Demographic Structure, Adjustment Costs, and Asset Prices *

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Abstract

Population of advanced economies is aging rapidly while emerging countries follow closely the same transformation. This paper investigates the impact of demographic structure changes on asset prices. Demographic structure is mainly driven by three phenomena: the global fertility rate, the increase in life expectancy and the postponing of motherhood. We study the impact of these three factors on asset prices in a three period overlapping generations closed economy with adjustment costs. We consider two working adulthood periods during which agents may have children. We simulate the demographic transitions and their consequences on asset prices. The early and later birth rates, as well as the life expectancy are picked from UN’s historical and projection data for France. An increase of 15% of asset prices is observed in the 3 generations simulation. In order to give a more detailed analysis of demographic effects, we extend the simulations to 16 generations with different behaviors for each generation of five years.

Keywords: Aging, Asset Prices, Overlapping generations

JEL Classification: J1, G12, E21

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1 Introduction

Over the last several decades, major advanced economies have experienced significant demographic changes. In the post second world war period, these economies exhibit important rises in the number of births. This baby-boom is a consequence of a positive transitory shock on the total fertility rate. Since the mid-sixties, however, subsequent decline in the fertility rate has occurred. Combined with this baby bust, gains in life expectancy has caused the aging of population. In addition, the postponement of motherhood is observed as a recent common trend. Longer education and higher women labor force participation have increased the age at first birth. Aging of population is rapidly becoming a global phenomenon. For the next few decades many currently developing economies will experience similar demographic transitions.

How does demographic structure affect asset prices? A common idea is that the asset prices is influenced by the relative cohort size of young generation and old generation, which are considered correspondingly the buyers and sellers of assets. The asset demands of a large working generation, such as the baby boomers, drives asset prices up. When the large old generation reaches retirement, without salary income, baby boomers will need to sell assets to support retirement consumption. The economy is aging and the ensuing young generation is relatively smaller. This mismatch between asset supply and demand makes the asset prices decline. In this view, the demographic changes have consequences at the market-wide level.

In the literature, a great variety of economic models suggest a link between demographic structure and asset prices. A common framework to investigate the demographic effect on asset prices is an overlapping generations (OLG) model with production and capital accumulation as in Diamond (1965). In standard neo classical growth model, there are no capital adjustment costs. In this case, investment is determined as output unconsumed. Therefore, the price of capital is constantly equal to one unit of consumption goods. Introduction of the adjustment cost in the model leads the capital to be priced endogenously. Abel (2003) studies the theoretical effects of a baby boom on stock prices and capital accumulation in an OLG model with adjustment cost. He founds that after an increase in stock prices during their working age, the retirement of baby-boomers leads to a meltdown in stock prices.

Empirical researches have found evidence that population age structure affects asset prices, but the consistency of this evidence is not overpowering. Even if theoretical models suggest an association between population age structure and asset prices, it is difficult to find robust evidence of their correlation. Poterba (2001) analyzes the historical relationship between population structure and real returns from investment in Treasury bills, long-term government bonds and stocks. His estimates offered weak evidence of a demographic impact on returns for the U.S, Canada, and the U.K. The complication of these empirical researches is the choice of the dependent variables that represent the asset prices and the independent variables that represent demographic factors. Besides econometric specification problems, the estimated effects are usually sensitive to the periods and to the countries included in the sample. These unfortunately unsatisfactory findings in empirical studies make the simulated models an important instrument for understanding and evaluating the conceiv-
able consequences of demographic transition.

In this paper we study the impact of demographic structure changes on asset prices. Demographic structure is mainly driven by three phenomena: the total fertility rate, the increase in life expectancy and the postponement of motherhood. The aging of the population may result either from a decrease in the total birth rate, an increase in life expectancy or a postponing of the average age of motherhood. Baby boom has transitory consequences on population structure dynamics, but contrarily to what is usually thought, it has no particular impact on aging as the larger number of baby-boomers have a larger number of children.

We build on Abel (2003) by adding another generation to his model. We have therefore a three period OLG model with adjustment cost. Our population structure is based on Pestieau and Ponzière (2014) and Momota and Horii (2013) where workers are divided in two categories: young and old. This allows us to incorporate the postponement of motherhood in the model with two fertility rates: one for young workers and another for old workers.

The main goal of this paper is to study the impact of the postponing of maternity and the increase in life expectancy on asset prices. The rest of the paper is organized as follows. The second section gives a presentation of the model with a complete demographic structure characterized by three demographic parameters (two birth rates plus a surviving probability at 60 years old). The third section is an numerical exercise on the relative effects of the three demographic parameters on asset prices. In the fourth section, we calibrate a more detailed model on France in order to compare our results to stylized facts. Last section concludes.

2 The Model

Consider a three overlapping generations (OLG) economy in which agents live for 4 periods: childhood, young working age, old working age, and retirement. Only the last three generations make decisions. Children only consume a fixed share of their parent’s consumption. During the two working periods, agents can have children, supply labor in-elastically, receive a wage, consume part of their incomes and save the rest. The demographic structure is a little more complicated as usual because all workers (young and old) have children. Old workers pass to retirement period with a probability \( p \). When retired, they consume the totality of their savings with interest.

2.1 Demography

The demographic structure is as follows, at time \( t \), there are four groups of agents in the economy, where \( N_t \) is the number of children, \( N_{t-1} \) the number of young workers, \( N_{t-2} \) the number of old workers, and \( pN_{t-3} \) the number of retirees.

The model will thus have three demographic parameters: the number of child per young workers \( (n) \), the number of child per old workers \( (m) \), and the surviving probability at around 60 years old \( (p) \). This will allow us to describe all aging phenomena: i) baby-booms with transitory increases of both fertility rates, ii) the postponing of the average age of motherhood with a combination of permanent decrease in young’s fertility rates and a permanent increase in
old's fertility rate, iii) permanent decrease in the birth rate with permanent decrease in both fertility rates, and iv) the increase in expected life at 60. The impact of the first phenomena on asset prices has been studied in Abel (2003).

The individual has two reproduction periods, the second and the third periods of life with respectively two exogenous birth rates \( n_t \) and \( m_t \). So the total births at the period \( t \) is:

\[
N_t = n_t N_{t-1} + m_t N_{t-2}
\]

The cohort size growth factor is:

\[
G_t = \frac{N_t}{N_{t-1}} = n_t + m_t G_{t-1}
\]

### 2.2 Consumers

An adult consumer in \( t \) born as a child at the beginning of period \( t-1 \) chooses consumptions when young \((c_t)\), in middle age \((d_{t+1})\), and when old \((e_{t+2})\) in order to maximize his lifetime welfare subject to the inter-temporal budget constraint. Let \( p \) be the probability of survival from old active period to retirement. Assume that \( u() \) is the utility function, the agent born at the beginning of period \( t-1 \) has the following expected lifetime utility in \( t \):

\[
U(c_t,d_{t+1},e_{t+2}) = u(c_t) + \beta u(d_{t+1}) + p\beta^2 u(e_{t+2})
\]

where \( \beta \) is a time preference factor with \( 0 < \beta < 1 \). Assume that consumers have logarithmic utility,

\[
u(c) = ln(c)
\]

The present value of the lifetime resources of this consumer is:

\[
\Omega_t \equiv w_t + \frac{w_{t+1}^{e}}{R_t^{e}}
\]

The instantaneous budget constraints for the three periods are:

\[
c_t = w_t - s_t
\]

\[
d_{t+1} = w_{t+1}^{d} + R_t^{e} s_t - z_{t+1}
\]
\[ c_{t+2} = \frac{R_{t+2}^e}{p} z_{t+1} \]

assuming a perfect annuity market.

Note that \( s_t \) is the saving of the young active agent and \( z_{t+1} \) is not the saving of the old active agent but his wealth. His saving is written as:

\[ w_{t+1}^e + (R_{t+1}^e - 1)s_t - d_{t+1} = z_{t+1} - s_t \]

The inter-temporal budget constraint of an agent born at the beginning of period \( t-1 \) is:

\[ c_t + \frac{d_{t+1}}{R_{t+1}} + \frac{pc_{t+2}}{R_{t+1}R_{t+2}} = \Omega_t \]

By solving the optimization problem, we get the optimal consumptions as:

\[ c_t = \frac{1}{1 + \beta + p\beta^2} \Omega_t \quad (2) \]
\[ d_{t+1} = \frac{\beta R_{t+1}^e}{1 + \beta + p\beta^2} \Omega_t \quad (3) \]
\[ e_{t+2} = \frac{\beta^2 R_{t+1}^e R_{t+2}^e}{1 + \beta + p\beta^2} \Omega_t \quad (4) \]

and the optimal wealth as:

\[ s_t = \frac{1}{1 + \beta + p\beta^2} ((\beta + p\beta^2)w_t - \frac{w_{t+1}^e}{R_{t+1}^e}) \quad (5) \]
\[ z_{t+1} = \frac{p\beta^2}{1 + \beta + p\beta^2} (R_{t+1}^e w_t + w_{t+1}^e) \quad (6) \]

2.3 Production

Production technology uses both capital and labor. Capital must be installed in our model thus is subject to adjustment costs as in Abel(2003). The capital also depreciates at rate \( \delta \). Assume that the consumption good technology use a Cobb-Douglas production function:

\[ Y_t = AK_t^\alpha L_t^{1-\alpha} \]

If we note \( k_t \equiv \frac{K_t}{L_t} \), consumption good production per worker will be:

\[ f(k_t) = Ak_t^\alpha \quad (7) \]

The depreciation is assumed to be linear thus:

\[ K_{t+1} = (1 - \delta)K_t + I_t \quad (8) \]

where \( 0 \leq \delta \leq 1 \), and investment, \( I_t \), is the aggregate amount of consumption goods diverted from the consumption goods technology and used as an input in the capital accumulation.
The adjustment cost function is assumed to be quadratic in investment:

\[ C_t = \left( \frac{a}{2} \right) \frac{I_t^2}{K_t} \]  

(9)

where the parameter \( a \geq 0 \) measures the magnitude of quadratic adjustment cost.

Let \( q_t \) be the price of capital at the end of period \( t \), which is the quantity of consumption goods in the period \( t \) that needed to acquire an additional unit of capital for use in period \( t + 1 \), \( w_t \) the wage rate and \( R_t \) the gross return on capital held from period \( t - 1 \) to \( t \). The producer maximizes its profits through the following program:

\[
\max_{K_t, L_t, I_t} \pi = Y_t + q_t K_{t+1} - w_t L_t - R_t q_{t-1} K_t - I_t - C(I_t, K_t)
\]

(10)

\( q_{t-1} K_t \) is the value of the capital used in \( t \) and the firm has to borrow to acquire capital.

Assuming that the economy is perfectly competitive, the firm take \( q_t, w_t, \) and \( R_t \) as given to maximize its profit. First order condition are given by:

\[
\frac{\partial \pi_t}{\partial L_t} = 0 \iff \frac{\partial Y_t}{\partial L_t} - w_t = 0
\]

\[
\frac{\partial \pi_t}{\partial K_t} = 0 \iff \frac{\partial Y_t}{\partial K_t} + q_t - \frac{\partial C_t}{\partial K_t} - R_t q_{t-1} = 0
\]

\[
\frac{\partial \pi_t}{\partial I_t} = 0 \iff q_t - 1 - \frac{\partial C_t}{\partial I_t} = 0
\]

and one gets

\[
w_t = f(k_t) - f'(k_t)k_t
\]

(10)

\[
R_t = \frac{f'(k_t) + q_t(1 - \delta) + \frac{a}{2} \left( \frac{I_t}{K_t} \right)^2}{q_{t-1}}
\]

(11)

\[
q_t = 1 + a \left( \frac{I_t}{K_t} \right)
\]

(12)

We can check that the profit in \( t \) of the firm \( F(K_t, L_t) + q_t K_{t+1} - w_t L_t - R_t q_{t-1} K_t - I_t - C(I_t, K_t) = 0 \).

2.4 Market equilibria

The labor market is at its equilibrium when the labor demand \( L_t \) is equal to the working-age population:

\[
L_t = N_{t-1} + N_{t-2}
\]

(13)

The output of the good technology is used as consumption and investment. So the equilibrium of the good market is given by:

\[
F(K_t, L_t) = N_{t-1} c_t + N_{t-2} d_t + p_t N_{t-3} c_t + I_t + C(I_t, K_t)
\]

(14)
The wealth market is written:

\[ N_{t-1} s_t + N_{t-2} z_t = q_t K_{t+1} \]

per head of workers, one has:

\[ \frac{N_{t-1}}{L_t} s_t + \frac{N_{t-2}}{L_t} z_t = q_t \frac{L_{t+1}}{L_t} k_{t+1} \]

(15)

3 A numerical exercise

In this section, we examine the quantitative effects on asset prices of demographic parameters in the model of previous section. The child generation is thus 0 to 24 years old. The young workers are 25 to 49 years old. The old workers are 50 to 74 years old. And the retiree are 75 and more. The numerical exercise gives an idea of the different expected magnitude of price variations. A simulation closer to stylized facts is given in section 4.

3.1 Numerical parameters

In order to simulate the long-run effects of the demographic change, we need to give the parameters for the model. For a generation of 25 years, we chose a complete depreciation, i.e. \( \delta = 1 \). The time preference factor is set at \( \beta = 0.366 = 0.9^{(25/4)} \), corresponding to a quarterly time preference factor of 0.99. Parameters are summarized in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total factor productivity</td>
<td>( A = 10 )</td>
</tr>
<tr>
<td>Capital share in production</td>
<td>( \alpha = 0.33 )</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>( \delta = 1 )</td>
</tr>
<tr>
<td>Adjustment cost parameter</td>
<td>( a = 1.5 )</td>
</tr>
<tr>
<td>Time preference factor</td>
<td>( \beta = 0.366 )</td>
</tr>
</tbody>
</table>

3.2 Sensitivity of capital prices to demographic variables

We simulate the model under four scenarios by changing the three demographic variables: fertility rates for young parents \( (n) \) alone, fertility rate for old parents \( (m) \) alone, survival probability \( (p) \) alone and all the three at the same time.

The four scenarios consider the following cases:

- a decrease of \( n \) from 0.75 to 0.5;
- an increase of \( m \) from 0.25 to 0.5;
- an increase of \( p \) from 0.2 to 0.5;
- all the three previous fluctuations.
In all simulations, the initial condition is given as $N(t) = 1, \forall t \leq 0$. We assume a complete capital depreciation ($\delta = 1$). The figure 2 gives us the variations of the demographic parameters and the reaction of the capital price.

Here, we show pure demographic effects on the capital price in the transitional dynamics. A decrease of $n$ induces a decrease in capital price. The mechanism is due to a pure quantitative effect of the cohort size. A decrease in $n$, reduces the labor size at the following period, thus the needs in capital. So the demand falls leading to a lower price. An increase in $m$ has the opposite effect with a longer lag on the increase of the labor force size. A growing probability of survival leads to a slight increase of prices compared to the fertility effects. We can also find that the effects of the two fertility rates are neutralized after combining the three effects.

### 3.3 French demographic evolution

If we assume that the four periods of life have the same length and thus childhood is from 0 to 24, young adult are aged 25-49, older workers are between 50 to 74, and retired people are more than 75. Computing $n$ and $m$ for France from UN demographic data leads to a positive $n$ and $m=0$ as people don’t have children after 50.

Before turning to a real calibration exercise, we compute situation where $n$ is equal to the observed values for 15 to 24 years old parents ($n$) and for the 25-49 years old parents ($m$). All the data are from the UN “World Population Prospects: The 2012 Revision”. In the empirical case, the variation of the capital prices is more important since the fluctuations of the demographic parameters are more impressive. The results are given in the figure 3. As before, we try to decompose the impact of each of the demographic variables on the capital price. The decrease of the early birth rate give a negative impact on the capital price while the increase of the later birth rate influences positively. The improvement of the survival probability have at the first a positive and then a negative effect on capital prices. Finally, the total effect of all the three demographic variables forms an inverted u shaped curve of asset prices.
We now turn to a more realistic model with shorter periods (5 years instead of 25 years).

4 Calibration and stylized facts

4.1 Demographic evolutions

In this section, we want to compare the relative impact on asset prices of the demographic parameters. With five years for each period, we therefore have a 16 generation model (20 with the children). The behaviours of individual in each period of life are summarized in the following table.
Table 2: The periods of life

<table>
<thead>
<tr>
<th>age</th>
<th>period</th>
<th>fertility</th>
<th>consumption</th>
<th>saving</th>
<th>participation rate</th>
<th>survival probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-4</td>
<td>children</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5-9</td>
<td>children</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10-14</td>
<td>children</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>15-19</td>
<td>children</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20-24</td>
<td>worker and parent</td>
<td>50,4</td>
<td>c0,4</td>
<td>s0,4</td>
<td>ϕ0,4 = 1</td>
<td>p0,4 = 1</td>
</tr>
<tr>
<td>25-29</td>
<td>worker and parent</td>
<td>51,5</td>
<td>c1,5</td>
<td>s1,5+1</td>
<td>ϕ1,5+1 = 1</td>
<td>p1,5+1 = 1</td>
</tr>
<tr>
<td>30-34</td>
<td>worker and parent</td>
<td>52,6</td>
<td>c2,6</td>
<td>s2,6+2</td>
<td>ϕ2,6+2 = 1</td>
<td>p2,6+2 = 1</td>
</tr>
<tr>
<td>35-39</td>
<td>worker and parent</td>
<td>53,7</td>
<td>c3,7+3</td>
<td>s3,7+3</td>
<td>ϕ3,7+3 = 1</td>
<td>p3,7+3 = 1</td>
</tr>
<tr>
<td>40-44</td>
<td>worker and parent</td>
<td>54,8</td>
<td>c4,8+4</td>
<td>s4,8+4</td>
<td>ϕ4,8+4 = 1</td>
<td>p4,8+4 = 1</td>
</tr>
<tr>
<td>45-49</td>
<td>worker and parent</td>
<td>55,9</td>
<td>c5,9+5</td>
<td>s5,9+5</td>
<td>ϕ5,9+5 = 1</td>
<td>p5,9+5 = 1</td>
</tr>
<tr>
<td>50-54</td>
<td>worker</td>
<td>56,10</td>
<td>c6,10+6</td>
<td>s6,10+6</td>
<td>ϕ6,10+6 = 1</td>
<td>p6,10+6 = 1</td>
</tr>
<tr>
<td>55-59</td>
<td>worker</td>
<td>57,11</td>
<td>c7,11+7</td>
<td>s7,11+7</td>
<td>ϕ7,11+7 = 1</td>
<td>p7,11+7 = 1</td>
</tr>
<tr>
<td>60-64</td>
<td>worker</td>
<td>58,12</td>
<td>c8,12+8</td>
<td>s8,12+8</td>
<td>ϕ8,12+8 = 1</td>
<td>p8,12+8 = 1</td>
</tr>
<tr>
<td>65-69</td>
<td>worker</td>
<td>59,13</td>
<td>c9,13+9</td>
<td>s9,13+9</td>
<td>ϕ9,13+9 = 1</td>
<td>p9,13+9 = 1</td>
</tr>
<tr>
<td>70-74</td>
<td>retiree</td>
<td>60,14</td>
<td>c10,14+10</td>
<td>s10,14+10</td>
<td>ϕ10,14+10 = 0</td>
<td>p10,14+10 = 1</td>
</tr>
<tr>
<td>75-79</td>
<td>retiree</td>
<td>61,15</td>
<td>c11,15+11</td>
<td>s11,15+11</td>
<td>ϕ11,15+11 = 0</td>
<td>p11,15+11 = 1</td>
</tr>
<tr>
<td>80-84</td>
<td>retiree</td>
<td>62,16</td>
<td>c12,16+12</td>
<td>s12,16+12</td>
<td>ϕ12,16+12 = 0</td>
<td>p12,16+12 = 1</td>
</tr>
<tr>
<td>85-89</td>
<td>retiree</td>
<td>63,17</td>
<td>c13,17+13</td>
<td>s13,17+13</td>
<td>ϕ13,17+13 = 0</td>
<td>p13,17+13 = 1</td>
</tr>
<tr>
<td>90-94</td>
<td>retiree</td>
<td>64,18</td>
<td>c14,18+14</td>
<td>s14,18+14</td>
<td>ϕ14,18+14 = 0</td>
<td>p14,18+14 = 1</td>
</tr>
<tr>
<td>95+</td>
<td>retiree</td>
<td>65,19</td>
<td>c15,19+15</td>
<td>s15,19+15</td>
<td>ϕ15,19+15 = 0</td>
<td>p15,19+15 = 1</td>
</tr>
</tbody>
</table>

The cohort size of generation $t$, born at the period $t-4$, is determined by:

$$N_t = n_{0, t-4}N_{t-4} + n_{1, t-4}N_{t-5} + n_{2, t-4}N_{t-6} + n_{3, t-4}N_{t-7} + n_{4, t-4}N_{t-8} + n_{5, t-4}N_{t-9}$$

(16)

4.2 A more general model

We keep the spirit of the model of section 2 with constant elasticity of substitution for both utility and production, but now we allow them to be different of one.

4.2.1 Consumers

We use here a constant inter-temporal elasticity of substitution (CIES) utility function,

$$u(c) = e^{(1 - \frac{1}{\sigma})}$$

where $\sigma > 0$, $\sigma \neq 1$, $\sigma$ measures the elasticity of inter-temporal substitution. For all $c > 0$, $u'(c) > 0$ and $u''(c) < 0$, so $u$ is strictly increasing and concave.

The expected lifetime utility in $t$ is:

$$U(c_{0, t}, c_{1, t+1}, c_{2, t+2}, ..., c_{15, t+15}) = \sum_{i=0}^{15} \left( \beta^i \cdot u(c_{i, t+i}) \cdot \prod_{j=0}^{i} p_{j, t+j} \right)$$

(17)

10
We now take into account the cost of children consumption in the budget constraints. In period 0, the consumer has no saving and income, so

\[(1 + \mu \nu_{0,t}) c_{0,t} + s_{0,t} = \varphi_{0,t} w_t\]

The budget constraints for each period \(i \in (1, 2, \ldots, 15)\) are 1:

\[(1 + \mu \nu_{i,t+i}) c_{i,t+i} + s_{i,t+i} = \varphi_{i,t+i} w_{t+i} + R_{t+i} \cdot s_{i-1,t+i-1}\]

with

\[\nu_{i,t+i} = n_{i-3,t+i-3} + n_{i-2,t+i-2} + n_{i-1,t+i-1} + n_{i,t+i}\]

Note that for \(i < 0\), \(n_i = 0\).

The agent expects to live in all periods, if he survives, the present value of the lifetime resource is \(\Omega_t\):

\[\Omega_t \equiv \varphi_{0,t} w_t + \sum_{i=1}^{9} \left( \varphi_{i,t+i} w_{t+i} \right) \prod_{j=1}^{i} R_{t+j} \]

(18)

The lifetime budget constraint is:

\[(1 + \mu \nu_{0,t}) c_{0,t} + \sum_{i=1}^{15} \left( \frac{1 + \mu \nu_{i,t+i}}{\prod_{j=1}^{i} R_{t+j}} \right) = \Omega_t\]

(19)

According to the first order condition, the optimal consumption for the generation 0 at period \(t\):

\[c_{0,t} = \frac{\Omega_t}{(1 + \mu \nu_{0,t}) + \sum_{i=1}^{15} \beta \sigma (1 + \mu \nu_{0,t})^\sigma (1 + \mu \nu_{i,t+i})^{1-\sigma} \left( \prod_{j=1}^{i} p_{j,t+j} \right)^\sigma \left( \prod_{j=1}^{i} R_{t+j} \right)^{\sigma-1}}\]

(20)

and the consumptions at other periods \(i \in (1, 2, \ldots, 15)\) of life are:

\[c_{i,t+i} = \left( \beta \prod_{j=1}^{i} p_{j,t+j} \prod_{j=1}^{i} R_{t+j} \right)^\sigma \left( \frac{1 + \mu \nu_{0,t}}{1 + \mu \nu_{i,t+i}} \right)^\sigma c_{0,t}\]

(21)

so, the optimal individual wealth are given by:

\[s_{0,t} = \varphi_{0,t} w_t - (1 + \mu \nu_{0,t}) c_{0,t}\]

(22)

and the wealth at other periods \(i \in (1, 2, \ldots, 14)\) of life are:

\[s_{i,t+i} = \varphi_{i,t+i} w_{t+i} + R_{t+i} \cdot s_{i-1,t+i-1} - (1 + \mu \nu_{i,t+i}) c_{i,t+i}\]

(23)

1We do not assume a perfect annuity market. Accidental bequest is taxed by the govern- ment to finance a public good. Indeed, perfect annuity market leads to very huge yields of savings at very old age. And this leads itself to a completely unrealistic consumption profile.
4.2.2 Production

We use here a constant elasticity of substitution (CES) production function,

\[ Y_t = F(K_t, L_t) = A (\alpha K_t^{1-\frac{1}{\rho}} + (1 - \alpha)(\Gamma_t L_t)^{1-\frac{1}{\rho}})^{\frac{1}{1-\rho}} \] (24)

where \(0 < \alpha < 1\), \(A > 0\), \(\rho > -1\), \(\rho \neq 0\). \(Y_t\) is the aggregate output of consumption goods, \(K_t\) is the aggregate capital stock at the beginning of period \(t\) and \(L_t\) is the aggregate labor in period \(t\). The elasticity of substitution between the factors of production, capital and labor, is given by \(\rho\).

\[ \Gamma_t = \prod_{i=1}^{t} (1 + \gamma_i) \]

where \(\gamma_t\) is the rate of labor productivity growth in period \(t\).

By maximizing the firm’s profit, we have the income per effective worker,

\[ \tilde{w}_t = (1 - \alpha) A \left( 1 - \alpha \right) + \alpha \left( \frac{K_t}{\Gamma_t L_t} \right)^{\frac{1}{1-\frac{1}{\rho}}} \] (25)

the rate of return,

\[ R_t = \frac{1}{q_{t-1}} \left( \alpha A \left( \alpha + (1 - \alpha) \left( \frac{K_t}{\Gamma_t L_t} \right)^{\frac{1}{1-\rho}} \right)^{-\frac{1}{1-\rho}} + q_t (1 - \delta) - \left( -\frac{\alpha}{2} \left( \frac{I_t}{K_t} \right)^2 \right) \right) \] (26)

and capital price

\[ q_t = 1 + a \left( \frac{I_t}{K_t} \right). \] (27)

Note that the income per worker is given by:

\[ w_t = \Gamma_t \tilde{w}_t \] (28)

4.2.3 Market equilibria

The labor market is at its equilibrium when the labor demand \(L_t\) is equal to the working-age population:

\[ L_t = \sum_{i=0}^{9} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) \varphi_{i,t} N_{t-i} \] (29)

The wealth market is written:

\[ \sum_{i=0}^{14} N_{t-i}s_{i,t} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) = q_t K_{t+1} \] (30)

The tax is:

\[ T_t = (1 - p_{0,t}) N_{t} s_{0,t} R_t + \sum_{i=1}^{14} \left( \prod_{j=0}^{i-1} p_{j,t-i+j} \right) (1 - p_{i,t}) N_{t-i}s_{i,t} R_t \] (31)
and the government budget balance is in equilibrium:

$$G_t = T_t$$  \hspace{1cm} (32)

The consumption good market is:

$$F(K_t, L_t) = \left( \sum_{i=0}^{15} \prod_{j=0}^i p_{j,t-i+j} \right) N_{t-i}(1 + \mu_{i,t})c_{i,t} + I_t + C(I_t, K_t) + G_t$$  \hspace{1cm} (33)

### 4.2.4 General equilibrium

The number of new born children who will be young workers in $t$ are

$$N_t = \sum_{i=0}^{5} n_{i,t-4} N_{t-i-4}$$  \hspace{1cm} (34)

$$Y_t = F(K_t, L_t) = A \left( \alpha K_t^{1-\frac{1}{\rho_t}} + (1 - \alpha)(\Gamma_t L_t)^{1-\frac{1}{\rho_t}} \right)^{-\frac{1}{\rho_t}}$$  \hspace{1cm} (35)

$$K_{t+1} = (1 - \delta) K_t + I_t$$  \hspace{1cm} (36)

$$L_t = \sum_{i=0}^{9} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) \varphi_{i,t} N_{t-i}$$  \hspace{1cm} (37)

$$w_t = (1 - \alpha) A \Gamma_t \left( (1 - \alpha) + \alpha \left( \frac{K_t}{\Gamma_t L_t} \right)^{1-\frac{1}{\rho_t}} \right)^{-\frac{1}{\rho_t}}$$  \hspace{1cm} (38)

$$R_t = \frac{1}{q_{t-1}} \left( \alpha A \left( \alpha + (1 - \alpha) \left( \frac{K_t}{\Gamma_t L_t} \right)^{\frac{1}{\rho_t}} \right) - q_t (1 - \delta) - \left( -\frac{a}{2} \left( \frac{I_t}{K_t} \right)^{1/2} \right) \right)$$  \hspace{1cm} (39)

$$q_t = 1 + a \left( \frac{I_t}{K_t} \right)$$  \hspace{1cm} (40)

$$\Omega_t \equiv \varphi_{0,t} w_t + \sum_{i=1}^{9} \left( \frac{\varphi_{i,t+1} w_{t+1}}{\prod_{j=1}^{i} R_{t+j}} \right)$$  \hspace{1cm} (41)

$$c_{0,t} = \frac{\Omega_t}{(1 + \mu_{0,t}) + \sum_{i=1}^{15} \beta^i (1 + \mu_{0,t})^\sigma (1 + \mu_{i,t+1})^{1-\sigma} \left( \prod_{j=1}^{i} p_{j,t+j} \right) \left( \prod_{j=1}^{i} R_{t+j} \right)^{\sigma-1}}$$  \hspace{1cm} (42)
\[ c_{i,t+i} = \left( \beta^{i} \prod_{j=1}^{i} p_{j,t+j} \prod_{j=1}^{i} R_{t+j} \right)^{\sigma} \left( \frac{1 + \mu \nu_{0,t}}{1 + \mu \nu_{i,t+i}} \right)^{\sigma} c_{0,t} \quad (43) \]

\[ s_{0,t} = \varphi_{0,t} w_{t} - (1 + \mu \nu_{0,t}) c_{0,t} \quad (44) \]

\[ s_{i,t+i} = \varphi_{i,t+i} w_{t+i} + R_{t+i} \cdot s_{i-1,t+i-1} - (1 + \mu \nu_{i,t+i}) c_{i,t+i} \quad (45) \quad i \in \{1, 2, \ldots, 14\} \]

\[ \left( \sum_{i=0}^{14} N_{t-i}s_{i,t} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) \right) = q_{t} K_{t+1} \quad (46) \]

4.3 Calibration

For the production function, we choose the most used allocation rule between Capital and Labor (1/3 and 2/3). We use a relatively low elasticity of substitution at 0.85. For a period of five years, we choose a depreciation rate at 25%. We use an annual time preference factor at 0.99 so $0.99^{5} = 0.95$ for five years.

To try to calibrate our model in the French case for the period of 2010-2015.

4.4 Results

To be completed

5 Conclusions

Conclusion To be completed

References


A Deflated General equilibrium

To stationarize the model, we deflate $Y_t, K_t, I_t, w_t, \Omega_t, c_t, s_{c_t, s_{d_t}, s_{f_t}, s_{g_t}}$ by 
$\Gamma_t = \prod_{i=1}^{t}(1 + \gamma_t)$. The stationarized variable $X_t$ is noted $\tilde{X}_t$

Thus the general equilibrium of the economy is characterized by: $N_t$, $\tilde{Y}_t$, $\tilde{K}_t$, $L_t$, $\tilde{I}_t$, $\tilde{w}_t$, $R_t$, $q_t$, $\tilde{\Omega}_t$, $\tilde{c}_{i,t+i}$, $\tilde{s}_{i,t+i}$ for $i \in (0, 2, ..., 14)$. (39 variables, 39 equations)

$$N_t = \sum_{i=0}^{5} n_{i,t-4} N_{t-i-4} \tag{47}$$

$$\tilde{Y}_t = F(\tilde{K}_t, L_t) = A \left( \alpha \tilde{K}_t^{1-\frac{1}{\rho}} + (1-\alpha)(L_t)^{1-\frac{1}{\rho}} \right)^{\frac{1}{1-\rho}} \tag{48}$$

$$\tilde{K}_{t+1} + 1 = (1-\delta)\tilde{K}_t + \tilde{I}_t \tag{49}$$

$$L_t = \sum_{i=0}^{9} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) \varphi_{i,t} N_{t-i} \tag{50}$$

$$\tilde{w}_t = (1-\alpha) A \left( 1 + \alpha \left( \frac{\tilde{K}_t}{L_t} \right)^{1-\frac{1}{\rho}} \right)^{-\frac{1}{1-\rho}} \tag{51}$$

$$R_t = \frac{1}{q_{t-1}} \left( \alpha A \left( \alpha + (1-\alpha) \left( \frac{\tilde{K}_t}{L_t} \right)^{\frac{1}{\rho} - 1} \right)^{-\frac{1}{1-\rho}} + q_t(1-\delta) - \frac{a}{2} \left( \frac{\tilde{I}_t}{K_t} \right)^2 \right) \tag{52}$$

$$q_t = 1 + a \left( \frac{\tilde{I}_t}{K_t} \right) \tag{53}$$
\[ \tilde{\Omega}_t \equiv \varphi_{0,t} \tilde{w}_t + \sum_{i=1}^{9} \left( \varphi_{i,t+i} \tilde{w}_{t+i} \prod_{j=1}^{i} \frac{1 + \gamma_{t+j}}{R_{t+j}} \right) \]  \hspace{1cm} (54)

\[
\tilde{c}_{0,t} = \frac{\tilde{\Omega}_t}{(1 + \mu_{0,t}) + \sum_{i=1}^{15} \beta^\sigma (1 + \mu_{0,t})^\sigma (1 + \mu_{i,t+i})^{1-\sigma} \left( \prod_{j=1}^{i} p_{j,t+j} \right)^\sigma \left( \prod_{j=1}^{i} R_{t+j} \right)^{\sigma-1}} \]  \hspace{1cm} (55)

\[
\tilde{c}_{i,t+i} = \left( \beta^\sigma \prod_{j=1}^{i} p_{j,t+j} \prod_{j=1}^{i} R_{t+j} \right)^\sigma \frac{(1 + \mu_{0,t})^\sigma}{\prod_{j=1}^{i} (1 + \gamma_{t+j})} \frac{\tilde{c}_{0,t}}{\prod_{j=1}^{i} (1 + \gamma_{t+j})} \]  \hspace{1cm} (56)

\[
\tilde{s}_{0,t} = \varphi_{0,t} \tilde{w}_t - (1 + \mu_{0,t})\tilde{c}_{0,t} \]  \hspace{1cm} (57)

\[
\tilde{s}_{i,t+i} = \varphi_{i,t+i} \tilde{w}_{t+i} + R_{t+i} \frac{\tilde{s}_{i-1,t+i-1}}{1 + \gamma_{t+i}} - (1 + \mu_{i,t+i})\tilde{c}_{i,t+i} \]  \hspace{1cm} (58)

\[
\left( \sum_{i=0}^{14} N_{t-i} \tilde{s}_{i,t} \left( \prod_{j=0}^{i} p_{j,t-i+j} \right) \right) = q_t \tilde{K}_{t+1}(1 + \gamma_{t+1}) \]  \hspace{1cm} (59)