The formation of partnerships in social networks

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Abstract

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1 Introduction

2 The Model

2.1 Partnerships

We consider a society of \( n \) agents who are organized in a social network \( g \). The social network evolves over time, as agents will delete links and leave the network. At any discrete time \( t = 1, 2, \ldots \), one agent is chosen with probability \( \frac{1}{n} \) to request a favor from a partner. If the favor is granted, the agent who receives the favor obtains a flow payoff of \( v \) and the agent who grants the favor pays a flow cost \( c \). All agents discount the factor using the same discount factor \( \delta \). We define the value of a partnership as the expected discounted payoff obtained by an agent when he has found a partner with whom he reciprocates favors,

\[
V = \frac{v - c}{n(1 - \delta)}.
\]

Partnerships are formed according to the following decentralized procedure. Suppose that an agent \( i \) needs a favor at date \( t \). Two situations may arise:

- Either agent \( i \) is already in a partnership
- Or agent \( i \) is not yet in a partnership

In the former case, the favor is offered by agent \( i \)'s partner. In the latter case, agent \( i \) turns to his direct neighbors in the current social network \( g_t \) and asks them sequentially for a favor. The sequence in which an agent approaches his neighbors is chosen at random. If neighbor \( j \) is approached by agent \( i \), he responds by Yes or No to the offer to form a partnership. If agent \( j \) responds Yes, the partnership \( \{ij\} \) is formed, and the two partners leave the social network, deleting all their links. We let

\[
g_{t+1} = g_t \setminus i, j,
\]

denote the network obtained after agents \( i \) and \( j \) have left. If agent \( j \) rejects the formation of the partnership, the link \( ij \) is destroyed, the new social network is \( g_t \setminus ij \), and agent \( i \) turns to the next neighbor in the sequence. If all agents reject \( i \)'s request, the network at next period is

\[
g_{t+1} = g_t \setminus i,
\]
the network obtained from $g$ by deleting $i$ and all his links. A strategy for player $i$ is thus a mapping associating to each history of the game a response in $\{Y, N\}$. We focus attention on Markov strategies which only depend on the current social network and the identity of the agent requesting a favor. A Markov equilibrium is a collection of Markov strategies such that all agents choose their optimal response after every history.

An outcome of the partnership formation process is a list of partnerships formed or agents leaving the network at every time period as a function of the realization of "needs"—the list of agents who request a favor at each period. Given a fixed realization of needs, an outcome is efficient if it maximizes the sum of payments of all agents. Given that agents are homogeneous, the sum of payments of all agents is maximized when the number of partnerships is maximized. We say that a social network $g$ supports efficient equilibria if and only if, for all realizations of needs, the equilibrium outcomes of the process of partnership formation starting from $g$ are efficient.

2.2 Matchings and bipartite graphs

In this subsection, we collect definitions in graph theory pertaining to matchings and bipartite graphs which will prove useful in our analysis. (See Lovasz and Plummer (1986) for an excellent monograph on matchings and bipartite graphs.) Given a network $g$, a matching $M$ is a collection of edges in $g$ such that no pair of edges in $M$ has a common vertex. A matching $M$ is maximal if there is no matching $M' \supset M$ in $g$. A matching $M$ is a maximum matching if there is no matching $M'$ in $g$ such that $|M'| > |M|$. For any graph $g$, we let $\mu(g)$ denote the matching number of graph $g$, i.e. the size of any maximum matching in $g$. If $n$ is even, and $\mu(g) = \frac{n}{2}$, there exists a matching covering all the vertices in $g$. This is called a perfect matching and any graph admitting a perfect matching is a perfect graph.

A graph $g$ is bipartite if the set of vertices can be partitioned into two subsets $A$ and $B$ such that there is no edge among any two vertices in $A$ and no edge among any two vertices in $B$. A bipartite graph is complete if all vertices in $A$ are related to all vertices in $B$. If $|A| = |B|$, a partite graph is perfect if and only if it satisfies Hall’s condition: for any subset $C \subseteq A$, the set of vertices in $B$ which are connected to vertices in $C$, $f(C)$ satisfies $|f(C)| \geq |C|$.
3 Partnerships with costly favors

In this Section, we analyze the equilibrium of the process of partnership formation. We first illustrate the equilibrium in a simple four-player line. We then introduce the concept of essential players in the network, and prove a Lemma on the effect of link deletion on essentiality. Using this definition, we characterize the optimal behavior of agents in the partnership formation game. We then prove the main Theorem of this section, establishing that all social networks support efficient equilibria for $\delta$ sufficiently close to 1. Finally, we discuss equilibrium behavior when players are less patient in the complete network and in the line.

3.1 Equilibrium in a four player line $L_4$

Let $n = 4$ and suppose that $g = L_4$, the four-player line as illustrated in Figure 1.

Figure 1: The line $L_4$

The matching number of the line $L_4$ is two: the maximum number of partnerships formed is two. We claim that, when $\delta$ is sufficiently close to 1, in equilibrium, the maximum number of matchings is achieved in equilibrium. Suppose that player 1 has a need and approaches player 2. If player 2 rejects the offer, he becomes a peripheral agent in the line $L_3$. In $L_3$, if a peripheral agent requests a favor from the central agent, the central agent will decline the request, as he can form a partnership in the line $L_2$ and economize on the cost of giving the favor. Hence, with a positive probability, agent 2 ends up being disconnected if he rejects the request of player 1. For $\delta$ sufficiently close to 1, the cost of being disconnected exceeds the economy in the cost $c$, so agent 2 always accepts agent 1’s request. Suppose that agent 2 has a request. If he meets agent 3, agent 3 declines the offer to form the partnership, as he can form a partnership later with agent 4 and economize on the cost $c$. If agent 2 meets agent 1, agent 1 always accepts the formation of the partnership.

In the line $L_4$, we can thus characterize the optimal response of agents for $\delta$ sufficiently close to 1 as follows:

- Agent 1 (4) accepts to form a partnership with agent 2 (3)
Agent 2 (3) accepts to form a partnership with agent 1 (4)
Agent 2 (3) declines to form a partnership with agent 3 (2)

Given this equilibrium behavior, the two partnerships 12 and 34 are always formed in equilibrium: the line $L_4$ supports efficient equilibria. We will now show that the construction of equilibrium can be extended to any graph $g$, and introduce the concept of essential nodes to characterize equilibrium behavior.

### 3.2 Essential nodes

A node $i$ in graph $g$ is called essential if it belongs to all maximum matchings of the graph $g$. It is called inessential otherwise. Clearly, all nodes are essential in a perfect graph. As illustrated in Figure 2, all nodes are inessential in the odd cycle $C_3$, and in the line $L_5$, nodes 2 and 4 are essential, but not nodes 1, 3 and 5. In the line $L_5$, node 3 is the most central node according to all measures of node centrality, but is inessential. This example shows that there is no connection between centrality and essentiality of nodes in a graph.

![Graphs](image)

Figure 2: Essential and inessential nodes

The next Lemma establishes properties on essential nodes which will prove useful in the characterization of equilibrium.
Lemma 3.1  

1. If $i$ is an essential node in $g$, there exists $ij \in g$, such that $j$ is inessential in $g \setminus i$.

2. If $i$ is not an essential node in $g$ and $ij \in g$, $j$ is an essential node in $g \setminus i$.

3. If $i$ is a essential node in $g$ and $j$ is inessential in $g$, $i$ is essential in $g \setminus j$.

4. If $i$ is an essential node in $g$, $jk \in g$, and $\mu(g) = \mu(g \setminus j, k) + 1$, then $i$ is essential in $g \setminus j, k$.

Proof:

1. Let $M$ be a maximum matching in $g$ and $E_1 = (ij)$ the edge covering $i$ in $M$. Then $(E_2, \ldots, E_M)$ is a maximum matching in $g \setminus i$ which does not contain $j$. Hence, $j$ is not essential in $g \setminus i$.

2. Suppose by contradiction that $j$ is not essential in $g \setminus i$. Then there exists a maximum matching of $g \setminus i$ with no edge covering $j$, $M = (E_1, \ldots, E_M)$. Consider then the matching $(M, ij)$ in $g$. This is a matching of cardinality $\mu(g \setminus i) + 1$, contradicting the fact that $i$ is not essential in $g$.

3. Suppose by contradiction that there exists a maximum $M$ matching of $g \setminus j$ where $i$ is not covered. Because $j$ is inessential in $g$, $\mu(g) = \mu(g \setminus j)$. So $M$ has the same cardinality as a maximum matching in $g$ and hence is a maximum matching of $g$, contradicting the fact that $i$ is essential in $g$.

4. Suppose that $i$ is inessential in $g \setminus j, k$. Then there exists a maximum matching $M$ in $g \setminus j, k$ not covering $i$. As $\mu(g) = \mu(g \setminus j, k) + 1$, $M' = (M, jk)$ is a maximum matching of $g$ not covering $i$, contradicting the fact that $i$ is essential in $g$.

Lemma 3.1 shows that any essential node $i$ must be connected to some node which is inessential in $g \setminus i$. On the other hand, all neighbors of an inessential node $i$ are essential in $g \setminus i$. When an inessential agent is removed from the network, all essential agents remain essential. When a pair of agents leaves the network, without disrupting the total number of matchings, all essential agents remain essential as well.
3.3 Equilibrium behavior

With the help of Lemma 3.1, we now characterize the optimal response of agents in the game of partnership formation.

**Proposition 3.2** Let j receive a request from i in the social network g.

- If j is the only neighbor of i in g, j accepts the request if and only if he is inessential in $g \setminus i$.
- If j is not the only neighbor of i in g, given the known sequence in which i contacts his neighbors, either there exists k who accepts the request of agent i or all subsequent agents reject i’s request. Hence, if j rejects i’s request, the social network evolves to $g'$ where either $g' = g \setminus i, k$ or $g' = g \setminus i$. Agent j accepts i’s request if and only if j is inessential in $g'$.

**Proof:** The proof is by induction on the number of agents in a connected component. For $n = 2$, both agents are essential and the statement is trivially satisfied. For $n = 3$, we distinguish between two cases:

- $g = L_3$
- $g = C_3$.

If $g = L_3$, the two peripheral agents become isolated if they reject the offer of the central agent and hence always accept it. The central agent rejects the request of any of the peripheral agents as he remains connected in the line $L_2$. If $g = C_3$, the request of the first agent is always rejected as the other two agents remain connected and essential if that agent is removed from the network.

Suppose now that the statement is true for all $n' < n$ and consider a component with n agents. Let $g'$ be the component formed if j rejects i’s request. As $|g'| < n$, we use the induction hypothesis to compute the continuation payoff of agent j if he rejects i’s request. Suppose first that j is inessential in $g'$. If j is chosen next period to have a request, by Lemma 3.1, (statement 2), all neighbors of j remain essential in $g' \setminus j$. By the induction hypothesis, they reject j’s request and hence j obtains a continuation payoff of 0 with positive probability. For $\delta$ sufficiently close to 1, agent j thus has an incentive to accept i’s request.

Suppose next that j is essential in $g'$. If j has a request next period, at $t + 1$, by Lemma 3.1 (statement 1), one of his neighbors, say k, becomes inessential
in $g' \setminus j$. This implies that $j$ must form a partnership. If not, the graph $g' \setminus j$ would form, and by the induction hypothesis, agent $k$ must have accepted $j$’s request – a contradiction.

Next suppose that $j$’s first requests happens at some period $t' > t + 1$. At period $t'$, the social network has evolved to $g_{t'} \equiv g''$. We claim that $j$ remains essential in $g''$. In the light of Lemma 3.1 (statements 3 and 4), we need to prove that (i) if an agent $k$ is removed from the network, he must be inessential and (ii) if a pair of agents $(k, l)$ leaves the network, they do not disrupt the number of matchings.

To prove statement (i) suppose by contradiction that there exists an agent $k$ and a period $t$ such that $g_{t+1} = g_t \setminus k$ and $k$ is essential in $g_t$. We have already argued that agent $k$ must have a neighbor who becomes inessential in $g_{t+1}$ and must have accepted $k$’s request – a contradiction. We now consider statement (ii).

Claim 3.3 If along the equilibrium path, at some period $t$, a pair $(k, l)$ of agents forms a partnership, then $\mu(g_t \setminus k, l) = \mu(g_t) - 1$.

Proof of the claim: Suppose that agent $k$ places the request and agent $l$ responds. Agent $k$ must be essential in $g_t$ – otherwise, as argued earlier, his request will be denied along the equilibrium path. Agent $l$ must be inessential in the graph $g'$ formed after his rejection. Suppose first that all agents following $l$ reject $k$’s request in equilibrium so that $g' = g_t \setminus k$. As $k$ is essential in $g_t$, $\mu(g_t \setminus k) = \mu(g_t) - 1$. Let $M$ be a maximum matching of $g_t \setminus i$ not containing $j$. Then $M$ is a maximum matching of $g \setminus i, j$ and $|M| = \mu(g_t) - 1$. Next suppose that there exist a sequence of agents following $l$ who accept $k$’s request and let $l_1, ..., l_N$ denote the agents in the sequence. By the preceding argument, $\mu(g_t \setminus k, l_N) = \mu(g_t) - 1$. We now argue that whenever $\mu(g_t \setminus k, l_{n-1}) = \mu(g_t) - 1$, then $\mu(g_t \setminus k, l_{n-1-1}) = \mu(g_t) - 1$. If agent $l_{n-1}$ accepts $k$’s request, he must be inessential in $\mu(g_t \setminus k, l_n)$. Pick a maximum matching $M$ of $g_t \setminus k, l_n$ not containing $l_{n-1}$. This is a maximum matching of $g_t \setminus k, l_{n-1}$ and as $|M| = \mu(g_t) - 1$, $\mu(g_t \setminus k, l_{n-1}) = \mu(g_t) - 1$. This inductive step shows that $\mu(g_t \setminus k, l) = \mu(g_t) - 1$, concluding the proof of the Claim.

To finish the proof of the Proposition, notice that, as $j$ remains essential in $g''$, if $j$ has not yet formed at partnership at $t'$, one of his neighbors will accept his request because he would become inessential in $g \setminus j$. Hence, whenever $g$ remains essential after rejecting $i$’s request, his next request will be accepted and he will form a partnership in equilibrium. Thus agent $j$ has an incentive to reject $i$’s request if he remains essential in the continuation game.  

3.4 The main theorem

We now use the characterization of equilibrium behavior to prove our main theorem: when players are sufficiently patient, the maximum number of pairs are formed in equilibrium, and any network supports efficient equilibria.

**Theorem 3.4** There exists \( \delta > 0 \) such that for all \( \delta \geq \delta \), all social networks support efficient equilibria.

**Proof:** For a fixed realization of needs, an equilibrium is efficient if and only if the maximum number of pairs is formed in equilibrium and no agent delays the formation of a partnership. In the equilibrium characterized in subsection 3.3, at any period \( t \) either a partnership is formed or an agent is isolated from the network. Hence, there is no delay in the formation of partnerships. Furthermore, by Claim 3.3, along the equilibrium path, whenever a pair of agents forms a partnership, it does not disrupt the formation of partnerships by the remaining agents. Hence the total number of partnerships formed in equilibrium is \( \mu(g) \) the maximum number of partnerships in the original social network.

3.5 Exact conditions for the efficient formation of partnerships

Theorem 3.4 establishes that all social networks support efficient equilibria for sufficiently large values of \( \delta \). However, the exact condition on parameters for which efficient equilibria are supported depends on the architecture of the social network. In this subsection, we explicitly compute this condition for two specific networks; the line \( L_n \) and the complete network \( K_n \) where \( n \) is an even number. Both networks are perfect so that the maximum number of matchings is equal to \( \frac{n}{2} \).

3.5.1 Conditions for efficient partnership formation in the line \( L_n \)

We determine the condition for existence of an efficient equilibrium – where the maximum number of matchings is formed for the line \( L_n \) and any line \( L_k \) of length \( k \). Let \( V^k \) denote the continuation value of a peripheral agent in a line with \( k \) agents. If \( k \) is even, \( V^k = V \) as we consider an efficient equilibrium; if \( k \) is odd, \( V^k < V \) and we provide an explicit computation below. When player \( j \) receives a request from agent \( i \), he accepts the request if and only if
\[ V \geq -c + \delta V^k, \]

where \( k \) is the size of the component containing \( j \) after the link \( ij \) is severed. Clearly, if \( k \) is even, \( j \) always has an incentive to reject \( i \)'s request. This guarantees that, whenever a partnership \( ij \) is formed, \( \mu(g \setminus i, j) = \mu(g) - 1 \) so that the total number of matchings formed in equilibrium is equal to the matching number of \( L_n \). We now compute the continuation value of a peripheral agent, say agent 1, in the line \( L - k \) where \( k \) is odd.

If an agent in \( n - k \) has a need, agent 1's value is \( \delta V^k \). If agent 1 has a need, his request is rejected and his value is 0. If any other inessential agent \( j = 3, 5, \ldots, k \) has a need, this request is rejected and the component containing agent 1 becomes an even line so that agent 1’s value is \( \delta V \). If an essential agent \( j = 2, \ldots, k - 1 \) has a request, in an efficient equilibrium, the request will be accepted by one of his neighbors. With probability \( \frac{1}{2} \), the neighbor is to the left of agent \( j \) and the component containing agent 1 becomes an even line so that agent 1’s value is \( \delta V \). With probability \( \frac{1}{2} \), the request is accepted by an agent to the right of agent \( j \), and agent 1 becomes a peripheral agent in an odd line of size \( j - 1 \). If agent 2 has a request and addresses it to agent 1, agent 1 accepts it and pays the cost \( c \). Hence, we write the value as

\[ V^k = \frac{1}{n} \left[ (n - k) \delta V^k + \frac{3(k - 1) \delta V}{4} - \frac{c}{2} + \sum_{j=0}^{k-1} \frac{\delta V^{2j+1}}{2} \right]. \quad (1) \]

As \( V^k < V \) for all \( k \), we observe that \( V^k \) is increasing in \( k \): the value of a peripheral agent in an odd line increases with the size of the line. This implies that the condition for existence of an efficient equilibrium in the line is the most stringent when \( k = n - 1 \). Hence the condition for existence of an efficient equilibrium is

\[ V \geq -c + \delta V^{n-1}, \]

where \( V^{n-1} \) is defined recursively through equation (1).

### 3.5.2 Conditions for efficient partnership formation in the complete network \( K_n \)

In any complete network \( K_k \), the continuation value is identical for all agents as they are all symmetric in the continuation network. Let \( k \) denote the continuation value of any agent in the complete network \( K - k \). We claim
that, whenever \( i \) requests a need from a sequence of agents \( j_1, \ldots, j_{k-1} \), all agents but the last agent \( j_{k-1} \) are going to reject the request. If \( k \) is odd, \( K_{k-1} \) is an even complete graph, and in an efficient equilibrium, all agents obtain a value \( V \) and reject the request. If \( k \) is even, in an efficient equilibrium, the last agent accepts the request, so that the continuation graph is the even complete graph \( K_{k-2} \). All agents preceding \( j_{k-1} \) thus have an incentive to reject the request, anticipating that the graph \( K_{k-2} \) will be formed.

An efficient equilibrium thus exists if and only if

\[
V \geq -c + \delta W^k,
\]

where \( W^k \) is the value of an agent, say agent 1, in an odd complete graph \( K_k \), which is computed as follows.

If an agent in \( n - k \) has a need, agent 1’s value is \( \delta W^k \). If agent 1 has a need, his request is rejected and he obtains a value 0. If any other agent has a need, his request is rejected and agent 1 obtains a value \( \delta V \). Hence

\[
W^k = \frac{1}{n}[(n - k)\delta W^k + (k - 1)\delta V].
\]

We thus obtain

\[
W^k = \frac{(k - 1)\delta V}{n - \delta(n - k)},
\]

which is increasing in \( k \) so that the most stringent equilibrium condition is

\[
V \geq -c + \delta W^{n-1}.
\]

Interestingly, we observe that, as \( V > V^k \) for all odd \( k \), \( V^k < W^k \) for all odd \( k \) and hence \( V^{n-1} < W^{n-1} \). The continuation value of an agent in an odd complete graph is always larger than in an odd line of the same cardinality. This implies that it is easier to sustain efficient partnership formation in the line than in the complete graph. Hence, an increase in the number of links in the social networks may be detrimental to the efficient formation of partnerships. Increasing the number of social links increases the number of potential matchings, but may also increase the continuation value of agents after a link is severed, making it more difficult to sustain the efficient formation of partnerships.
4 Partnerships with joint values

4.1 Joint values

We now consider a model where the formation of a partnership results in positive values for both agents. When an agent responds to a request, he obtains a positive flow payoff \( w > 0 \) rather than incurring a negative cost \( c < 0 \). The equilibrium response of an agent is obvious: every agent has an incentive to accept the formation of a partnership immediately. As opposed to the model in the previous section, where agents try to delay the formation of a partnership, when agents obtain joint values, they want to rush to form partnerships. This behavior may result in the inefficient formation of matches. For example, in the line \( L_4 \), if agent 2 approaches agent 3 with a request, agent 3 accepts immediately, and the pair (23) is the only partnership formed, short of the maximum number of matches which is equal to 2. Hence, when agents rush to form partnerships, not every social network supports efficient equilibria, and our objective in this Section is to characterize those social networks for which the maximum number of matchings is always formed in equilibrium.

4.2 Elementary social networks

Following Lovasz and Plummer (1986), we call a social network \( g \) elementary if any edge in \( g \) appears in some maximum matching. The line \( L_4 \) is not elementary because the edge 23 does not appear in any maximum matching. On the other hand, the line \( L_5 \) is elementary. Any cycle \( C_k \) is elementary. Any complete graph \( K_k \) is elementary. Lovasz and Plummer (1986) provide the following characterization of elementary social networks.

**Lemma 4.1** (Lovasz and Plummer) A connected social network \( g \) is elementary if and only if for all \( ij \in g \), \( \mu(g \setminus i, j) = \mu(g) - 1 \).

**Proof:** Pick an edge \( ij \in g \). If \( g \) is elementary, there exists a maximum matching containing \( ij \), \( M = (ij, M') \). As \( M' \) is a maximum matching of \( g \setminus i, j \), \( \mu(g \setminus i, j) = |M'| = |M| - 1 = \mu(g) - 1 \). Conversely, pick a maximum matching \( M' \) of \( \mu(g \setminus i, j) \), then as \( \mu(g) = \mu(g \setminus i, j) + 1 \), \( M = (M', ij) \) is a maximum matching of \( g \) and thus any edge of \( g \) appears in some maximum matching.

When a network is elementary, whenever a pair of agents \( i, j \) leaves the network, the maximum number of pairs formed in \( g \setminus i, j \) is equal to the
matching number of \( g \) minus one, so that the formation of the partnership \((ij)\) does not result in the disruption of the matchings in the graph. However, this argument only works if one considers the formation of a single partnership \((ij)\) and not of a sequence of partnerships \((ij_1)(j_1j_2)\).... Consider for example the cycle \( C_6 \). This cycle is elementary: when a partnership \((ij)\) forms, the remaining network is the line \( L_4 \) and the matching number of the line \( L_4 \) satisfies \( \mu(L_4) = 2 = \mu(C_6) - 1 \). However, as we argued earlier, the line \( L_4 \) is not elementary: the formation of the partnership \((23)\) results in a single matching formed. Hence, in order to guarantee that the maximum number of partnerships is formed at some social network \( g \) we require that the social network formed after any sequence of pairs have left is itself elementary. This is a very strong property and we call graphs satisfying this condition completely elementary.

A social network \( g \) is completely elementary if after any sequence of pairs \((ij_1), (j_1j_2), \ldots, (j_kj)\) leaves the network, the resulting network \( g \setminus i, j_1, j_2, \ldots, j_k, j \) is elementary. Any complete network \( K_k \) is completely elementary. The line \( L_5 \) is completely elementary because if a pair leaves, it results either in the formation of the line \( L_3 \) which is elementary or in the formation of the lines \( L_2 \) and \( L_1 \) which are elementary. The line \( L_7 \) is not elementary because if a pair leaves, it may result in the formation of the line \( L_4 \) which is not elementary. The cycles \( C_5 \) and \( C_7 \) are completely elementary as the formation of a partnership results in the formation of the line \( L_3 \) and \( L_5 \) which are themselves completely elementary. But the cycle \( C_9 \) which leads to the formation of the line \( L_7 \) is not completely elementary.

### 4.3 Completely elementary perfect networks

### 5 Conclusion
6 References

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