Spatial Wage Inequality within and across Chinese Cities

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Abstract

Individuals hold heterogeneous beliefs on future housing price. How does this behavior heterogeneity influence housing price’s response to a negative productivity shock in the economy? This paper develops a dynamic general equilibrium framework, where agents are exogenously divided into two behavior groups, namely "contrarian" and "momentum" agents. Agents within each behavior group hold a different housing price reference point. For contrarian, if observed housing price rises above agent’s reference point, they expect future housing price to converge back to their reference point, while on the contrary momentum expect future housing price to deviate further more from their reference point. Our theoretical framework also distinguishes the demand of housing for utility and investment purposes, respectively. Therefore, agents who expect future housing price to decline will not have demand for investment housing, but agents who expect future housing price to go up may face an exogenous borrowing constraint. We theoretically prove the existence, uniqueness and stability of the equilibrium. We also characterize some analytical properties of the equilibrium. The fraction of contrarian agents will have a deterministic impact on steady-state housing price level, but the sign depends on some fundamental economic characteristics. Our simulation results also quantitatively exam how this behavior heterogeneity affects the magnitude of housing price crash and housing market recovery speed given a negative productivity shock.

Keywords: Behavior Heterogeneity, Housing Prices, General Equilibrium

JEL classification: D58; E03; G12
1 Introduction

Some people tend to be more optimistic than others. How do people with different beliefs on future housing price interact with each other in the housing market? How does this behavior heterogeneity influence housing price’s response to a negative productivity shock in the economy? What are the effective policy instruments that can enable a fast recovery in the housing market? These are important questions for us to understand housing price dynamics in the real world. This paper intends to developed a dynamic general equilibrium framework to tackle these issues.

This paper develops a dynamic general equilibrium framework, where agents are exogenously divided into two behavior groups, namely "contrarian" and "momentum" agents. Agents within each behavior group hold a different housing price reference point. For contrarian agents, if observed housing price rises above their reference point, they expect future housing price to converge back to their reference point, on the contrary, momentum agent expects future housing price to deviate further more from their reference point. Our theoretical framework also distinguishes the demand of housing for utility and investment purposes, respectively. This guarantees a positive demand for housing even though no one in the economy has demand for investment housing. Given an alternative risk-free asset in the economy, so agents who expect future housing price to decline will not have demand for investment housing, but agents who expect future housing price to go up may face an exogenous borrowing constraint. We theoretically exam some properties of the steady-state equilibrium and housing price dynamics within this framework.

In the existing literature, potential factors that may affect the housing price dynamics can be divided into two main sources: fundamental and non-fundamental factors. These fundamental factors include demographics, income trends, and government policy. In one well-known study, Mankiw and Weil(1989) argue that population demographics are the prime determinants. They pre-
dicted that prices would fall in the subsequent two decades with the maturation of the baby boom generation and the resulting decline in the growth rate of the prime home-owning age group, but their prediction proved inaccurate. Martin (2005) renews the argument by favoring an important role for demographics. Glaeser et al. (2005) argues that price increase since 1970 largely reflect artificial restrictions. Gyouko et al. (2006) also cite inelastically supplied land as a key driver of the phenomenon they call "super cities." Van Nieuwerburgh and Weill (2006), however, argue that regional markets are likely to be modest, in other words, they primarily affect the cross-sectional distribution of housing prices as opposed to the aggregate. Iacoviello and Neri (2006) examine the role of monetary policy under credit market frictions. In line with these research, our paper also incorporate fundamental factors such as new housing supply, downpayment requirement and interest rate.

"Non-fundamental" factors include speculative bubbles and self-fulfilling beliefs, as argued by Case and Shiller (2003) and Himmelberg et al. (2005). These are important elements, especially for the recent crisis. Burnside et.al. (2014) developed a heterogenous agent model where agents hold different expectations on long-run fundamental housing price, but they may change their beliefs when they interact other agents in the economy. Chen and Wen (2014) constructs a self-fulfilling growing housing bubble because high capital returns driven mainly by resource allocation cannot be sustained in the long-run, so productive entrepreneurs instead speculate in housing during economic transition.

The notion of "contrarian" and "momentum" agents in our paper stems from a stream of deterministic heterogeneous agent models (HAM) as in Huang et.al (2010), where they divide agents into "chartists" and "fundamentalists". Chartist always expect housing price will follow the current trend while fundamentalists expect the price will return to the fundamental value. Our paper distinguishes from the "HAM" approach mainly in the following aspects: (i) we provide a micro foundation for agents’ demand of different types of housing, in contrast with a reduced-form demand function in the "HAM" approach. (ii)
Housing price is determined via a general equilibrium model (iii) We extend the "HAM" approach by having a continuum of heterogenous chartists, each of them holds different belief regarding to the future housing prices.

The main results are summarized as follows: under some mild assumptions, an unique steady-state equilibrium exists where housing price remains constant in the long-run. Given a negative productivity shock, housing price will converge back to the initial steady-state price level without any government intervention. The model predicts that the fraction of contrarian agents has a deterministic impact on steady-state housing price. But whether more contrarian agents imply higher or lower housing price level may depend on some preference parameter or utility housing depreciation rate. Moreover, the simulation results quantitatively show how the composition of each type of agents affects steady-state housing price level, and the housing market recovery speed after a negative productivity shock.

The rest of the paper is organized as follows. Section 2 describes the theoretical model. We explore some analytical properties of the equilibrium in Section 3. Numerical exercise are implemented in Section 4. Section 5 concludes.

2 Theoretical Framework

In this section, we will present the theoretical framework. We first briefly describe the demographic structure and production in the model, then we introduce the preferences and timeline for representative agent in the economy. Before we characterize agent’s optimization problem, we introduce the notion of heterogeneous beliefs among agents. We then explicitly solve housing demand for different types of agents. Aggregate housing demand together with aggregate housing supply give rise to equilibrium housing price in our setup.
2.1 Demographic Structure and Production

Time is discrete and infinite with \( t = 0, 1, 2, \ldots \). In each period \( t \), there is a mass of two-period lived agents born in the economy. They live in period \( t \) and \( t + 1 \), namely youth and old age. There is a single type of consumption goods and agents can gain utilities from consuming it. Moreover, agents can also derive utilities from housing. We will describe the housing demand and supply in details in later section.

Each period, youth and old agent inelastically provides \( 1 \) and \( 0 < \theta < 1 \) unit of labor, respectively. They both earn competitive wage accordingly. Labor is assumed to be the only input for producing consumption goods and the production function is linear in labor:

\[
Y_t = A_t (1 + \theta)
\]

where \( A_t \) denotes the labor productivity in period \( t \), and \( (1 + \theta) \) is the total labor supply by young and old agents. Therefore, young and old agent’s wage rate in \( t \) are:

\[
w_t = A_t \\
w^o_t = \theta A_t
\]

2.2 Preferences and Timeline

All agents are assumed to share identical preferences. A representative youth agent derives utilities from housing services in youth and the expected consumption in his old-age\(^1\). Utility is assumed to be time additive and specifically, the lifetime utility of an agent born in \( t \) is given as:

\[
U_t = E[c^o_{t+1}] + \beta log(h^u_t)
\]

\(^1\)To simplify the analysis, we assume agents do not consume in youth.
where $h_t^u$ denotes housing services that youth agent enjoys, and $\beta > 0$ measures the agent’s preference over housing services.

Housing can also be served as an asset to smooth consumption in the old-age. However, we assume that those housing services that bring utilities to agents are subjected to a depreciation rate $0 < \delta < 1$. That is, only $(1 - \delta)h_t^u$ can be sold in agent’s old-age. On the contrary, housing service will not depreciate if it does not bring agent utilities. We justify this assumption by arguing that agent can enjoy housing services only when they live in it, but this may deteriorate the house. In addition to housing, there is also one risk-free bond in the economy. The interest rate is exogenously set at $r^* > 0$. We explicitly characterize the budget constraint for an agent born in $t$ as follows:

$$
p_t(h_t^u + h_t^i) + s_t = w_t
$$

$$
E[c_{t+1}^c] = p_{t+1}^c [h_t^u (1 - \delta) + h_t^i] + w_{t+1}^p + s_t (1 + r)
$$

where $h_t^u$ and $h_t^i$ denote the housing services that agent purchases for utility purpose and investment purpose, respectively. Hence, in agent’s old age, he can in total sell $h_t^u (1 - \delta) + h_t^i$ unit of housing services. Consumption goods is numeraire, and $p_t$ is the housing price in period $t$. $s_t$ refers to agent’s saving in period $t$. Note that $s_t$ can be negative, in which case agent is borrowing. But agent may face a borrowing constraint: the amount of debt repayment cannot exceed a fraction $0 < \lambda < 1$ of agent’s old-age wage income. This implies a fraction $\lambda$ of agent’s old-age wage income is seizable. $\lambda$ captures the degree of financial friction in the economy. Specifically, the borrowing constraint can be written as:

$$
-s_t (1 + r) \leq \lambda w_{t+1}^p
$$

The timeline for a representative agent born in $t$ can be summarized as follows: They consume $c_t$, save $s_t$, and purchase houses for living (denoted as utility demand $h_t^u$) as well as for speculative purposes (denoted as investment
demand $h_t^i$) by the end of $t$. At time $t+1$, agents who were born in $t$ earn wages $w_{t+1}^o$ as elder workers. They consume $c_{t+1}^o$ and sell their houses by the end of $t+1$ and die.

### 2.3 Heterogeneous Beliefs

Each agent has a unique reference point of the housing price, $x$, which is assumed to follow a cumulative distribution function (CDF) $F(\cdot)$. If the current housing price is above or below agent’s reference point, agent may expect future housing price to deviate even further or converge back to his reference point. Suppose agents’ beliefs on future housing price can be divided into two types in general: the contrarian belief and the momentum belief. Let a fraction $0 < \pi < 1$ of the agents be the so-called "contrarian" agents, and the rest $1 - \pi$ of agents be the so-called "momentum" agents. In the following, we describe in details the expected housing price for each individual agent.

**Contrarian Belief**

A contrarian agent with reference point $x$ believes that future housing price will converge towards $x$. In another words, if current housing price $p_t$ is higher than his reference point $x$ ($p_t > x$), the contrarian agent believes housing price(trend) will reverse in the next period ($p_{e,t+1}^c(x) < p_t$), and similarly if $p_t < x$, then a contrarian agent believes $p_{e,t+1}^c(x) > p_t$. Explicitly we assume a contrarian agent with reference point $x$ forms his or her expectation according to:

$$p_{e,t+1}^c(x) = p_t - \tau (p_t - x), \text{ where } 0 < \tau < 1$$

$\tau$ measures how sensitive agent is to the price bias against his reference point. A larger value of $\tau$ implies agent believes housing price will adjust more dramatically towards to his reference point in the next period. We restrict the value of $\tau$ to be less than 1 so that when $p_t > (\langle \rangle)x$, we always have $p_{e,t+1}^c(x) > (\langle \rangle)x$. This implies that when current price is higher (lower) than contrarian agent’s
reference agent, their expectation on future housing price is also higher (lower) than his reference point.

**Momentum Belief**

Momentum agents are assumed to be the "trend follower". If current housing price deviates from his reference point, a momentum agent believes that future housing price will deviate even further from his reference point. In other words, for a momentum agent with reference point \( x \), if \( p_t \) is larger than \( x \), he believes housing price will rise even more in the next period \( p_{t+1}^m(x) > p_t \), and similarly if \( p_t < x \), then he believes housing price will decline even more in the future \( p_{t+1}^m(x) < p_t \). Explicitly we assume a momentum agent with reference point \( x \) forms his expectation according to:

\[
p_{m,t+1}^e(x) = p_t + \tau (p_t - x), \quad \text{where } 0 < \tau < 1
\]

Where we have assumed momentum agents have the same value of \( \tau \) as contrarian agents. \( \tau \) measures how much the future housing price will deviate further from agent’s reference point. Similarly, when \( p_t > (<)x \), we always have \( p_{e,t+1}^c(x) > (<)x \) holds. ²

**2.4 Agent’s Optimization Problem**

We formally characterize the optimization problem for agent born in period \( t \) as follows. The agent needs to decide \( c_t, h_t^u, h_t^l, \) and \( s_t \) to maximize lifetime utility subject to a budget constraint in period \( t \) and \( t+1 \), and a borrowing constraint:

\[
\max_{h_t^u, h_t^l, s_t} E[c^o_{t+1}] + \beta \log(h_t^u)
\]

\[
p_t(h_t^u + h_t^l) + s_t = w_t
\]

\[
E[c^o_{t+1}] = p_{t+1}^e[h_t^u(1-\delta) + h_t^l] + w_{t+1}^o + s_t(1+r)
\]

\[
-s_t(1+r) \leq \lambda w_{t+1}^o
\]

where \( p_{t+1}^e \) denotes the agent’s expected housing price in \( t + 1 \). Agent’s saving and housing demand decisions greatly depend on \( p_{t+1}^e \). In the following we ²We do not restrict the analysis to the case where \( p_{m,t+1}^m(x) \geq 0 \).
study two different situations under which agent may borrow or lend in youth.

**Case 1:** $p_{t+1}^e > p_t(1 + r)$ If there were no borrowing constraint, the agent would borrow an infinite amount to purchase investment housing $h_t^i$ when $p_{t+1}^e > p_t(1 + r)$ holds, because he believes doing this can deliver him an infinite amount of consumption in old-age. This also implies given the borrowing constraint, the agent will always borrow up to the limit, and thus savings in youth should satisfy: $s_t = -\lambda \theta A_{t+1}/(1 + r)$. Therefore, the optimization problem under this case can be written as:

$$\max_{h_t^i, h_t^u \geq 0} E[c_{t+1}^o] + \beta \log(h_t^u)$$

$$p_t(h_t^u + h_t^i) = A_t + \frac{\lambda \theta A_{t+1}}{1 + r}$$

$$E[c_{t+1}^o] = p_{t+1}^e [h_t^u (1 - \delta) + h_t^i] + \theta A_{t+1} (1 - \lambda)$$

The optimal utility demand and investment demand for houses, and saving are solved as:

$$h_t^i = \frac{1}{p_t} \left( A_t + \frac{\lambda \theta A_{t+1}}{1 + r} \right) - \frac{\beta}{\delta p_{t+1}}$$

$$h_t^u = \frac{\beta}{\delta p_{t+1}}$$

$$s_t = 0$$

The condition to guarantee that $h_t^i > 0$ is:

**Condition 1 (A1)**

$$\delta > \frac{\beta}{A(1 + r + \lambda \theta)}$$

**Lemma 1:** Under assumption A1, for agents with housing price expectation satisfying $p_{t+1}^e > p_t(1 + r)$, they do not hold any risk-free bond. Total housing
demand \((h_t^i + h_t^u)\) is independent on \(p_{t+1}^e\). Moreover, the following properties with respect to \(h_t^u\) and \(h_t^i\) hold:

\[
\frac{\partial h_t^u}{\partial p_{t+1}^e} < 0, \quad \frac{\partial h_t^i}{\partial A_t} = 0, \quad \frac{\partial h_t^u}{\partial A_{t+1}} = 0
\]

\[
\frac{\partial h_t^i}{\partial p_{t+1}^e} > 0, \quad \frac{\partial h_t^i}{\partial A_t} > 0, \quad \frac{\partial h_t^i}{\partial A_{t+1}} > 0
\]

The results can be easily obtained by taking comparative statics of equation 2-4, so we skip the proof there. The results suggests that when expected future housing price is higher, agents will tend to substitute current utility housing demand with investment demand. Moreover, current and future productivity have no impact on current utility demand, since income and substitution effects exactly offset each other. But they will increase the demand for investment housing.

**Case 2:** \(p_{t+1}^e < p_t(1 + r)\) If we allow housing short sale, the agent would have an infinite short position on houses to finance an infinite lending when \(p_{t+1}^e < p_t(1 + r)\) holds. But it is prohibited in the model. Therefore, it is not optimal for the agent to invest in housing market at all \((h_t^i = 0)\) when \(p_{t+1}^e < p_t(1 + r)\). The optimization problem under this situation can then be written as:

\[
\max_{h_t^u \geq 0, s_t} E[c_{t+1}^o] + \beta \log(h_t^u)
\]

s.t. \(p_t h_t^u + s_t = A_t\)

\[
E[c_{t+1}^o] = p_{t+1}^e h_t^u (1 - \delta) + \theta A_{t+1} + s_t (1 + r)
\]

\[-s_t (1 + r) \leq \lambda \theta A_{t+1}\]

The optimal utility demand and investment demand for houses are solved as:

\[
h_t^i = 0 \tag{5}
\]

\[
h_t^u = \frac{\beta}{p_t (1 + r) - p_{t+1}^e (1 - \delta)} \tag{6}
\]

\[
s_t = \frac{\beta p_t}{p_t (1 + r) - p_{t+1}^e (1 - \delta)} \tag{7}
\]
Note that in this situation even though demand for investment housing is zero, but we still cannot rule out the possibility that agent may borrow to purchase utility housing. However, we can show that if the following condition holds, borrowing constraint will not be binding:

**Condition 2 (A2)**

\[ A_t \delta (1 + r + \lambda \theta) \geq \beta \]

**Lemma 2:** Under assumption A2, for agents with housing price expectation satisfying \( p_{t+1}^e < p_t(1+r) \), they do not have any investment demand for housing. The demand for utility housing satisfy the following properties:

\[ \frac{\partial h_t^u}{\partial p_t^e} < 0, \quad \frac{\partial h_t^u}{\partial p_{t+1}^e} > 0, \quad \frac{\partial h_t^u}{\partial A_t} = 0, \quad \frac{\partial h_t^u}{\partial A_{t+1}} = 0 \]

Similarly, the results in Lemma 2 can be obtained by taking comparative statics with equation (5)-(7). Intuitively, demand for utility housing decreases with current price level and increases with expected future price level. Income and substitution effect still offset each other in this case.

Lemma 1 and 2 summarizes individual’s housing demand. We also compare a cross-sectional of agents’ housing demand in Lemma 3 to investigate how agents’ demand for different types of housing varies against their reference points, \( x \).

**Lemma 3:** For each type of agents, their utility demand is continuous in their reference point \( x \). Moreover, \( h_t^u(x) \) exhibits an inverse "U" pattern for contrarian agents and a "U" pattern for momentum agents.

**Proof:** When \( p_{t+1}^e < p_t(1+r) \), we have \( \frac{\beta}{p_t(1+r) - p_{t+1}^e(1-\delta)} \), on the other hand, when \( p_{t+1}^e > p_t(1+r) \), we have \( \frac{\beta}{p_{t+1}^e} \). They are equal when \( p_{t+1}^e = p_t(1+r) \).
For contrarian agents, they do not have investment housing demand when $x$ is lower than $\hat{x}_{ct}$. When $x$ increases towards $\hat{x}_{ct}$, utility housing demand increases as agents with higher $x$ replace more savings in bonds by utility housing, since they increase their current utilities. When $x$ overpasses $\hat{x}_{ct}$, utility demand starts to gradually decline since now agents with a larger $x$ have a higher expectation on future housing price, so they substitute their utility housing demand into investment demand. Similar arguments apply for momentum agents and the only difference lies in that momentum agents have investment housing demand when $x$ is less than $\hat{x}_{mt}$. 

Figure 1: Individual Agent’s Utility Housing Demand
2.5 Aggregate Housing Demand

In the previous section, we have derived individual’s housing demand for each type of housing as a function their expected future housing price. Since agents may have different housing price reference point, in addition, contrarian and momentum agents will have different housing price expectation even though they share the same reference point. Therefore, it is worthwhile to find out the aggregate demand within the group of contrarian and momentum agents first.

The previous analysis suggest that agents will have investment demand for housing if and only if \( \frac{p_{t+1}}{p_t} > 1 + r \). Since contrarian agent with reference point \( x \) forms housing price expectation according to \( p_{c,t+1}^c(x) = p_t - \tau (p_t - x) \), and thus it is straightforward to show that contrarian agents will have positive investment demand for housing if and only if \( x > (1 + \frac{r}{\tau})p_t \). Similarly, it can be shown that momentum agents will invest in housing if and only if \( x < (1 - \frac{r}{\tau})p_t \).

We summarize into the following lemma.

**Lemma 4:** A contrarian agent with reference point \( x \) has investment demand if and only if the following holds:

\[
\begin{align*}
x > \hat{x}_{ct}, & \text{ where } \quad \hat{x}_{ct} = (1 + \frac{r}{\tau})p_t. \\
\end{align*}
\]

Similarly, a momentum agent with reference point \( x \) has investment demand if and only if:

\[
\begin{align*}
x < \hat{x}_{mt}, & \text{ where } \quad \hat{x}_{mt} = (1 - \frac{r}{\tau})p_t. \\
\end{align*}
\]

Denote \( h_{ct}(x) \) to be the housing demand for a contrarian with a reference point \( x \), then we can derive it by summing up individual’s demand for utility and investment housing:

\[
h_{ct}(x) = \begin{cases} 
\frac{\beta}{|r+\delta+\tau(1-\delta)||p_t-\tau(1-\delta)x|} & \text{if } x \leq \hat{x}_{ct} \\
\frac{1}{p_t} \left( A_t + \frac{\beta A_{t+1}}{1+\tau} \right) & \text{if } x > \hat{x}_{ct}
\end{cases}
\]
Corollary 1: Under assumption A1-A2, housing demand among contrarian agents are increasing in their reference point $x$. In addition, the housing demand function is continuous and differentiable in every point except $x = \hat{x}_{ct}$. If condition (BC) holds, there is a sudden jump in housing demand level at $x = \hat{x}_{ct}$.

Similarly, denote $h_{mt}(x)$ to be the housing demand for a momentum with a reference point $x$, then we can derive it by summing up individual’s demand for utility and investment housing:

$$h_{mt}(x) = \begin{cases} \beta & \text{if } x \leq \hat{x}_{mt} \\ \frac{1}{p_x} \left( A_t + \frac{\lambda_0 h_{mt}}{1+r} \right) & \text{if } x > \hat{x}_{mt} \end{cases}$$

Corollary 2: Under assumption A1-A2, housing demand among momentum agents are decreasing in their reference point $x$ when $x < \hat{x}_{mt}$. In addition, the housing demand function is continuous and differentiable in every point except $x = \hat{x}_{mt}$. When condition 1 holds, there is a sudden jump in housing demand level at $x = \hat{x}_{mt}$.
Figure 2 presents how individual’s total housing demand varies against their reference point $x$. Let $H^u_{ct}$ and $H^i_{ct}$ denote contrarian agents’ total utility demand and investment demand for housing in $t$. We can express them as follows:

\[
H^u_{ct} = \int_0^{\hat{x}_{ct}} \frac{\beta}{p_t(1+r) - [p_t - \tau(p_t-x)](1-\delta)} dF(x) + \int_{\hat{x}_{ct}}^P \frac{\beta}{\delta [p_t - \tau(p_t-x)]} dF(x)
\]

\[
H^i_{ct} = \int_{\hat{x}_{ct}}^P \frac{1}{p_t} \left( A_t + \frac{\lambda \theta A_{t+1}}{1+r} \right) - \frac{\beta}{\delta [p_t - \tau(p_t-x)]} dF(x)
\]

Similarly, let $H^u_{mt}$ and $H^i_{mt}$ denote contrarian agents’ total utility demand.
and investment demand for housing in $t$. We can express them as follows:

$$H_{mt}^u = \int_0^{\hat{x}_{mt}} \frac{\beta}{\delta [\pi_t + \tau (\pi_t - x)]} dF(x) + \int_{\hat{x}_{mt}}^{P_t} \frac{\beta}{p_t (1 + r) - [\pi_t + \tau (\pi_t - x)] (1 - \delta)} dF(x)$$

$$H_{mt}^i = \int_0^{\hat{x}_{mt}} \frac{1}{p_t} \left( A_t + \frac{\lambda \theta A_{t+1}}{1 + r} \right) - \frac{\beta}{\delta [\pi_t + \tau (\pi_t - x)]} dF(x)$$

We can now derive the aggregate housing demand ($H_t^d$) by summing up aggregate demand for utility ($H_t^u$) and investment housing ($H_t^i$) from the two groups of agents as follows. Some properties on the aggregate housing demand function are summarized in the Lemma 5.

$$H_t^u = \pi H_{ct}^u + (1 - \pi) H_{mt}^u$$
$$H_t^i = \pi H_{ct}^i + (1 - \pi) H_{mt}^i$$
$$H_t^d = H_t^u + H_t^i$$

$$= \frac{\pi \beta}{(1 - \delta) \tau \pi} \ln \left( 1 + r - (1 - \delta) (1 - \tau) \right) + \frac{(1 - \pi) \beta}{(1 - \delta) \tau \pi} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{(1 - \delta) \tau P}{p_t} \frac{1}{p_t}$$

$$+ \pi \left( \frac{1}{p_t} - \frac{\tau + r}{\tau P} \right) \left( A_t + \frac{\lambda \theta A_{t+1}}{1 + r} \right) + (1 - \pi) \frac{\tau - r}{\tau P} \left( A_t + \frac{\lambda \theta A_{t+1}}{1 + r} \right)$$

**Lemma 5:** The aggregate housing demand $H_t^d$ satisfies the following properties:

1. $\partial H_t^d / \partial p_t < 0$, $\partial H_t^d / \partial A_t > 0$, $\partial H_t^d / \partial A_{t+1} > 0$, $\partial H_t^d / \partial \beta > 0$, $\partial H_t^d / \partial \lambda > 0$, $\partial H_t^d / \partial \theta > 0$
2. The sign of $\partial H_t^d / \partial r$, $\partial H_t^d / \partial \delta$ and $\partial H_t^d / \partial \tau$ are ambiguous.

Intuitively, lower current housing price, or higher current and future productivity, or a stronger preference against housing will all lead to higher aggregate housing demand. When collateral constraint is tighter, agents are allowed to borrow more, and this will increase investment demand for those who expect future price to grow faster than interest rate. When efficiency labor supply in old-age is higher, those agents who expect future housing price to grow slower than interest rate do not need to save much bonds, and this may increase their
demand for utility housing. Hence, aggregate housing demand increases accordingly.

2.6 Aggregate Housing Supply

In period $t$, old agents who are born in period $t - 1$ sell their undepreciated houses before they exit. Therefore, there will be in total $(H_{t-1}^d - \delta H_{t-1}^u)$ units of houses on sale in period $t$. In addition, we also assume that in each period an additional $\bar{H}$ units of houses are provided. Therefore, the aggregate housing supply in $t$ can be summarized as:

$$H^s_t = \bar{H} + (H_{t-1}^d - \delta H_{t-1}^u)$$

The expression above implies that a large housing demand from previous period will stimulate a large housing supply in the current period.

2.7 Competitive Equilibrium

We study a competitive equilibrium in which each agent makes consumption and investment decisions based on his or her own expectation to maximizes his lifetime utility. Market clearing condition determines the equilibrium housing price.

Definition: Given government policy $\bar{H}, r$ and productivity path $\{A_t\}$, a competitive equilibrium consists a sequence of allocations $\{h_{ct}(x), h_{mt}(x), s_{ct}(x), s_{mt}(x), c_{t+1}^c(x)\}$ and a series of prices $\{p_t\}$, such that:

- Given price sequence $\{p_t\}$, agent makes optimal consumption, saving and housing demand decisions to maximize his lifetime utility.

- Housing market clears: $H_t^d = H_t^s$
3 Steady-state Equilibrium Analysis

In this section, we intend to further explore some analytical properties of the steady-state equilibrium housing price. We first examine the conditions under which steady state equilibrium exists, we then examine the stability of the steady-state equilibrium and investigate the potential factors that might affect the equilibrium housing price. We first define steady-state equilibrium as follows.

In order to gain analytically tractable solutions, in the following analysis we assume agents’ reference points follow an uniform distribution defined on \([0, \bar{P}]\) with cumulative distribution function (CDF) as follows:

\[
F(x) = \frac{x}{\bar{P}}, \quad \text{for} \ x \in [0, \bar{P})
\]  

(8)

**Definition:** A steady-state equilibrium is a competitive equilibrium where productivity and housing price stay constant over time: \(A_t = A_t, p_t = p\).

In steady-state equilibrium when price is constant we should have \(H^d_t = H^d_{t-1}\) and \(H^u_t = H^u_{t-1}\). From housing market clearing condition in \(t\): \(\bar{H} + (H^d_{t-1} - \delta H^u_{t-1}) = H^d_t\), this implies \(\bar{H} = \delta H^u\) holds in steady-state equilibrium.

Define \(G(p)\) and \(\kappa\) as follows:

\[
G(p) = \frac{\pi \beta}{P \tau} \ln \left( \frac{1 - \tau}{1 + \tau} + \frac{\tau \bar{P}}{1 + \tau \bar{P}} \right) + \frac{(1 - \pi) \beta \delta}{(1 - \delta) \tau \bar{P} \delta} \ln \left( \frac{1 + r - (1 - \delta)(1 + \tau)}{\delta (1 + r)} \right) + \frac{(1 - \delta) \tau \bar{P}}{P \delta (1 + r)} \right) \]  

\[\kappa = \frac{(1 - \pi) \beta}{P \tau} \ln \left( \frac{1 + \tau}{1 + r} \right) + \frac{\pi \beta \delta}{(1 - \delta) \tau \bar{P}} \ln \left( \frac{1 + r - (1 - \delta)(1 - \tau)}{\delta (1 + r)} \right) \]  

(10)

Then steady-state housing market clearing condition can then be written as:

\[
G(p_s) = \bar{H} - \kappa
\]

In the following analysis, we maintain the following assumption to guarantee the uniqueness of the equilibrium:

**Condition 3 (A3)**

\[
G(\bar{P} \frac{\tau}{\tau + r}) < \bar{H} - \kappa
\]
Proposition 1: Under assumptions A1-A3, there exists an unique steady-state equilibrium, where the steady state housing price $p_s$ satisfies $\frac{r+\kappa}{r}p_s < \bar{P}$.

Proof: It is straightforward to show that $G(p)$ is strictly decreasing in $p$. Therefore, to guarantee the existence of an unique solution $p_s$, we simply require $\lim_{p \to -\infty} G(p) < \bar{H} - \kappa$ holds. Therefore, if $G(\bar{P} \frac{r}{r+\kappa}) < \bar{H} - \kappa$ holds, then the uniqueness of solution is guaranteed. Moreover, if $\frac{r+\kappa}{r}p_s < \bar{P}$ is satisfied, then demand for investment and utility housing is non-empty within the group of contrarian and momentum agents. This essentially requires $G(\bar{P} \frac{r}{r+\kappa}) < \bar{H} - \kappa$ due to the property that $G(p)$ is strictly decreasing in $p$. Q.E.D

Evaluating the expression for $p_s$, it is straightforward to derive the following comparative statics.

Proposition 2: Steady-state housing price $p_s$ satisfies the following properties: $\partial p_s / \partial A = 0$, $\partial p_s / \partial \lambda = 0$, $\partial p_s / \partial \beta > 0$, $\partial p_s / \partial \bar{H} < 0$.

Proof: Differentiate $G(p_s) = \bar{H} - \kappa$ with respect to $\bar{H}$, we have $G'(p_s) \frac{\partial p_s}{\partial \bar{H}} = 1$, and thus $\frac{\partial p_s}{\partial \bar{H}} < 0$. Q.E.D

Our results in Proposition 2 suggest that both productivity and financial frictions have no impact on steady-state housing price. Therefore, if the economy is hit by a productivity or financial shock, the model implies that the housing price will return to its "pre-shock" steady-state level itself without any government intervention, if everything else in the economy remains the same. We will further discuss the housing price dynamics given a negative productivity shock in the next section.

The remaining of this section aims to exam how the fraction of contrarian agents, $\pi$, affects the steady-state price $p_s$. For any given price $\pi$, we differentiate $G(p_s) = \bar{H} - \kappa$ with respect to $\pi$:

$$G'(p_s) \frac{\partial p_s}{\partial \pi} + \frac{\partial G(p_s)}{\partial \pi} = -\frac{\partial \kappa}{\partial \pi}$$
where it can be shown that:

\[ \frac{\partial \kappa}{\partial \pi} = \frac{\beta \delta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta)(1 - \tau)}{\delta (1 + r)} \right) - \frac{\beta}{P_r} \ln \left( \frac{1 + \tau}{1 + r} \right) \]

\[ \frac{\partial G(p_s)}{\partial \pi} = \frac{\beta}{P_r} \ln \left( \frac{1 - \tau + \frac{\tau P_{\frac{1}{Pt}}}{1 + r}}{1 + r - (1 - \delta)(1 - \tau)} \right) - \frac{\beta \delta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta)(1 + \tau)}{\delta (1 + r)} + \frac{(1 - \delta) \tau P_{\frac{1}{Pt}}}{P_r} \right) \]

Since \( G'(p_s) < 0 \), so the sign of \( \frac{\partial p_s}{\partial \pi} \) is equivalent to the sign of \( -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} \).

We write \( -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} \) as follows:

\[ -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} = \frac{\beta}{P_r} \left[ \ln \left( \frac{1 + \tau}{1 - \tau + \frac{\tau P_{\frac{1}{Pt}}}{1 + r}} \right) + \ln \left( \frac{1 + r - (1 - \delta)(1 + \tau)}{1 + r - (1 - \delta)(1 - \tau)} \right) \right]^{\frac{\delta}{1 - \delta}} \]

So \( -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} > 0 \) if and only if the following inequality holds:

\[ (1 + \tau) \left( 1 + \frac{2 \delta \tau - 2 \tau + (1 - \delta) \tau P_{\frac{1}{Pt}}}{r + \delta + \tau - \delta \tau} \right)^{\frac{\delta}{1 - \delta}} > 1 - \tau + \frac{\tau P_{\frac{1}{Pt}}}{P_r} \quad (11) \]

We define \( f_1 \left( \tau P_{\frac{1}{Pt}} \right) \) and \( f_2 \left( \tau P_{\frac{1}{Pt}} \right) \) to be as follows, respectively:

\[ f_1 \left( \tau P_{\frac{1}{Pt}} \right) = (1 + \tau) \left( 1 + \frac{2 \delta \tau - 2 \tau + (1 - \delta) \tau P_{\frac{1}{Pt}}}{r + \delta + \tau - \delta \tau} \right)^{\frac{\delta}{1 - \delta}} \]

\[ f_2 \left( \tau P_{\frac{1}{Pt}} \right) = 1 - \tau + \tau P_{\frac{1}{Pt}} \]

It can be shown that \( f_1 \left( \tau P_{\frac{1}{Pt}} \right) \) is increasing and convex in \( \tau P_{\frac{1}{Pt}} \) when \( \delta > \frac{1}{2} \); increasing and concave in \( \tau P_{\frac{1}{Pt}} \) when \( \delta < \frac{1}{2} \). Similarly, \( f_2 \left( \tau P_{\frac{1}{Pt}} \right) \) is linear in \( \tau P_{\frac{1}{Pt}} \). Given the properties of both curves, we reach the following proposition.

**Proposition 3:** Steady-state housing price \( p_s \) is monotone in \( \pi \) for any \( \pi \in [0, 1] \). However, whether \( p_s \) is increasing or decreasing in \( \pi \) is ambiguous.

**Proof:** It is straightforward to show that \( f_1 \left( \tau P_{\frac{1}{Pt}} \right) \) intersects with \( f_2 \left( \tau P_{\frac{1}{Pt}} \right) \) at most twice. In the following analysis, we discuss the sign of \( -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} \) in each case.

(i) If the two curves do not intersect, so either \( f_1 \left( \tau P_{\frac{1}{Pt}} \right) > f_2 \left( \tau P_{\frac{1}{Pt}} \right) \) or \( f_1 \left( \tau P_{\frac{1}{Pt}} \right) \leq f_2 \left( \tau P_{\frac{1}{Pt}} \right) \) always holds, which implies \( -\frac{\partial \kappa}{\partial \pi} - \frac{\partial G(p_s)}{\partial \pi} \) is either
always positive or negative for any \( \pi \in [0,1] \), so \( \frac{\partial p_s}{\partial \pi} \) is either always negative or positive for any \( \pi \in [0,1] \).

(ii) If the two curves intersect once, so there exists an unique \( p^* \) such that \( f_1 \left( \tau \bar{P} \frac{1}{p} \right) = f_2 \left( \tau \bar{P} \frac{1}{p} \right) \). This implies \( \frac{\partial p_s}{\partial \pi} = 0 \) when \( p = p^* \). Moreover, if \( f_1 \left( \tau \bar{P} \frac{1}{p} \right) > f_2 \left( \tau \bar{P} \frac{1}{p} \right) \) for any \( p > p^* \), then \( f_1 \left( \tau \bar{P} \frac{1}{p} \right) < f_2 \left( \tau \bar{P} \frac{1}{p} \right) \) holds for any \( p < p^* \). In the following, we show if there exists some \( \pi \in [0,1] \), such that \( \frac{\partial p_s}{\partial \pi} > 0 \), then \( \frac{\partial p_s}{\partial \pi} > 0 \) holds for any \( \pi \in [0,1] \). If not, suppose there exists \( \pi_o \in [0,1] \) such that \( \frac{\partial p_s}{\partial \pi} < 0 \) when \( \pi = \pi_0 \), since \( \frac{\partial p_s}{\partial \pi} \) is continuous and differentiable in \( \pi \in [0,1] \), and thus there exists a \( \pi^* \in [0,1] \) such that \( \frac{\partial p_s}{\partial \pi} = 0 \) when \( \pi = \pi^* \). Moreover, since we have shown there is an unique \( p^* \) such that \( \frac{\partial p_s}{\partial \pi} = 0 \), holds so \( p_s = p^* \) when \( \pi = \pi^* \). Therefore, \( p_s \) arrives its global maximum value at \( p^* \) when \( \pi = \pi^* \). But when \( \pi = \pi_0, \frac{\partial p_s}{\partial \pi} < 0 \) implies that \( f_1 \left( \tau \bar{P} \frac{1}{p} \right) > f_2 \left( \tau \bar{P} \frac{1}{p} \right) \), which holds only when \( p > p^* \). This contradicts with that \( p^* \) is the global maximum value. Similarly, we can also show if there exists some \( \pi \in [0,1] \), such that \( \frac{\partial p_s}{\partial \pi} < 0 \), then \( \frac{\partial p_s}{\partial \pi} < 0 \) holds for any \( \pi \in [0,1] \).

(iii) If the two curves intersect twice, so there exists \( p_1^* \) and \( p_2^* \) such that \( f_1 \left( \tau \bar{P} \frac{1}{p_1^*} \right) = f_2 \left( \tau \bar{P} \frac{1}{p_2^*} \right) \) and \( f_1 \left( \tau \bar{P} \frac{1}{p_1^*} \right) = f_2 \left( \tau \bar{P} \frac{1}{p_2^*} \right) \). We can apply similar argument as in (ii) to show that if there exists some \( \pi_o \in [0,1] \) such that \( p_s \) falls in one of the three intervals: \( [0, p_1^*], [p_1^*, p_2^*] \) and \( [p_2^*, \infty] \) when \( \pi = \pi_0 \), then for any \( \pi \in [0,1] \), \( p_s \) will fall in the same interval, otherwise it will violate \( p_1^* \) and \( p_2^* \) are the locally maximum or minimum value. This implies \( \frac{\partial p_s}{\partial \pi} \) is either always positive or negative for any \( \pi \in [0,1] \). Q.E.D

Given the proposition above, we intend to derive the condition under which \( \frac{\partial p_s}{\partial \pi} < 0 \) holds. It is equivalent to find the condition \( \frac{\partial p_s}{\partial \pi} < 0 \) holds when \( \pi = 1 \). We solve \( p_s \) when \( \pi = 1 \) in the following from market clearing condition:

\[
\frac{\tau \bar{P}}{1+r} e^{H \frac{p_s}{\bar{P}}} \left( \frac{1+r-(1-\delta)(1-r)}{\delta (1+r)} \right)^{\frac{\tau}{1+r}} - \frac{1-\tau}{1+r} = p_s
\]
Substitute the price above into equation (11) we have:

\[
\frac{(1 + \tau)}{(1 + r)} \left( r + 2\delta - 1 + (1 - \delta) e^{\frac{\delta}{r (1 + \tau)}} \frac{1 + r - (1 - \delta)(1 - \tau)}{\delta r (1 + \tau)} \right) > b \frac{\delta}{r (1 + r)}
\]

(12)

The inequality above gives the condition under which \( \frac{\partial p_s}{\partial \pi} < 0 \). In the appendix, we also show when we solve steady-state price \( p_s \) when \( \pi = 0 \), and substitute it into equation (11), we reach the same condition as above. This further supports our argument in Proposition 3. We summarize our conclusion in the following proposition.

**Proposition 4:** If equation (12) holds, then \( \frac{\partial p_s}{\partial \pi} < 0 \) on \( \pi \in [0, 1] \), otherwise \( \frac{\partial p_s}{\partial \pi} > 0 \) always holds.

The previous analysis have discussed how steady-state housing price responds to changes in productivity \( A \), collateral constraint \( \lambda \), preference \( \beta \), housing supply \( \tilde{H} \), and the fraction of contrarian agents, \( \pi \). We skip the discussion of comparative statics with respect to \( \tilde{P} \) due to the following reason: To support any equilibrium price level \( p, \) the fraction of agents who expect price to rise in the future is given as: \( (1 - \pi) \frac{p - \tilde{P}}{\tilde{P}} + \pi \left( \frac{p + \tilde{P}}{\tilde{P}} - \frac{p - \tilde{P}}{\tilde{P}} \right) \). Therefore, when we change the value of \( \tilde{P}, \) it is equivalent to changing the value of \( \pi, \) when all other parameters remain same.

## 4 Dynamic Analysis

In this section, we focus on the dynamics of the economy. We first exam the stability of the steady-state equilibrium. Then we explore how housing price evolves over time when the economy is hit by a productivity shock.

**Proposition 5:** The unique steady-state housing price \( p^* \) is globally stable.

**Proof:** To show steady state housing price \( p^* \) is stable, it is equivalent to show \( \left. \frac{\partial p_s}{\partial p \mid \tau \rightarrow \infty} \right|_{p_s = p^*} < 1 \). Our proof follows two steps:
step 1 First, we prove \( \frac{dp_t}{dp_{t-1}} > 0 \) for any \( p_{t-1} > 0 \). Housing market clearing condition in \( t \) can be written as:

\[
\begin{align*}
&\frac{(1-\pi)\beta}{(1-\delta)\pi P} \ln \left( \frac{1+r-(1-\delta)(1+\tau)+(1-\delta)\tau\frac{p_t}{p_{t-1}}}{\delta(1+r)} \right) + \frac{\pi A(1+r+\lambda\theta)}{1+r} \left( \frac{1}{p_t} - \frac{\tau+r}{\pi P} \right) + G(p_{t-1}) \\
= & \bar{H} \frac{(1-\pi)\beta}{(1-\delta)\pi P} \ln \left( \frac{1+r-(1-\delta)(1+\tau)+(1-\delta)\tau\frac{p_t}{p_{t-1}}}{\delta(1+r)} \right) + \pi \left( A + \frac{\lambda\theta A}{1+r} \right) \left( \frac{1}{p_t} - \frac{\tau+r}{\pi P} \right)
\end{align*}
\]

The definitions of \( G(p) \) and \( \kappa \) still follow (9) and (10). Moreover, we denote:

\[
L(p_t) = \frac{(1-\pi)\beta}{(1-\delta)\pi P} \ln \left( \frac{1+r-(1-\delta)(1+\tau)+(1-\delta)\tau\frac{p_t}{p_{t-1}}}{\delta(1+r)} \right) + \pi \left( A + \frac{\lambda\theta A}{1+r} \right) \left( \frac{1}{p_t} - \frac{\tau+r}{\pi P} \right)
\]

The condition can be rewritten into:

\[
L(p_t) = \bar{H} + L(p_{t-1}) - G(p_{t-1}) - \kappa
\]

Differentiate the above equation with respect to \( p_{t-1} \) gives:

\[
L'(p_t) \frac{dp_t}{dp_{t-1}} = L'(p_{t-1}) - G'(p_{t-1})
\]

where

\[
\begin{align*}
L'(p_t) &= \frac{-\pi(1-\pi)\beta}{(1+r-(1-\delta)(1+\tau))p_t^2 + (1-\delta)\tau P p_t} - \frac{\pi A + \lambda\theta A}{1+r} \frac{1}{p_t^2} \\
G'(p_{t-1}) &= \frac{-\pi\beta}{(1-\tau)p_{t-1}^2 + \tau P p_{t-1}} + \frac{-\pi(1-\pi)\beta\delta}{(1+r-(1-\delta)(1+\tau))p_{t-1}^2 + (1-\delta)\tau P p_{t-1}}
\end{align*}
\]

To have \( \frac{dp_t}{dp_{t-1}} > 0 \), we need \( L'(p_{t-1}) - G'(p_{t-1}) < 0 \), which is equivalent to:

\[
\frac{(1-\pi)\beta(1-\delta)p_{t-1}}{(1+r-(1-\delta)(1+\tau))p_{t-1} + (1-\delta)\tau P p_{t-1}} - \frac{\pi\beta p_{t-1}}{(1-\tau)p_{t-1} + \tau P} > -\frac{\pi A + \lambda\theta A}{1+r}
\]

When \( A \) is sufficiently large, the above inequality always holds, and thus \( \frac{dp_t}{dp_{t-1}} > 0 \) holds. (The minimum value exists and depends on \( \pi \)).

Step 2 We then prove \( \frac{dp_t}{dp_{t-1}}|_{p_{t-1}=p^*} < 1 \), then it is straightforward to show the steady state is stable since \( p_t \) is monotonically increasing in \( p_{t-1} \) and the steady-state \( p^* \) is unique.
Since \( \frac{dp_t}{dp_{t-1}} = \frac{L'(p_{t-1}) - G'(p_{t-1})}{L'(p_t)} = 1 - \frac{G'(p_{t-1})}{L'(p_t)} \), and thus when \( p_{t-1} = p_s \), we have \( \frac{dp_t}{dp_{t-1}} = 1 - \frac{G'(p_s)}{L'(p_s)} < 1 \) since we have shown before that \( L'(p_{t-1}) - G'(p_{t-1}) < 0 \) holds on \((0, \bar{P})\) and both \( L'(p_{t-1}) \) and \( G'(p_{t-1}) \) are negative.

Q.E.D

Now, let the economy initially be in a steady-state with equilibrium housing price \( p^*_s \). Suppose there is an unexpected negative productivity shock to \( A \) in period \( T \). \( A \) permanently decreases to \( A' \). Denote housing price in \( T \) to be \( p_T \), and new steady-state housing price to be \( p^*_s' \). We summarize how the housing market responds to the shock in the following proposition:

**Proposition 6:** Housing price suddenly drops to \( p_T \) and then gradually returns back to \( p^*_s \). In other words, we have: \( p_T < p^*_s \) and \( p^*_s' = p_s \).

**Proof:** Housing market clearing condition at \( T \) is: \( H^d_T = \bar{H} + (H^d_{T-1} - \delta H^u_{T-1}) \). Since the economy is in steady-state in \( T-1 \), this implies \( \bar{H} = \delta H^u_{T-1} \), and thus the condition can be rewritten as: \( H^d_T = H^d_{T-1} \), where \( H^d_T \) and \( H^d_{T-1} \) are as follows:

\[
H^d_T = \frac{\pi \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{(1 - \pi) \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{(1 - \delta) \tau \bar{P}}{\bar{P}_T} \\
+ \pi \left( \frac{1}{p_T} - \frac{\tau + r}{\tau P} \right) \left( A' + \frac{\lambda \theta A'}{1 + r} \right) + (1 - \pi) \frac{\tau - r}{\tau P} \left( A' + \frac{\lambda \theta A'}{1 + r} \right)
\]

\[
H^d_{T-1} = \frac{\pi \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{(1 - \pi) \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{(1 - \delta) \tau \bar{P}}{\bar{P}_s} \\
+ \pi \left( \frac{1}{p^*_s} - \frac{\tau + r}{\tau P} \right) \left( A + \frac{\lambda \theta A}{1 + r} \right) + (1 - \pi) \frac{\tau - r}{\tau P} \left( A + \frac{\lambda \theta A}{1 + r} \right)
\]

\( H^d_T = H^d_{T-1} \) implies:

\[
(1 - \pi) \frac{\tau - r}{\tau P} (A' - A) \left( 1 + \frac{\lambda \theta}{1 + r} \right) = L(p^*_s, A) - L(p_T, A')
\]

where

\[
L(p, A) = \frac{(1 - \pi) \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \frac{\lambda \theta}{\tau P} \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} \right) + \pi \left( \frac{1}{p} - \frac{\tau + r}{\tau P} \right) \left( A + \frac{\lambda \theta A}{1 + r} \right)
\]
When \( A' < A \), the equation above implies \( L(p_s^*, A) < L(p_T, A') \). Since \( L(p, A) \) is increasing in \( A \), and thus \( L(p_T, A') < L(p_T, A) \). Moreover, \( L(p, A) \) is also decreasing in \( p \), and thus \( L(p_s^*, A) < L(p_T, A) \) implies \( p_T < p_s^* \).

The previous analysis summarizes housing market clearing condition in steady-state as \( G(p_s) = \bar{H} - \kappa \). The value of \( G(p_s) \) and \( \kappa \) do not depend on \( A \). Therefore, the new steady-state price level \( p_s^* \) should remain the same level as \( p_s^* \). This completes the proof. \( \text{Q.E.D} \)

We are also interested to see how the composition of contrarian versus momentum agents affect the housing market’s response to the productivity shock.

**Proposition 7:** Define \( R = \frac{p_s^*}{p_T} \), then we have

(i) when \( \pi = 1 \), \( \frac{\partial R}{\partial H} > 0, \frac{\partial R}{\partial \lambda} = 0, \frac{A - A'}{p_T} > 0 \), the sign of \( \frac{\partial R}{\partial \pi} \) is ambiguous

(ii) when \( \pi = 0 \), \( \frac{\partial R}{\partial H} < 0, \frac{\partial R}{\partial \lambda} > 0, \) the sign of \( \frac{A - A'}{p_T} \) and \( \frac{\partial R}{\partial \pi} \) is ambiguous.

(iii) when \( 0 < \pi < 1 \), the sign of \( \frac{\partial R}{\partial \pi} \) and \( \frac{A - A'}{p_T} \) are ambiguous.

**Proof:** We have shown before that housing market clearing condition in \( T \) is given as:

\[
(1 - \pi) \frac{\tau - r}{\tau P} (A - A') \left( 1 + \frac{\lambda \theta}{1 + r} \right) = L(p_s^*, A) - L(p_T, A')
\]

where \( L(p, A) = \frac{(1-\pi)\beta}{(1-\delta)\tau P} \ln \left( \frac{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}}{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}} \right)^{\pi} \left( \frac{1}{\beta} \right)^{\left( \frac{\tau + r}{\tau P} \right)} \left( A + \frac{\lambda \theta A}{1 + r} \right) T.

Moreover, \( L(p_s^*, A) - L(p_T, A') \) can be written as:

\[
\frac{(1 - \pi) \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}}{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}} \right)^{\pi} \left( \frac{1 + r + \lambda \theta}{1 + r} \right) \left[ \left( \frac{A - A'}{p_T} \right) - \frac{\tau + r}{\tau P} (A - A') \right]
\]

Therefore, housing market clearing condition can be rewritten as:

\[
\left( 1 + \frac{\lambda \theta}{1 + r} \right) \left[ A' - A \right] \frac{1}{\tau P} (\tau - r - 2\pi \tau) =
\]

\[
\frac{(1 - \pi) \beta}{(1 - \delta) \tau P} \ln \left( \frac{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}}{1 + r - (1 - \delta)(1 - \tau) + (1 - \delta) \tau P \frac{1}{p_T}} \right)^{\pi} \left( \frac{A - A'}{p_T} \right) \left( 1 + \frac{\lambda \theta}{1 + r} \right)
\]

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(i) When $\pi = 1$, we have:

$$[A' - A] \frac{1}{\tau P} (\tau - r - 2\pi\tau) = \left( \frac{A}{p^*_s} - \frac{A'}{p^*_r} \right)$$

It can be shown that $\frac{A}{p^*_s} - \frac{A'}{p^*_r} > 0$, and this implies price drops more significantly compared with productivity decline. Moreover, since $\frac{\partial p^*_s}{\partial H} < 0$ and $\frac{\partial p^*_s}{\partial \lambda} = 0$, it is straightforward to have $\partial R/\partial \tilde{H} > 0$ and $\partial R/\partial \lambda = 0$ from the equation above.

(ii) When $\pi = 0$, we have:

$$\left( 1 + \frac{\lambda \theta}{1 + r} \right) [A' - A] \frac{(1 - \delta)}{\beta} = \ln \left( \frac{1 + r - (1 - \delta) (1 - \tau) + (1 - \delta) \tau \tilde{P} \frac{1}{p^*_r}}{1 + r - (1 - \delta) (1 - \tau) + (1 - \delta) \tau P \frac{1}{p^*_r}} \right)$$

since $\frac{\partial p^*_s}{\partial H} < 0$ and $\frac{\partial p^*_s}{\partial \lambda} = 0$, we can have $\partial R/\partial \tilde{H} < 0, \partial R/\partial \lambda > 0$ holds. Moreover, we express $\frac{A'}{p^*_r} - \frac{A}{p^*_s}$ into the following way:

$$\frac{A'}{p^*_r} - \frac{A}{p^*_s} = \left[ 1 - e^{(1 + \frac{\lambda \theta}{1 + r}) [A' - A] (\tau - r) (1 - \delta)} \right] \left[ A' p^*_s \frac{1 + r - (1 - \delta) (1 - \tau)}{\delta (1 + r)} + \frac{(1 - \delta) \tau \tilde{P} A}{\delta (1 + r)} \right] + \frac{(1 - \delta) \tau \tilde{P} (A' - A)}{\delta (1 + r)}$$

where $\frac{\tau \tilde{P}}{p^*_s} = e^{(1 - \delta) \frac{\lambda \theta}{1 + r} \tilde{H}} \left( 1 + \frac{r}{1 + r} \right)^{\frac{2 + \frac{\theta}{1 + r}}{1 - \delta} - \frac{\delta (1 + r) (1 + r)}{(1 - \delta)} + (1 - \delta) \frac{1 + r - (1 - \delta) (1 + r)}{(1 - \delta)}}$. Therefore, the sign of $\frac{A'}{p^*_r} - \frac{A}{p^*_s}$ is ambiguous. 

**Q.E.D**

5 Numerical Analysis

5.1 Parameterization

To quantitatively assess the impact of $\pi$ on steady-state and the dynamics of housing prices, we perform some numerical analysis in this section. First, we assign benchmark value to the following parameters in Table 1.
Table 1: Benchmark Parameterization

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ratio of old-age to middle-age income</td>
<td>0.2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>subjective discount rate</td>
<td>0.8</td>
</tr>
<tr>
<td>$A$</td>
<td>productivity</td>
<td>2</td>
</tr>
<tr>
<td>$\delta$</td>
<td>utility housing depreciation rate</td>
<td>0.8</td>
</tr>
<tr>
<td>$\tau$</td>
<td>housing price expectation parameter</td>
<td>0.5</td>
</tr>
<tr>
<td>$H$</td>
<td>new housing supply</td>
<td>0.5</td>
</tr>
<tr>
<td>$P$</td>
<td>upper bound of reference points</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>collateral constraint</td>
<td>0.8</td>
</tr>
<tr>
<td>$r$</td>
<td>interest rate</td>
<td>0.2</td>
</tr>
</tbody>
</table>

What we have in mind is that each period corresponds to 20 years. So $\delta = 0.8$ corresponds to 7.7 percent of depreciation rate annually. All other parameter values are also set within the reasonable range of the data counterparts in the real world.

5.1.1 Steady-state Analysis

The benchmark parameter value are set so that the condition in Proposition 1 is satisfied, and thus an unique steady-state housing price $p_s$ exists and $\frac{r + r}{r} p_s < \bar{P}$ holds. In the following, we let $\pi$, the fraction of contrarian agents, take value from 0 to 1, and then evaluate how steady-state housing price responds to changing of $\pi$. We have proved in Proposition 3 that $p_s$ is monotone in $\pi$, but the sign depends on the condition in Proposition 4. In the following, we present two different scenarios, where $p_s$ is increasing in $\pi$ in the left panel and decreasing in $\pi$ in the right panel, when we change the value of $\delta$. 

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A plausible explanation for why $p_s$ increases in $\pi$ for relatively larger value of $\beta$, and decreases for smaller value of $\beta$ is offered as follows. Since steady-state housing price is derived from $\delta H^u = \bar{H}$, so we shall mainly focus on demand for utility housing. Intuitively, when housing preference parameter $\beta$ is low, we may expect a low housing price and little demand for utility housing, which implies that more individuals have reference point $x$ greater than housing price. Therefore, when there are more contrarian agents, more agents might expect housing price will increase in the next period, and this may induce more agents to replace their utility demand by investment demand for housing. Hence, housing price decreases in $\pi$. On the other hand, when $\beta$ is high, demand for utility housing and thus housing price may still be high even though the depreciation rate is high, and this implies more agents have reference point lower than housing price. Therefore, more momentum agents means less demand for investment housing and more demand for utility housing, and thus housing price is increasing in $\pi$. 

Figure 1: steady-state housing price

$\beta = 1$ 

$\beta = 0.5$
5.1.2 Dynamic Analysis

When $0 < \pi < 1$, it is hard to derive tractable analytical solutions. We perform numerical analysis in the following. We let productivity $A$ decline from its benchmark value 2 to 1.5, which corresponds to a 25 percent of drop. We have normalized the benchmark housing price to be 1. Figure 3 presents the degree of housing price drop in the period that the productivity shock hits as a function of $\pi$, the fraction of contrarian agents. As shown in Figure 3, housing price declines less significantly when $\beta$ becomes larger. Moreover, $p_T/p_s$ seems to be concave in $\pi$ when $\beta$ is relatively low, but convex in $\pi$ when $\beta$ is large.

**Figure 3:** steady-state housing price

\[ \beta = 0.5 \quad \beta = 1 \]

Intuitively, a negative productivity shock may induce more contrarian agents replace their utility housing by investment housing, since given low housing price in the current period, more contrarian agents expect future price to grow. However, more momentum agents instead replace their utility demand by saving since fewer momentum agents expect the price to grow in the future. Therefore
the total housing demand and the housing price declines most when \( \pi = 0 \) and least when \( \pi = 1 \). As shown in proposition 8, \( \frac{A}{p_T} - \frac{A'}{p_T} > 0 \) always holds when \( \pi = 1 \), but the sign is ambiguous for relatively low value of \( \pi \).

Figure 4 also plots the economy’s recovery path after the economy is hit by a negative productivity shock. When the shock hits in \( t = 4 \), housing price drops dramatically, then gradually recovers till it converges back to the original steady-state, which we have shown in Proposition 2 that productivity shock will not change the steady-state level of housing price.

Overall, housing market recovers faster when \( \beta = 1 \) compared with \( \beta = 0.5 \). Intuitively, when housing brings a larger utility to agents, agents will demand more when its price is low, and thus housing price recovers faster. Consistent with the previous analysis, housing price declines least in \( T \) when \( \pi = 1 \). Moreover, housing price also recovers slower for a larger value of \( \pi \). The reasoning is the following: as argued before, when housing price is low, more contrarian agents replace utility housing by investment housing, and thus total demand for housing changes little. When the price recovers gradually, contrarian agents simply switches back to demand more utility housing. In contrast, when price is low, momentum agents have little demand for investment housing, but when price recovers momentum agents replace their saving by investment housing, and thus housing demand and housing price goes up quickly.
6 Conclusions

This paper develops a dynamic general equilibrium framework, where agents are exogenously divided into two behavior groups, namely "contrarian" and "momentum" agents. Agents within each behavior group hold a different housing price reference point. For contrarian agents, if observed housing price rises above their reference point, they expect future housing price to converge back to their reference point, on the contrary, momentum agent expects future housing price to deviate further more from their reference point. Agents demand housing for utility and investment propose, respectively. Housing that bring agents utilities are subjected to a positive depreciation rate. There is an alternative risk-free asset in the economy, so agents who expect future housing price to decline will not have demand for investment housing, but agents who expect future housing price to go up may face an exogenous borrowing constraint. We theoretically
prove under some mild assumptions, an unique steady-state equilibrium exists where housing price remains constant in the long-run. Given a negative productivity shock, housing price will converge back to the initial steady-state price level without any government intervention. The model also predicts that the fraction of contrarian agents has a deterministic impact on steady-state housing price. But whether more contrarian agents imply higher or lower housing price level may depend on some preference parameter or utility housing depreciation rate. Moreover, the simulation results quantitatively show how the composition of each type of agents affects steady-state housing price level, and the housing market recovery speed after a negative productivity shock.

The model can be easily extended to a life-cycle framework to allow for a richer quantitative analysis. Moreover, there is no "learning" effect in the current setup. We may extend our framework into a multiple period model, where agents’ beliefs may be affected by someone else, and also by the historical events. Some further empirical and behavioral study on how people form their expectation on future housing price, and how these affect housing markets are also highly desirable.
References


7 Appendix (Not Intended for Publication)

In this appendix, we show the condition proposed in Proposition 4 can also be derived by solving steady-state price $p_s$ when $\pi = 0$ and substituting into equation (11), and finally check $p_s|_{\pi=0} > p_s|_{\pi=1}$ also holds.

First, steady-state housing price $p_s$ when $\pi = 0$ satisfies follows:

$$\frac{\tau \bar{P}}{p_s} = e^{\frac{(1-\delta)\tau}{\pi H}} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta \frac{(1+r)}{(1-\delta)} - \frac{1+r-(1-\delta)(1+\tau)}{(1-\delta)}$$

Substitute the equation above into (11) we have:

$$(1+r) \left( r + 2\delta - 1 + (1-\delta) e^{\frac{\lambda}{\delta (1+r)}} (1+r) \right) \left( \frac{\delta}{\delta (1+r)} \right)^{\delta} > e^{\frac{\lambda}{\delta (1+r)}} (r + \delta + \tau - \delta \tau) \left( \frac{\delta}{\delta (1+r)} \right)^{\delta}$$

We can show the inequality above is equivalent to

$$(1+r) \left( \frac{\frac{(1-\delta)\tau}{\pi H} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta (1+r)}{r + \delta + \tau - \delta \tau} \right) > 1 - \frac{2\delta - r}{1-\delta} + e^{\frac{(1-\delta)\tau}{\pi H}} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta (1+r)$$

It is easy to verify that $e^{\frac{\lambda}{\delta (1+r)}} (1+r) = (1+\tau)$, therefore the inequalities above is equivalent to (12) in Proposition 4. Moreover, we also need to check the condition under which $p_s|_{\pi=0} > p_s|_{\pi=1}$. This is equivalent to show:

$$e^{\frac{(1-\delta)\tau}{\pi H}} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta (1+r) > e^{\frac{\lambda}{\delta (1+r)} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta (1+r)}$$

which is also equivalent to:

$$e^{\frac{\lambda}{\delta (1+r)} (1+r) \left( \frac{r + \delta + \tau - \delta \tau}{\delta (1+r)} \right)^{\frac{\delta}{\delta (1+r)}}} > 1 - \frac{2\delta - r}{1-\delta} + e^{\frac{(1-\delta)\tau}{\pi H}} \left( \frac{1+\tau}{1+r} \right)^{\frac{\delta-1}{\delta}} \delta (1+r)$$

The above condition also coincides with (12).