^t$ and if $x_i \in \hat{D}_i(p^t)$, then $p_i^{t+1}(x_i) = p_i^t(x_i) + 1$. Otherwise, let $p_i^{t+1}(x_i) = p_i^t(x_i)$.

3. The process terminates (without loss of generality) at $T$ when $I^T = \emptyset$. The auctioneer selects an allocation $x \in X$ and determines bidders’ payments $p \in \mathbb{R}^n$.

During the auction, bidders are restricted by the following Activity Rule so that there exists a valuation function consistent with bidder’s reports.

**Assumption 4 (Activity Rule; Mishra and Parkes, 2007)** Each bidder must satisfy the following:

1. For all $t$, $\hat{D}_i(p^t) \subseteq \hat{D}_i(p^{t+1})$.

2. If $x_i \subseteq x_i'$ and $x_i \in \hat{D}_i(p^t)$, then $x_i' \in \hat{D}_i(p^t)$.

A strategy for $i$ is a mapping from his decision nodes to demand sets satisfying the Activity Rule. The revealed valuation function $\hat{v}_i : X_i \rightarrow \mathbb{R}_+$ is defined by $\hat{v}_i(\cdot) \equiv p_i^T(\cdot)$. The provisional valuation function $v_i^t(\cdot)$ is defined in the same manner. Given $v^t = (v_i^t)_{i \in I}$, $V^t$, $X^t$, and $\pi^t_i$ are defined in the same manner. The Vickrey-target strategy for the general valuation case is as follows.
Definition 5 A strategy is the Vickrey-target strategy if \( \forall t \) and \( p' \),

\[
\hat{D}_i(p') = \begin{cases} 
D_i(p') & \text{if } p'_i(x_i) < v_i(x_i) - \bar{\pi}^{t-1}_i \text{ for all } x_i(\neq \hat{x}_i) \in X_i^{t-1} \text{ or } X_i^{t-1} = \{\hat{x}\} \\
X_i & \text{otherwise.}
\end{cases}
\]

B Proofs

B.1 Proof of Proposition 1

We prove by induction. Suppose there are \( m \) active bidders at round \( t \).

Step 1 Suppose that \( m = 1 \) and bidder \( i \) is active. Since \( p'_i \leq v_i \) for all \( s \) by assumption, \( X^s = X^{s-1} \) for all \( s \geq t \) as long as \( i \) is active.

If \( X_t^i = \{\hat{x}_i\} \), clearly \( X^s = X^t \) for all \( s \geq t \), thus we have done. Suppose that there exists \( x'^{t-1} \in X'^{t-1} \) with \( x'_{i-1} = y_i \) and that \( p'_i \leq v_i - \bar{\pi}^{t-1}_i \). Consider that bidder \( i \) stops at \( s \geq t \) and that \( d \equiv v_i - \hat{v}_i \). Note that every other bidder has already stopped, so \( v_{t-i} = \hat{v}_{-i} \). By DM and Sano (2013), \( \hat{v}_i = v_i - \bar{\pi}_i \) is among the best responses given \( \hat{v}_{-i} \), but any \( \tilde{v}_i < v_i - \bar{\pi}_i \) is not. Hence, \( d \leq \bar{\pi}_i^{t-1} \) in every SPE. Then,

\[
\max_{x \in X} [(v_i - d)1(x_i = y_i) + \sum_{j \neq i} \hat{v}_j(x_j)] = V^{t-1}(N) - d.
\]

On the other hand, the total value of \( x'^{t-1} \) is

\[
v_i - d + \sum_{j \neq i} \hat{v}_j(x'^{t-1}_j) = V^{t-1}(N) - d.
\]

Therefore, \( x'^{t-1} \in X^s \).

It is trivial in the case that \( p'_i > v_i - \bar{\pi}_i^{t-1} \).

Step 2 Suppose that \( m \geq 2 \) and the proposition is true for \( \forall m' \leq m - 1 \). Let \( i \) be the bidder taking an action at round \( t \). Hence, \( v^t_{t-i} = v_{t-i}^{t-1} \).

Step 2.1 Suppose \( X_i^{t-1} = \{\hat{x}_i\} \). Then,

\[
\hat{v}_i + \sum_{j \neq i} v'_j(\hat{x}_j) < \max_{j \neq i} v'_j(x_j) = V^t(N_{-i}).
\]
for all $\hat{v}_i \leq v_i$ and for all $\hat{x} \in \{x \in X | x_i = y_i \}$. Hence, we have $X_i^\dagger = \{\hat{x}_i\}$ and $X^\dagger = X^\dagger - 1$.

**Step 2.2** Suppose $y_i \in X_i^\dagger - 1$. Suppose that bidder $i$ stops at $t$. Let $\hat{v}_i$ be the revealed value and $\hat{d} \equiv v_i - \hat{v}_i$.

If $d > \bar{\pi}_i^t - 1$, then, for any $\hat{x} \in \{x \in X | x_i = y_i \}$,

\[
v_i - d + \sum_{j \neq i} v_j(x_j) \leq V^t - 1(N) - d < V^t - 1(N - i) = V^t(N - i).
\]

Hence, $X_i^\dagger = \{\hat{x}_i\}$. The induction hypothesis and Assumption 2 imply that in any SPE, $i$ must obtain $\hat{x}_i$.

Suppose $d \leq \bar{\pi}_i^t - 1$. Then,

\[
\max_{x \in X} [(v_i - d)1_{X_i = y_i} + \sum_{j \neq i} v_j(x_j)] = V^t - 1(N) - d.
\]

On the other hand, the total value of $x^t - 1$ is

\[
v_i - d + \sum_{j \neq i} v_j(x^t - 1) = V^t - 1(N) - d.
\]

Therefore, $x^t - 1 \in X^t$. By the induction hypothesis, $x^t - 1 \in X^*(\hat{v})$.

Hence, it is not optimal to stop in the case of $d > \bar{\pi}_i^t - 1 > 0$. In addition, by Assumption 3, bidder $i$ does not stop in the case of $d > \bar{\pi}_i^t - 1 = 0$ either. Therefore, $x^t - 1 \in X^*$ for all $s \geq t$.

**B.2 Proof of Theorem 1**

We suppose every bidder chooses $S$ if $p_i^t > v_i - \bar{\pi}_i^t - 1$ and $I^t = \{i\}$. We verify that this also constitutes a part of an equilibrium.

We prove by induction. Suppose there are $m$ active bidders at round $t$.

**Step 1** Suppose $m = 1$ and bidder $i$ is active. Note that every other bidder has stopped and that we have $v_i = \hat{v}_i - \bar{\pi}_i$. By DM and Sano (2013), $\hat{v}_i = v_i - \bar{\pi}_i$ is among the best responses given $\hat{v}_i$. Hence, the Vickrey-target strategy is obviously optimal.
if \( p_t^i \leq v_i - \bar{\pi}^{t-1}_i \). If \( p_t^i > v_i - \bar{\pi}^{t-1}_i \), it is optimal to choose \( S \) because the auction is monotone.

**Step 2** Suppose that there are \( m \geq 2 \) active bidders at \( t \). Further, suppose that every active bidder follows the Vickrey-target strategy \( \sigma^*_i \) after \( m' \leq m - 1 \) bidders remain active and that this constitutes an equilibrium for \( m - 1 \) bidders.

Consider \( \bar{\pi}^s_i = V^s(N) - V^s(N_{-i}) \) for any \( s > t \). Suppose that all the active bidders at \( t \) except for \( i \) follow the Vickrey-target strategy. Then, \( V^s(N) \) decreases at \( s \) by \( \bar{\pi}^{s-1}_i \) if and only if some \( j \) chooses \( S \). At the same time, \( V^s(N_{-i}) \) decreases at \( s \) by at most \( \bar{\pi}^{s-1}_j \). Hence, \( \bar{\pi}^s_i \) is nonincreasing in \( s \) as long as every other bidder follows the Vickrey-target strategy \( \sigma^*_j \).

Suppose that \( X^t_{i-1} = \{x_i\} \). No bidder overstates the values when he follows the Vickrey-target strategy. Hence, by the proof of Proposition 1, \( x_i = x_{i, d} \) for \( \forall x \in X^*(\hat{v}) \) for any strategy such that \( \hat{v}_i \leq v_i \). Suppose that bidder \( i \) continues bidding until \( p_t^i > v_i \) and that for some \( t \), \( y_i \in X_t^i \). Let \( t \) be the minimum of such \( t \). Then, \( \{x_i, y_i\} = X_t^i \) since the bid increment is unity. Hence, \( V^t(N) = V^t(N_{-i}) \). Bidder \( i \) has to pay at least \( p_t^V(h) \) for \( y_i \) and his payoff is at most

\[
v_i - p_t^V(h) = v_i - \hat{v}_i + \hat{V}(N) - \hat{V}(N_{-i}) \\
\leq v_i - p_t^i + V^t(N) - V^t(N_{-i}) \\
< 0.
\]

The weak inequality comes from the fact that \( \bar{\pi}^s_i \) does not increase in \( s \) in which bidders except for \( i \) make decisions. Hence, it is never optimal to bid over the true valuations, and thus, the Vickrey-target strategy is an optimal strategy.

Suppose that \( \exists x^{t-1} \in X^{t-1} \) and \( x^{t-1}_{i-1} = y_i \). Suppose that bidder \( i \) stops bidding at \( t \). If \( d = v_i - \hat{v}_i > \bar{\pi}^{t-1}_i \), then \( X^t_i = \{x_i\} \) by the consideration in Proposition 1. By the induction hypothesis and Assumption 2, \( i \) obtains \( x_i \) in the end and \( \pi_i = 0 \). On the other hand, if \( d \leq \bar{\pi}^{t-1}_i \), then \( x^{t-1} \in X^t \) and \( i \) obtains \( y_i \) in the end. Since
bidder $i$ pays at least the Vickrey payment $p_i^V(h)$, bidder $i$'s payoff $\pi_i$ is

$$\pi_i \leq v_i - p_i^V(h)$$

$$= v_i - \hat{\pi}_i + \hat{V}(N) - \hat{V}(N_{-i})$$

$$\leq V^{t-1}(N) - V^{t-1}(N_{-i}) = \bar{\pi}_i^{t-1}.$$ 

The second inequality comes from the weak monotonicity of $\bar{\pi}_i^s$ in $s$. If $d = \bar{\pi}_i^{t-1}$, then $x_i^{t-1} \in X^s(\hat{v})$ and bidder $i$ wins $y_i$.

Bidder $i$ earns at least $\bar{\pi}_i^{t-1}$ by individual rationality. Since $\bar{\pi}_i^s$ is nonincreasing in $s$, it is suboptimal to stay active whenever $m$ bidders are active. If the current price satisfies $p_i^t > v_i - \bar{\pi}_i^{t-1}$, the optimal action is not determined but depends on the detail of the auction mechanism. However, in any case, the SPE payoff for $i$ is bounded at $\bar{\pi}_i^{t-1}$. Hence, it is optimal to choose $S$ at $t$ when $p_i^t = v_i - \bar{\pi}_i^{t-1}$, and such cases that $p_i^t > v_i - \bar{\pi}_i^{t-1}$ do not arise on path. Therefore, the Vickrey-target strategy is among the best responses for $i$. ■

**B.3 Proof of Theorem 2**

Let $x^* \in X^*(\hat{v})$ be the resulting allocation. In the SPE of Theorem 1, every losing bidder earns zero payoff and every winning bidder pays the amount of his stopping price. Thus, $\hat{\pi}_i = 0$ for all $i \in I$. Hence, for all $J \subseteq N$,

$$\hat{V}(J) \leq \hat{V}(N) = \pi_0^*.$$ 

(4)

\[\frac{17}{17}\text{If there is another allocation } \bar{x} \in X^{t-1}, \text{ where } i \text{ obtains } \bar{x}_i, \text{ and if it is selected by the tie-breaking rule, then } \bar{\pi}_i^{t-1} = 0. \text{ It is still optimal to follow the Vickrey-target strategy by the same consideration as the case of } X_i^{t-1} = \{x_i\}.\]
Note that each bidder’s true payoff $\pi^*_i = v_i - \hat{v}_i$ for both winners and losers. Therefore, for any coalition $J$ including the seller,

$$\sum_{j \in J} \pi^*_j \geq \hat{V}(J) + \sum_{j \in J_0} \pi^*_j$$

$$= \max_{x \in X} \left[ \sum_{j \in J_0} \hat{v}_j(x_j) \right] + \sum_{j \in J_0} \pi^*_j$$

$$\geq \max_{x \in X} \left[ \sum_{j \in J_0} (v_j - \pi^*_j) \right] + \sum_{j \in J_0} \pi^*_j$$

$$\geq V(J).$$

The first inequality is from (4). The second inequality comes from

$$\hat{v}_i(x_i) = \begin{cases} v_i - \pi^*_i & \text{for } x_i = y_i, \\ 0 & \text{for } x_i = \bar{x}_i. \end{cases}$$

By Theorem 1, any winner $i$ is blocked (will obtain nothing) if he stops one round earlier. This implies that if a bidder’s payment is decreased by unity, then the seller chooses a different revenue-maximizing allocation. Hence, $\pi^*$ is bidder-optimal. ■

**B.4 Proof of Theorem 3**

Since winning bidders pay the terminal prices, no bidder has incentive to raise his price over the true value in equilibrium. Thus, any SPE is efficient by Proposition 1. Since all losers reveal their true valuations in equilibrium, for any winning bidder $i$, $\hat{\pi}_t^i$ is nonincreasing in $t$ in any equilibrium by the argument in the proof of Theorem 1.

By Proposition 1, every winner $i$ stops bidding only if $p^i_t \geq v_i - \hat{\pi}_t^{i-1}$. If it is optimal to choose $C$ when $p^i_t = v_i - \hat{\pi}_t^{i-1}$, then it should be the terminal price for $i$. That means that it is indifferent for $i$ to choose $C$ or $S$ at $t$. We need to show that $i$’s action at $t$ does not affect the equilibrium price path.

In what follows, we consider iBundle auction and use the term “round” as its original sense. That is, each round $t$ has $n$ subrounds. Bidder $i$ makes a choice
at subround $i$ of round $t$. In addition, if $p_i^t = p_i^{t-1}$, then subround $i$ of round $t$ is assumed to be skipped.

Suppose that $p_i^t = v_i - \pi_i^{t-1}$ and that bidder $i$ chooses $C$ at $t$ in an SPE. For each feasible allocation $x$, let $U(x) \equiv \{j \in I|x_j = x_j\}$, which denotes the set of unsatisfied bidders under $x$. Then, the set of bidders facing price increase at $t+1$, $J^t(h^t)$, is denoted by

$$J^t(h^t) = J(h^t) \equiv U(x(t)) \cap I^t.$$  

We are going to show $J(h^s)$ is independent from $a_i^t$ for all $s \geq t$. Since $x(t) \in \arg \max \sum p_i^t(x_i)$, bidder $i$’s action $a_i^t$ at $t$ does not affect $U(x(t))$. Since $i$’s terminal price is $p_i^t$ regardless of $a_i^t$, we have $i \notin U(x(t))$. Hence, $J(h^t)$ is the same for each of $a_i^t \in \{C, S\}$.

Given that $a_i^t = C$ and $p_i^t$ is the terminal price for $i$, we have $i \notin U(x(s))$ for all $s > t$ in equilibrium by $i \notin J(h^s)$. Hence,

$$x(s) \in \arg \max_{\{x \in X|y_i\}} p_i^t + \sum_{j \neq i} p_j^s(x_j).$$

When bidder $i$ chooses $a_i^t = S$ at $t$, tentative allocation at $s > t$ is selected from

$$\arg \max p_i^t(x_i) + \sum_{j \neq i} p_j^s(x_j).$$

Hence, we have $x(s)$ is independent from $a_i^t \in \{C, S\}$ for all $s > t$, and $J(h^s)$ is not affected by $a_i^t$. □

**References**


