Liquidity Preference and Liquidity Traps: A Dynamic Optimization Approach*

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Abstract

Using a dynamic optimization model in which money and bond holdings yield utility and nominal wages sluggishly adjust, this paper shows the condition for a liquidity trap with zero nominal interest rates and unemployment to persist in steady state. If the liquidity trap occurs, there is a continuum of equilibrium paths: a path along which full employment eventually holds and multiple paths that converge to the liquidity trap. Both current and future monetary expansions have no effect on the economic activity and government purchases crowd out steady-state consumption.

Keywords: Liquidity preference, Liquidity trap, Unemployment, Zero bound on nominal interest rates.

JEL Classification Numbers: E24, E32, E41.

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*The authors would like to thank Yoshiyasu Ono and seminar participants at Okinawa International University, Osaka University, and Osaka University of Economics for their helpful comments and suggestions.

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1 Introduction

Japan has experienced persistent stagnation with unemployment since the early 1990s. Although the Bank of Japan has expanded the monetary base and then short-run interest rates has reached nearly zero (see Figure 1), the economy has not shown signs of genuine recovery. The United States and many European countries also have trodden the same tragic path as Japan since the global economic crisis triggered by the Lehman shock in September 2008. These economic situations can be interpreted as a typical liquidity trap. As in Figure 2, the short-run nominal interest rate ceases to fall as the monetary base has increased between January 1990 to June 2014 in Japan.¹

Using a dynamic optimization model with satiable liquidity preference and sluggish nominal wage adjustment, this paper clarifies the condition of a liquidity trap where nominal interest rates come down to zero and unemployment persists. If the liquidity trap arises, there are both a path along which full employment eventually attains and multiple paths that approach the liquidity trap. The resulting liquidity trap is thus a self-fulfilling phenomenon. Policy implications are somewhat against the Keynesian wisdom presented by e.g., the traditional IS-LM and AD-AS analyses and the recent contributions of Krugman (1998) and Ono (1994, 2001). Government purchases deteriorate steady-state welfare, whereas not only current but also future monetary expansions have no welfare effect in the liquidity trap.

There are many theoretical attempts that aim to revive Keynes’ (1936) idea of liquidity traps using a modern dynamic optimization framework. Among them, Krugman (1998) employs an essentially two-period model that imposes the cash-in-advance constraint on consumption. In the first period, a commodity demand shortage takes place due to price rigidity and, at the same time, expansionary monetary policy is impotent because the nominal interest rate hits a zero lower bound. In the second and subsequent periods, prices flexibly adjust so that full employment holds. In this setting, future monetary expansions raise future price levels and in turn leads to a rise in the current inflation rate that induces households to increase current consumption. Inspired by this seminal work, a growing body of theoretical studies analyze such a temporary liquidity trap assuming

¹The similar relationship to Figure 2 is obtained if the real monetary base, which is divided by the consumer price index, is used as substitute for the nominal monetary base.
that money demand is satiated at some finite level. This paper in contrast focuses on a permanent liquidity trap rather than the temporary one, thereby showing that not only current but also future monetary policy is neutral to the real economic activity.

In this regard, Benhabib et al. (2001a, b, 2002) and Schmitt-Grohe and Uribe (2009) are closely related to mine. Assuming satiable money preference, they emphasize Taylor’s (1993) interest-rate rule as a potential source of self-fulfilling and persistent liquidity traps, whereas this paper obtains a similar outcome without imposing any interest-rate rule. More importantly, this paper takes into account the utility of bond holdings and unemployment dynamics in order to reconsider policy implications shown by Benhabib et al. (2002). Under a balanced-budget rule, negative monetary expansion rates enhance steady-state welfare in the full-employment equilibrium but, on the other hand, generate the Pareto-inferior liquidity trap. In the presence of public debt issuance, if the government declares the Ponzi game, the liquidity trap is ruled out but a resulting decrease in the liquidity of public bonds may vanish not only the liquidity-trap paths but also the full-employment one. Therefore, it is hard to prevent the economy from falling into the liquidity trap.

While Krugman (1998), Benhabib et al. (2001a, b, 2002) and Schmitt-Grohe and Uribe (2009) consider satiable money demand, Ono (1994, 2001) and Ono and Ishida (2013) point out a crucial role of insatiable money demand in causing persistent liquidity traps. This paper analyzes macroeconomic dynamics under these polar assumptions and compares them. In the case of insatiable money demand, if the degree of money demand is enough strong, a path that leads to the liquidity trap may become a unique equilibrium

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2For example, Ireland (2005) investigates an intergenerational redistribution effect of monetary policy using an overlapping-generations model. Although Krugman (1998) and Ireland (2005) impose the cash-in-advance constraint, the money-in-the-utility-function model is functionally equivalent to the cash-in-advance model under some conditions, as shown by Feenstra (1986). For this reason, many studies (e.g., Eggertsson and Woodford 2003, Eggertsson 2006, Jeanne and Svensson 2007) use the money-in-the-utility-function model to treat zero nominal interest rates.

3Recently, insatiable money preference is assumed to address various issues in the persistent liquidity trap. For example, Johdo and Hashimoto (2009) introduce direct foreign investment into a two-country model and examine the effects of the corporation tax on employment. Murota and Ono (2012) take account of the utility of holding bank deposits to explain the declining money multiplier during periods with zero nominal interest rates. See also Matsuzaki (2003) and Rodriguez-Arana (2007).
path and government purchases stimulate steady-state consumption.

The paper is organized as follows. Section 2 presents the basic structure of the model. Section 3 analyzes macroeconomic dynamics and examines policy implications in the case of satiable liquidity preference. Section 4.1 investigates the relationship between economic policy and the transversality condition. Section 4.2 compares implications derived under satiable and insatiable liquidity preferences. Section 5 summarizes and concludes.

2 The Model

There are identical firms and households of which sizes are normalized to unity, respectively. Both money and bond holdings yield the utility of liquidity. Nominal wages sluggishly adjust depending on demand shortages in the labor market, so that unemployment may exist.

2.1 Firms

A representative firm employs labor $l_t$ to produce output $y_t$ according to the following linear-homogeneous technology:

$$y_t = \theta l_t,$$

where $\theta(> 0)$ is an input-output coefficient. Given the production function, nominal commodity price $P_t$ and nominal wage rate $W_t$, the competitive firm maximizes profit $\theta l_t - \frac{W_t}{P_t} l_t$. Thus, the labor demand satisfies $l_t = \infty$ if $\theta > \frac{W_t}{P_t}$; $l_t = 0$ if $\theta < \frac{W_t}{P_t}$; and $0 < l_t < \infty$ if

$$\theta = \frac{W_t}{P_t}. \tag{1}$$

2.2 Households

Total financial asset $a_t$ consists of money $m_t$ and bond $b_t$, all of which are measured in real terms:

$$a_t = m_t + b_t. \tag{2}$$
The flow budget constraint of a representative household is

\[ \dot{a}_t = (i_t - \pi_t) a_t + w_t X_t - c_t - i_t m_t - \tau_t, \tag{3} \]

where \( i_t \) denotes the nominal interest rate, \( \pi_t (\equiv \dot{P}_t / P_t) \) the inflation rate, \( w_t (\equiv W_t / P_t) \) the real wage rate, \( X_t \) the realized amount of labor supply, \( c_t \) consumption, and \( \tau_t \) the lump-sum tax. In the presence of nominal wage rigidities, labor demand may be less than the full-employment level of labor supply, which is normalized to unity:

\[ X_t = \min\{t, 1\}. \]

In this paper, I define liquidity as

\[ q_t \equiv m_t + \chi b_t, \tag{4} \]

where \( 0 \leq \chi \leq 1 \). If \( \chi = 0 \), bond holdings yield no liquidity, whereas if \( \chi = 1 \), bonds are comparable with money in the degree of liquidity. The lifetime utility depends on consumption and liquidity:

\[ U_0 = \int_0^\infty [\ln c_t + v(q_t)] e^{-\rho t} dt, \tag{5} \]

where \( \rho (> 0) \) is the subjective discount rate. The function \( v(\cdot) \) satisfies the regularity conditions:

\[ v'(\cdot) > 0, \quad v''(\cdot) < 0, \quad v'(0) = \infty, \quad \lim_{q \to 0} v'(q) q > 0. \tag{6} \]

The household maximizes (5) subject to (2)–(4). The first-order optimal conditions are given by

\[ \frac{\dot{c}_t}{c_t} = [i_t + \chi v'(m_t + \chi b_t)c_t] - \pi_t - \rho, \tag{7} \]

\[ i_t = (1 - \chi) v'(m_t + \chi b_t)c_t. \tag{8} \]

The transversality condition is

\[ \lim_{t \to \infty} c_t^{-1} a_t e^{-\rho t} = 0. \tag{9} \]

Equation (7) is the Euler equation. Equation (8) represents the liquidity demand function that makes the nominal interest rate equal to the marginal rate of substitution between money holdings and consumption minus that between bond holdings and consumption.\(^4\)

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2.3 The government and markets

The equilibrium conditions in the money and bond markets are

\[ m_t = \frac{M_t}{P_t}, \quad (10) \]

\[ b_t = 0, \quad (11) \]

where \( M_t \) stands for the nominal money supply. Letting \( \mu(\equiv \dot{M}_t/M_t \geq -\rho) \) denote the monetary expansion rate, the time differentiation of (10) generates

\[ \frac{\dot{m}_t}{m_t} = \mu - \pi_t. \quad (12) \]

In the absence of public debt issuance, the government finances government purchase \( g(\in [0, \theta)) \) by imposing taxes and printing money:

\[ g = \tau_t + \mu m_t. \]

Because commodity prices are flexible, demand always equals supply in the commodity market:

\[ c_t + g = \theta X_t, \quad (13) \]

In the labor market, unemployment may exist because of sluggish wage adjustment. To be more precise, this paper considers a mechanism proposed by Ono and Ishida (2014) in which workers’ concern for fairness determines nominal wages. In their model, employed workers get fired at Poisson rate \( \alpha(>0) \), so that employment \( X_t \) changes according to

\[ \dot{X}_t = x_t - \alpha X_t, \quad (14) \]

where \( x_t \) is the number of newly hired workers.

Consider small time interval \( \Delta t \) from \( t - \Delta t \) to \( t \). Keeping in mind economic conditions of outsiders who are unemployed, insiders who are employed form the fair wage as a social average of rightful wages. More precisely, zero income is assumed to be rightful for unemployed workers, so that the fair wage at time \( t - \Delta t, W_{t-\Delta t} \), is given by

\[ W_{t-\Delta t} = \nu_{t-\Delta t} X_{t-\Delta t} + 0 \times (1 - X_{t-\Delta t}), \quad (15) \]

where \( \nu_{t-\Delta t} \) is the nominal wage rate that workers believe fair if they are employed.
The evolution of the fair wage from \( t - \Delta t \) to \( t \) is as follows. In time \( t \), workers who remain employed believe that the most recent rightful wage \( v_{t-\Delta t} \) is still fair, whereas newly hired workers follow the incumbent worker’s conceptions, \( W_t \). Therefore, the fair wage as a social average of rightful wages evolves according to

\[
W_t = v_{t-\Delta t}(1 - \alpha \Delta t)X_t - \Delta t + W_t x_t \Delta t + 0 \times (1 - X_t).
\]

Eliminating \( v_{t-\Delta t}X_t - \Delta t \) from this equation by the use of (15) and letting \( \Delta t \to 0 \) gives

\[
\frac{\dot{W}_t}{W_t} = \lim_{\Delta t \to 0} \frac{W_t - W_{t-\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{x_t - \alpha}{1 - x_t \Delta t} = x_t - \alpha.
\]

From (13) and (14), it reduces to

\[
\pi_t = \frac{\dot{W}_t}{W_t} = \alpha \left( \frac{\dot{c}_t + g}{\theta} - 1 \right) + \frac{\dot{c}_t}{\theta},
\]

where the first equality comes from (1). Note that if \( \dot{c}_t = 0 \), (16) corresponds to the conventional Walrasian price adjustment in which output gaps determine the speed of price adjustment.\(^5\)

### 3 Satiable Liquidity Preference

This section investigates the dynamic properties in the case of satiable liquidity preference and finds that a self-fulfilling liquidity trap with unemployment may persist in steady state. More precisely, in addition to (6), the utility function of liquidity \( v(\cdot) \) satisfies:

**Assumption 1. (Satiable liquidity demand)** \( v(\cdot) \) reaches a positive upper bound at \( q_t = \bar{q} \):

- For \( q_t < \bar{q} \), \( v(q_t) < \bar{v} \), \( v'(q_t) > 0 \), \( v''(q_t) < 0 \);
- For \( q_t \geq \bar{q} \), \( v(q_t) = \bar{v} \), \( v'(q_t) = 0 \).

Figure 3 depicts the function \( v(\cdot) \) under Assumption 1. The money demand function (8) to which (11) is applied is illustrated in Figure 4, showing that the nominal interest rate reaches a zero lower bound when \( m_t \) exceeds the satiated level of liquidity, \( \bar{q} \). This theoretical finding seems consistent with the data in Figure 2.

\(^5\)Numerous studies (for example, Blanchard and Sachs 1982 and van der Ploeg 1993) employ this type of price/wage adjustment to analyze disequilibrium dynamics in various contexts.
3.1 Dynamics and steady states

In the presence of unemployment, the inflation rate follows (16). The consumption dynamics are then obtained by substituting (8), (11) and (16) into (7):

\[
\frac{\dot{c}_t}{c_t} = \frac{\theta}{c_t + \theta} \left[ v'(m_t)c_t - \alpha \left( \frac{c_t + g}{\theta} - 1 \right) - \rho \right].
\] (17)

Substituting (16) into (12) and using (17) to eliminate \( \dot{c}_t \) from the result yields

\[
\frac{\dot{m}_t}{m_t} = \mu - \frac{c_t}{c_t + \theta} \left[ v'(m_t)c_t + \frac{\alpha \theta}{c_t} \left( \frac{c_t + g}{\theta} - 1 \right) - \rho \right].
\] (18)

These equations constitute an autonomous dynamic system with respect to \( c_t \) and \( m_t \), giving the following boundary curves:

\[
\dot{c}_t = 0 : \quad v'(m) = \frac{\alpha}{\theta c} \left\{ c - \left( \theta - g - \frac{\rho \theta}{\alpha} \right) \right\},
\] (19)

\[
\dot{m}_t = 0 : \quad v'(m) = \frac{1}{c} \left[ \frac{(\theta - g)\alpha}{\theta} + (\rho - \alpha) + \frac{\mu (c + \theta)}{c} \right].
\] (20)

Appendix A shows that these curves have the following properties. The slope of (19) is zero for \( m_t \in [\bar{q}, \infty) \) and negative (positive) for \( m_t \in (0, \bar{q}) \) depending on \( \theta - g > (<) \frac{\theta}{\alpha} \). On the other hand, in the neighborhood of \( \mu = 0 \), (20) is positively sloped and is located on the left-hand side of (19).

Figure 5 illustrates the phase diagram in the case where \( \theta - g > \frac{\theta}{\alpha} \) and \( \mu = 0 \). There are three kinds of paths:

1. Paths along which \( m_t \) eventually decreases and \( c_t \) reaches \( \theta - g \) within a finite time.

2. A path that converges to the full-employment equilibrium given by

\[
c^f = \theta - g \ (> 0), \quad v'(m^f) = \frac{\theta + \mu}{\theta - g} \ (\geq 0),
\] (21)

where the superscript \( f \) represents variables in the full-employment equilibrium.

3. Paths along which \( m_t \) diverges to infinity and \( c_t \) approaches

\[
c^u = (\theta - g) - \frac{\rho \theta}{\alpha}, \quad \text{where} \ c^u < c^f,
\] (22)

where the superscript \( u \) denotes variables in the liquidity-trap equilibrium.
All paths of type 1 are not equilibrium because $m_t$ becomes negative within a finite time along them.\(^6\)

Not only the full-employment path of type 2 but also all paths of type 3 are possible candidates for equilibria. Although $m_t$ diverges to infinity along the paths of type 3, the transversality condition (9) is valid if the growth rate of $m_t$ is less than $\rho$:

$$-\frac{\dot{c}_t}{c_t} + \frac{\dot{m}_t}{m_t} - \rho = \mu < 0 \quad \text{if } \mu < 0,$$

which is derived by keeping in mind $\dot{c}_t = 0$ and substituting $v'(m_t) = 0$ and (22) into (18). As a result, if $\mu < 0$, the equilibrium path is indeterminate and almost all paths converge to the liquidity-trap steady state where the nominal interest rate falls to zero and unemployment persists even though deflation continues to occur, $\pi^u = -\rho$.

The steady-state welfare in the full-employment and liquidity-trap equilibria are respectively calculated by applying (11) and either (21) or (22) to (5):

$$U^f = \ln(\theta - g) + v(m_f), \quad \text{(24)}$$

$$U^u = \ln \left( \left( \theta - g \right) - \frac{c'}{\alpha} \right) + v(\bar{q}). \quad \text{(25)}$$

Linearizing the former equation around the liquidity-trap equilibrium and taking $v'(\bar{q}) = 0$ into account provides

$$U^f - U^u = \frac{\theta}{\alpha c^u} > 0.$$

Hence, the liquidity trap is a tragic fact. From (21) and (24) and for given $g$, $U^f$ takes its maximum value at $\mu = -\rho(< 0)$, which is known as the Friedman rule. Such a deflationary policy achieves the zero nominal interest rate in the full-employment equilibrium but, simultaneously, makes the transversality condition valid along the liquidity-trap path (see (23)).

The preceding result is summarized in the following proposition:

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\(^6\)After $c_t$ reaches $\theta - g$, inflation follows $\pi_t = v'(m_t)(\theta - g) - \rho$ from (7) in which $\dot{c}_t = 0$, (8) and (11). Then, (12) implies $\dot{m}_t = (\rho + \mu)m_t - (\theta - g)v'(m_t)m_t$. As proved in Obstfeld and Rogoff (1983), under the last property of (6), $m_t$ eventually goes into negative for $\forall m_t \in (0, m_f)$. All paths of type 1 are thus not equilibrium. On the path of type 2, in contrast, $m_t$ eventually reaches $m_f$, which satisfies (21) and attains $\dot{m}_t = 0$. This means that the path of type 2 is equilibrium.
Proposition 1. Suppose that the utility of holding liquidity has a positive upper bound (Assumption 1). If $\mu < 0$, there are multiple equilibrium paths that approach the liquidity trap where unemployment and deflation persist at zero nominal interest rates. Evaluated in steady state, the liquidity-trap equilibrium is Pareto-inferior to the full-employment equilibrium.

While Benhabib et al. (2001a, b, 2002) emphasize the role of Taylor’s (1993) interest-rate rule as a source of persistent and self-fulfilling liquidity traps, this paper finds that satiable liquidity demand per se rather than monetary policy rules is essential. In fact, the self-fulfilling liquidity trap is shown to persist without imposing any interest-rate rule.

3.2 Policy implications

Let me examine the effects of monetary and fiscal policy in the liquidity-trap steady state. Since the economy permanently falls into the liquidity trap with the zero nominal interest rates, not only current but also future monetary expansions have no effect on consumption and welfare: from (22) and (25),

$$\frac{dc^u}{dM_t} = \frac{dc^u}{d\mu} = 0, \quad \frac{dU^u}{dM_t} = \frac{dU^u}{d\mu} = 0.$$

The result sharply differs from Krugman (1998) where inflation generated by future monetary expansions raises current consumption since the liquidity trap temporarily occurs and full employment holds in the future. In contrast, the liquidity trap obtained in this paper is a permanent phenomenon, therefore making both current and future monetary expansions impotent to generate inflation.

The dynamic analysis however offers a new insight into monetary policy. In the present context, increasing $\mu$ up to a nonnegative value makes the transversality condition invalid along the liquidity-trap paths of type 3 in Figure 5, so that the full-employment path of type 2 becomes a unique equilibrium.\(^7\) It is however noted that there is a dilemma since

\(^7\)Note that there is a difficulty in controlling $\mu$ in the liquidity trap where an increase in the monetary base may not correspond to that in monetary aggregates because of zero nominal interest rates. Using a dynamic optimization model in which both cash and deposit holdings yield utility, Murota and Ono (2012) shows that under zero nominal interest rates a commercial bank holds excess reserve, which decreases monetary aggregates.
an increase in $\mu$ reduces the steady-state welfare in the full-employment equilibrium from (21) and (24).

Implications of fiscal policy to overcome the liquidity trap is opposite to the conventional Keynesian view, e.g., the IS-LM and AD-AS analyses. As implied (17), government purchases affect households’ intertemporal decision on consumption by mitigating deflation. It induces households to increase current consumption relative to future consumption but ends up reducing steady-state consumption and welfare from (22) and (25):

$$\frac{dc^u}{dg} = -1, \quad \frac{dU^u}{dg} < 0.$$

A further increase in $g$ sufficient to satisfy $\theta - g < \frac{\theta}{\alpha}$ shifts the phase diagram from Figure 5 to Figure 6 in which consumption converges to zero on the path of type 3.

These policy implications are summarized as follows:

**Proposition 2.** Suppose that the utility of holding liquidity has a positive upper bound (Assumption 1). In the liquidity-trap steady state, government purchases deteriorates the steady-state welfare, whereas monetary expansions have no effect. Decreasing the monetary expansion rate up to negative increases the welfare in the full-employment steady state but, at the same time, generates the equilibrium paths that converge to the liquidity trap.

Note that it is difficult to calculate accurate policy effects including transitional paths because the equilibrium path is indeterminate. Any policy and structural shock, whether it is fundamental or non-fundamental, can potentially move the economy from the liquidity-trap equilibria to the full-employment equilibrium, and vice versa.

4 Discussion

This section first incorporates public bonds into the model so as to explore the role of fiscal policy in making the transversality condition invalid along the liquidity-trap path. Next, satiable liquidity preference (Assumption 1) is replaced with insatiable one, which brings down a different type of liquidity traps from that in the previous section.
4.1 Fiscal policy and the transversality condition

As stressed in Benhabib et al. (2002), fiscal policy is also helpful in selecting equilibrium. For this purpose, this subsection introduces public bonds into the basic model. For simplicity, the analysis is limited to the case of $\chi = 0$ so that the dynamic system is still characterized by (17) and (18).

The government finances $g$ also by issuing public bond $b_t$ and thus the budget equation of the government is modified as

$$\dot{b}_t = (i_t - \pi_t)b_t + g - \tau_t - \mu m_t.$$  

For purpose of comparison, I follow Benhabib et al. (2002) in assuming that the government adjusts $\tau_t$ so as to satisfy $g - \tau_t - i_t m_t = \psi a_t$. Together with (2) and (12), it rewrites the government budget equation as

$$\dot{a}_t = (i_t - \pi_t + \psi)a_t. \quad (26)$$

The case of $\psi (>)0$ implies that the government reduces (increases) taxes as total government liability $a_t$ accumulates.

Letting be $\dot{c}_t = 0$ and using (26) in which $i^u = 0$ and $\pi^u = -\rho$, the transversality condition (9) is valid along the liquidity-trap path if $\psi < 0$:

$$-\frac{\dot{c}_t}{c_t} + \frac{\dot{a}_t}{a_t} - \rho = \psi < 0 \quad \text{if } \psi < 0.$$  

Which type of monetary and fiscal policy is conducted behind being $\psi < 0$? Since (12) and $\pi^u = -\rho$ indicate that $\dot{m}_t = (\mu + \rho)m_t$ holds in the liquidity-trap path, (2) and (26) in which $i^u = 0$ and $\pi^u = -\rho$ bring the following motion of $b_t$:

$$b_t = b_s e^{(\rho + \psi)(t-s)} + m_s \left[ e^{(\rho + \psi)(t-s)} - e^{(\rho + \mu)(t-s)} \right] \quad \text{for } t \geq s.$$  

Along the liquidity-trap path in which the transversality condition is valid ($\psi < 0$), if $b_s \geq 0$ and $\mu$ satisfies $-\rho \leq \mu \leq \psi < 0$, $b_t$ keeps positive —i.e., the deflationary policy ($\mu < 0$) makes the government possible to issue public bonds permanently. By contrast, if $\mu > \psi$ and $\psi < 0$, $b_t$ becomes negative with a finite time —i.e., households become borrowers and the government becomes a lender. This means that the inflationary policy ($\mu \geq 0$) is consistent with the transversality condition if the government imposes taxes so
that the government accumulates assets. In other words, either the deflationary monetary policy \((\mu < 0)\) or the fiscal consolidation that eventually achieves \(b_t < 0\) causes the self-fulfilling liquidity trap.

On the other hand, the full-employment steady state satisfies \(i^f - \pi^f = \rho - \chi v'(mf)(\theta - g)\) from (7) in which \(c_t = 0\). Substituting it into (26) gives

\[
-\frac{\dot{c}_t}{c_t} + \frac{\dot{a}_t}{a_t} - \rho = \psi - \chi v'(mf)(\theta - g) < 0 \quad \text{if} \quad \psi < \chi v'(mf)(\theta - g).
\]

Unless \(\chi = 0\), controlling \(\psi\) so as to satisfy \(0 \leq \psi < \chi v'(mf)(\theta - g)\) preserves the full-employment equilibrium but, simultaneously, eliminates the liquidity-trap path by making the transversality condition invalid.

In the literature, the case of \(\psi \geq 0\) can be interpreted as a declaration of future financial collapse since the government does not raise taxes as the government liability accumulates. If \(\chi > 0\), such a declaration is useful to rule out only the liquidity trap since in the full employment equilibrium the liquidity value of holding bonds, \(\chi v(m^f)(\theta - g)\), is still positive and enables the government to promise the non-Ponzi game. However, in the actual economy, it creates difficulty in maintaining the liquidity of public bonds, \(\chi\). If the government puts \(\psi > 0\) into practice and thereby \(\chi\) becomes zero, the transversality condition is invalid on any path, so that all equilibrium paths including the full-employment one disappear. As a consequence, the government may choose \(\psi < 0\) that causes the self-fulfilling liquidity trap unfortunately. An importance of considering the bond liquidity \(\chi\) is not pointed out in the existing literature, e.g., Benhabib et al. (2002).

These results are summarized in the following proposition:

**Proposition 3.** Suppose that the utility of holding liquidity has a positive upper bound (Assumption 1) and the government issues public bonds. If \(\psi < 0\), there are multiple equilibrium paths that approach either the full-employment or liquidity-trap steady state. If \(\psi \geq 0\) and bond holdings yield liquidity \((\chi > 0)\), there is a unique equilibrium path that converges to the full-employment steady state. If \(\psi \geq 0\) and \(\chi = 0\), there is no equilibrium path.
4.2 Insatiable Liquidity Preference

Ono (1994, 2001) and Ono and Ishida (2014) makes the following alternative assumption to analyze a liquidity trap:

**Assumption 2. (Insatiable liquidity demand)** \( v'(\cdot) \) has a positive lower bound:

\[
\lim_{q \to \infty} v'(q) = \beta(> 0).
\]

The utility of liquidity under Assumption 2 is illustrated in Figure 7. From (8), the nominal interest rate has nonnegative lower bound \((1 - \chi)\beta c_t\), which potentially becomes zero when bond holdings yield sufficient liquidity, \( \chi = 1 \) (see Figure 8). This theoretical outcome is similar to that obtained under Assumption 1 and also seems to agree with the data in Figure 2. The aim of this subsection is to clarify the difference between the liquidity traps which arise under Assumption 1 and 2.

Dynamic equations and boundary curves, (17)–(20), remain unchanged. In the case where the lower bound of the marginal utility of holding liquidity, \( \beta \), is low enough to satisfy \( \rho > \beta(\theta - g) \), the phase diagram are given by Figure 5 (Figure 6) depending on \( \theta - g > (\leq) \frac{\rho \theta}{\alpha} \) although \( c^n \) in Figure 5 is replaced with

\[
c^n = \frac{\alpha}{\alpha - \beta \theta} \left[ (\theta - g) - \frac{\rho \theta}{\alpha} \right],
\]

which comes from (17) in which \( \dot{c}_t = 0 \) and \( v'(m) = \beta \). Hence, dynamic properties and policy implications in this case are basically the same as those obtained under Assumption 1 although there is a minor difference in the condition on the validity of the transversality condition. On the path where \( m_t \) diverges to infinity, \( m_t \) evolves according to (18) in which \( v'(m_t) = \beta \) and \( c^a \) is given by (27). Thus, even if the government keeps \( \mu \) positive under a balanced-budget rule, the liquidity-trap path may satisfy the transversality condition (9):

\[
-\frac{\dot{c}_t}{c_t} + \frac{\dot{m}_t}{m_t} - \rho = \mu - \beta c^a < 0 \quad \text{if} \quad \mu < \beta c^a.
\]

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\(^8\)Ono et al. (2004) gives empirical evidence on insatiable money preference using both parametric and non-parametric methods. Using a Bayesian likelihood approach, Matsumae et al (2014) estimates a dynamic stochastic general equilibrium model and finds the existence of a positive lower bound of the marginal utility of money, \( \beta \).
More important differences arise when $\beta$ is large enough to satisfy $\rho < \beta(\theta - g)$. Figure 9 depicts the phase diagram in the case where $\theta - g < \frac{\rho}{\alpha}$. The full-employment path disappears. A path along which $m_t$ diverges to infinity and $c_t$ converges to (27) is equilibrium if (28) is satisfied. Paths along which $c_t$ approaches zero also becomes equilibrium if (28) in which $c^u = 0$ holds, that is, if $\mu < 0$. Therefore, there is a unique equilibrium path that leads to the liquidity trap if $0 \leq \mu < \beta c^u$ under a balanced-budget rule. Ono (1994, 2001) and Ono and Ishida (2014) provide a detail analysis for this case.

Although monetary expansions are ineffective in the liquidity-trap steady state under both Assumption 1 and 2, government purchases can stimulate steady-state consumption (27):

$$\frac{dc^u}{dg} > (\cdot 0) \quad \text{if} \quad \alpha < (\cdot) \beta \theta, \quad \frac{dc^u}{dM_t} = \frac{dc^u}{d\mu} = 0.$$

Remember that under Assumption 1, government purchases reduce steady-state consumption by mitigating deflation and increasing current consumption. In addition to this effect, under Assumption 2, the nominal interest rate $i_t = (1 - \chi) \beta c_t$ elastically rises responding to an increase in current consumption and hence households have an incentive to increase future consumption relative to current consumption. If the latter effect dominates the former one, i.e., if $\alpha < \beta \theta$, steady-state consumption (27) increases.\(^9\) Due to such a beneficial effect, a further increase in $g$ enough to satisfy $\rho > \beta(\theta - g)$ shifts the phase diagram from Figure 9 to Figure 6 in which the full-employment equilibrium emerges.

The results are summarized as follows

**Proposition 4.** Suppose that the marginal utility of holding liquidity has a positive lower bound (Assumption 2) and the government keeps a balanced-budget rule. If $\rho > \beta(\theta - g)$ and $\mu < 0$, there are multiple equilibrium paths that converge to the liquidity-trap steady state. In contrast, if $\rho < \beta(\theta - g)$, $\theta - g < \frac{\rho}{\alpha}$ and $0 \leq \mu < \beta c^u$, there is a unique equilibrium path that approaches the liquidity-trap steady state. Government purchases crowd out steady-state consumption in the former case but raises it in the latter case,\(^9\)\(^10\)

\(^9\)In the case where $\rho < \beta(\theta - g)$ and $\theta - g > \frac{\rho}{\alpha}$, any equilibrium path, whether the full-employment or liquidity-trap paths, vanishes. See Ono (1994) for this point.

\(^{10}\)The conditions of $\rho < \beta(\theta - g)$ and $\theta - g < \frac{\rho}{\alpha}$, which are imposed in Figure 9, imply that $\alpha < \beta \theta$. Similarly, the conditions of $\rho > \beta(\theta - g)$ and $\theta - g > \frac{\rho}{\alpha}$ in Figure 5 reduce to $\alpha > \beta \theta$. These signs determine the dynamic property and the effect of government purchases.
whereas monetary expansions have no effect in both cases.

5 Conclusion

This paper shows the condition where the liquidity trap with zero nominal interest rates and unemployment persists when the liquidity demand is satiated. Differently from Benhabib et al. (2001a, b, 2002), this is a consequence not from monetary policy such as Taylor’s (1993) interest-rate rule, but instead stems from satiable liquidity demand per se.

Both current and future monetary expansions are neutral to the steady-state consumption and welfare. This differs from Krugman (1998) in which a liquidity trap is a temporary phenomenon and therefore an increase in the future money supply stimulates current consumption by raising current inflation. In contrast with the conventional Keynesian views, government purchases end up crowding out the steady-state consumption. This also differs from the finding of Ono and Ishida (2014) in which the liquidity demand is insatiable.

The policy implications obtain in this paper depend on the fact that in steady state the nominal interest rate $i$ is zero and equals the sum of the inflation rate $\pi$ and the real interest rate determined by the subjective discount rate $\rho$:

$$i = \pi + \rho = 0.$$  

Accordingly, the results may alter if $\rho$ is endogenous with respect to consumption, or if $\pi$ depends on the monetary expansion rate due to a different type of the Phillips curve. These extensions are my future work.

The dynamic optimization approach offers a new perspective on economic policy through the validity of the transversality condition. Under a balanced-budget rule, reducing the monetary expansion rate to negative enhances welfare in the full-employment steady state but, on the other hand, generates the self-fulfilling liquidity trap. In the presence of public debt issuance, a declaration of future financial collapse eliminates the liquidity trap by making the transversality condition invalid but a resulting decrease in the liquidity of public bonds may vanish all equilibria including the full-employment one. Such dilemmas make it difficult to prevent the economy from going into the liquidity trap.
Appendix A. Properties of Boundary Curves

This appendix derives the properties of boundary curves (19) and (20). First, the differentiation of (19) provides

\[
\frac{dm}{dc} = \frac{\alpha}{v''(m)\theta c} \left[ (\theta - g) - \frac{\rho \theta}{\alpha} \right],
\]

which implies that (19) is negatively (positively) sloped if \( \theta - g > (<) \frac{\rho \theta}{\alpha} \).

Next, (20) around \( \mu = 0 \) generates

\[
\frac{dm}{dc} = \frac{(\rho - \alpha)c + 2(\theta - g)\alpha}{v''(m)c^3}.
\]

Due to \( \theta - g > 0 \), the slope is always positive if \( \rho \geq \alpha \). On the contrary, if \( \rho < \alpha \), the the right-hand side of this equation is monotonically decreasing with respect to \( c \) because the numerator is positive for \( c \in (0, \theta - g] \). Hence, it takes the minimum value at \( c = \theta - g \):

\[
\frac{dm}{dc} = -\frac{\alpha + \rho}{v''(m)(\theta - g)^2} > 0 \quad \text{for} \quad c = \theta - g.
\]

Consequently, (20) always has a positive slope within the range where \( 0 < c \leq \theta - g \).

Finally, (20) is located on the left-hand side of (19). This is because \( v''(m) < 0 \) and subtracting the right-hand side of (19) from that of (20) in which \( \mu = 0 \) yields

\[
\frac{\alpha}{\theta c} \left\{ c - \left[ (\theta - g) - \frac{\rho \theta}{\alpha} \right] \right\} - \frac{1}{c} \left[ \frac{(\theta - g)\alpha}{c} + (\rho - \alpha) \right] = -\frac{\alpha(\theta + c)\left[(\theta - g) - c\right]}{\theta c^2} < 0.
\]
References


Figure 1: The monetary base and the uncollateralized overnight call rates between January 1990 to June 2014 in Japan

Source: The Bank of Japan.

Figure 2: The monetary base and the uncollateralized overnight call rates between January 1990 to June 2014 in Japan

Source: The Bank of Japan.
Figure 3: The utility of liquidity under Assumption 1.

Figure 4: The money demand function under Assumption 1.
Figure 5: Dynamics under Assumption 1 and $\theta - g > \frac{\omega^0}{\alpha}$.

Figure 6: Dynamics under Assumption 1 and $\theta - g < \frac{\omega^0}{\alpha}$. 
Figure 7: The utility of liquidity under Assumption 2.

Figure 8: The money demand function under Assumption 2.
Figure 9: Dynamics under Assumption 2, $\rho < \beta(\theta - g)$ and $\theta - g < \frac{c^g}{\alpha}$.