Central bank accountability under adaptive learning

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Abstract

Using a New Keynesian model, we examine the accountability issue in a delegation framework where private agents form expectations through adaptive learning while the central bank is rational and should optimally set monetary policy under discretion. The deviation from rational expectations induces an intertemporal tradeoff and hence an incentive for the central bank to influence private expectations through the setting of an appropriate policy interest rate to improve future intratemporal tradeoff between inflation and the output gap. To help the central bank to better manage the intertemporal tradeoff, the government should set a negative optimal inflation penalty according to the value of learning coefficient. By reducing the deviation of the feedback effects of inflation expectations and cost-push shocks on inflation, the policy interest rate and the output gap from the corresponding ones under rational expectations, the optimal inflation penalty allows the economy to be more efficient and improves the social welfare. The main conclusions are valid with both constant gain learning and decreasing gain learning.

Keywords: Adaptive learning, optimal monetary policy, accountability, inflation penalty, rational expectations.

JEL Classification: E42, E52, E58

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1 Introduction

Over the past several decades, most of the research on monetary policy and central banking relies upon the hypothesis of model-consistent or rational expectations (RE). The RE benchmark has permitted the theoretical literature in monetary economics to make major advances and improvements in its analysis of dynamics.

However, the assumption that private agents always form rational expectations and exclusively base their economic decisions on such expectations seems to be too strong and is even heroic under some circumstances for effective monetary policy decision-making. According to Bernanke (2004), modelling adaptive learning is highly relevant to the understanding of modern U.S. monetary history. Indeed, the literature on adaptive learning attracts a growing interest essentially because of the needs for central bankers to well communicate about their goal in a context where conventional monetary policy instruments become less effective in stabilizing the real economy.\(^1\) On the other hand, in a world where private agents could make mistakes inherent to their learning process, it is important for the central bank to properly consider private agents’ expectations. Introducing expectations with learning has three main consequences (Gaspar, Smets and Vestin 2006). First, agents do their own regressions and forecasts, and inflation is no longer only caused by the game between the central bank and the private sector. Second, promises of a future policy do not affect the behavior of private agents because the latter focus on the past value of inflation. Finally, expectations with learning produce a non-linear model. These consequences should be seriously taken into account in monetary policy decisions and central banking given that models with constant and decreasing gain learning seem to provide a good fit for the expectations of professional forecasters for a range of variables (Markiewicz and Pick 2014).

\(^1\)See Evans and Honlapohja (2009), Zumpe (2011), and Woodford (2013) for a survey of the literature.
This paper contributes to the literature that focuses on the role of deviations from RE equilibrium and studies the consequences of non-rational expectations for monetary policy by considering the issue of central bank governance. In a framework where private agents form expectations with adaptive learning and where the central bank is rational and should optimally set discretionary monetary policy, we examine how adaptive learning affects the way the government should delegate monetary policy decision to an independent central bank. In this process, the latter’s incentives are affected by the weight on the target objective, which can be interpreted as the degree of central bank conservatism (Rogoff 1985), optimally set by the government. Under RE, central bank accountability can neither be ensured by insufficiently powered incentive schemes nor by excessively powered one, meaning that there is an optimal weight to place on the achievement of inflation target (Walsh 2003).

Our main findings are that, when private agents form expectations with adaptive learning, a higher inflation penalty weakens the feedback effects of inflation expectations and cost-push shocks on inflation and the policy interest rate while strengthening those on the output gap. Moreover, as long as the learning algorithm is characterized by a positive learning coefficient, the government sets a negative optimal inflation penalty that is negatively correlated with the learning coefficient, implying that the government should nominate a more liberal central banker than under RE. This contrasts with the result found by Walsh (2003) who shows that inflation penalty should be positive if the central bank is subject to unobservable political pressures for greater economic expansion, while it is comparable to that obtained by Dai and Spyromitros (2010) who find that when private agents form RE, the optimal inflation penalty should be negative in response to non-persistent cost-push shocks and the central bank’s preference for policy robustness.

We have obtained closed-form solutions that provide a better understanding of pol-
icy tradeoffs in the presence of accountability issue. In our framework, the government could affect both the well-known intratemporal tradeoff between inflation and the output gap but also an intertemporal tradeoff introduced by a slight departure from RE. In the current period, the central bank stabilizes the economy in a way to better anchor future inflation expectations, thus reducing the future intratemporal trade-off. The institutional design introduced in our analysis helps the central bank to achieve the intertemporal tradeoff and could substantially improve the social welfare.

Our paper is related to the large growing literature that applies learning to macroeconomic models, in particular several strands of literature that examine the consequences of adaptive learning applied to monetary policy. These studies demonstrate the relevance of introducing adaptive learning for monetary policy analysis and design. Marceet and Nicolini (2003) have shown that the process of learning matches remarkably well some major stylized facts observed during the hyperinflations of the 1980’s, while Slobodyan and Wouters (2012) have reported that expectations based on small forecasting models are closely related to the survey evidence on inflation expectations, and the adaptive learning model with an inertial Taylor rule fits the data better than the one with RE. A number of studies (Bullard and Mitra, 2002, Evans and Honkapohja 2003, 2006) find that Taylor rules, which are optimal or ensure determinacy under RE, can lead to instability if private expectations slightly deviate from rationality by following for example adaptive learning. Machado (2013) suggests that, under adaptive learning, a direct monetary policy response to asset prices is not desirable under common instrumental rate rules. In general, departures from rational expectations increase the potential for instability in the economy, strengthening hence the importance of anchoring inflation expectations. Ferrero (2007) find that by reacting strongly to private agents’ inflation expectations with adaptive learning, a central bank increases the speed of convergence and thus shortens the transition length to the RE equilibrium. Gaspar,
Smets and Vestin (2010) find that the commitment rule under rational expectations is robust when expectations are formed with adaptive learning. Marzioni (2014) shows that the economic dynamics is less volatile if the central bank takes into account the impact of signals, i.e. the communication of its own forecasts, on private agents’ prior expectations estimated in conformity with the adaptive learning scheme. Our paper is closely related to Molnár and Santoro (2014) who explore issues of intertemporal and intratemporal tradeoffs arising when a rational central bank should optimally conduct monetary policy while private agents form expectations with adaptive learning. To the difference of existing studies, we take into account the institutional design of the central bank.

The remainder of the paper is organized as follows. The next section presents the model. Section 3 solves the model for the equilibrium under monetary policy discretion. Section 4 analyzes the impact of inflation penalty on the effects of constant gain learning on the feedback coefficients of endogenous variables to inflation expectations and cost-push shocks and determines the optimal level of inflation penalty. Section 5 discusses the implications of decreasing gain learning for central bank accountability. The last section concludes.

2 The model

The theoretical framework is based on a standard New Keynesian model that is widely used in the recent literature on monetary policy (Clarida, Galí and Gertler 1999). It is characterized by optimizing private-sector behavior and nominal rigidities. This model consists of an aggregate demand specification (or IS equation) derived from the representative household’s optimal consumption decision and a forward-looking inflation adjustment (or Phillips curve) equation.
2.1 Aggregate demand and supply

The New Keynesian IS equation is given by

\[ x_t = E_t^s x_{t+1} - \sigma^{-1}(r_t - E_t^s \pi_{t+1}), \tag{1} \]

where \( x_t \) stands for the output gap, \( r_t \) the nominal short-term interest rate and \( \pi_t \) inflation. Here, \( \sigma \) represents the risk aversion for households. The expectation operator \( E_t^s \) represents private agents’ expectations conditional on the information set available at time \( t \). The asterisk on the expectations operator in (1) reflects the fact that the private sector may form expectations which could be rational or not. To simplify the analysis, we assume that there is no demand shock in the IS equation. Indeed, since the central bank can neutralize a demand shock by setting the interest rate, introducing such shocks does not modify the analytical results.

The forward-looking New Keynesian Phillips curve is:

\[ \pi_t = \beta E_t^s \pi_{t+1} + \kappa x_t + e_t, \tag{2} \]

where \( 0 < \beta < 1 \) is the discount factor. The composite parameter \( \kappa \) measures the output-gap elasticity for inflation and captures the effects of the output gap on real marginal costs and thus on inflation, and it is a function of structural parameters, i.e.,

\[ \kappa \equiv \frac{(1-\vartheta)(1-\vartheta \Bar{\beta})}{\vartheta}(1 + \varphi), \]

where \( \vartheta \) represents the share of firms that do not optimally adjust but simply update in period \( t \) their previous price by the steady-state inflation rate. The noise \( e_t \sim N(0, \sigma_e^2) \) is an iid cost-push or supply shock. Assuming that shocks are serially uncorrelated allows the model to be tractable. Furthermore, this assumption is justified in the context of learning since as shown by Milani (2006, 2007), learning represents the main cause of persistence in inflation.
2.2 Institutional settings and policy objectives

We assume that the central bank (agent) is independent and is delegated by the government or the public (the principal) to implement the monetary policy without any external political (but short sighted) interference. This institutional setting would be credible for private agents and could avoid the inflation bias if the nominated central banker was conservative, with conservativeness referring to the relative importance that he/she assigns to price stability objective. To describe the relationship in this delegation framework, we distinguish the objective function of the government from that of the central bank. The government designs the targeting regime, by setting the central bank’s target and the penalty associated with a failure to achieve the target, under which the central bank conducts monetary policy.\(^2\)

The expected social loss function is assumed to take a standard form:

\[
L_t^s = \frac{1}{2} E_t \sum_{i=0}^{+\infty} \beta^i [\pi_{t+i}^2 + \alpha x_{t+i}^2],
\]

(3)

where \(\alpha > 0\) is the relative weight assigned to the objective of output-gap stabilization. Social loss, that could be micro-founded by deriving the utility function of representative agent as in Woodford (2003), is a function of the variance of both inflation and the output gap. The overly ambitious output target, which is common in the Barro-Gordon framework, is here absent in the formulation given in (3). Thus, discretionary monetary policy set to minimize social loss (3) would avoid an average inflation bias.

The central bank implements discretionary monetary policy to minimize the conditional expectation of the loss function:\(^3\)

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\(^2\)See Eijffinger and Masciandaro (2014) for a survey of the literature on central bank governance.

\(^3\)Issues of learning under monetary policy with commitment have been studied by Evans and Honkapohja (2006), and Mele, Molnár and Santoro (2012). The first study shows that both rational expectations commitment equilibrium (RECE) and rational expectations discretionary equilibrium (REDE) are attainable, while the second suggests that the optimal monetary policy drives the economy far from the RECE, and to the REDE.
$$L_t^{CB} = \frac{1}{2} E_t \sum_{i=0}^{+\infty} \beta^i [(1 + \tau)^2 \pi_{t+i}^2 + \alpha x_{t+i}^2], \quad 1 + \tau > 0,$$

where $\tau$ is the penalty (weight) inflicted by the government on the central banker for deviations from inflation target. The loss function represents a weighted average of the variance of inflation and the output gap around their respective target. The inflation target is set to zero for simplicity even though it characterizes, together with $\tau$, alternative inflation targeting regimes. Under RE, setting inflation target at a positive level influences the linear penalty associated with inflation (Svensson 1997) and can offset any average inflation bias as in the optimal contract formulated by Walsh (1995). The condition $1 + \tau > 0$ implies that any deviation from the inflation target is a loss.

### 2.3 Learning rules of private expectations

The private agents' expectations are assumed to be formed according to an adaptive learning algorithm. This assumption relies on the idea that agents have no knowledge of the exact process governing the evolution of endogenous variables. However, to improve their decisions, they recursively make the estimation of a Perceived Law of Motion (PLM), i.e., a steady-state noise in the terminology of Evans and Honkapohja (2001), consistent with the law of motion that the central bank follows under rational expectations. More precisely, private agents believe that steady-state levels of inflation and the output gap only depend on $i.i.d$ cost-push shocks and hence perceive their expected levels as constants knowing that the conditional and unconditional expectations of these variables are identical. This provides a justification for private agents to estimate these variables using sample means.

In accordance with the literature on learning algorithms (Marcet and Sargent 1989, Evans and Honkapohja 2001, Marcet and Nicolini 2003), private agents’ expectations
are assumed to be formed with following learning algorithms:

\[ E_t \pi_{t+1} \equiv a_t = a_{t-1} + \gamma_t (\pi_{t-1} - a_{t-1}), \] (5)

\[ E_t x_{t+1} \equiv b_t = b_{t-1} + \gamma_t (x_{t-1} - b_{t-1}), \] (6)

where \( 0 \leq \gamma_t \leq 1 \), with \( \gamma_t \), represents a deterministic sequence of learning gains that defines the speed of integration of new data into expectations, which are initially set at \( a_0 \) and \( b_0 \). This learning mechanism implies that inflation (output-gap) expectations are increasing with last period inflation (output gap).\(^4\) In the case of decreasing gain learning, as time goes by, private agents assign a decreasing importance to past values of inflation and the output gap in the formation of expectations. When \( t \to \infty \), i.e., \( \gamma \to 0 \), the policymakers cannot manipulate future expectations by changing the current policy.

As Preston (2005) highlights it, without the rational expectations hypothesis, the micro-economic structural equations forming the New Keynesian model should also include the forecasting of macroeconomic conditions in many time periods to come (infinite horizon learning) and not only one period ahead (corresponding to the Euler equation learning). Following the learning literature, we adopt the Euler equation learning to maintain analytical tractability.

Using (5), we rewrite (2) as

\[ \pi_t = \beta [a_{t-1} + \gamma_t (\pi_{t-1} - a_{t-1})] + \kappa x_t + e_t. \] (7)

This equation shows the dependence of inflation on the current shock, the current output-gap value, and the past values of inflation and especially of expected inflation. The past expectations based on learning themselves contain a share of past inflation shocks. This will be explicitly shown below by defining the Actual Law of Motion

\(^4\)The learning process formulated in (5) and (6) is limited by the fact that they focus on past information and the forecasting with one period ahead.
3 The equilibrium under monetary policy discretion

Assuming expectations based on learning is an alternative way of conceiving how private agents interact with monetary policy compared to the rational expectations hypothesis. Expectations with learning modify the central bank’s trade-off between inflation and the output gap. Notably, they give rise to an incentive for the central bank to increase the current inflation but also a larger room to maneuver to reduce the latter through manipulating the output gap. Thus, there could be good grounds for an inflation penalty to be inflicted to the central bank if the latter does not respect the inflation target fixed by the government. To put into evidence the role of learning in the presence of accountability issue, in the following, we solve the model first under rational expectations and then under adaptive learning.

3.1 Benchmark equilibrium with rational expectations

We concisely show the rational expectations equilibrium solution when the central bank sets optimal monetary policy given private agents’ expectations. We use then this solution as a benchmark to illustrate how the equilibrium is modified by an optimal policy designed with private agents’ beliefs being taken into account.

The central bank minimizes its loss function (4) subject to (2) taking inflation expectations as given. This leads to the optimal trade-off rule between inflation and the output gap:

$$x_t = -\left(1 + \tau \right) \kappa \frac{\pi_t}{\alpha}.$$  \hspace{1cm} (8)

The targeting rule defined in (8) implies the equilibrium solution of inflation and...
the output gap now depends on inflation penalty and the central bank's preferences for output-gap stabilization. The tradeoff between inflation and the output gap is affected by the presence of inflation penalty in the sense that higher the inflation penalty is, the costlier it is for the central bank to adjust the output gap.

Solving (1), (2) and (8) yields the ALM for inflation and the output gap, and the interest rate rule that implements the optimal monetary policy as follows:

\[
\pi_t = \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)} E_t^* \pi_{t+1} + \frac{\alpha}{\alpha + \kappa^2 (1 + \tau)} e_t, \tag{9}
\]

\[
x_t = - \frac{\beta \kappa (1 + \tau)}{\alpha + \kappa^2 (1 + \tau)} E_t^* \pi_{t+1} - \frac{\kappa (1 + \tau)}{\alpha + \kappa^2 (1 + \tau)} e_t, \tag{10}
\]

\[
r_t = \sigma E_t^* x_{t+1} + \left[ 1 + \frac{\sigma \beta \kappa (1 + \tau)}{\alpha + \kappa^2 (1 + \tau)} \right] E_t^* \pi_{t+1} + \frac{\sigma \kappa (1 + \tau)}{\alpha + \kappa^2 (1 + \tau)} e_t. \tag{11}
\]

The ALMs defined by (9)-(11) correspond to the anticipated utility policy set by a policymaker who does not take into account the way private agents revise in the future their beliefs.\(^5\)

The system of equations (1), (2) and (8) has a unique non-explosive rational expectations equilibrium (REE) solution only in terms of exogenous state variable \(e_t\), which is known as the "minimal state variable" solution (McCallum, 1983). Thus, in the case of rational expectations, i.e., \(E_t^* = E_t\), the solution of \(\pi_t\) takes the following form:

\[
\pi_t = \zeta e_t.
\]

\(^5\)The anticipated utility (Kreps 1998) is commonly used in the learning literature. It is similar to expected utility except for two properties. First, private agents have no knowledge of the true model. Second, despite that private agents know that they are learning about the parameters or the state of the economy, they choose actions today in a way that is myopic with respect to the updating of their information set while ignoring that they will continue to learn in the future. Under rational expectations, anticipated utility coincides with expected utility, implying that the current beliefs of private agents reflect the true model and the optimal anticipated utility policy would also maximize expected utility.
The formation of rational expectations conditional on the available information at \( t \) leads to:

\[
E_t \pi_{t+1} = \zeta E_t e_{t+1} = 0. \tag{12}
\]

Substituting \( E_t \pi_{t+1} = 0 \) into (9)-(11), we obtain the REE solution corresponding to optimal discretionary monetary policy:

\[
\pi_t = \frac{\alpha}{\alpha + \kappa^2(1 + \tau)} e_t, \tag{13}
\]

\[
x_t = -\frac{(1 + \tau)\kappa}{\alpha + \kappa^2(1 + \tau)} e_t, \tag{14}
\]

\[
r_t = \frac{\sigma \kappa (1 + \tau)}{\alpha + \kappa^2(1 + \tau)} e_t. \tag{15}
\]

The optimal level of inflation penalty is determined by minimizing (3) taking account of the solutions of \( \pi_t \) and \( x_t \) given by (13)-(14) as:

\[
\tau = 0 \tag{16}
\]

This result is implied by the rational expectations hypothesis and the absence of inflation bias.

### 3.2 Equilibrium with learning

The deviation of private expectations from rationality implies that they become state variables and hence their law of motion could affect monetary policy. More precisely, when private agents adopt learning algorithms to form expectations, the central bank has the opportunity to deal with an additional intertemporal tradeoff (between optimal behavior in \( t \) and in later periods), generated by its ability to manipulate future inflation expectations. Current monetary policy decision, given its effects on future inflation
expectations, also has to consider future intratemporal tradeoffs between inflation and the output gap. Here, we assume that the central bank knows the exact learning algorithm followed by private agents and takes it into account when setting monetary policy. This hypothesis, even though it is quite strong, allows us to appreciate how the policy design could change if private agents depart from rationality.

The optimal choice of the central bank

If the central bank takes into account the exact learning algorithm followed by private agents to form their expectations, its optimization problem consists of minimizing (4) subject to (1)-(2), with $E_t^* x_{t+i+1}$ being substituted by $b_{t+i}$ and $E_t^* \pi_{t+i+1}$ by $a_{t+i}$, and (5)-(6). The Lagrangian of the central bank’s minimization problem is:

$$
\mathcal{L}^{CB}_t = E_t \sum_{i=0}^{+\infty} \beta^i \left\{ \frac{1}{2} \left[ (1 + \tau) \pi_{t+i}^2 + \alpha x_{t+i}^2 \right] - \lambda_1 \pi_{t+i} - \lambda_2 a_{t+i} - \lambda_3 x_{t+i} - \lambda_4 a_{t+1} \right\} + \lambda_2 \left[ \pi_{t+i} - \beta a_{t+i} - \kappa x_{t+i} - \mu_{t+i} \right] - \lambda_3 \left[ a_{t+i+1} - a_{t+i} - \gamma_{t+i+1}(\pi_{t+i} - a_{t+i}) \right] + \lambda_4 \left[ b_{t+i+1} - b_{t+i} - \gamma_{t+i+1}(x_{t+i} - b_{t+i}) \right],
$$

(17)

where the $\lambda_i$, $i = 1, ..., 4$ are Lagrangian multipliers associated with (1), (2), (5) and (6), respectively. The first-order conditions of the central bank’s optimization problem are obtained by deriving (17) with respect to $r_t$, $\pi_t$, $x_t$, $a_{t+1}$ and $b_{t+1}$:

$$
\lambda_{1,t} = 0,
$$

(18)

$$
(1 + \tau) \pi_t - \lambda_{2,t} + \gamma_{t+1} \lambda_{3,t} = 0,
$$

(19)

$$
\alpha x_t - \lambda_{1,t} + \kappa \lambda_{2,t} + \gamma_{t+1} \lambda_{4,t} = 0,
$$

(20)

$$
\lambda_{3,t} = E_t \left[ \frac{\beta}{\sigma} \lambda_{1,t+1} + \beta^2 \lambda_{2,t+1} + \beta \lambda_{3,t+1}(1 - \gamma_{t+2}) \right],
$$

(21)

$$
\lambda_{4,t} = E_t \left[ \lambda_{1,t+1} + \beta \lambda_{4,t+1}(1 - \gamma_{t+2}) \right].
$$

(22)

Substituting $\lambda_{1,t} = 0$ given by (18) into (22) leads to $\lambda_{4,t} = \beta(1 - \gamma) E_t [\lambda_{4,t+1}]$. The
only bounded forward-looking solution is $\lambda_{4,t} = \lambda_{4,t+1} = 0$. Using these results into (20) yields $\lambda_{2,t} = -\alpha x_t$ and $\lambda_{2,t+1} = -\frac{\alpha}{K} x_{t+1}$. Substituting $\lambda_{2,t} = -\frac{\alpha}{K} x_t$ into (19), we get:

\[(1 + \tau)\pi_t + \frac{\alpha}{K} x_t + \gamma_{t+1} \lambda_{3,t} = 0.\]  

(23)

According to (23), only $\lambda_{3,t}$, the Lagrangian multiplier associated with the evolution of inflation expectations plays a role in the choice of optimal monetary policy. We note that when $\gamma = 0$, i.e. the case where expectations are constant, the rule defined by (23) becomes identical to (8), which is the optimality condition derived for setting the optimal monetary policy under discretion with the central bank considering private rational expectations as given.

Besides intratemporal tradeoff between the output gap and inflation observed in the benchmark with rational expectations, the learning effect induces an intertemporal tradeoff due to feedback between monetary policy and inflation expectations. The term $\gamma_{t+1} \lambda_{3,t}$ in (23) distinguishes the optimal policy rule with learning from the one with rational expectations by the fact that the optimal decision should now depend on inflation expectations. According to (5) and (23), $\gamma_{t+1}$ represents the marginal effect of an increase in inflation on inflation expectations at $t + 1$, i.e., $\alpha_{t+1}$; and $\lambda_{3,t}$ the marginal effect of an increase in inflation expectations on welfare loss. For $\gamma_{t+1} > 0$, the sign of $\lambda_{3,t}$ depends on current inflation expectations. A change in inflation in the current period impacts future inflation expectations and thus will result in a variation of social welfare. Thus, the learning effect introduces an intertemporal tradeoff for the central bank between the stabilization in the current period and the stabilization in the following periods. This tradeoff is generated by the ability of the central bank to influence inflation expectations in future periods.

The sign of $\lambda_{3,t}$ depends on the sign of current inflation expectations $\alpha_t$. Given that inflation target is set to zero, $\alpha_t$ could be either positive or negative, depending on the
nature of past shocks. When $a_t$ is positive, an increase in $a_t$ drives it further away from the target and hence reduces social welfare, implying that $\lambda_{3,t}$ is positive, and vice versa.

In the following, we first consider constant gain learning and then decreasing gain learning. Indeed, constant gain learning is more suitable for time-varying environments but decreasing gain learning can be considered as the first step in the expectations process adopted by most economic agents (Berardi and Galimberti, 2013).

4 Inflation penalty and the effects of constant gain learning

As extensively discussed in the learning literature (Evans and Honkapohja 2009), private agents would be more inclined to use a constant gain learning algorithm if they believe in possible structural changes in the near future. In this section, we first analyze how constant gain learning and inflation penalty interact with macroeconomic stabilization compared to the benchmark case where private agents form rational expectations, and then examine how the government should set inflation penalty to improve the social welfare.

4.1 Equilibrium solution

There exists a unique solution of the ALMs corresponding to the control problem of the central bank under constant gain learning (see Appendix A and B for the proof). The ALM of inflation is given by:

$$\pi_t = c^{\pi}_t a_t + d^{\pi}_t e_t$$  \hspace{1cm} (24)
where

\[ c_{\pi}^{cg} = -\frac{p_0 + p_2 (c_{\pi}^{cg})^2}{p_1}, \]  
(25)

\[ d_{\pi}^{cg} = \frac{\alpha}{\kappa^2(1 + \tau) + \alpha + \alpha^2 \gamma (\beta - c_{\pi}^{cg}) + \beta \gamma (1 - \gamma) \left\{ \alpha \beta - [\alpha + \kappa^2 (1 + \tau)] c_{\pi}^{cg} \right\}}, \]  
(26)

with

\[ p_2 = \gamma \left\{ \alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma)(1 + \tau) \right\} > 0, \]

\[ p_1 = (1 - \gamma) \left\{ \alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma)(1 + \tau) \right\} \]

\[ - \left\{ \kappa^2 (1 + \tau) + \alpha + \alpha^2 \gamma [1 - \gamma (1 - \beta)] \right\} < 0, \]

\[ p_0 = \alpha \beta \left\{ 1 - \beta (1 - \gamma) [1 - \gamma (1 - \beta)] \right\} > 0. \]

The solution for \( c_{\pi}^{cg} \) ensuring a non-explosive evolution of inflation is given by:

\[ c_{\pi}^{cg} = \frac{-p_1 - \sqrt{p_1^2 - 4p_2p_0}}{2p_2}. \]  
(27)

Given that the value of \( c_{\pi}^{cg} \) is in the interval \( [0; \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)}] \), current inflation increases with inflation expectations \( (a_t) \) less than proportionally. Current inflation is indirectly influenced by the central bank’s policy responses to past shocks. Thus, an increase in the learning coefficient \( \gamma \) has two opposite effects on \( c_{\pi}^{cg} \). According to (5), a higher \( \gamma \) increases the positive correlation between current inflation \( \pi_t \) and future inflation expectations \( a_{t+1} \), and hence the incentive for the central bank to lower \( c_{\pi}^{cg} \), i.e., the feedback from \( a_t \) to \( \pi_t \) in (24). However, according to the same learning algorithm, an increase in \( \gamma \) attenuates the effect of \( a_t \) on \( a_{t+1} \), thus allowing a greater \( c_{\pi}^{cg} \) without deteriorating social welfare. When the learning coefficient is zero, (24) is reduced to the form given by (9) with exogenous inflation expectations.
Regarding the effect of cost-push shocks on current inflation, we notice that the higher and closer to 1 the learning coefficient is, the more inflation is influenced by current cost-push shocks. The denominator of $d_{\pi}^{cg}$ is clearly decreasing in $\gamma$ if the sign of the derivative of $c_{\pi}^{cg}$ with respect to $\gamma$ is negative.

According to (26) and (27), at first sight, a rise in inflation penalty has ambiguous effects on the correlation between current inflation and inflation expectations and that between current inflation and cost-push shocks. This is because inflation penalty incites the central bank to reduce inflation and this decreases future inflation expectations through learning process, hence diminishing the incentive for the central bank to fight against inflation. The influence of inflation penalty on inflation will be analyzed in detail in subsection 4.2.

The optimal level of the output gap is obtained by substituting $\pi_t$ given by (24) into (2) as:

$$x_t = c_{x}^{cg} a_t + d_{x}^{cg} e_t,$$

with

$$c_{x}^{cg} = -\frac{\beta - c_{\pi}^{cg}}{\kappa},$$

$$d_{x}^{cg} = -\frac{1 - d_{\pi}^{cg}}{\kappa}.$$  

In response to an increase in private inflation expectations, the central bank sets a monetary policy that reduces the output gap, given that $c_{\pi}^{cg} < \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)}$ implies $c_{x}^{cg} < -\frac{\beta \kappa (1 + \tau)}{\alpha + \kappa^2 (1 + \tau)}$. If the policy involves a contraction in the output gap, equation (2) implies an increase in current inflation smaller than that of inflation expectations and hence lower future inflation expectations.

Using (24) and (28) to eliminate $\pi_t$ and $x_t$ in equation (1), we get the ALM ruling the evolution of the interest rate:
\[ r_t = \delta^{cg}_r b_t + c^{cg}_r a_t + d^{cg}_r e_t \]

where \( \delta^{cg}_r = \sigma \), \( c^{cg}_r = 1 + \frac{\sigma(\beta - c^{cg}_\pi)}{\kappa} \) and \( d^{cg}_r = \frac{\sigma(1 - d^{cg}_\pi)}{\kappa} \). It is to notice that \( c^{cg}_r > 1 \) since \( \forall \gamma, c^{cg}_\pi \leq \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} < \beta \), meaning that the Taylor principle is verified.

The ALM (31) highlights that the interest rate is affected by both inflation and output-gap learning with their impact depending on the risk aversion of households. Thus, a higher \( \sigma \) mechanically increases the interest rate, all things being equal. Moreover, the feedback coefficient \( (c^{cg}_\pi) \) of inflation expectations associated with inflation in (24) is negatively correlated with the feedback coefficient \( (c^{cg}_r) \) of inflation expectations on the interest rate. A positive cost-push shock increases the interest rate. The higher the learning coefficient \( \gamma \) is, the greater the effect of the shock on the interest rate (smaller \( d^{cg}_\pi \)). The central bank sets the coefficient associated with output-gap expectations \( b_t \) in (31) to \( \delta^{cg}_r = \sigma \), thus fully neutralizing the effect of output-gap expectations on the output gap and hence inflation. Finally, if inflation penalty is higher, the feedback coefficient \( d^{cg}_\pi \) is lower so the net effect of the cost-push shock on the interest rate is higher, all things being equal. Indeed, a high inflation penalty allows the central bank to focus on reducing inflation and to pay less attention to the feedback phenomenon between inflation and inflation expectations in its interest rate decision.

The learning gain coefficient determines the time horizon of private agents’ expectations and hence the persistence of inflation and the flexibility in the central bank’s discretionary policy. For \( \gamma = 0 \), i.e., when inflation expectations are constant over time with \( a_t = a_{t-1} \) and \( b_t = b_{t-1} \), we obtain:

\[ c^{cg}_\pi = \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)}, \]
\[ d^{cg}_\pi = \frac{\alpha}{\alpha + \kappa^2(1 + \tau)}. \]
Thus, the feedback coefficients of the ALMs for inflation, the output gap and the interest rate are identical to those associated with anticipated utility policy given by (9)-(11).

For $\gamma = 1$, i.e., inflation expectations are static or naive with $a_t = \pi_{t-1}$ and $b_t = x_{t-1}$, we have

$$c^g_{\pi} = \frac{\kappa^2(1 + \tau) + \alpha + \alpha \beta^3 - \sqrt{[\kappa^2(1 + \tau) + \alpha + \alpha \beta^3]^2 - 4\alpha^2 \beta^3}}{2\alpha \beta^2},$$

$$d^g_{\pi} = \frac{\alpha}{\alpha + \kappa^2(1 + \tau) + \alpha \beta^2(\beta - c^g_{\pi}).}$$

As $\gamma$ is equal to 1, inflation is self-sustained because private agents’ inflation expectations are depending on past inflation. The effect is similar for $\gamma$ near to 1, since in this case agents form expectations on the basis of a very short horizon. Introducing inflation penalty could significantly influence private expectations by affecting current inflation and output gap, implying that inflation penalty represents an extra incentive for the central bank to improve the tradeoﬀ between inflation and the output gap.

4.2 The effects of inflation penalty on the feedback coefficients of ALMs

The response of monetary policy to cost-push shocks depends on learning and inflation penalty. According to (16), the optimal inflation penalty under rational expectations is equal to $\tau = 0$. Setting $\tau = 0$ and comparing (13)-(15) with their corresponding equations (24), (28) and (31), we find that the feedback effect of inflation expectations on the ALM of inflation (the output gap) is lower (higher) in the case of learning than under rational expectations, i.e., $c^g_{\pi} < \frac{\alpha \beta}{\alpha + \kappa}$ (i.e., $c^g_{x} < -\frac{\beta \kappa}{\alpha + \kappa^2}$ respectively). To make this possible, the interest rate under learning must react more strongly to inflation
expectations, i.e., \( c^g_r > 1 + \frac{\sigma \beta \kappa}{\alpha + \kappa^2} \). The ALMs of inflation and the output gap are independent of output-gap expectations under both learning and rational expectations, while the interest rate under learning responds to output-gap expectations with the same coefficient as under rational expectations and is independent of inflation penalty.

Regarding the feedback coefficients associated with \( \epsilon_t \) in the ALMs, it is straightforward to show that \( d^c_g > \frac{\alpha}{\alpha + \kappa^2} \), \( d^c_g > -\frac{\kappa}{\alpha + \kappa^2} \) and \( d^c_g < \frac{\sigma \kappa}{\alpha + \kappa^2} \), meaning that under learning, the current inflation and output gap respond more strongly while the interest rate responds less to current cost-push shocks than under rational expectations.

We examine how an increase in inflation penalty could affect the feedback coefficients in the ALMs by deriving the feedback coefficients in (24), (28) and (31) with respect to \( a_t \) and \( \tau \) yields (Appendix C):

\[
\frac{\partial c^g_g}{\partial \tau} = \kappa \frac{\partial c^g_g}{\partial \tau} = -\frac{\kappa}{\sigma} \frac{\partial c^g_g}{\partial \tau} = -\frac{p_0}{p_2} \left[ 2p_2 \frac{\partial p_1}{\partial \tau} - (1 + 2p_2) \frac{\partial p_2}{\partial \tau} \right] < 0,
\]

\[
\frac{\partial d^c_g}{\partial \tau} = \kappa \frac{\partial d^c_g}{\partial \tau} = -\frac{\kappa}{\sigma} \frac{\partial d^c_g}{\partial \tau} = \frac{\alpha \left( \kappa^2 - \alpha \beta^2 \gamma^2 \frac{\partial c^g_g}{\partial \tau} - \beta \gamma (1 - \gamma) [\alpha + \kappa^2 (1 + \tau)] \frac{\partial c^g_g}{\partial \tau} \right)}{\left( \kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma^2 (\beta - c^g_g) + \beta \gamma (1 - \gamma) \{ \alpha \beta - [\alpha + \kappa^2 (1 + \tau)] c^g_g \} \right)^2} < 0,
\]

\[
\frac{\partial \delta^c_g}{\partial \tau} = 0,
\]

where \( \frac{\partial p_1}{\partial \tau} = \kappa^2 [\beta (1 - \gamma)^2 - 1] < 0 \), \( \frac{\partial p_2}{\partial \tau} = \gamma \beta \kappa^2 (1 - \gamma) > 0 \). The ALMs of inflation and the output gap are independent of output-gap expectations and the feedback effect of output-gap expectations on the interest rate is not affected by inflation penalty. An increase in inflation penalty induces a decrease in the feedback effects of inflation expectations and cost-push shocks on current inflation and the policy interest rate while
it strengthens the feedback effects on the current output gap. Thus, a higher inflation penalty incites the central bank to focus more on reducing inflation due to current positive cost-push shocks but reduces the possibility for it to control future inflation expectations through the feedback between inflation and inflation expectations.

Figure 1 shows how these coefficients evolve with $\tau$. Notably, a given past inflation or a positive cost-push shock induces lower current inflation and interest rate but a higher output gap if the government increases inflation penalty from zero to a positive value.

**Result 1.** *An increase in inflation penalty weakens the positive feedback effect of inflation expectations and cost-push shocks on inflation and the policy interest rate, and strengthens the negative feedback effects on the output gap.*

A positive inflation penalty set by the government incites the central bank to keep inflation at a lower level while accepting a greater output gap to minimize its loss function. By reducing the feedback effect of inflation expectations on current inflation, a positive inflation penalty could have a contractionary effect on future inflation expectations, with its importance depending on the learning coefficient according to (5). In the case of positive inflation expectations, the output moves away from its potential level especially since the central bank, undergoing a positive inflation penalty, must implement a restrictive policy that reduces not only current inflation but also the persistence of inflation. This leads to decreasing inflation expectations, which eventually lower the central bank’s loss due to inflation penalty. We notice that since $\kappa$ is very small, the impact of a positive inflation penalty on the feedback effects of inflation expectations and cost-push shocks in ALM of the output gap is largely greater than that on the corresponding feedback effects in the ALM of inflation.

Indeed, if $\gamma = 0$, the central bank cannot influence private agents’ expectations by varying the actual values of inflation since private agents behave as if their expectations
Figure 1: Feedback coefficient associated with inflation and cost-push shocks in the ALMs.
were fixed. Since the action of the central bank is limited in influencing private expectations, the effect of inflation penalty will be smaller in this case than when \( \gamma > 0 \). When \( \gamma = 1 \), an increase in inflation penalty makes the largest impact on the feedback effects of inflation expectations and cost-push shocks on the ALMs of endogenous variables. We notice that numerical simulations show that for \( \gamma > 0.2 \), the impact of inflation penalty on the feedback coefficients of ALMs is very close to the ones obtained when \( \gamma = 1 \).

Figure 1 shows, using the calibration of Woodford (1999), i.e., \( \beta = 0.99 \), \( \kappa = 0.024 \), \( \alpha = 0.048 \) and \( \sigma = 0.157 \), that the feedback coefficients \( c_{\pi}^{cg} \), \( d_{\pi}^{cg} \), \( c_{x}^{cg} \) and \( d_{x}^{cg} \) (\( c_{r}^{cg} \) and \( d_{r}^{cg} \)) are increasing (decreasing respectively) in inflation penalty. All feedback coefficients are increasing in the learning coefficient \( \gamma \), meaning that a higher learning coefficient reinforces the effects of inflation penalty. The values of \( c_{\pi}^{cg} \), \( c_{x}^{cg} \) and \( c_{r}^{cg} \) are very sensitive to the value of \( \gamma \) for \( \gamma = 0.2 \), and are close to their values when \( \gamma = 1 \), while for \( \gamma = 0.2 \), the feedback coefficients \( d_{\pi}^{cg} \), \( d_{x}^{cg} \) and \( d_{r}^{cg} \) stay close to their corresponding curves when \( \gamma = 0 \).

**Result 2.** An increase in learning coefficient enlarges the deviation of feedback coefficients of inflation expectations and cost-push shocks associated with the actual law of motion of inflation, the output gap and the policy interest rate from the corresponding ones under rational expectations.

Our simulation exercises show that when the learning coefficient is higher, the deviation from the REE is larger. Consequently, a decrease in inflation penalty allows the central bank to manipulate the private sector’s expectations in a more significant manner, thus permitting a larger correction of the deviation.
4.3 The optimal level of inflation penalty

The feedback coefficients of inflation and the output gap are function of inflation penalty and learning coefficient. This implies that the contribution of their respective volatility to the social welfare loss also depends on these two parameters. Using (24) and (28), the volatility of inflation and the output gap are respectively given by

\[ \text{var}(\pi_{t+1}) = (c^g_\pi)^2 E_t(a_{t+i}^2) + (d^g_\pi)^2 E_t(e_{t+i}^2) \]

and

\[ \text{var}(\xi_{t+1}) = (c^g_\pi - \beta)^2 E_t(a_{t+i}^2) + (d^g_\pi - 1)^2 E_t(e_{t+i}^2). \]

Given that \( \beta < c^g_\pi \) and \( d^g_\pi < 1 \), the volatility of inflation is decreasing in inflation penalty while the volatility of the output gap increasing with it. Thus, the social loss function (3) can be rewritten as

\[
L_s^* = \frac{1}{2} \sum_{i=0}^{+\infty} \beta^i \left\{ \left[ (c^g_\pi)^2 + \frac{\alpha(c^g_\pi - \beta)^2}{\kappa^2} \right] E_t(a_{t+i}^2) + \left[ (d^g_\pi)^2 + \frac{\alpha(d^g_\pi - 1)^2}{\kappa^2} \right] E_t(e_{t+i}^2) \right\}
\]

(34)

In the case where \( \gamma = 0 \), using (29)-(30) and (32)-(33), the social loss function (34) becomes

\[
L_s^* = \frac{1}{2} E_t \sum_{i=0}^{+\infty} \beta^i \left\{ \frac{\alpha^2 \beta^2}{[\alpha + \kappa^2(1 + \tau)]^2} + \frac{\alpha \kappa^2 (1 + \tau)^2}{[\alpha + \kappa^2(1 + \tau)]^2} \right\} \left[ \beta E_t(a_{t+i}^2) + E_t(e_{t+i}^2) \right]
\]

(35)

The government’s optimal decision obtained by the minimization of (35) is to set \( \tau = 0 \).

**Result 3.** When the learning algorithms (5)-(6) are characterized by a learning coefficient equal to zero, i.e., \( \gamma = 0 \), the government sets the optimal inflation penalty to zero.

The result 3 shows that the optimal inflation penalty set by the government when private agents ignore their expectations errors is the same as when private agents form
rational expectations. This is because under rational expectations, inflation expectations are taken as given by the central bank and consequently not affected by the latter, the government cannot influence inflation expectations and hence current inflation by imposing an inflation penalty on the central bank. In the case of adaptive learning with zero learning coefficient, the fact that private agents form exogenous inflation expectations implies a similar decision problem for the government. Under rational expectations as in the absence of learning, the central bank cannot make an intertemporal tradeoff by influencing private future expectations. Thus, the intratemporal tradeoff, made by the central bank when setting its monetary policy following the imposition of an inflation penalty other than zero, will deteriorate the social welfare.

For $\gamma > 0$, the minimization of the social loss function (34) does not allow an analytical solution of $\tau$. As the choice of $\tau$ for a period is independent of those in other periods, to determine the optimal solution of $\tau$, it is sufficient to consider the expected unconditional (or average) social loss function per period. We proceed to simulate numerically the social loss function by setting $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$, $\text{var}(a) = 0.5$, and $\text{var}(e) = 0.5$. It follows that for $\gamma = 0.1$, $\gamma = 0.5$ and $\gamma = 1$, the optimal inflation penalty $\tau^*$ is $-0.86$, $-0.94$ and $-0.96$ respectively (Figure 2). We notice that the existence of an optimal negative inflation penalty does not depend on the initial past inflation.

**Result 4.** When the learning algorithm is characterized by a positive constant learning coefficient, the government sets an optimal inflation penalty such that $\tau \in [-1; 0]$. The higher is the learning coefficient, the lower is the optimal inflation penalty.

Under learning with a positive learning coefficient, since private agents take into account the expectations errors when forming their expectations, the central bank can influence their future expectations by setting monetary policy. Compared to the rational expectations equilibrium, the equilibrium with adaptive learning is suboptimal
given that inflation and output-gap expectations based on past information deviate from correct expectations formed with the knowledge of the distribution of cost-push shocks. Thus, when private agents are learning, choosing a central banker with the same preferences as the society is not socially optimal. In the delegation framework, the government can set a negative inflation penalty on the central bank for it to mimic the social optimal equilibrium with rational expectations, given that the difference between the feedback coefficients in ALMs with adaptive learning and those corresponding to the anticipated utility policy is decreasing in inflation penalty.

5 Implications of decreasing gain learning

The main results obtained above are based on the assumption of constant gain learning. However, agents could begin to learn with a decreasing gain coefficient before stabilizing the latter’s value. Thus, we relax in this section the assumption of constant gain learning
to show that these results remain valid when learning gain is decreasing over time.\footnote{The relaxation of the assumption of constant gain learning could be justified by the study of Milani (2014) who shows that private agents appear to have often switched to constant-gain learning, with a high constant gain, during most of the 1970s and until the early 1980s, while reverting to a decreasing gain later on.}

Assume that learning gain characterizing algorithms (5) and (6) decreases with time such that $\gamma_t = \frac{1}{t}$. There exists a unique solution of the ALMs corresponding to the control problem of the central bank under the decreasing gain learning, which is given by (see Appendix D and E for the proof):

$$\pi_t = \frac{\partial G}{\partial \pi_t} a_t + \frac{\partial G}{\partial \pi_t} e_t$$

(36)

where

$$\frac{\partial G}{\partial \pi_t} = \frac{\beta \alpha - (1 - \gamma_{t+1}) \left[ \alpha (1 + \gamma_{t+1} \beta) (\beta - \frac{\partial G}{\partial \pi_t}) - \kappa^2 (1 + \tau) \frac{\partial G}{\partial \pi_t} \right]}{\alpha + \alpha \beta \gamma_{t+1} (1 + \gamma_{t+1} \beta) (\beta - \frac{\partial G}{\partial \pi_t}) + \kappa^2 (1 + \tau) (1 - \gamma_{t+1} \beta \frac{\partial G}{\partial \pi_t})},$$

(37)

$$\frac{\partial G}{\partial \pi_t} = \frac{\beta \alpha}{\alpha + \alpha \beta \gamma_{t+1} (1 + \gamma_{t+1} \beta) (\beta - \frac{\partial G}{\partial \pi_t}) + \kappa^2 (1 + \tau) (1 - \gamma_{t+1} \beta \frac{\partial G}{\partial \pi_t})},$$

(38)

In the limit, as $t \to +\infty$, equation (37) could be approached by

$$\frac{\partial G}{\partial \pi_t} = \beta \frac{\partial G}{\partial \pi_t} + (1 - \beta) \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)}, \quad \text{with} \quad \left\{ \begin{array}{ll}
\frac{\partial G}{\partial \pi_t} = \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)} & \text{if } \gamma_{t+1} \to 0 \\
0 < \frac{\partial G}{\partial \pi_t} < \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)} & \text{if } \gamma_{t+1} \in [0, 1],
\end{array} \right.$$

Under the optimal discretionary policy, current inflation increases with inflation expectations ($a_t$) less than proportionally given that $\frac{\partial G}{\partial \pi_t} < \frac{\alpha \beta}{\alpha + \kappa^2 (1 + \tau)}$. Current inflation is indirectly influenced by the central bank’s policy responses to past shocks. The level of expectations and mostly the value of the learning coefficient $\gamma_{t+1} = \frac{1}{t+1}$ are crucial to
the determination of current inflation. In the first period, we have \( t = 1 \) and \( \gamma_t = 0.5 \). The learning coefficient \( \gamma_{t+1} \) rapidly decreases over the time. As \( t \to +\infty \), we obtain \( \gamma_\infty \to 0 \). This corresponds to the steady state where expectations are constant or the case where private agents do not correct their expectations errors (absence of learning). This also leads to a constant value for inflation expectations. The feedback coefficients in ALM of inflation, \( c^\pi_{\pi,t} \) and \( d^\pi_{\pi,t} \), will be identical to \( c^g_\pi \) and \( d^g_\pi \) given by (32)-(33).

The ALM of the output gap is obtained as:

\[
x_t = c^d_x a_t + d^d_x e_t \tag{39}
\]
with \( c^d_{x,t} = -\frac{\beta - c^\pi_{\pi,t}}{\kappa} \) and \( d^d_{x,t} = -\frac{1 - d^\pi_{\pi,t}}{\kappa} \).

The ALM ruling the interest rate is given by:

\[
r_t = \delta^d r b_t + c^d r a_t + d^d r e_t \tag{40}
\]
where \( \delta^d r = \sigma \), \( c^d r = 1 + \sigma \frac{\beta - c^\pi_{\pi,t}}{\kappa} \) and \( d^d r = \sigma \frac{1 - d^\pi_{\pi,t}}{\kappa} \).

Following Molnár and Santoro (2014), we can show that \( c^d_{\pi,t} \), \( d^d_{\pi,t} \), \( c^d_{x} \) and \( d^d_{x} \) are decreasing in \( \gamma_t \) while \( c^d_{r,t} \) and \( d^d_{r,t} \) are increasing in \( \gamma_t \) so that \( c^d_{\pi,t} < \frac{\alpha \beta}{\alpha + \kappa^2} \), \( d^d_{\pi,t} > \frac{\alpha}{\alpha + \kappa^2} \), \( c^d_{x} < -\frac{\beta \kappa}{\alpha + \kappa^2} \), \( d^d_{x} > -\frac{\kappa}{\alpha + \kappa^2} \), \( c^g_{r} > 1 + \frac{\sigma \beta \kappa}{\alpha + \kappa^2} \) and \( d^g_{r} < \frac{\sigma \kappa}{\alpha + \kappa^2} \). Thus, the higher and closer to 1 the learning gain is, the less inflation and the output gap are influenced by inflation expectations and current cost-push shocks while the inverse is true with the feedback coefficients in the ALM of the interest rate. Therefore, decreasing gain learning leads the equilibrium solution to deviate from a more efficient REE, implying the possibility for the government to improve the social welfare by setting an inflation penalty different from zero.

**Result 5.** As time goes on, the learning coefficient \( \gamma_t \) decreases from 1 to 0. For
a given volatility of inflation and cost-push shocks, the optimal inflation penalty will increase from a value not far away from \(-1\) to \(0\).

Under decreasing gain learning, as long as the economy is not in the steady state where \(\lim_{t \to +\infty} \gamma_t \to 0\), private agents will adjust their expectations by correcting expectations errors, making possible for the central bank to influence their future expectations. Given the sub-optimality of transitory learning equilibria with regard to the REE, there is an incentive for the government to impose time-varying negative inflation penalty so that the learning equilibrium could be as close as possible to the REE.

The effects of cost-push shocks and expected inflation on inflation, the output gap and the interest rate under decreasing gain learning are similar to those observed under constant gain learning. Given that under decreasing gain learning, the learning coefficient is decreasing with the time, the effect of learning on the equilibrium is also decreasing with time. As a consequence, inflation penalty increases from a negative value to zero as the economy approaches its steady state where, with \(\gamma_{t+1} \to 0\), private agents do not revise their previous expectations and the central bank is no more able to manipulate expectations, thus eliminating the possibility for the government to improve the social welfare by nominating a liberal central banker. During this process, as the learning coefficient decreases while inflation penalty rises, the impacts of inflation penalty on the equilibrium tends to approach those observed at the REE.

6 Conclusion

In this paper, we consider the issue of accountability of an independent central bank when private agents form inflation expectations with learning algorithms. When private agents are learning, the central bank faces both the intratemporal tradeoff between inflation and the output gap and the intertemporal tradeoff between the stabilization
in current and future periods. The central bank should stabilize the economy in a way to better anchor inflation expectations, thus easing future intratemporal tradeoffs. Introducing inflation penalty helps the central bank to better manage the intertemporal tradeoff and this could substantially improve the social welfare.

We have shown that under adaptive learning, the optimal inflation penalty set by the government should be negative. This strengthens the positive feedback of inflation expectations and cost-push shocks on inflation and the policy interest rate, and weakens the negative feedback effects on the output gap, thus reducing the deviations of the feedback coefficients from their corresponding ones under rational expectations and making the economy more efficient.

The above results are valid even for a slight departure from rational expectations. Moreover, the higher the learning gain coefficient is, the larger the deviation of feedback coefficients from the corresponding ones under rational expectations, reinforcing the need for a more liberal central banker. It is to notice that in the limit case where the learning gain coefficient is zero, the adaptive learning replicates the results under rational expectations. We have also shown that the main conclusions obtained with constant gain learning remain valid under the assumption of decreasing gain learning.

APPENDIX

A. The equilibrium solution of inflation under learning

Using (23) to obtain $\lambda_{3,t}$ and $\lambda_{3,t+1}$ and substituting their expressions as well as $\lambda_{1,t+1} = 0$, $\lambda_{2,t+1} = -\frac{\alpha}{k}x_{t+1}$ into (21), we get:

$$\left(1 + \frac{\tau}{\gamma}\right) \pi_t + \frac{\alpha}{\gamma \kappa} x_t = \frac{\alpha \beta^2}{\kappa} E_t x_{t+1} + \frac{(1 + \tau) \beta (1 - \gamma)}{\gamma} E_t \pi_{t+1} + \frac{\alpha \beta (1 - \gamma)}{\gamma \kappa} E_t x_{t+1}. \quad (41)$$
Using (2) and (5), we obtain:

\[ x_t = \frac{1}{\kappa} \pi_t - \frac{\beta}{\kappa} a_t - \frac{1}{\kappa} e_t, \]  
\( (42) \)

\[ x_{t+1} = \frac{1}{\kappa} \pi_{t+1} - \frac{\beta}{\kappa} [a_t + \gamma(\pi_t - a_t)] - \frac{1}{\kappa} e_{t+1}. \]  
\( (43) \)

Substituting \( x_t \) and \( x_{t+1} \) respectively given by (42) and (43) into (41) and arranging the terms lead to

\[ E_t \pi_{t+1} = A_{11,t} \pi_t + A_{12,t} a_t + P_{1,t} e_t, \]  
\( (44) \)

where

\[ A_{11} = \frac{\kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma [1 - \gamma (1 - \beta)]}{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma) (1 + \tau)}, \]  
\( (45) \)

\[ A_{12} = -\frac{\alpha \beta \{1 - \beta (1 - \gamma) [1 - \gamma (1 - \beta)]\}}{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma) (1 + \tau)}, \]  
\( (46) \)

\[ P_1 = -\frac{\alpha}{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma) (1 + \tau)}. \]  
\( (47) \)

The solution of the ALM of inflation takes the following form:

\[ \pi_t = c_{\pi} a_t + d_{\pi} e_t. \]  
\( (48) \)

We obtain using (5) and (48):

\[ E_t \pi_{t+1} = c_{\pi} [(1 - \gamma) a_t + \gamma \pi_t]. \]  
\( (49) \)

Using equations (44) and (49) to eliminate \( E_t \pi_{t+1} \) and arranging the terms yield:

\[ \pi_t = \frac{A_{12} - c_{\pi} (1 - \gamma)}{c_{\pi} \gamma - A_{11}} a_t + \frac{P_1}{c_{\pi} \gamma - A_{11}} e_t. \]  
\( (50) \)

This implies that:

\[ c_{\pi} = \frac{A_{12} - c_{\pi} (1 - \gamma)}{c_{\pi} \gamma - A_{11}}, \]  
\( (51) \)

and

\[ d_{\pi,t} = \frac{P_1}{c_{\pi} \gamma - A_{11}}. \]  
\( (52) \)
We gather equations (44), (5) and (6) while using (42) to substitute \( x_t \) to obtain the system of three equations:

\[
E_t y_{t+1} = A_t y_t + P_t e_t,
\]

where

\[
y_t \equiv [\pi_t, a_t, b_t], \quad A \equiv \begin{bmatrix}
A_{11} & A_{12} & 0 \\
\gamma & 1 - \gamma & 0 \\
\frac{1}{\kappa} & -\beta \gamma & 1 - \gamma
\end{bmatrix}, \quad \text{and} \quad P \equiv \begin{bmatrix}
P_1 \\
0 \\
-\gamma
\end{bmatrix}.
\]

The above system are subject to three boundary conditions: \( a_0, b_0, \) and \( \lim_{s \to \infty} |E_t \pi_{t+s}| < \infty. \) The eigenvalues of \( A_t \) are given by \( 1 - \gamma \) and by the two eigenvalues of \( A_1: \)

\[
A_1 = \begin{bmatrix}
A_{11} & A_{12} \\
\gamma & 1 - \gamma
\end{bmatrix}
\]

(53)

We can show that \( A_{1,t} \) has an eigenvalue inside and one outside the unit circle. This implies that we can invoke the proposition 1 from Blanchard and Kahn (1980) to conclude that the subsystem admits two real eigenvalues.\( \Box \)

B. The single stable solution

Among infinite stochastic sequences satisfying equation (51), we focus on a non-explosive solution, i.e., the solution corresponding to the eigenvalue of \( A_1 \) given (53) inside the unit circle.

It is straightforward to show that its trace and determinant are both positive. Thus, for \( A_1 \) to have two real eigenvalues \((\mu_1, \mu_2)\), one inside and one outside the unit circle, it is sufficient to show that \( (1 - \mu_1)(1 - \mu_2) < 0. \) This is equivalent to:

\[
\mu_1 + \mu_2 > 1 + \mu_1 \mu_2.
\]

(54)
Knowing that $\mu_1 + \mu_2$ is equal to the trace of $A_1$ and $\mu_1 \mu_2$ equal to its determinant, we can rewrite (54) as:

$$\kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma [1 - \gamma (1 - \beta)] + 1 - \gamma > 1 + \frac{\kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma [1 - \gamma (1 - \beta)]}{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma)(1 + \tau)} (1 - \gamma)$$

After simplification, we get:

$$\kappa^2 (1 + \tau) [1 - \beta (1 - \gamma)] + \alpha \{1 + \beta (\gamma^2)\} > 0,$$

which is always verified given that $\beta \in [0,1]$ and $\gamma \in [0,1]$.

Rewriting (51) as $c^g_{\pi} c^g_{\pi} \gamma - c^g_{\pi} A_{11} - A_{12} + c^g_{\pi} (1 - \gamma) = 0$ and substituting $A_{11}$ and $A_{12}$ by their expressions, we obtain:

$$p_2 (c^g_{\pi})^2 + p_1 c^g_{\pi} + p_0 = 0$$

with

$$p_0 = [\alpha \beta [1 - \beta (1 - \gamma) [1 - \gamma (1 - \beta)]]] > 0,$$

$$p_1 = (1 - \gamma) \{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma)(1 + \tau)\}$$

$$- \{\kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma [1 - \gamma (1 - \beta)]\},$$

$$p_2 = \gamma \{\alpha \beta [1 - \gamma (1 - \beta)] + \beta \kappa^2 (1 - \gamma)(1 + \tau)\} > 0.$$

We rewrite $p_1$ as $p_1 = -\kappa^2 (1 + \tau) [1 - \beta (1 - \gamma)] - \alpha (1 - \beta) \{1 - \beta [1 - \gamma (1 - \beta)]\} - p_0 - p_2$, it follows immediately that $p_1 < 0$. Then, it is straightforward to show that the discriminant of the polynomial (55) $\Delta$ is positive.

To characterize the two solutions of $c^g_{\pi}$, we rewrite (55) as:

$$c^g_{\pi} = \frac{-p_0 + p_2 (c^g_{\pi})^2}{p_1} \equiv f(c^g_{\pi})$$

As $f(c^g_{\pi})$ is strictly increasing for $c^g_{\pi} \in [0,1]$ with $f'(c^g_{\pi}) = -\frac{2p_2}{p_1} c^g_{\pi} > 0$ for $c^g_{\pi} \in [0,1]$. To prove $f(c^g_{\pi}): [0,1] \rightarrow [0,1]$, it is sufficient to show that $f(0) > 0$ and that
\( f(1) < 1 \). It is straightforward to see that \( f(0) = -\frac{p_0}{p_1} > 0 \) and

\[
\begin{align*}
f(1) &= -\frac{p_0 + p_2}{p_1} \\
&= \frac{p_0 + p_2}{\kappa^2(1 + \tau)[1 - \beta(1 - \gamma)] + \alpha(1 - \beta)[1 - \beta[1 - \gamma(1 - \beta)]]} + p_0 + p_2 < 1.
\end{align*}
\]

Since \( f(c^g_{\pi}) : [0, 1] \to ]0,1[ \) and \( f(c^g_{\pi}) \) is strictly increasing, it follows from the theorem of Brouwer that there exists one unique solution of \( c^g_{\pi} \) in the interval \( ]0,1[ \). This solution corresponds to

\[
c^g_{\pi} = -\frac{p_1 - \sqrt{\frac{p_1^2}{2} - 4p_2p_0}}{2p_2} \quad (57)
\]

The other possible solution \( c^g_{\pi} = -\frac{p_1 + \sqrt{\frac{p_1^2}{2} - 4p_2p_0}}{2p_2} \) is larger than unit, which is excluded to avoid an explosive evolution of inflation.

Substituting \( A_{11} \) and \( P_1 \) into (52) leads to:

\[
d^g_{\pi} = \frac{\alpha}{\kappa^2(1 + \tau) + \alpha + \alpha\beta^2\gamma^2(\beta - c^g_{\pi}) + \beta\gamma(1 - \gamma)\{\alpha\beta - [\alpha + \kappa^2(1 + \tau)]c^g_{\pi}\}}. \quad (58)
\]

We now show that \( f(c^g_{\pi}) : ]0; \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)}[ \to ]0; \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)}[ \). Knowing that \( f(0) > 0 \) and substituting \( c^g_{\pi} \) by \( \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)} \) into the function \( f(c^g_{\pi}) \) defined by (59), we find

\[
\begin{align*}
f\left(\frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)}\right) &= -\frac{p_0 + p_2 \left[\frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)}\right]^2}{p_1} \\
&= \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)} \left\{\frac{\alpha + \kappa^2(1 + \tau)}{\alpha\beta}p_0 + \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)p_2}\right\} \\
&= -p_1
\end{align*}
\]

Using \( p_2 = \frac{\alpha(1 - \beta) + \kappa^2(1 + \tau)}{\alpha + \kappa^2(1 + \tau)}p_2 + \frac{\alpha\beta}{\alpha + \kappa^2(1 + \tau)p_2} \), \( p_0 = -\frac{\alpha(1 - \beta) + \kappa^2(1 + \tau)}{\alpha\beta}p_0 + \frac{\alpha + \kappa^2(1 + \tau)}{\alpha\beta}p_0 \) and the definition of \( p_0 \), \( p_1 \), and \( p_2 \) given above, we rewrite the denominator as
\[-p_1 = \kappa^2(1 + \tau) [1 - \beta (1 - \gamma)] + \alpha (1 - \beta) \{1 - \beta [1 - \gamma (1 - \beta)]\} + p_0 + p_2 \]
\[
= \kappa^2(1 + \tau) [1 - \beta (1 - \gamma)] + \alpha (1 - \beta) \{1 - \beta [1 - \gamma (1 - \beta)]\} + \\
- \frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)} p_2 + \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} p_2 \\
= - (1 - \beta) p_2 + \frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} p_2 \\
= \frac{\beta \kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)} p_2 + \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} p_2. \\
\]

Substituting the above expression of \(-p_1\) into (59), we obtain:

\[
f(\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}) = \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} \left\{ \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta} p_0 + \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} p_2 \right\} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}. \\
\]

Given that \(f'(c_{\pi}^g) = -\frac{2p_2}{p_1} c_{\pi}^g > 0\) for \(c_{\pi}^g \in [0, 1]\), \(f(c_{\pi}^g)\) is strictly increasing in the interval \(\left[0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\right]\). This property and the fact that \(f(c_{\pi}^g) : \left[0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\right] \to \left(0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\right]\) imply that there is a unique solution for \(c_{\pi}^g\) so that \(0 < c_{\pi}^g < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\).

The case where \(\gamma = 0\). We obtain by substituting \(\gamma = 0\) into (45)-(47):

\[
A_{11} \equiv \frac{1}{\beta}, \\
A_{12} \equiv -\frac{\alpha(1-\beta)}{\alpha+\kappa^2(1+\tau)}, \\
P_1 \equiv -\frac{\alpha}{\alpha\beta + \beta \kappa^2(1+\tau)}. \\
\]

It follows from (51)-(52) that

\[
c_{\pi}^g = \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}, \\
d_{\pi,t}^g = \frac{\alpha}{\alpha+\kappa^2(1+\tau)}. \\
\]
The case where $\gamma = 1$. Inserting $\gamma = 1$ into (45)-(47) yields

$$A_{11} \equiv \frac{\kappa^2(1 + \tau) + \alpha + \alpha \beta^3}{\alpha \beta^2},$$

$$A_{12} \equiv -\frac{1}{\beta},$$

$$P_1 \equiv -\frac{1}{\beta^2}.$$

Substituting the latter into (45)-(47) leads to

$$c_{\pi}^g = \frac{\{\kappa^2(1 + \tau) + \alpha + \alpha \beta^3\} - \sqrt{\{\kappa^2(1 + \tau) + \alpha + \alpha \beta^3\}^2 - 4\alpha^2 \beta^3}}{2\alpha \beta^2},$$

$$d_{\pi}^g = \frac{\alpha}{\kappa^2(1 + \tau) + \alpha + \alpha \beta^2(\beta - c_{\pi}^g)}.$$

C. The effects of inflation penalty

Deriving $p_0, p_1$ and $p_2$ with respect to $\tau$ gives:

$$\frac{\partial p_0}{\partial \tau} = 0,$$

$$\frac{\partial p_1}{\partial \tau} = \kappa^2 [\beta(1 - \gamma)^2 - 1] < 0,$$

$$\frac{\partial p_2}{\partial \tau} = \gamma \beta \kappa^2(1 - \gamma) > 0.$$

Deriving $c_{\pi}^g$ given by (57) with respect to $\tau$ yields:

$$\frac{\partial c_{\pi}^g}{\partial \tau} = \left[ -\frac{\partial p_1}{\partial \tau} - \frac{p_1 \frac{\partial p_1}{\partial \tau} - 2p_0 \frac{\partial p_2}{\partial \tau}}{\sqrt{p_1^2 - 4p_2 p_0}} \right] p_2 + \left[ p_1 + \sqrt{p_1^2 - 4p_2 p_0} \right] \frac{\partial p_2}{\partial \tau} \Rightarrow$$

$$35$$
\[
\frac{\partial c_\pi^g}{\partial \tau} = \frac{-p_2 \left(1 + \frac{p_1}{\sqrt{p_1^2 - 4p_2p_0}}\right) \frac{\partial p_1}{\partial \tau} + \left[\frac{4p_2p_0 + 2p_0}{(p_1 - \sqrt{p_1^2 - 4p_2p_0})\sqrt{p_1^2 - 4p_2p_0}}\right] \frac{\partial p_2}{\partial \tau}}{2p_2^2}.
\]

The fact that \(p_1 < 0\) implies \(\frac{p_1}{\sqrt{p_1^2 - 4p_2p_0}} < -1\). Using this result as well as \(p_2 > 0\), \(p_0 > 0\), \(p_1 - \sqrt{p_1^2 - 4p_2p_0} < 0\), \(\frac{\partial p_1}{\partial \tau} < 0\) and \(\frac{\partial p_2}{\partial \tau} > 0\), yields:

\[
\frac{\partial c_\pi^g}{\partial \tau} < 0.
\]

Using this result and deriving \(d_\pi^g\) given by (58) with respect to \(\tau\), we get:

\[
\frac{\partial d_\pi^g}{\partial \tau} = \frac{-\alpha \left\{ \kappa^2 - \alpha \beta^2 \kappa^2 \frac{\partial c_\pi^g}{\partial \tau} - \beta \gamma (1 - \gamma) \left[ \alpha + \kappa^2 (1 + \tau) \right] \frac{\partial c_\pi^g}{\partial \tau} \right\}}{\left\{ \kappa^2 (1 + \tau) + \alpha + \alpha \beta^2 \gamma^2 (\beta - c_\pi^g) + \beta \gamma (1 - \gamma) \left[ \alpha \beta - \left( \alpha + \kappa^2 (1 + \tau) \right) c_\pi^g \right] \right\}^2} < 0.
\]

D. The equilibrium solution of inflation under decreasing gain learning

Using (23) to obtain \(\lambda_{3,t}\) and \(\lambda_{3,t+1}\) and substituting their expressions as well as \(\lambda_{1,t+1} = 0\), \(\lambda_{2,t+1} = -\frac{\alpha}{\kappa} x_{t+1}\), \(\gamma_{t+1} = \frac{1}{t+1}\) and \(\gamma_{t+2} = \frac{1}{t+2}\) into (21), we get:

\[
(1 + \tau)(t + 1)\pi_t + \frac{\alpha (t + 1)}{\kappa} x_t = E_t \left[ \frac{\alpha \beta^2}{\kappa} x_{t+1} + \beta (1 + \tau)(t + 1)\pi_{t+1} + \beta \frac{\alpha (t + 1)}{\kappa} x_{t+1} \right]
\]

(60)

Using (2), (5) and \(\gamma_{t+1} = \frac{1}{t+1}\), we write

\[
x_t = \frac{1}{\kappa} \pi_t - \frac{\beta}{\kappa} a_t - \frac{1}{\kappa} e_t \quad \text{ (61)}
\]

\[
x_{t+1} = \frac{1}{\kappa} \pi_{t+1} - \frac{\beta}{\kappa} a_t + \frac{1}{t + 1} (\pi_t - a_t) - \frac{1}{\kappa} e_{t+1}. \quad \text{ (62)}
\]

Substituting \(x_t\) and \(x_{t+1}\) respectively given by (61) and (62) into (60) and arranging
the terms lead to

\[ E_t \pi_{t+1} = A_{11,t} \pi_t + A_{12,t} a_t + P_{1,t} e_t \]  

(63)

where

\[ A_{11,t} \equiv \frac{\alpha + \kappa^2 (1 + \tau) + \frac{1}{(t+1)} \alpha \beta^2 (1 + \frac{1}{t+1})}{\alpha \beta (1 + \frac{1}{t+1}) + \beta \kappa^2 (1 + \tau)}, \]

\[ A_{12,t} \equiv \frac{-\alpha \beta + \alpha \beta^2 (1 - \frac{1}{t+1}) (1 + \frac{1}{t+1})}{\alpha \beta (1 + \frac{1}{t+1}) + \beta \kappa^2 (1 + \tau)}, \]

\[ P_{1,t} \equiv -\frac{\alpha}{\alpha \beta (1 + \frac{1}{t+1}) + \beta \kappa^2 (1 + \tau)}. \]

The solution of the ALM of inflation takes the following form:

\[ \pi_t = c_{\pi,t}^{dg} a_t + d_{\pi,t}^{dg} e_t. \]  

(64)

Using (5) and (64), we obtain:

\[ E_t \pi_{t+1} = c_{\pi,t+1}^{dg} [(1 - \gamma_{t+1}) a_t + \gamma_{t+1} \pi_t] \]  

(65)

Using equations (65) and (63) to eliminate \( E_t \pi_{t+1} \) and arranging the terms yield:

\[ \pi_t = A_{12,t} - \left(1 - \frac{1}{t+1}\right) c_{\pi,t+1}^{dg} \left[A_{11,t} - \frac{P_{1,t}}{1 + \frac{1}{t+1}} \right] a_t + \frac{P_{1,t}}{1 + \frac{1}{t+1}} e_t. \]  

(66)

This implies that:

\[ c_{\pi,t}^{dg} = \frac{A_{12,t} - \left(1 - \frac{1}{t+1}\right) c_{\pi,t+1}^{dg}}{1 + \frac{1}{t+1} c_{\pi,t+1}^{dg} - A_{11,t}} \]  

(67)

and

\[ d_{\pi,t}^{dg} = \frac{P_{1,t}}{1 + \frac{1}{t+1} c_{\pi,t+1}^{dg} - A_{11,t}}. \]  

(68)

We gather equations (63), (5) and (6) while using (61) to substitute \( x_t \) to obtain the system of three equations:
\[ E_t y_{t+1} = A_t y_t + P_t e_t \]

where

\[ y_t \equiv [\pi_t, a_t, b_t], \quad A_t \equiv \begin{bmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{1+t} & \frac{t}{1+t} & 0 \\ \frac{1}{\kappa(t+1)} & -\frac{\beta}{\kappa(t+1)} & \frac{t}{1+t} \end{bmatrix}, \quad \text{and} \quad P_t \equiv \begin{bmatrix} P_{1,t} \\ 0 \\ -\frac{1}{\kappa(t+1)} \end{bmatrix}. \]

The above system is subject to three boundary conditions: \( a_0, b_0, \text{ and } \lim_{s \to \infty} |E_t \pi_{t+s}| < \infty \). The eigenvalues of \( A_t \) are given by \( \frac{t}{1+t} \) and by the two eigenvalues of \( A_{1,t} \):

\[ A_{1,t} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{1+t} & \frac{t}{1+t} \end{bmatrix} \quad (69) \]

We can show that \( A_{1,t} \) has a real eigenvalue inside and one outside the unit circle. \( \square \)

E. The single stable solution under decreasing gain learning

Among infinite stochastic sequences satisfying equation (67), we focus on a non-explosive solution. To characterize the properties of this solution, we consider the value of \( c_{\pi,t}^{dg} \) when \( t \to +\infty \). Using the boundary conditions \( \lim_{t \to +\infty} A_{11,t} = \frac{1}{\beta} \text{ and } \lim_{t \to +\infty} A_{12,t} = \frac{-\alpha(1-\beta)}{\alpha + \kappa^2(1+\tau)} \), we find that in the limit, \( c_{\pi,t}^{dg} \) evolves according to:

\[ \lim_{t \to +\infty} c_{\pi,t}^{dg} = \beta \lim_{t \to +\infty} c_{\pi,t+1}^{dg} + (1-\beta) \frac{\alpha \beta}{\alpha + \kappa^2(1+\tau)}. \quad (70) \]

Given the boundary condition \( \lim_{s \to \infty} |E_t \pi_{t+s}| < \infty \), we have \( \lim_{n \to +\infty} \beta^n c_{\pi,t+n}^{dg} = 0. \) Using this condition and solving (70) forward yield one and only one bounded solution of \( c_{\pi,t}^{dg} \):
\[
\lim_{t \to +\infty} c_{\pi,t}^g = \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)}.
\]

Therefore, for any \( t > T \), using (67) and (70), we have:

\[
c_{\pi,T}^g = \frac{A_{12,T} - \frac{T}{T+1} c_{\pi,T+1}^g}{A_{11,T}} \geq \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \Rightarrow \]

\[
\left[ \frac{T}{T + 1} + \frac{1}{T + 1} \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \right] c_{\pi,T+1}^g \geq A_{12,T} + \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} A_{11,T} \tag{71}
\]

The right hand of (71) can be transformed taking account of \((1 + \frac{1}{T+1}\beta)^{-1} < 1\) as follows:

\[
A_{12,T} + \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} A_{11,T} = \frac{\alpha \beta \left\{ (1 - \frac{1}{T+1})\beta \left[ 1 + \frac{1}{T+1}\beta \right] - 1 + \frac{[\kappa^2(1 + \tau) + \alpha] + \alpha \beta^2 \left( \frac{1}{T+1}\beta + \frac{1}{T+1} \right)}{\alpha + \kappa^2(1 + \tau)} \right\}}{\alpha \beta \left( \frac{1}{T+1}\beta + 1 \right) + \beta \kappa^2(1 + \tau)}
\]

\[
= \frac{\alpha \beta^2 \left[ 1 + \frac{1}{T+1}\beta \right]}{\alpha \beta \left( \frac{1}{T+1}\beta + 1 \right) + \beta \kappa^2(1 + \tau)} \left[ 1 - \frac{1}{T+1} + \frac{1}{T+1} \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \right]
\]

\[
= \frac{\alpha \beta^2 \left[ 1 + \frac{1}{T+1}\beta \right]}{\alpha \beta \left( \frac{1}{T+1}\beta + 1 \right) + \beta \kappa^2(1 + \tau)} \left[ \frac{T}{T+1} + \frac{1}{T+1} \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \right]
\]

\[
> \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \left[ \frac{T}{T+1} + \frac{1}{T+1} \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \right].
\]

Substituting this result into (71), we get \( c_{\pi,t+1}^g > \frac{\alpha \beta}{\alpha + \kappa^2(1 + \tau)} \).

References


