A dynamic calculus of consent: repeated ballots and delayed elections

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Abstract

Elections to appoint a pope/president/chief executive take place over time. We examine a model where an electorate must vote to select an alternative from a finite set of candidates. Starting from the set of candidates that receive some supports at the onset, repeated ballots take place over time and the set of candidates decreases over time until one receives (super)majoritarian support. Our motivating example is the election of the pope, and other ceos in organizations. The analysis applies generally to collective decision where constructing a majoritarian coalition is a dynamic process. We show that identifying two main subsets of alternatives - the electable candidates and the core - determines the outcome of each election. We establish conditions assuring that a unique core candidates is immediately elected. We also display electorates with a unique core candidate for which the election is delayed, and non-core candidates are elected with positive probability. For elections that are not resolved immediately, we characterize the equilibrium with tools and insights borrowed from the war of attrition literature.

Key words: Voting, Coalition Formation, Pope, Vatican Conclave, War of attrition, Repeated Ballots.

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1. Introduction

On March 12, 2013, 115 members of the College of Cardinals meet in the Sistine Chapel in Rome to choose a successor to Pope Benedict XVI. 77 votes ("duae ex tribus partes suffragiorum omnium praesentium requiruntur", a 2/3 majority) were necessary to elect a new pope. No time limit was set. Two days and five rounds of voting later, Jorge M. Bergoglio, a candidate ignored in all "top 10" bets, was elected as Pope Francis I. The Vatican conclave follows precise rules developed and refined over centuries. Those rules very explicitly describe a game of voting over time: Repeated ballots take place for as long as it takes (possibly forever) until one of the candidates attains the required (super)majority.\(^1\) Unless the Holy Spirit reveals a 2/3 super-majority at the onset of the conclave, electing a pope will require that cardinals initially giving support a candidate eventually switch to vote another. In the conclave that elected Pope Francis, according to several media accounts, in the first vote cardinals Scola and Ouellet led with roughly equal numbers of votes, Bergoglio was third, and the rest of the votes were scattered among several others. The two rounds of voting on the morning of the second day proved inconclusive. After that, Scola’s candidacy stalled and votes began to converge around the candidacies of Ouellet and Bergoglio. Sources report that at some point, Ouellet threw his support behind Bergoglio, and by the first afternoon ballot—the fourth ballot of the conclave—Bergoglio became the clear front runner. On the fifth ballot, the cardinals voted overwhelmingly in favor of Bergoglio, reportedly giving him more than 90 votes.

Although not so structured, similar procedures apply to (s)elect top-ranking appointees in organizations, such as NATO, the WTO, and many others. WTO Director-general Roberto Azevedo was selected by consensus in May 2013 after a complex and secretive process that spanned months of consultations among 159 members. The initial nine candidates narrowed to five after a first round of consultations, and only two candidates made it to the last round: Azevedo, the candidate favored by the Brics (Brazil, Russia, India, China and South Africa) defeated the Mexican candidate that had been favored by the US, Japan, Canada and the EU. The recent appointment of NATO’s Secretary General is another example. Selection is carried through informal diplomatic consultations among member countries, which put forward candidates for the post. No decision is confirmed until consensus is reached on one candidate. At the end of his 4th year term, the incumbent might be offered to stay on for a fifth year.

Sometimes the supports to make an appointment fail to materialize. Just a few weeks after

\(^{1}\) After 33 rounds only the two top vote-getters remain as candidates.
pope Francis’ election, the appointment of a successor to Italian President Giorgio Napolitano was due. Negotiations and repeated ballots dragged on for weeks, but no candidate attained the required super-majority of the Chambers. The election was unsolvable until incumbent President Napolitano, holder of the post since 2006, agreed to run for another term.

In all these situations an election ensues from a dynamic process combining votes, negotiations and concessions: There are negotiations across groups supporting different candidates, one side trying to convince another to vote differently. And there are negotiations within groups supporting a given candidate, to decide whether to yield to another, and to coordinate the timing of concessions. As negotiations develop, voters receive and transmit information. Hence voters’ disposition to change who to vote may adjust as a result.

What candidates emerge, which ones persist over time, how are votes cast and eventually converge to attain the required majority? To address these questions we formalize elections with repeated ballots as a dynamic non-cooperative game and examine its equilibrium outcomes. Our results describe foreseeable outcomes in dynamic voting process such as the Vatican conclave and provide insights that apply more generally to many other elections in which repeated ballots, and deliberations over time are important.

Our model describes the sequence of ballots as follows. A the onset, the set of candidates and the split of voters into parties supporting each candidate are known. Inside each party, members agree that their candidate is the best, but they may disagree in the ranking of what other candidate may be acceptable, and on their valuation of a compromise relative to an impasse. The members of each faction, party comrades that agree on the second acceptable candidate, meet separately to deliberate whether to withdraw support from their party candidate and favour their second, or to hold out - thereby incurring delay costs - in the hope that voters from the other parties give in. Inside each faction, over time, the more compromising individuals will express their desire to be flexible and switch vote. We refer to these individuals as dissidents. When dissidence reaches a critical level, the dissident faction defects from the party. This switch of support may be sufficient to elect an opponent candidate. Otherwise, parties change, and the ballots will continue under the new party configuration, and so on. The game goes on, forever, or until one of the candidates attains the required (super)majority $Q \geq \frac{n+1}{2}$. From the initial set of candidates and parties, groups of dissidents will sequentially switch their support, parties will be restructured, eventually leading to races with fewer candidates/parties, and so on. This process may be in quick steps, or take a
long time. As in the election of Pope Francis I, a candidate with very few supports at the onset may end up elected.

Our results are as follows.

First, we show that in equilibrium a voter strategy is independent of the actions of comrades in the same faction; she must play as she would if she alone could dictate the faction defection date. This is a drastic simplification that plays a major role in rendering the equilibrium characterization a tractable problem.

Second, we examine the basic properties of an electorate that support dominance arguments. These properties depend on the composition of two important, distinct, subsets of the candidates: the electables and the core. Electable candidates are those that are ranked first or second by at least $Q$ voters. In equilibrium, elections with a non-empty set of electables turn immediately into either a bipartisan or a tripartite election. Core candidates are those that are not beaten in bilateral choices when a decision requires $Q$ votes. We describe the variety of possible links between these two sets and show how their intersection is determinant for the equilibrium.

Third, we examine the power of dominance arguments to resolve elections. Dominance will be sufficient to solve the game for a large class of electorates; these are immediate elections. We establish conditions assuring that a unique core candidates is immediately elected. Dominance arguments are also effective to identify a tripartite elections that reduce immediately to a bipartisan election.

Fourth, for games that are not dominance solvable, we provide a precise characterization that relies on the insights and techniques of the war of attrition. Delayed elections are random realizations arising from continuous distributions of concession dates, where the most stubborn voters concede arbitrarily late. We supply precise characterizations of these. We compute delayed equilibria of bipartisan elections, and we build on this computation to characterize delayed tripartite elections. We display electorates with a unique core candidate for which the election is delayed, and non-core candidates are elected with positive probability.

Discussion of the literature here.

The rest of the paper is organized as follows. Section 2 presents the model. In section 3 we show that equilibrium strategies do not depend on faction specific histories. Section 4 studies the sets of electable and core candidates. Section 5 deals with bipartisan elections that are resolved quickly. In section 6 we discuss elections with three candidates and identify the conditions assuring dominance
solvability, i.e. immediate resolution. Section 7 discusses delayed bipartisan elections. Section 8 describes delayed tripartite elections for symmetric electorates. Section 9 displays an example with a unique core candidate where the election is delayed and electing a non-core candidate is possible.

2. A model of repeated ballots

A set of voters $i = 1, \ldots, n$, $n \geq 3$ must make an appointment from a set $\mathcal{A}$ containing $K \geq 2$ candidates. Repeated rounds of voting take place over time $t = 0, \Delta, 2\Delta, \ldots, \infty$ until a candidate obtains $Q$ votes, $Q \in \mathbb{N}_{\geq 2}$, $Q \geq \frac{n+1}{2}$. Whenever $Q = \frac{n+1}{2}$ we will assume $n$ odd.

A generic candidate is denoted by $X$, two generic candidates are denoted by $X$ and $Y$ and three generic candidates are denoted by $X$, $Y$, $Z$. These always satisfy $X \in \mathcal{A}$, $Z \in \mathcal{A}$, $Z \in \mathcal{A}$, $X \neq Y \neq Z \neq X$. The cardinality of $\mathcal{A}$ is $\#\mathcal{A}$.

Voter’s preferences are fully characterized by a pair of two acceptable candidates $r_i = (i_1, i_2)$, and a valuation $s_i > 0$. Upon election of $X$ at date $t$, voter $i$ payoffs are

$$U_i(r_i, s_i, X, t) = \begin{cases} 
(1 + s_i) e^{-t}, & \text{if } X = i_1 \\
 s_i e^{-t}, & \text{if } X = i_2 \\
0, & \text{otherwise.}
\end{cases}$$

Payoffs from perpetual disagreement are 0. Thus, voters get large positive utility when their top candidate is elected and a smaller positive utility $s_i > 0$ from electing their second acceptable candidate. And voters are impatient, so that the election of a given acceptable candidate is preferred earlier rather than later.

Pairs $r_i = (i_1, i_2)$ are publicly known. Given their top candidate, voters are initially split into $K \geq 2$ parties. Their second acceptable candidate is one of the other $K - 1$ candidates. Thus, inside each party, members are split into up to $K - 1$ factions. The party that supports candidate $X$ is denoted $X$, the number of $X$ supporters is $n_X = \#X < Q$, and $d_X$, $n_X + d_X = Q$ denotes the number of non-members necessary for $X$ to win the election. The $Y$- faction in $X$, the set of members of $X$ that rank $Y$ second, is denoted $D_{XY}$ and $n_{XY} = \#D_{XY}$. We say that faction $D_{XY}$ is strong if $n_Y + n_{XY} \geq Q$, that is, $n_{XY} \geq d_Y$ its members alone can decide the election of $Y$; otherwise we say that $D_{XY}$ is a weak faction. We say that faction $D_{XZ}$ is necessary in party $X$ if $n_X + \sum_{Y \neq X} n_{YX} \geq Q$ and $n_{XY} + \sum_{Y \neq X} n_{YX} < Q$, and otherwise we say that it is unnecessary.
We write $P$ to denote a partition of voters into parties and factions.

The value $s_i$ is private information to each voter. They are i.i.d. realizations drawn from a probability distribution $F$ with a positive density $f$ over $(0, 1]$. Thus, individuals prefer to elect their second acceptable candidate over disagreement, but for low values of $s_i$ the advantage of the second candidate over disagreement is low, and can be arbitrarily close to zero.\footnote{We think of this setup as the limit (as $\epsilon$ approaches 0) of situations where $s_i$ have positive density over $[-\epsilon, 1]$. We discuss this in more detail in section 7.}

The rules of the game are as follows. The game takes place over time $t = 0, \Delta, \ldots$ Within each period $t$, play is sequential; each period is divided into $n$ stages ($t_1, \ldots, t_n$) and player $i$ moves at $t_i$.\footnote{The sequential moves are for convenience. Under simultaneous moves results apply for appropriate refinement of equilibrium.} A subgame $h_t$, is the history of play observed by $i$ at $t$. $P(h_t)$ denotes the party-faction composition at subgame $h_t$ which is publicly observed as described next.\footnote{We write $h_t$ if the players moving at the subgame is of no relevante and no confusion arises.} Additionally each player $i \in D_{XY}$ observes the full history of moves in his faction. The game starts $t = 0$ with initial party-faction composition $P(\emptyset) = P$, at 0i voter $i \in D_{XY}$ chooses between delay or concession. If she delays she supports the party candidate for one more period. If she concedes she becomes an $Y$-dissident in party $X$; that is, she he declares - secretly to faction comrades - willingness to cast a vote for $Y$. Concessions observed within each faction remain secret as long as the number of dissidents stays below $d_Y$ or $n_{XY}$ for all $D_{XY}$. In this case, after a full round, the game moves to $t = \Delta$, with $P(h_\Delta) = P(\emptyset)$, a new round of moves takes place, where each $i$ that has not yet conceded\footnote{Concession to a candidate that has lost all supports is not allowed. Given a subgame $h_t$ we refer to voters that have not conceded at the current $P(h_t)$ (and whose second candidate party is non-empty) as active voters.}, can delay or concede, and so on. This goes on forever or until play reaches a subgame $h_t$ where the number of $Y$-dissidents from a some faction $D_{XY}$ is $d_Y$ or $n_{XY}$, at this subgame $h_t$ dissidences become public, delivering an election or a defection. When the number of $Y$-dissidents from a strong faction $D_{XY}$ reaches $d_Y$, $d_Y$ new votes go to $Y$, $Y$ is elected, and the game ends. When all $n_{XY} < d_Y$ members of a weak faction $D_{XY}$ become $Y$-dissidents at some $t$, faction $D_{XY}$ defects from party $X$, and no further moves are allowed at $t$. At $t' = t + \Delta$, the game starts again under the new party-faction composition $P(h_{t'}) = P'$ where $X$’s party is $X' = X/D_{XY}$, and $Y$’s party is $Y' = Y \cup D_{XY}$; a new round of delay or concession takes place (dissidences declared at the preceding $P$ no longer bind), and so on, until play delivers an election, another defection, or perpetual delay. Concession to a candidate that has lost all supports is not allowed. Given a subgame $h_t$ we refer to voters that have not conceded at the current $P(h_t)$ (and whose second candidate party is non-empty) as active voters.
For each \( P \) we refer to the corresponding vector \( v \) of numbers of voters in each faction as an **electorate**, and to a pair \((v, Q)\) as an **election**. Scenarios where \( K \leq 3 \) play a major role in the analysis. We will write
\[
v = ((n_{XY}, n_{XZ}), (n_{YX}, n_{YZ}), (n_{ZX}, n_{ZY}))
\]
to denote an initial electorate, and when non-active voters are present we will make it explicit and write
\[
u = ((x, n_{XY}), (x, n_{XZ}), (y, n_{YX}), (y, n_{YZ}), (z, n_{ZX}), (z, n_{ZY})).
\]

A strategy for voter \( i \), \( \tau_i \), specifies, for each of type \( s_i \), a decision to delay, or to concede for each subgames \( h_{ti} \) in which \( i \) is active. After any subgame \( h_t \), a strategy profile, \( \tau \), and a type profile, \( s \), uniquely determine an outcome: the election of candidate \( X(\tau(s, h_t)) \) at time \( t(\tau(s, h_t)) < \infty \), or perpetual disagreement. A belief system, \( \beta \) in a candidate equilibrium specifies a (joint) probability distribution over the types of the other voters, \( \beta^i(\cdot | s_{-i} | h_t) \) for each voter \( i \), conditional on \( h_t \). Given a strategy-belief profile \((\tau, \beta)\), let \( V_i(s_i, \tau, \beta, h_t) \) denote the expected payoff to voter \( i \) of type \( s_i \) conditional on his beliefs at \( h_t \).

**Definition 1 (Equilibrium).** A pair \((\tau, \beta)\) is a perfect Bayesian equilibrium (PBE) if and only if for all \( t \), \( \beta^i(\cdot | s_i, h_t) \) is consistent with \( F \) and \( \sigma \) according to Bayes’ rule, and for all \( h_t, s_i \) and \( i = 1, 2, ..., N \), \( V_i(s_i, \tau_i, \tau_{-i}, \beta, h_t) \geq V_i(s_i, \tau_i', \tau_{-i}, \beta, h_t) \) for all \( \tau_i' \).

The game is a standard extensive form game, which we have formalized in discrete time for tractability. However, as we are interested in the game where voters can react to each others moves arbitrarily fast, we think of \( \Delta \), the interval between successive periods, as arbitrarily small. Whenever no confusion arises, we will abuse language and when a candidate \( X \) is elected thanks to consecutive defections at \( t \) and \( t + \Delta \), we will say that \( X \) is elected *immediately* at \( t \). Furthermore, in sections 8 and 9, it will be convenient to focus on a continuous time formulation that will describe equilibrium strategies in the limit as \( \Delta \to 0 \).

A fundamental assumption of our formulation is that individuals do not unilaterally withdrawn support form their party candidate. What they can do is to express, secretly to their comrades, their willingness to do so; when they do they become a dissident in their faction/party. Then, only once number of dissidents reaches the threshold to determine the election, or if all faction members become dissident, their votes are "automatically" cast. This formalization, greatly simplifies the analysis and makes our model tractable. We feel that this is a good description even for situations...
where party meetings are not explicitly organized (as in the Vatican conclave). The strategic interactions will fit our model as long as individuals of the same party/faction (a) have means to communicate each other their disposition to yield, and (b) maintain party discipline as long as the set of dissidents is not the whole faction, or their number is below the threshold to call the election or restructure the party-faction composition.

Moreover, we feel that our assumption is a natural constraint that imposes no loss of generality. Consider a game where the voters can choose between a) unilaterally and publicly change their vote and b) communicate secretly to their comrades their willingness to vote differently when sufficiently many others are ready. A voter of party $X$ ready to accept the victory of $Y$ has nothing to gain by taking action a) and publicly voting $Y$ too early – when his vote is still not decisive – and he may well lose in doing so: on the one hand, a increase in the support for $Y$ can only delay the possible concession of dissidents in party $Y$, and changing sides too early excludes the dissident from the benefits of party membership if the (still) possible victory of candidate $X$ materializes. Hence, a) a dominated action. Individuals wishing to yield will express flexibility privately to their party (perhaps hoping to encourage others to join them), but would refrain from changing votes until this change is decisive.

3. Independence of Comrades

In this section we expose a powerful property of equilibrium strategies that will greatly simplify our analysis. We argue that in equilibrium a voter strategy is independent of the actions of comrades in the same faction; she must play as she would if she alone could dictate the time of defection for her faction. For 2 candidates, this observation is Proposition 1 in Ponsatí and Sákovics (1996). We argue that their insight extends directly, essentially with the same argument, to setups with more candidates.

Consider a (sub)game with party-faction composition $P$. Each active voter starts out with a vote in favor of her top candidate in $P$. Over time, the public support for each candidate remains at the initial count until dissidence in some strong faction $D_{XY}$ reaches the critical level $d_Y$ to elect candidate $Y$, or until some weak faction $D_{XZ}$ defects from party $X$. As long as $P$ remains unchanged, voters observe only dissidence moves inside their faction and choose the time at which to concede. As "negotiations" proceed, some information is transmitted across factions, but it is very limited: the pass of time reveals only that the dissidents of other factions, if any, have not
reached sufficient supports to defect. On the other hand, when a weak faction defects, this may reveal information about the types of defecting voters, but since they will no longer be active in the continuation, such information is irrelevant to evaluate expected payoffs in the continuation.

Given party composition $\mathbf{P}$ we denote by $C_{XY}(\mathbf{P})$ the set of party compositions $\mathbf{P}'$ that follow from $\mathbf{P}$ by defections of factions other than $D_{XY}$, and by $\mathbf{P}'_{XY}$ the party composition that follows when $D_{XY}$ defects. Given a strategy profile $\tau$ continuation expected payoffs for $i \in D_{XY}$ upon a defection to $\mathbf{P}' \in C_{XY}(\mathbf{P})$ at date $t$ depend on $\mathbf{P}'$, $s_i$, and $t$ (dissidences in $D_{XY}$ prior to $t$ are irrelevant). So we will denote them as $V_i(s_i, \tau, \mathbf{P}')$. Obviously if $\mathbf{P}'$ is terminal, say candidate $X$ is elected at $t$, then $V_i(s_i, \tau, \mathbf{P}') = U_i(r_i, s_i, X, t)$.

For each faction $D_{XY}$ and strategy profile $\tau$, let

$$
\Pi_{\mathbf{P}}^\tau(t, \mathbf{P}') = \Pr(\tau \text{ induces } \mathbf{P}' \text{ at } t \mid \mathbf{P}(\emptyset) = \mathbf{P})
$$

each $\mathbf{P}'$ is induced by one faction $D_{YZ}$, and hence $\Pi_{\mathbf{P}}^\tau(t, \mathbf{P}')$ depends only on $(\tau_j)$ and the distribution of types $s_j$ for $j \in D_{YZ}$.

Given strategy profile $\tau$ and an initial party-faction composition $\mathbf{P}$, the optimal unilateral plans voters defined as follows play a fundamental role in our analysis.

**Definition 2.** Fix $\mathbf{P}(\emptyset) = \mathbf{P}$ and a strategy profile $\tau$. The optimal unilateral plan of $i \in D_{XY}$ of type $s_i$ is

$$
t_{XY}(s_i, \mathbf{P}) = \arg \max_{T \geq 0} \sum_{t=0}^T \sum_{\mathbf{P}' \in C_{XY}(\mathbf{P})} \Pi_{\mathbf{P}}^\tau(t, \mathbf{P}') V_i(s_i, \tau, \mathbf{P}')
$$

$$
+ (1 - \sum_{t=0}^T \sum_{\mathbf{P}' \in C_{XY}(\mathbf{P})} \Pi_{\mathbf{P}}^\tau(t, \mathbf{P}')) s_i e^{-T}, \text{ for } D_{XY} \text{ strong},
$$

and

$$
t_{XY}(s_i, \mathbf{P}) = \arg \max_{T \geq 0} \sum_{t=0}^T \sum_{\mathbf{P}' \in C_{XY}(\mathbf{P})} \Pi_{\mathbf{P}}^\tau(t, \mathbf{P}') V_i(s_i, \tau, \mathbf{P}')
$$

$$
+ (1 - \sum_{k=0}^T \sum_{\mathbf{P}' \in C_{XY}(\mathbf{P})} \Pi_{\mathbf{P}}^\tau(t, \mathbf{P}')) V_T(s_i, \tau, \mathbf{P}'_{XY}), \text{ for } D_{XY} \text{ weak.}
$$

That is, $t_{XY}(s_i)$ is the time at which voter $i \in D_{XY}$ of type $s_i$ would chose to defect if she could choose unilaterally the defection date for her faction $D_{XY}$.
Of course, when defection by $D_{XY}$ requires $n_{XY} > 1$ or $d_Y > 1$ concessions, individual $i \in D_{XY}$ cannot choose the defection date unilaterally. At each subgame $h_{ti}$ player $i$ observes previous concessions in the same faction, and can therefore condition her plans to this information: potentially there are "internal" negotiation within each faction to organize the coalition of dissidents to jointly deliver their votes. A priori, internal faction interactions may be very complex. In equilibrium, however, they turn out to be irrelevant: the date at which an individual chooses to become a dissident is independent of the dates chosen by others in the same faction. Consequently, individuals concede at the same time they would if they were alone -controlling all the votes of the faction-facing the "conglomerate player" that aggregates the strategies in the opposite factions. We state this important observation as Proposition 1.

**Proposition 1 (Independence of Comrades).** In every PBE, for each party-faction composition, strategies specify concession times that are independent of the actions of the rest of the players in the same faction. Moreover, these concession times correspond to the ones each voter would choose if he were the dictator of their faction. That is, for $i \in D_{XY}, s_i$ and $h_{ti}$ such that $P(h_{ti}) = P$ and $t \leq t_{XY}(s_i, P)$

$$\tau_i(s_i, h_{ti}) = \begin{cases} 
\text{delay} & t < t_{XY}(s_i, P) \\
\text{concede} & t = t_{XY}(s_i, P)
\end{cases}$$

To understand why the result holds, note that, given the strategies in other factions, every player optimal unilateral plan depends solely on her type $s_i$. During the game she will not learn more about the types of voters in the other factions, since she does not observe their individual concessions and since the types in other factions are independent of comrades’ types (about which she may learn). Assume that she sees a comrade concede (these are the only observable actions as long as no faction in $P$ defects). This will not change her preferred date of defection since it does not have an effect on the behavior of voters in other factions (because they do not observe it). Thus, it is optimal for her to concede at the optimal unilateral plan date, since conceding at any other time is either outcome irrelevant or suboptimal.

Proposition 1 has major implications for equilibrium strategies: First, within each faction all voters play the same (type dependent) strategy. Consequently, at each party composition $P$, the aggregate behavior of each faction only depends on the numbers in each faction, i.e. the electorate $v$, and the realization of types. This drastic simplification plays a major role in rendering the
equilibrium characterization a tractable problem.

Building on Proposition 1 our characterization of equilibrium will proceed in two main steps: First, we will examine basic properties of the electorate that support dominance arguments. These properties will be sufficient to solve the game for a large class of electorates. Otherwise, for games that are not dominance solvable, dominance arguments simplify the game to setups where the tools of analysis for the war of attrition can be applied to characterize delayed election.

4. The electables and the core

In this section we examine the basic properties of an electorate that provide grounds for dominance arguments.

4.1. Electable candidates

A candidate $X$ cannot obtain more votes than those of all his party plus all of those in different parties that rank $X$ second, that is $\bar{v}_X = n_X + \sum_{Y \neq X} n_{YX}$. This upper bound must be above $Q$ for some candidates if the election is to be resolved at a finite date.

**Definition 3 (Electables).** Given election $(v, Q)$ candidate $X$ is electable if

$$n_X + \sum_{Y \neq X} n_{YX} \geq Q,$$

and he is unelectable otherwise. The subset of electable candidates is $E$. If $E \neq \emptyset$ we say that the election is solvable, and we say the election is unsolvable otherwise.

The following remarks are obvious, but very important.

**Proposition 2.** In an unsolvable election the unique equilibrium outcome is perpetual disagreement.

**Lemma 1 (Unelectables out).** Consider a solvable election. Fix an equilibrium. (a) Candidates that are elected with positive probability must be electable. (b) Consider $i \in D_{XY}$ an active voter moving at subgame $h_{ti}$. If $X$ is unelectable at $h_{ti}$, and $Y$ is electable, $i$ concedes. (c) Perpetual disagreement is not an equilibrium.

Next we remark that at most 3 candidates are electable:
Lemma 2 (At most 3 electables). \( \#E \leq 3 \).

**Proof.** See Appendix. 

Therefore, in a solvable election with \( K > 3 \) initial candidates, an unelectable candidate will immediately lose support. If \( K - 1 > 3 \), another unelectable candidate will immediately lose support in the continuation, and so on. Whatever the initial number of candidates, every election will immediately move to a continuation subgame where at most three parties remain relevant.

In the remainder of the paper we restrict attention to elections with \( K \leq 3 \).

Even with \( K \leq 3 \), unsolvable elections are a plausible scenario if \( Q \) is too demanding. Our next result relates values of \( Q \) to the existence, and size, of the electables.

Lemma 3 (Q and \( \#E \)). Let \( P, P' \) and \( R \) be statements

\[
\begin{align*}
P & = \{ \exists v \text{ such that } \#E = 3 \land \exists v \text{ such that } \#E = 0 \} \\
P' & = \{ \exists v \text{ such that } \#E = 3 \land \exists v \text{ such that } \#E = 0 \} \\
R & = \{ \exists v \text{ such that } \#E = 3 \land \exists v \text{ such that } \#E = 0 \}.
\end{align*}
\]

1. If \( \text{mod } (n, 3) = 0 \), then \( Q \leq \frac{2}{3}n \Rightarrow P \) and \( Q \geq \frac{2}{3}n + 1 \Rightarrow P' \).

2. If \( \text{mod } (n, 3) \neq 0 \), then \( Q \leq \lfloor \frac{2}{3}n \rfloor \Rightarrow P \), \( Q \geq \lceil \frac{2}{3}n + 1 \rceil \Rightarrow P' \) and \( Q = \lfloor \frac{2}{3}n \rfloor = \lfloor \frac{2}{3}n + 1 \rfloor \Rightarrow R \).

**Proof.** See Appendix. 

Lemma 3 already hints that the "Vatican" rule \( Q^V = \lfloor \frac{2}{3}n \rfloor = \lfloor \frac{2}{3}n + 1 \rfloor \) is special; it is is the minimal \( Q \) assuring that all elections are solvable and that no election has more than two electables.

4.2. Core candidates

Binary votes, (hypotetical) elections between two candidates, play a fundamental role in our analysis:

**Definition 4 (Binary Vote).** The number of votes that \( X \) receives if the choice is restricted to \( X \) and \( Y \) is \( v_{XY} = n_{XY} + n_{XZ} + n_{ZX} \). We write \( E(X,Y) \) to denote the outcome of this binary vote:

\[
E(X,Y) = \begin{cases} 
\{X\} & \text{if } v_{XY} \geq Q \\
\{Y\} & \text{if } v_{YX} \geq Q \\
\{X,Y\} & \text{if } v_{XY} < Q \land v_{YX} < Q
\end{cases}
\]
Note \( n_{XY} + n_{XZ} + n_{YX} + n_{ZX} = v_{XY} + n_{YZ} + v_{XZ} + n_{ZX} \). \( \{X\} = E(X,Y) \) rewrites as \( v_{XY} \geq Q \) so that \( X \in \mathcal{E} \). We state this as Lemma 4:

**Lemma 4.** \( \{X\} = E(X,Y) \Rightarrow X \in \mathcal{E} \).

**Definition 5 (Core).** A candidate is a core candidate if she is not beaten in any binary vote. The subset of core candidates is \( \mathcal{C} \).

If \( \#A = 2 \), \( \mathcal{C} = \{X\} \) if and only if \( n_X \geq Q \), and otherwise \( \mathcal{C} = \{X,Y\} \). If \( \#A = 3 \), \( X \in \mathcal{C} \) if and only if \( x \in E(X,Y) \land x \in E(X,Z) \).

There are no general straightforward inclusion relations between \( \mathcal{E} \) and \( \mathcal{C} \).

**Lemma 5.** (a) \( X \in \mathcal{E} \Rightarrow X \in \mathcal{C} \) is false. (b) \( X \in \mathcal{C} \Rightarrow X \in \mathcal{E} \) is false. (c) \( \mathcal{E} = A \) and \( \mathcal{C} \neq \mathcal{E} \) is possible.

**Proof.** See Appendix. ■

The following Lemma describes the possible scenarios.

**Lemma 6.** (a) \( \mathcal{C} = \emptyset \Rightarrow \mathcal{E} = A \). (b) \( \mathcal{E} = \emptyset \Rightarrow \mathcal{C} = A \). (c) \( \#\mathcal{E} = 1 \Rightarrow \mathcal{E} \subseteq \mathcal{C} \). (d) \( \#\mathcal{E} = 2 \Rightarrow \#\mathcal{C} \in \{1,2,3\} \), and \( \mathcal{E} \cap \mathcal{C} \neq \emptyset \). (e) \( \#\mathcal{E} = 3 \Rightarrow \#\mathcal{C} \in \{0,1,2,3\} \), and \( \mathcal{C} \neq \emptyset \Rightarrow \mathcal{E} \cap \mathcal{C} \neq \emptyset \).

**Proof.** See Appendix. ■

Summarizing Lemma 6, a solvable election must fit into one the following scenarios:

<table>
<thead>
<tr>
<th>( #C ) ( #\mathcal{E} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( \mathcal{E} = {X} )</td>
<td>( \mathcal{E} = {X,Y}, \mathcal{C} = {X} )</td>
<td>( \mathcal{E} = A, \mathcal{C} = {X} )</td>
</tr>
<tr>
<td>2</td>
<td>( \mathcal{E} = {X}, \mathcal{C} = {X,Y} )</td>
<td>( \mathcal{E} \cap \mathcal{C} = {X} ) or ( \mathcal{E} = \mathcal{C} = {X,Y} )</td>
<td>( \mathcal{E} = A, \mathcal{C} = {X,Y} )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{E} = {X}, \mathcal{C} = A )</td>
<td>( \mathcal{E} = {X,Y}, \mathcal{C} = A )</td>
<td>( \mathcal{E} = \mathcal{C} = A )</td>
</tr>
</tbody>
</table>

Next we relate \( \mathcal{C} \) and \( \mathcal{E} \) to the values of \( Q \). Proposition 3 assures that a core candidate exists for \( Q > \frac{2}{3}n \). Proposition 4 describes the possible scenarios for the minimum \( Q \), that is \( Q = \frac{n+1}{2} \).

And finally, Proposition 5 establishes that for a fixed electorate \( \mathcal{C} \) and \( \mathcal{E} \) change monotonically in \( Q \).

**Proposition 3 (\( Q > \frac{2}{3}n \) assures a core candidate).** \( Q > \frac{2}{3}n \Rightarrow \#\mathcal{C} > 0 \).
Proof. Assume, towards contradiction, that \( Q > \frac{2}{3}n \) and \( \#C = 0 \). The latter implies that all binary votes have unique outcomes that form a cycle. There are two cycles and it suffices to consider implication of one, say \( E(X, Y) = \{X\} \), \( E(X, Z) = \{Z\} \) and \( E(Y, Z) = \{Y\} \). This can be rewritten as

\[
\begin{align*}
n_{XY} + n_{XZ} + n_{ZX} & \geq Q \\
n_{ZX} + n_{ZY} + n_{YZ} & \geq Q \\
n_{XY} + n_{YZ} + n_{XY} & \geq Q
\end{align*}
\]

and sums to \( n + n_{ZX} + n_{YZ} + n_{XY} \geq 3Q \). With \( Q > \frac{2}{3}n \) we have \( n + n_{ZX} + n_{YZ} + n_{XY} \leq 2n < 3Q \), a contradiction. ■

Proposition 4 (Simple Majority). If \( Q = \frac{n+1}{2} \) then \( X \in C \Rightarrow X \in E \), \( \#C \in \{0,1\} \) and \( \#E \in \{2,3\} \).

Proof. Binary votes have a unique outcome \((n \text{ odd by assumption})\). Therefore if \( X \in C \), then \( E(X, Y) = E(X, Z) = \{X\} \) and hence \( X \in E \). To see \( \#C \in \{0,1\} \), assume \( X \in C \) and \( Y \in C \). By \( E(X, Y) \in \{\{Y\}, \{Y\}\} \) this yields immediate contradiction. To see \( \#E \in \{2,3\} \), since \( \frac{n+1}{2} \leq \frac{2}{3}n \) for \( n \geq 3 \), we know \( \#E > 0 \). Hence we need to prove that \( \#E \neq 1 \). To see that this holds note that each binary vote has a unique outcome that belongs to \( E \) and that the set of unique outcomes of all binary elections includes at least two distinct alternatives. ■

Proposition 5 (Monotonicity). Given an electorate \( v \), if \( Q' > Q \), \( E(v, Q') \subset E(v, Q) \) and \( C(v, Q') \subset C(v, Q) \).

Proof. Clearly if \( X \in E \) under \( Q \), then \( X \in E \) under \( Q' < Q \). \( X \in C \) implies that either i) \( E(X, Y) = \{X, Y\} \) and \( E(X, Z) = \{X, Z\} \), or ii) \( E(X, Y) = \{X\} \) and \( E(X, Z) = \{X, Z\} \), or iii) \( E(X, Y) = \{X, Y\} \) and \( E(X, Z) = \{X\} \), or iv) \( E(X, Y) = \{X\} \) and \( E(X, Z) = \{X\} \). If the first case applies under \( Q \), then it clearly applies under \( Q' > Q \). If the second or third case applies under \( Q \), then it either applies under \( Q' > Q \) or case one applies under \( Q' \), which potentially enlarges \( C \) by adding \( Y \) (case ii) or \( Z \) (case iii). If the fourth case applies under \( Q \), then it either applies under \( Q' > Q \) or case two or three applies under \( Q' \), which potentially enlarges \( C \) by adding \( Z \) (switch to case ii) or \( Y \) (switch to case iii). ■
5. Bipartisan immediate elections

Consider the election at a (sub)game where only two candidates run. This may be because only two candidates run initially, or because one of the 3 initial runners loses all supports.

**Lemma 7.** Assume a two-party electorate \( v = (n_{YX}, n_{YZ}, n_{ZX}, n_{ZY}) \), \( n_{YX} + n_{YZ} > 0 \) and \( n_{ZX} + n_{ZY} > 0 \). Then \( \#C \in \{1, 2\} \) and \( \#E \in \{0, 1, 2\} \). If \( \#C = 2 \), then \( E \subseteq C \). If \( \#C = 1 \), then \( C \subseteq E \).

**Proof.** See Appendix. ■

By Lemma 7, solvable elections fit in one of these 4 scenarios:

<table>
<thead>
<tr>
<th>#(C) #(E)</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E = C = {Y})</td>
<td>(E = {Y}, C = {Y, Z})</td>
<td></td>
</tr>
<tr>
<td>(E = {Y, Z}, C = {Y})</td>
<td>(E = {Y, Z})</td>
<td></td>
</tr>
</tbody>
</table>

Examples:

<table>
<thead>
<tr>
<th>#(C) #(E)</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3, 2)(1, 1), Q = 5)</td>
<td>((2, 3)(3, 1), Q = 7)</td>
<td></td>
</tr>
<tr>
<td>((3, 3)(2, 3), Q = 6)</td>
<td>((0, 4)(0, 4), Q = 5)</td>
<td></td>
</tr>
</tbody>
</table>

Hence \(E \cap C = \{Y\}\) is the necessary and sufficient condition for an immediate election, which must elect a core candidate. This is our next proposition:

**Proposition 6 (K=2, Immediate Election).** Assume a two-party electorate \( v = (n_{YX}, n_{YZ}, n_{ZX}, n_{ZY}) \), \( n_{YX} + n_{YZ} > 0 \) and \( n_{ZX} + n_{ZY} > 0 \). Candidate Y wins immediately if and only if \(E \cap C = \{Y\}\):

(a) If \(C = \{Y\}\) then the election is uncontested \((n_Y > Q)\), (b) if \(E = \{Y\}\) then the active voters of Z concede immediately. Otherwise, both Y and Z are electable-core candidates, and the election is delayed.

When the conditions for an immediate election are not met, because both parties support electable-core candidates, both candidates have a strong faction in the opposing party. The election is in fact a war of attrition. Ponsati and Sákovics (1996) supply the techniques and the insights to characterize the unique equilibrium. In equilibrium the active members of both parties play strategies in which they wait for a while before they concede: Since concession is relatively more attractive for higher \(s_i\), the date of concession depends on types. Hence, the election takes time to resolve, and both candidates can win with positive probability. We postpone the detailed discussion of this scenarios to section 7.
6. Tripartite Elections

In what follows we assume that \( A = \{X, Y, Z\} \), and \( n_{XY} > 0 \) for some \( Y \) for all \( X \in A \); that is there are 3 candidates and 3 parties. If only one candidate is electable the election is trivial. So we consider in turn elections where two or three candidates are electable.

6.1. An unelectable candidate

In elections with two electable candidates, by Lemma 6, the range of possible scenarios is as follows.

<table>
<thead>
<tr>
<th>(^{#C}) (^{#E})</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( E = C = {X} )</td>
<td>( E = {X, Y}, C = {X} )</td>
</tr>
<tr>
<td>2</td>
<td>( E = {X}, C = {X, Y} )</td>
<td>( E \cap C = {X} ) or ( E = C = {X, Y} )</td>
</tr>
<tr>
<td>3</td>
<td>( E = {X}, C = A )</td>
<td>( E = {X, Y}, C = A )</td>
</tr>
</tbody>
</table>

Lemma 8. (a)\(^{#E} = 2 \land #C = 1 \) implies that when voters in party \( A \setminus E \) concede, then the candidate in \( C \) wins. (b)If \( #E = 2 \land #C = 2 \), then either \( E = C \) or \( E \neq C \). In the latter case, when voters in party \( A \setminus E \) concede, then the candidate in \( C \cap E \) wins.

Proof. See Appendix. ■

We are now ready to characterize the equilibrium in three party elections where \( #E = 1, 2 \).

Proposition 7 (Equilibrium Outcomes when \( #E = 1, 2 \)). Consider elections with \( #E = 1, 2 \). Fix an equilibrium. (a) If \( E \cap C = \{X\} \) then \( X \) is elected immediately. (b) If \( E \cap C = \{X, Y\} \) then \( Z \) loses supports immediately and a delayed 2 party election follows.

Proof. (a) The case \( #E = 1 \) is trivial. For \( #E = 2 \) the results follow from Lemma 8.

(b) Since \( Z \) is unelectable all voters of party \( Z \) concede immediately. The two party continuation is a war of attrition. ■

Proposition 7 is quite powerful. For \( Q > \frac{2}{3} n \) it describes equilibrium outcomes for all solvable electorates. Recall that Lemmas 3 and 3, imply that for \( Q > \frac{2}{3} n \) the range of possible scenarios is
restricted to the following:

\[
\begin{array}{c|c|c}
\#C \setminus \#E & 1 & 2 \\
\hline
1 & \mathcal{E} = \mathcal{C} = \{X\} & \mathcal{E} = \{X, Y\}, \mathcal{C} = \{X\} \\
2 & \mathcal{E} = \{X\}, \mathcal{C} = \{X, Y\} & \mathcal{E} \cap \mathcal{C} = \{X\} \text{ or } \mathcal{E} = \mathcal{C} = \{X, Y\} \\
3 & \mathcal{E} = \{X\}, \mathcal{C} = \mathcal{A} & \mathcal{E} = \{X, Y\}, \mathcal{C} = \mathcal{A} \\
\end{array}
\]

Consequently, the following is immediate.

**Proposition 8 (Pope Election).** Consider the pope election \( Q^V = \left\lceil \frac{5}{3}n \right\rceil \) (assume \( \text{mod} \ (n, 3) \neq 0 \)). Then either there is a unique electable-core candidate who is elected immediately, or there is a pair of electable-core candidates and the election immediately moves into a delayed election between this pair.

**Proof.** By Lemma 3 \#E \in \{1, 2\} and by Lemma 3 \( \mathcal{E} \cap \mathcal{C} \neq \emptyset \) and the result follows by Proposition 7. ■

Proposition 7 also has important implications for elections under \( Q \leq \frac{2}{3}n \). Electorates where Proposition 7 applies for the simple majority rule are remarkably simple, they are described in Proposition 9. And Proposition 10 describes electorates where Proposition 7 applies for intermediate supermajorities \( \frac{n+1}{2} < Q \leq \frac{2n}{3} \).

**Proposition 9 (#E < 3, Simple Majority).** Consider elections with \( Q = \frac{n+1}{2} \). If not all candidates are electable, then \( \mathcal{E} = \{X, Y\}, \mathcal{C} = \{X\} \) and \( X \) is elected immediately.

**Proof.** By Proposition 4 the possible scenarios for \( Q = \frac{n+1}{2} \) are only

\[
\begin{array}{c|c|c}
\#E \setminus \#C & 0 & 1 \\
\hline
2 & \mathcal{E} = \{X, Y\}, \mathcal{C} = \{X\} \\
3 & \mathcal{E} = \mathcal{A}, \mathcal{C} = \emptyset & \mathcal{E} = \mathcal{A}, \mathcal{C} = \{X\} \\
\end{array}
\]

Therefore \( \mathcal{E} = \{X, Y\}, \mathcal{C} = \{X\} \), and the immediate election of \( X \) follows by Lemma 8. ■

**Proposition 10 (#E < 3, Intermediate Q).** Consider an election \((v, Q)\) such that \#E(v, \frac{n+1}{2}) < 3 and \( Q > \frac{n+1}{2} \). If \( v \) is solvable under \( Q \) then \#E(v, Q) \leq 2 , \#C(v, \frac{n+1}{2}) \in \{1, 2\}, \) and \( \mathcal{E} \cap \mathcal{C} \neq \emptyset \).

Therefore either (a) \( \mathcal{E} \cap \mathcal{C} = \{X\} \) and \( X \) is elected immediately, or (b) \( \mathcal{E} = \mathcal{C} = \{X, Y\} \), so that \( Z \) loses supports immediately and a delayed 2 party election follows.
Proof. Since \( \#\mathcal{E}(v, \frac{n+1}{2}) < 3 \), then \( \mathcal{E}(v, \frac{n+1}{2}) \subset \{X, Y\} \) and \( \mathcal{C}(v, \frac{n+1}{2}) = \{X\} \). For \( Q > \frac{n+1}{2} \), by Lemma 5, the set of electables must be included in \( \{X, Y\} \) and the core must include \( X \). Hence Proposition 7 applies and the result follows immediately.  

6.2. All electable

Let us now address elections where all candidates are electable \( \mathcal{E} = \mathcal{A} \).

Recall that a weak faction is one that does not have enough votes to elect their second candidate. When a weak faction defects, the continuation is a subgame with 3 parties with restructured party-faction memberships. Our equilibrium characterization cannot ignore strategies at these subgames since they arise with positive probability: in a party with two distinct factions at least one is a weak. This observation is the following Lemma.

Lemma 9 (Weak Factions Exist). If faction \( D_{XY} \) is strong and \( D_{XZ} \neq \emptyset \), then \( D_{XZ} \) is weak.

Proof. See Appendix.  

Next we note that a weak faction that is necessary never defects:

Proposition 11 (Weak Necessary Factions do not Defect). Consider an election where faction \( D_{ZX} \) is weak and necessary (and \( D_{ZY} \) is strong). Fix equilibrium profile \( \sigma \). For all \( i \in D_{ZX} \) all \( s_i \) and all subgames \( h_i \) with the initial party structure, \( i \) delays.

Proof. Because \( D_{ZX} \) is necessary to elect \( Z \), after \( D_{ZX} \) concedes, \( Z \) becomes unelectable. Hence faction \( D_{ZY} \) immediately concedes and elects \( Y \), the worse possible outcome for \( i \). Hence concession is a dominated action for \( i \in D_{ZX} \), regardless of \( s_i \).  

As a consequence of Proposition 11 identifying the weak necessary factions in an electorate is a crucial step in the analysis of equilibrium. The following two lemmas constraint the possible scenarios.

Lemma 10 (Symmetric Factions). Take a pair of factions with the same first and second alternative, except in reverse order. Then these factions can be neither both unnecessary nor, if in addition \( Q = \frac{n+1}{2} \), necessary.

Proof. See Appendix.  

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Lemma 11 (Weak factions and Empty Core). If \#C = 0 and all factions are non-empty, then there is exactly one strong faction and exactly one weak faction in each party and, and least one of the weak factions in \( v \) is unnecessary.

Proof. See Appendix. ■

Simple majority Consider elections under simple majority rule.

By Proposition 4 there are two possible scenarios, either (a) \( E = A, C = \emptyset \) or (b) \( E = A, \#C = 1 \). We will describe the possible faction compositions in each case, and their equilibrium implications.

Lemma 12 \((Q = \frac{n+1}{2} \text{ and } E = A \Rightarrow 3 \text{ strong and } \geq 2 \text{ weak})\). Assume \( Q = \frac{n+1}{2} \), \( E = A \), and \#C = 1. Then there are three non-empty strong factions and either two or three non-empty weak factions. Moreover

1. If there are two non-empty weak factions, then there exist two alternatives \( X \) and \( Y \), such that \( X \in C, E(Y, Z) = \{Y\}, n_{ZY} = 0 \), the set of weak factions is \( \{n_{XZ}, n_{YZ}\} \) and \( n_{YZ} \) is necessary.

2. If the are three non-empty weak factions, then if \( X \in C \) the two weak factions in \( Y \) and \( Z \) are neither both necessary nor both unnecessary.

Proof. See Appendix. ■

Lemma 13 (Electorates for \( Q = \frac{n+1}{2}, E = A, C = \{Z\} \)). Assume \( Q = \frac{n+1}{2}, E = A \) and \( C = \{Z\} \). Then there are at least two non-empty weak factions, and either one or two are necessary.

Proof. See Appendix. ■

Hence, for \( Q = \frac{n+1}{2} \) electorates where \( E = A \) and \( C = \{Z\} \) are one of the following 4:

<table>
<thead>
<tr>
<th></th>
<th>strong</th>
<th>weak</th>
<th>w. nec.</th>
<th>Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_{21} )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>Delay</td>
</tr>
<tr>
<td>( v_{22} )</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>( D_{XZ} ) defects, ( Z ) elected</td>
</tr>
<tr>
<td>( v_{31} )</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>Delay</td>
</tr>
<tr>
<td>( v_{32} )</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>( D_{XZ} ) defects, ( Z ) elected</td>
</tr>
</tbody>
</table>
Proposition 12 (Simple Majority and $C = \{Z\}$, Equilibrium). Let $Q = \frac{n+1}{2}$ and assume an electorate such that $E = A$ and $C = \{Z\}$. Fix an equilibrium. (a) If there are two weak necessary factions, the unique equilibrium outcome is immediate election of $Z$. (b) Otherwise, the election is delayed with positive probability and candidates other than $Z$ are elected with positive probability.

Proof. i) Assume an electorate $v_{22}$, where there is party with a single faction $X = D_{XZ}$. At subgames with the initial electorate electorate, the 2 weak necessary factions never defect, and consequently $X$ cannot be elected. Therefore all $i \in D_{XZ}$ must concede, $D_{XZ}$ defects and elects $Z$. ii) Assume an electorate $v_{32}$, where there is a weak unnecessary faction $D_{XY}$. We claim that $D_{XY}$ must not defect from the initial electorate. Assume otherwise, and consider the electorate induced by $D_{XY}$ defection, say $v'$. At $v'$ the original weak necessary factions remain weak and necessary and therefore do not defect; hence $D_{XZ}$ immediately elects $Z$. Hence, defection by $D_{XY}$ (indirectly) elects $Z$. Therefore, at $v_{32}$, concession is a dominated action for all $i \in D_{XY}$, proving the claim. Since no weak faction (necessary or unnecessary) defects from $v_{32}$, $X$ cannot be elected, then $D_{XZ}$ must defect, electing $Z$. This proves (a).

We prove (b) by means of the following examples.

Example 1 (Non-Core candidate may be elected 1). $n = 7, Q = 4, u = ((1, 1, 1), (1, 1), (0, 2))$. $E = A$ and $C = \{Y\}$: The core candidate $Y$ is not elected immediately.

Proof. Observe that $E(X, Y) = Y, E(Y, Z) = \{Y, Z\}, E(X, Z) = X$, so that $C = \{Y\}$. Faction properties are $(wu, wu, sn, wn, \emptyset, sn)$.

(a) Note that $D_{YZ}$ is $wu$ while $D_{YX}$ is $sn$, hence $D_{YZ}$ must stay firm.

(b) Faction $D_{XZ}$ is $wu$, and must stay firm. To prove this claim, will argue that concession is a dominated action for $i \in D_{XZ}$ regardless of $s_i$. Consider the electorate after $D_{XZ}$ defects, $w = ((1, 1, 0), (1, 1), (1, 0, 2))$. Note that $X$ is not electable in $w$, so that $D_{XY}$ defects and elects $Y$. Hence, since defection by $D_{XZ}$ leads to $Y$ election, concession is dominated.

(c) Faction $D_{XY}$ is $wu$, and defects with positive probability. If $D_{XY}$ concedes the new electorate is $w' = ((1, 0, 1), (1, 1, 1), (0, 2))$, where $X$ is unelectable; therefore $D_{XZ}$ must defect and the result is $w'' = ((1, 0, 0), (1, 1, 1), (1, 0, 2))$. In $w''$ a 2-party election over $Y$ and $Z$, $Z$ elected with positive probability. Since $D_{YZ}$ and $D_{XZ}$ stay firm at $u$, if $D_{XY}$ stays firm forever, $Z$ cannot be elected, and $D_{ZY}$ elects $Y$. Hence $i \in D_{XY}$ is better off if $D_{XY}$ defects from $u$, and the game moves to $w''$ where $Z$ is elected with positive probability.
Example 2 (Non-Core candidate may be elected 2). Let $n = 7, Q = 4, v = ((1, 1), (1, 1), (1, 2))$. The unique core candidate $Z$ is not elected immediately.

Proof. First let us check that $E = A, C = \{Z\}$. Candidates $X$ and $Y$ need 2 votes, candidate $Z$ needs only 1 vote. All candidates are electable: $X$ can obtain 4 votes, $Y$ and $Z$ can obtain 5 votes. To see that $C = \{Z\}$ observe that $E(X, Y) = Y, E(Y, Z) = Z, E(X, Z) = Z$. Next note that the properties of factions are $(wn, sn, wu, su, wu, sn)$. Now observe the following:

(a) Note that $D_{XY}$ is $wn$ while $D_{XZ}$ is $sn$, hence $D_{XY}$ must stay firm.

(b) Faction $D_{YX}$, is $wu$, and stays firm. To prove this claim we argue that concession is a dominated action for all $i \in D_{YX}$ regardless of $s_i$. The electorate upon $D_{YX}$ defecting, is $v' = ((1, 1, 1), (0, 1), (1, 2))$, where factions are $(wn, sn, \emptyset, sn, sn, wn)$ and $wn$ factions $D_{XY}$ and $D_{ZY}$ stay firm. The consequence is that at $v'$, $Y$ cannot be elected, so $D_{YZ}$ must concede and elect $Z$. Hence, concession is a dominated action for $i \in D_{YX}$.

(c) Faction $D_{ZX}$, is $wu$, and defects with positive probability. Consider the electorate if $D_{ZX}$ defects, that is $u = ((1, 1, 1), (1, 1), 0, 2)$. In Example 1 we have shown that immediate election of $Y$ is not the equilibrium outcome at $u$.

Next we examine electorates where the core is empty.

Lemma 14 (Electorates for $Q = \frac{n+1}{2}$, $E = A, C = \emptyset$). Assume $Q = \frac{n+1}{2}, E = A$ and $C = \emptyset$. There are three strong and necessary factions, and the weak factions can be any combination, except three non-empty and necessary.

Proof. See Appendix.

Proposition 13 (Simple majority and $C = \emptyset$, Equilibrium). Assume $Q = \frac{n+1}{2}$ and an electorate such that $E = A$ and $C = \emptyset$. At subgames with the initial electorate, voters in weak necessary factions never concede, and voters in strong and weak unnecessary factions play a type monotone concession strategy. When weak factions defect continuations are still 3 party elections.

Supermajorities For $\frac{n+1}{2} < Q \leq \frac{2n}{3}$ core scenarios range from $C = \emptyset$ to $C = A$.

Example 3 (No Strong Factions $E = C = A$). $n = 30, Q = 20, v = (1, 1, 9, 1, 9, 9)$. Observe that $E(X, Y) = \{X, Y\}, E(Y, Z) = \{Y, Z\}, E(X, Z) = \{X, Z\}$, so that $C = \{X, Y, Z\}$. All factions are weak and necessary. All faction defections induce party collapse and the continuation is a delayed two party election.
To complete our characterization of equilibria, we next turn our attention to delayed elections.

7. Bipartisan delayed elections

In this section we examine 2-party electorates that do not deliver an immediate election in equilibrium.

Assume (active) voters are split into two non-empty parties, \( X \) and \( Y \). Both candidates are electable \( n_X + n_{XY} \geq Q \) and \( n_Y + n_{XY} \geq Q \), and both are core candidates i.e. neither has sufficient supports \( n_X < Q \) and \( n_Y < Q \). If all voters are active, then \( n_X = N - n_Y < Q \); hence if \( Q = \frac{n+1}{2} \) the election is trivially resolved at \( t = 0 \). What follows applies to a) elections where only two candidates run initially and \( Q > \frac{n+1}{2} \); and b) subgames after one party collapses and the remaining two parties still have memberships below \( Q \geq \frac{n+1}{2} \).

Types are i.i.d. realizations drawn from a probability distribution \( F \) with a positive density \( f \) over \([0, a]\) for party \( X \) and \([0, b]\) for party \( Y \).\(^6\)

Since only two candidates run, the game will end at the first defection. Hence the strategic choice of active voters reduces to choosing the time at which to concede. To describe setups where the interval between successive periods is arbitrarily small, it is convenient to formulate the game in continuous time \( t \in [0, \infty) \). This is the setup of Ponsatí and Sákovics (1996) from where we borrow the tools to compute the equilibrium.

Strategy \( \tau_i(s_i) \) for \( i \in X \) selects the date to concede to \( Y \) as function of type \( s_i \). Since voters do not have dominant actions, they choose strategies that depend on their types, voters are more impatient the higher their \( s_i \); so in equilibrium they concede later the lower their \( s_i \) value. In equilibrium the following properties must hold:\(^7\):

E1 \( \tau_i(s_i) < \infty, \forall s_i > 0 \forall i \) active.

E2 \( \tau_i(s_i) \) differentiable, strictly decreasing, and \( \lim_{s \to 0} \tau_i(s_i) \rightarrow \infty \).

E3 The probability of defection at \( t = 0 \) is positive for at most one party.

\(^6\) We write \( F(.) \) to denote the cumulative distributions of types in either side. This is an abuse of notation, but arguments inside \( F \) are sufficiently informative.

\(^7\) See Ponsatí and Sákovics (1995) and (1996). Recall that \( s_i \) are i.i.d. drawn from a probability distribution \( F \) with positive density \( f \) over \([0, 1]\). Equilibria in strictly monotone strategies are robust to a minor perturbation of this information structure: Assume instead \( F \) has a positive density \( f \) over \([-\epsilon, 1]\), for a vanishing small \( \epsilon > 0 \). Thus, most individuals prefer electing their second candidate over disagreement, but there is a very small possibility that they will never vote for his second candidate. Our results will focus on the limit as \( \epsilon \) approaches 0.
Fix an equilibrium strategy profile $\tau$. Define $H_X(t) = \text{Prob}(X \text{ elected in } [0, t) \text{ under } \tau)$, when no confusion arises we will write $H_X(t)$. Since strategies are strictly decreasing in type and continuously differentiable, $H_X(t)$ is strictly increasing and continuously differentiable in $t$; let $H'_X(t) = h_X(t)$.

By Proposition 1, voters behave as if they were alone -controlling all the votes of the party-facing the "conglomerate player" that aggregates the strategies in the opposite party. That is, $\tau_i(s_i) = \tau_X(s_i)$ for all $i \in X$, where $\tau_X(s_i)$ is the "optimal unilateral plan" of a voter $i \in X$ given that voters $j \in Y$ play according to $\tau$ that is:

$$\tau_X(s_i) \equiv \arg \max_{t \in [0, \infty)} \int_0^t (1 + s_i)e^{-\tau}h_X(\tau)d\tau + (1 - H_X(t))e^{-\tau}s_i.$$ 

Thanks to the continuous time formulation optimal unilateral plans are easily described by first order conditions: the necessary first order condition that type $s_i$ conceeding at $\tau_X(s_i) = t > 0$, must satisfy is:

$$(1 + s_i)e^{-t}h_X(t) - (1 - H_X(t))e^{-t}s_i - h_X(t)e^{-t}s_i = 0,$$

that simplifies to

$$h_X(t) = (1 - H_X(t))s_i.$$  \hspace{1cm} (7.1) 

Let us next compute $1 - H_X(t) = \text{Prob}(X \text{ not elected in } [0, t) \text{ under } \tau)$. $X$ is not elected in $[0, t)$ if and only if $\tau_Y(s_i) > t$, or equivalently, if and only if $s_i > \sigma_Y(t)$, (where $\sigma_Y(t)$ denotes the inverse of $\tau_Y$ ) for at most $d_X - 1$ active voters of party $Y$,where $a_Y \leq \# Y$ is the number of active voters in party $Y$. So we can write $1 - H_X(t)$ as:

$$1 - H_X(t) = \sum_{k=0}^{d_x-1} \binom{a_Y}{k} F(\sigma_Y(t))^{a_Y-k} (1 - F(\sigma_Y(t)))^k.$$ 

Differentiating $H_X(t)$ we get

$$h_X(t) = -\sigma'_Y(t)f(\sigma_Y(t)) \sum_{k=0}^{d_x-1} \binom{a_Y}{k} F(\sigma_Y(t))^{a_Y-k-1}(1 - F(\sigma_Y(t)))^{k-1}(a_Y(1 - F(\sigma_Y(t))) - k)$$

$$= -\sigma'_Y(t)f(\sigma_Y(t)) \frac{a_Y}{d_X} d_X F(\sigma_Y(t))^{a_Y-d_X} (1 - F(\sigma_Y(t)))^{d_X-1}$$

$$= -\sigma'_Y(t)f(\sigma_Y(t)) \frac{a_Y - 1}{d_X - 1} a_Y F(\sigma_Y(t))^{a_Y-d_X} (1 - F(\sigma_Y(t)))^{d_X-1}$$

Substituting in condition (7.1) we conclude that strategies must satisfy the differential equation
The following is immediate.

Proposition 14 (2 party, uniform distributions). Consider a two-party election where the types of active players are independent draws from uniform distributions, with supports $[0, a]$ for party $X$ and $[0, b]$ for party $Y$. In the unique equilibrium $\tau_X(x_i) = t$ if and only if $x_i = x(t)$ and $\tau_Y(y_i) = t$ if and only if $y_i = y(t)$ where $x(\cdot)$ and $y(\cdot)$ solve

\[
\frac{dy}{dx} = \frac{xf(x)\binom{a_x}{dx} F(x)^{ax-dy} (1 - F(x))^{dy-1} \sum_{k=0}^{dy-1} \binom{ay}{k} F(y)^{ay-k} (1 - F(y))^k}{yf(y)\binom{a_y}{dy} F(y)^{ay-dy} (1 - F(y))^{dy-1} \sum_{k=0}^{dy-1} \binom{ax}{k} F(x)^{ax-k} (1 - F(x))^k}
\]

with boundary value $\rho(0) = 0$ (by properties E1-E2).

Property E3, requires that at $t = 0$ a defection may occur with strictly positive probability at most on one side $H_X(0)H_Y(0) = 0$, implies

\[
(1 - \sigma_X(0))(1 - \sigma_Y(0)).
\]

which in turn implies

\[
\sigma_X(0) = a, \sigma_Y(0) = \rho(a), \text{ if } \rho(a) \leq b; \tag{7.5}
\]

\[
\sigma_X(0) = \rho^{-1}(b), \sigma_Y(0) = b, \text{ if } \rho(a) > b.
\]

Hence, equilibrium strategies must solve (7.2) with boundary condition (7.5).

The following is immediate.
where $\rho$ solves $\rho(0) = 0$ and

\[
\frac{dy}{dx} = \frac{\binom{a_X}{d_Y} d_Y x^{a_X-d_Y+1} (a-x)^{d_Y-1} \sum_{k=0}^{d_X-1} y^{a_Y-k} (a_Y)^k (b-y)^k}{\binom{a_Y}{d_X} d_X y^{a_Y-d_X+1} (b-y)^{d_X-1} \sum_{k=0}^{d_Y-1} x^{a_X-k} (a_X)^k (a-x)^k}.
\] (7.8)

**Proof.** With $F(x) = \frac{x}{a}$ and $F(y) = \frac{y}{b}$, equation (7.2) simplifies to (7.6) and (7.3) simplifies to (7.8). Checking that strategies so constructed are indeed an equilibrium is immediate. □

For simple set-ups we can compute strategies in closed form:

**Example 4 (A single voter ends election).** Let $N = 7$, $Q = 4$, $w = ((1, 0, 2), (1, 1, 1), (1, 0, 0))$, types $x$ and $y$ uniformly distributed in $[0, a]$ and $[0, b]$. Since $d_X = d_Y = 1$, $a_X = 2$, $a_Y = 1$ condition (7.6) simplifies to

\[
\frac{y'(t)}{x(t)} = -\frac{\sum_{k=0}^{1-1} \binom{1}{k} y^{1-k} (b-y)^k}{\binom{1}{0} y^0 (b-y)^0} = -y
\]

\[
\frac{x'(t)}{y(t)} = -\frac{\sum_{k=0}^{1-1} \binom{2}{k} x^{2-k} (a-x)^k}{\binom{2}{1} x^{2-1} (a-x)^0} = \frac{1}{2} x
\]

which imply

\[
\rho'(x) = \frac{dy}{dx} = 2, \rho(0) = 0, \Rightarrow \rho(x) = 2x.
\]

Solving

\[
\frac{dx}{dt} = -x^2,
\]

we obtain

\[
\sigma_X(t) = \frac{1}{2 + t}, \tau_X(s) = \frac{1}{s} - 2,
\]

\[
\sigma_Y(t) = \frac{2}{2 + t}, \tau_Y(s) = \frac{2}{s} - 2.
\]

If both sides are uniform in $[0, 1]$ party $Y$ is elected at $t = 0$ with probability

\[
H_Y(0) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.
\]

After $t = 0$, if $\min(2x_1, 2x_2) = \underline{x} < y$, $Y$ is elected at $\tau_X(\underline{x})$, otherwise $X$ is elected at $\tau_Y(y)$.

Our next example is an asymmetric 2-party election where some voters are inactive. We can compute $\rho^{-1}$ in closed form.
Example 5 (An asymmetric 2-party electorate). Assume $n = 9, Q = 6$ and $v = ((3, 1), (2, 3))$.

Susbituting $a_X = 1, d_X = 2, a_Y = 2, and d_Y = 1$ in condition (7.6) we obtain

\[
\begin{align*}
\frac{y'}{x} &= -\sum_{k=0}^{2-1} \binom{2}{k} y^{2-k} (b - y)^k = -\frac{1}{2} 2b - \frac{y}{b - y} \\
\frac{x'}{y} &= -\sum_{k=0}^{1-1} \binom{a_X}{k} x^{1-k} (a - x)^k = -x
\end{align*}
\]

and therefore $\rho$ must solve

\[
\rho'(x) = \frac{dy}{dx} = \frac{(2b - y)}{2 (b - y)}.
\]

With the change of variable $y = b - z$, the differential equation can be rewritten as

\[
\begin{align*}
\frac{dz}{dx} &= -\frac{(b + z)}{2z} \\
\frac{dx}{dz} &= -\frac{2z}{b + z}
\end{align*}
\]

that has general solution

\[
x = K - 2z + 2b \ln (b + z);
\]

and with condition $x(0) = 0$ we obtain the inverse of $\rho$:

\[
\rho^{-1}(y) = x(y) = 2 \left( y + b \ln \left( \frac{2b - y}{2b} \right) \right).
\]

For $b = 1$, $\rho^{-1}$ is $x(y) = 2 \left( y + \ln \left( \frac{2 - y}{2} \right) \right)$. 

Since $x(1) = 2 \left(1 + \ln \left(\frac{1}{2}\right)\right) = 0.61371$, we conclude that in equilibrium $Y$ is elected at $t = 0$ with probability $1 - 0.61371 = 0.38629$.

Example 5 will be the major stepping stone to compute the equilibrium for Example 7, a 3-party electorate where $n = 9, Q = 6$, and $\mathbf{v} = ((1, 0), (2, 3), (3, 0))$. Indeed, if faction $D_{ZX}$ defects party $Z$ disappears and the continuation is Example 5.

8. Symmetric tripatite elections

Consider symmetric electorates $\mathbf{v} = ((k, k), (k, k), (k, k)), n = 6k, 3k + 1 \leq Q \leq 4k$.

It is immediate that $\mathcal{E} = \mathcal{C} = \mathcal{A}$. All factions are weak and necessary. Hence any defection from the initial electorate, is immediately followed by defection of the other faction in the same party, and thus collapses to a 2 party delayed election $((k, k), (k, k, k))$.

Consider an equilibrium. We write $\tau_X(s_i)$ with inverse $x(t)$ to denote the concessions from initial electorate. Initial concession at $t$ yields 2 party delayed election with types $\mathcal{U}[0, x(t)]$. We write $\tau_{x^a}(s_i)$ with inverse $x^a(t)$ to denote the concession in the 2 party electorate with types in $\mathcal{U}[0, a]$. Let $V_{kl}(a, s_i)$ denote value to player $s_i$ of getting into 2 party subgame with support $[0, a]$ when candidates $kl \in \{12, 13, 23\}$ compete.

Example 6 (3-party, symmetric electorate). Let $n = 6, Q = 4, \mathbf{v} = ((1, 1), (1, 1), (1, 1))$, and assume types are i.i.d. uniformly distributed in $[0, 1]$. The election is delayed and all three candidates

\[\text{For } Q > 4k, \mathcal{E} = \emptyset \text{ and the election is unsolvable.}\]
are elected with positive probability.

(Insert Figures 1 and 2 here)

**Continuation 2-party subgames**

Consider subgame \(((1, 1, 1) (1, 1, 1))\) with support \([0, a]\). The first order conditions are

\[
-y^a(t) = x^a(t)y^a(t),
\]

\[
-x^a(t) = x^a(t)y^a(t),
\]

with boundary condition \(x^a(0) = y^a(0) = a\) the solution is

\[
x^a(t) = \frac{a}{1 + at},
\]

\[
\tau_X^a(s) = \frac{a - s}{as}.
\]

The probability of victory for each candidate is \(\frac{1}{2}\) and the mean wait to the end of the game is \(\frac{1}{a}\).

Continuations values are:

\[
V_{12}(a, s) = \frac{(1 + 2s) \left(a(a - 1) - e^{\frac{s}{a}} A\left(-\frac{1}{a}\right)\right)}{2a^2}
\]

\[
V_{13}(a, s) = \frac{(1 + s) \left(a(a - 1) - e^{\frac{s}{a}} A\left(-\frac{1}{a}\right)\right)}{2a^2}
\]

\[
V_{23}(a, s) = \frac{s \left(a(a - 1) - e^{\frac{s}{a}} A\left(-\frac{1}{a}\right)\right)}{2a^2}
\]

\[
\frac{\partial}{\partial a} V_{23}(a, s) = V_{23}'(a, s) = \frac{s \left(a + a^2 + (1 + 2a)e^{\frac{s}{a}} A\left(-\frac{1}{a}\right)\right)}{2a^4}
\]

where \(A = -\int_{-z}^{\infty} \frac{e^{-u}}{u} du\) is exponential integral function.

**Initial 3-party electorate**
Player with type $x_i$ (w.l.o.g. member of faction $D_{11}$) maximizes

$$
\int_0^t e^{-u} [a + b + c] du + e^{-t} \cdot d \cdot e
$$

$$
a = -x^4 \cdot V_{23}(x(u), x_i)
$$
$$
b = -2x^4 \cdot V_{13}(x(u), x_i)
$$
$$
c = -2x^4 \cdot V_{12}(x(u), x_i)
$$
$$
d = x(t)^5
$$
$$
e = V_{23}(x(t), x_i)
$$

After some simplification, dropping $(t)$ from $x(t)$ notation and substituting $x_i = x(t) = x$ the first order condition writes is

$$
4x^4V_{23}(x, x) - 2x^4V_{13}(x, x) - 2x^4V_{12}(x, x) - x^5V_{23}(x, x) + x^5V_{23}'(x, x) = 0,
$$

and the solution needs to satisfy $x(0) = a$. This system cannot be solved for $x(t)$, but for $\tau_X$ the solution can be obtained in ‘closed form’. It is

$$
\tau_X(s) = \frac{4(a - s)}{as} + 4 \log \left( \frac{a}{s} \right) + \log \left( \frac{v(s)}{v(a)} \right)
$$

(8.1)

where $v(z) = z^2 - z - e^{\frac{1}{z}}A(-\frac{1}{z})$. $\lim_{s \to a} \tau_X(s) = 0$ is almost immediate, and $\lim_{s \to 0} \tau_X(s) = \infty$.

(Insert Figures 3 and 4 here).

9. Delay and election of a non-core candidate

Next we compute the equilibrium strategies for an example where a non-core candidate is elected with positive probability.

Example 7 (3-party, non-core elected). Let $n = 9, Q = 6$, $v = ((1, 0), (2, 3), (3, 0))$, and assume types are i.i.d. uniformly distributed in $[0, 1]$. The unique core candidate $Y$ is elected with at $t = 0$ if $i \in D_{ZX}$ is of type $x_i \geq .61$. Otherwise the election is delayed and all three candidates are elected with positive probability.

First check that $E = A$, and $C = \{Y\}$ because $E(X, Y) = \{X, Y\}, E(Y, Z) = \{Y\}, E(X, Z) = \{Z\}$ and note that factions are $(sn, \emptyset, wn, sn, wn, \emptyset)$.
Fix an equilibrium profile and consider concession plans from electorate $v$:

1. Since $D_{YX}$ is wn while $D_{YX}$ is sn, concession is a dominated action for $i \in D_{YX}$. Hence $D_{YX}$ never defects from $v$.

2. For $i \in D_{ZX}$ "always delay" for all types cannot be an equilibrium: If $D_{ZX}$ never defects, $X$ cannot be elected; consequently $D_{XY}$ would elect $Y$ immediately. Hence, in equilibrium $D_{ZX}$ defects with positive probability; that is, every $z_i > 0$ must concede at some finite date $\tau_Z(z_i)$.

3. For factions $D_{XY}$ and $D_{YZ}$ defection terminates the game, electing $Y$ and $Z$ respectively. Since the election of the top candidate is not assured, every type $x_i, y_i > 0$ must concede at some finite date $\tau_X(x_i), \tau_Y(y_i)$.

Standard arguments establish that, $\tau_X(x_i), \tau_Y(y_i)$ and $\tau_Z(z_i)$ are decreasing and differentiable; let $x(t), y(t)$ and $z(t)$ denote their inverses.

2-Party continuation Payoffs To compute optimal dates for concession from $v$, continuation payoffs at 2-party subgames must be evaluated. Such subgames arise only if $D_{ZX}$ defects inducing the 2-party electorate $v' = ((3, 1), (2, 3))$, at some $t \geq 0$. At this continuation types are distributed over $[0, a] = [0, x(t)]$ (because $i \in D_{XY}$ concedes from $v$ following $x(t)$) and $[0, b] = [0, 1]$ (because $i \in D_{XY}$ concedes from $v$ following $x(t)$).

Let $V(x_i, a), V(y_i, a), V(z_i, a)$ denote the continuation values of $D_{XY}, D_{YZ}, D_{ZX}$.

To compute payoffs at this continuations we recall Example 5: Since, $1 > \rho(a)$, all $i \in D_{YX}$ of type $y_i \geq \rho(a)$ must concede a 0. Therefore $X$ is immediately elected at $t$ with probability $(1 - \rho(a))^2$. With the complementary probability the game continues and and the active players of both sides concede following the strategies $\tau^a_Y$ and $\tau^a_X$ computed in Example 5.

$V(x_i, a)$ has meaning only for $x_i \leq a$. When $x_i > a$, player with $x_i$ from $D_{XY}$ consecedes and ends the game.

$$V(x_i, a) = (1 - \rho(a))^2(1 + x_i) + x_i \int_0^{\tau^a_X(x_i)} e^{-t} \cdot 2(1 - F^a_Y(\sigma^a_Y(t)))(-f^a_Y(\sigma^a_Y(t)) \int_0^{\tau^a_X(x_i)} dt + x_i \left[1 - (1 - F^a_Y(\sigma^a_Y(\tau^a_X(x_i)))(\tau^a_X(x_i))^2)\right]$$

Because $\sigma^a_X(0) = a, \tau^a_X(a) = 0, \sigma^a_Y(\tau^a_X(a)) = \rho(a)$ so that
\[ V(x_i = a, a) = (1 - \rho(a))^2(1 + x_i) + x_i \int_{t_0}^{t_0(x_i) = 0} e^{-t} \cdot 2(1 - F_Y^0(\sigma_Y(t)))(-f_Y^0(\sigma_Y(t))dt + x_i[1 - (1 - F_Y^0(\rho(a)))^2] \\
= (1 - \rho(a))^2(1 + x_i) + x_i(1 - (1 - \rho(a))^2) \\
= (1 - \rho(a))^2 + x_i \]

\[ V(y_i, a) = (1 + y_i) \int_0^\infty e^{-t} \cdot (-f_X^0(\sigma_X(t))) \cdot [1 - (1 - F_Y^0(\sigma_Y(t)))^2] \, dt \\
= (1 + y_i) \int_0^\infty e^{-t} \cdot \left( -\frac{\sigma_X^0(t)}{a} \right) \cdot [1 - (1 - \sigma_Y(t))^2] \, dt \]

\[ V(z_i, a) = (1 - \rho(a))^2 z_i + z_i \int_0^\infty e^{-t} \cdot 2(1 - F_Y^0(\sigma_Y(t)))(-f_Y^0(\sigma_Y(t)) \cdot F_X^0(\sigma_X(t))dt \\
= (1 - \rho(a))^2 z_i + z_i \int_0^\infty e^{-t} \cdot 2(1 - \sigma_Y^0(t))(-\sigma_Y^0(t)) \cdot \frac{\sigma_X^0(t)}{a} \, dt \]

We can now turn to examine optimal unilateral plans at in the initial electorate.

**Optimal Unilateral Plans** Consider \( x(t) \), \( y(t) \), and \( z(t) \) differentiable and strictly decreasing.

Let us examine the optimization problems that \( x(t) \), \( y(t) \), and \( z(t) \) must jointly resolve to be compatible with equilibrium.

Consider player \( i \in D_{XY} \). For type \( x_i = x(t) \) concession at \( t \) must be the best response to the concession plans \( y(t) \), and \( z(t) \) of players in factions \( D_{YZ} \) and \( D_{ZX} \). Thus \( t \) must maximize

\[ \int_0^t e^{-v} \cdot (a + b) \cdot dv + c \cdot e^{-t} x_i \]

where
\[a = 0 \cdot 3(1 - F_Y Z(y(v)))^2(-f_Y Zy'(v)) \cdot \left[1 - (1 - F_Z X(z(v)))^3\right]\]
\[= 0 \cdot 3(1 - y(v))^2(-y'(v)) \cdot \left[1 - (1 - z(v))^3\right],\]
\[b = V(x_i, x(v)) \cdot 3(1 - F_Z X(z(v)))^2(-f_Z Xz'(v)) \cdot \left[1 - (1 - F_Y Z(y(v)))^3\right]\]
\[= V(x_i, x(v)) \cdot 3(1 - z(v))^2(-z'(v)) \cdot \left[1 - (1 - y(v))^3\right],\]
\[c = [1 - (1 - F_Z X(z(t)))^3] \cdot [1 - (1 - F_Y Z(y(t)))^3]\]
\[= [1 - (1 - z(t)))^3] \cdot [1 - (1 - y(t))^3].\]

The first term integrates the payoffs that \(x_i\) receives from defections at all dates \(u < t\) weighted the p.d.f. of these events at each \(u\). A defection by \(D_{YZ}\) elects \(Z\) which delivers payoff 0. A defection by \(D_{ZX}\) at \(u\) induces a 2-party subgame with continuation payoffs \(V(x_i, x(v))e^{-u}\). Argue here how the density of these events is obtained! The second term is the payoff from concession at \(t\), i.e. electing \(Y\) at \(t\) pays \(e^{-t}x_i\), weighted by the probability that play reaches date \(t\).

Similarly, for player \(i \in D_{YZ}\) of type \(y_i = y(t)\) a concession at \(t\) that is best response to the concession plans of players in factions \(D_{XY}\) and \(D_{ZX}\) maximizes
\[
\int_0^t e^{-v} \cdot (a + b) \cdot dv + c \cdot e^{-t}y_i
\]
where
\[a = (1 + y_i) \cdot (-f_{XY}x'(v)) \cdot \left[1 - (1 - F_Z X(z(v)))^3\right]\]
\[= (1 + y_i) \cdot (-x^3),\]
\[b = V(y_i, x(v)) \cdot 3(1 - F_Z X(z(v)))^2(-f_Z Xz'(v)) \cdot F_{XY}(x(v))\]
\[= V(y_i, x(v)) \cdot 3(1 - z(v))^2(-z'(v)) \cdot x(v),\]
\[c = F_{XY}(x(t)) \cdot [1 - (1 - F_Z X(z(t)))^3]\]
\[= x(t) \cdot [1 - (1 - z(t)))^3].\]

And finally, for player \(i \in D_{ZX}\) of type \(z_i = z(t)\) concession at \(t\) that is best response to the
concession plans of players in factions $D_{YZ}$ and $D_{XY}$ maximize

$$\int_0^t e^{-v} \cdot (a + b) \cdot dv + c \cdot e^{-t}V(z_i, x(t))$$

\[a = 0 \cdot (-f_{XY}x'(v)) \cdot [1 - (1 - F_{YZ}(y(v)))^3]\]
\[= 0 \cdot (-x^3)\]

\[b = (1 + z_t) \cdot 3(1 - F_{YZ}(y(v)))^2(-f_{YZ}y'(v)) \cdot F_{XY}(x(v))\]
\[= (1 + z_t) \cdot 3(1 - y(v))^2(-y'(v)) \cdot x(v)\]

\[c = F_{XY}(x(t)) \cdot [1 - (1 - F_{YZ}(y(t)))^3]\]
\[= x(t) \cdot [1 - (1 - y(t))^3]\]

Hence $x(t), y(t), z(t)$ must jointly solve the the first order conditions:

$$-3(1 - z)^2 z^3 \cdot [1 - \rho(x)]^2 + 3(1 - y)^2 y^3 \cdot x - [1 - (1 - y)^3][1 - (1 - z)^3] \cdot x = 0$$
$$-\rho(x) \cdot (x^2 z'x' \cdot (y - V(y, x))) = 0$$
$$-3(1 - y)^2 y^3 \cdot (V'(z, x)x' - V(z, x)) = 0$$

10. Appendix

Proof of Lemma 2:

Assume $K \geq 4$ and $H = \{A, B, C, D\} \subset \mathcal{E}$

\[n_A + n_{BA} + n_{CA} + n_{DA} + \sum_{X \notin H} n_{XA} \geq Q\]
\[n_B + n_{AB} + n_{CB} + n_{DB} + \sum_{X \notin H} n_{XB} \geq Q\]
\[n_C + n_{AC} + n_{BC} + n_{DC} + \sum_{X \notin H} n_{XC} \geq Q\]
\[n_D + n_{AD} + n_{BD} + n_{CD} + \sum_{X \notin H} n_{XC} \geq Q\]
Summing inequalities yields

\[ Q \leq \frac{2(n - \sum_{x \not\in H} n_x c) + \sum_{x \not\in H} n_x}{4} < \frac{n}{2} \]

contradicting that \( Q \geq \frac{n+1}{2} > \frac{n}{2} \).

**Proof Lemma 3:**

To facilitate the proof below, let us first establish \( \#E = 3 \Rightarrow Q \leq \frac{2}{3}n \) and \( \#E = 0 \Rightarrow Q > \frac{2}{3}n \).

To show the former, \( \#E = 3 \) rewrites as

\[
\begin{align*}
n_{XY} + n_{XZ} + n_{YZ} + n_{Z} &\geq Q \\
n_{YX} + n_{YZ} + n_{XY} + n_{Y} &\geq Q \\
n_{ZX} + n_{Y} + n_{XZ} + n_{Y} &\geq Q
\end{align*}
\]

which sums to \( 2n \geq 3Q \). Replacing all \( \geq \) by \( < \) in the argument just made shows the latter claim.

We will use converse of the two implications that read \( Q > \frac{2}{3}n \Rightarrow \#E < 3 \) and \( Q \leq \frac{2}{3}n \Rightarrow \#E > 0 \), or, equivalently, \( Q > \frac{2}{3}n \Rightarrow \exists v \) such that \( \#E = 3 \) and \( Q \leq \frac{2}{3}n \Rightarrow \exists v \) such that \( \#E = 0 \).

Now assume \( \mod (n, 3) = 0 \). To prove \( Q \leq \frac{2}{3}n \Rightarrow \mathcal{P} \) it suffices to prove \( Q = \frac{2}{3}n \Rightarrow \mathcal{P} \). Since \( Q \leq \frac{2}{3}n \Rightarrow \#E > 0 \), all we need to show is that we can always construct \( v \) such that \( \#E = 3 \) under \( Q = \frac{2}{3}n \). Considering \( v'(n) = \{\frac{1}{3}n, 0, 0, \frac{1}{3}n, \frac{1}{3}n, 0\} \) shows that this is indeed that case. To prove \( Q \geq \frac{2}{3}n + 1 \Rightarrow \mathcal{P}' \) it suffices to prove \( Q = \frac{2}{3}n + 1 \Rightarrow \mathcal{P}' \). Since \( Q > \frac{2}{3}n \Rightarrow \#E < 3 \), all we need to show is that we can always construct \( v \) such that \( \#E = 0 \) under \( Q = \frac{2}{3}n + 1 \). Considering the same \( v'(n) \) as above shows that this is indeed the case.

Now assume \( \mod (n, 3) \neq 0 \). Denote \( Q_1 = \lfloor \frac{2}{3}n \rfloor \), \( Q_2 = \lfloor \frac{2}{3}n \rfloor + 1 \) and \( Q_3 = \lfloor \frac{2}{3}n \rfloor + 1 \). Because \( \mod (n, 3) \neq 0 \), \( Q_1 < \frac{2}{3}n \), \( Q_2 \in \left( \frac{2}{3}n, \frac{2}{3}n + 1 \right) \) and \( Q_3 > \frac{2}{3}n + 1 \). Trivially \( Q_2 = Q_1 + 1 = Q_3 - 1 \), if \( \mod (n - 1, 3) = 0 \) then \( Q_1 = \frac{2}{3}(n - 1) \) and if \( \mod (n - 2, 3) = 0 \) then \( Q_1 = \frac{2}{3}(n - 2) + 1 \).

**Proof.** To prove \( Q \leq \lfloor \frac{2}{3}n \rfloor \Rightarrow \mathcal{P} \) it suffices to prove \( Q = Q_1 \Rightarrow \mathcal{P} \). Since \( Q \leq \frac{2}{3}n \Rightarrow \#E > 0 \) and \( Q_1 < \frac{2}{3}n \), all we need to show is that we can always construct \( v \) such that \( \#E = 3 \) under \( Q_1 \).

If \( \mod (n - 1, 3) = 0 \), then \( Q_1 = \frac{2}{3}(n - 1) \) and we already know \( v'(n - 1) \) has \( \#E = 3 \) under \( Q_1 \).

Constructing \( v'' \) by adding one player to \( v'(n - 1) \) while keeping \( Q_1 \), \( \#E = 3 \) will hold for \( v'' \) as well. If \( \mod (n - 2, 3) = 0 \), then \( Q_1 = \frac{2}{3}(n - 2) + 1 = \frac{2}{3}(n + 1) - 1 \) and we already know \( v'(n + 1) \) has \( \#E = 3 \) under \( Q_1 + 1 \). Constructing \( v'' \) by dropping arbitrary player from \( v'(n + 1) \), \( \#E = 3 \) will hold for \( v'' \) under \( Q_1 \) as well. ■
To prove $Q = Q_2 \Rightarrow R$, since $Q > \frac{2}{3}n \Rightarrow \#E < 3$ and $Q_2 > \frac{2}{3}n$, all we need to show is that there does not exist $v$ such that $\#E = 0$. Suppose, towards contradiction, that such $v$ exists under $Q_2$. Note that if $X$ is electable under $Q_1$ but not under $Q_2 = Q_1 + 1$, then $n_{XY} + n_{XZ} + n_{YX} + n_{ZX} = Q_1$. In addition, we know from the previous part that under $Q_1$, $\#E \in \{1, 2, 3\}$. Case 1: $v$ is such that $\#E = 3$ under $Q_1$ and $\#E = 0$ under $Q_2$. Then

$$n_{XY} + n_{XZ} + n_{YX} + n_{ZX} = Q_1$$
$$n_{YX} + n_{YZ} + n_{XY} + n_{ZY} = Q_1$$
$$n_{ZX} + n_{YZ} + n_{XZ} + n_{YZ} = Q_1$$

so that $2n = 3Q_1$, contradiction to $\text{mod } (n, 3) \neq 0$. Case 2: $v$ is such that $\#E = 2$ under $Q_1$ and $\#E = 0$ under $Q_2$. Then

$$n_{XY} + n_{XZ} + n_{YX} + n_{ZX} = Q_1$$
$$n_{YX} + n_{YZ} + n_{XY} + n_{ZY} = Q_1$$
$$n_{ZX} + n_{YZ} + n_{XZ} + n_{YZ} < Q_1$$

where we can rewrite the last inequality as $n_{ZX} + n_{YZ} + n_{XZ} + n_{YZ} = Q_1 - c$ where $c \in \mathbb{N}_{>0}$. Summing the equalities, we get $Q_1 = \frac{2}{3}n + \frac{2}{3}$, contradicting either $Q_1 = \frac{2}{3}n - \frac{2}{3}$ when $\text{mod } (n - 1, 3) = 0$ or $Q_1 = \frac{2}{3}n - \frac{1}{3}$ when $\text{mod } (n - 2, 3) = 0$. Case 3: $v$ is such that $\#E = 1$ under $Q_1$ and $\#E = 0$ under $Q_2$. Then

$$n_{XY} + n_{XZ} + n_{YX} + n_{ZX} = Q_1$$
$$n_{YX} + n_{YZ} + n_{XY} + n_{ZY} < Q_1$$
$$n_{ZX} + n_{YZ} + n_{XZ} + n_{YZ} < Q_1$$

where we can rewrite the last two inequalities as $n_{YX} + n_{YZ} + n_{XY} + n_{ZY} = Q_1 - c_1$ and $n_{ZX} + n_{YZ} + n_{XZ} + n_{YZ} = Q_1 - c_2$ where $c_1 \in \mathbb{N}_{>0}$ and $c_2 \in \mathbb{N}_{>0}$. Summing the equalities, we get $Q_1 = \frac{2}{3}n + \frac{c_1 + c_2}{3}$, again contradicting $Q_1 < \frac{2}{3}n$.

Proof. To prove $Q \geq \lceil \frac{2}{3}n + 1 \rceil \Rightarrow P'$ it suffices to prove $Q = Q_3 \Rightarrow P'$. Since $Q > \frac{2}{3}n \Rightarrow \#E < 3$ and $Q_3 > \frac{2}{3}n$, all we need to show is that we can always construct $v$ such that $\#E = 0$ under $Q_3$. We construct such $v$ for $n \in \{4, 5, 7, 8\}$ by means of examples, where $\#E = 0$ for all the cases is
beats another candidate, call it $E$.

The lemma is obvious, so assume $\#C = 1$. By similar argument $\#C = 1$. Now for any $n \in \{10, 11, 13, 14\}$ and $Q_3 = \lceil \frac{3}{2}n + 1 \rceil$ we can take $v$ from the table with $n - 6$ and $Q_3 - 4$ and add one player to each faction. Since $\#C = 0$ for $n - 6$ under $Q_3 - 4$, $\#C = 0$ for $n$ under $Q_3$ as well. ■

Proceeding inductively for $n \in \{16, 17, 19, 20\}$ and so on concludes the proof.

**Proof of Lemma 5:**

Let $n = 6$ and $Q = 4$. To see part (a), with $(0, 1, 0, 2, 0, 3)$, $E(A, B) = \{B\}$, $E(A, C) = \{C\}$ and $E(B, C) = \{C\}$ so that $\mathcal{E} = \{B, C\}$ and $C = \{C\}$. To see part (b), with $(0, 1, 0, 3, 2, 0)$, $E(A, B) = \{A, B\}$, $E(A, C) = \{C\}$ and $E(B, C) = \{B, C\}$ so that $\mathcal{E} = \{C\}$ and $C = \{B, C\}$. For part (c), with $(0, 1, 2, 0, 1, 2)$, $E(A, B) = \{B\}$, $E(A, C) = \{A, C\}$ and $E(B, C) = \{C\}$ so that $\mathcal{E} = \{A, B, C\}$ and $C = \{C\}$. This example has $\#C = 1$. To see that (c) is possible with $\#C = 2$ as well, $(1, 1, 1, 0, 1, 2)$ implies $E(A, B) = \{A, B\}$, $E(A, C) = \{A, C\}$ and $E(B, C) = \{C\}$ so that $\mathcal{E} = \{A, B, C\}$ and $C = \{A, C\}$.

**Proof of Lemma 6:**

(a) $\mathcal{C} = \emptyset$ implies that all the binary votes have a unique outcome (otherwise the core would not be empty). $\mathcal{C} = \emptyset$ thus implies that $X \notin E(X, Y)$ or $X \notin E(X, Z)$ but not both (if both, then either $Y$ or $Z$ would be in $\mathcal{C}$). Take the first case, $X \notin E(X, Y) \land \{X\} = E(X, Z)$. Then $X \in \mathcal{E}$. The second case, $\{X\} = E(X, Y) \land X \notin E(X, Z)$ is symmetric.

(b) Suppose $X \notin \mathcal{E}$. Then $e_X = n_{XY} + n_{XZ} + n_{YX} + n_{ZX} < Q$ so that $v_{XY} < Q$ and $v_{XZ} < Q$. By similar argument $v_{YX} < Q$, $v_{YZ} < Q$, $v_{ZX} < Q$ and $v_{ZY} < Q$ so that $\mathcal{C} = A$.

(c) If $\#C = 1$ then $\mathcal{C} \neq \emptyset$; if indeed $\mathcal{C} = \emptyset$ then $\mathcal{E} = \mathcal{A}$, a contradiction. If $\#C = 3$ then the lemma is obvious, so assume $\#C \in \{1, 2\}$. Then there has to be a candidate, call it $X$, that beats another candidate, call it $Y$, in binary vote, that is $E(X, Y) = \{X\}$, and thus $X \in \mathcal{E}$. We also know $E(Y, Z) = \{Y, Z\}$; if not then either $Y \in \mathcal{E}$ or $Z \in \mathcal{E}$ and $\#\mathcal{E} \geq 2$. Similarly either $E(X, Z) = \{X\}$ or $E(X, Z) = \{X, Z\}$; if not then $Z \in \mathcal{E}$ and $\#\mathcal{E} \geq 2$. This implies that either

\[
\begin{array}{c|c|c}
 n & Q_3 & v \\
 4 & 4 & \{1, 1, 1, 0, 0, 1\} \\
 5 & 5 & \{1, 1, 1, 0, 1, 1\} \\
 7 & 6 & \{1, 1, 1, 1, 1, 2\} \\
 8 & 7 & \{1, 1, 1, 1, 2, 2\}
\end{array}
\]
$\mathcal{C} = \{X\}$ or $\mathcal{C} = \{X, Z\}$, which, because $\mathcal{E} = \{X\}$, proves the claim.

(b) and (c) are obvious.

Proof of Lemma 7:

With only $Y$ and $Z$ parties, $E(Y, Z) \in \{\{Y\}, \{Z\}, \{Y, Z\}\}$, hence $\#\mathcal{C} \in \{1, 2\}$. $\#\mathcal{E} \in \{0, 1, 2\}$ and $\mathcal{E} \subseteq \mathcal{C}$ under $\#\mathcal{C} = 2$ are obvious. When $\#\mathcal{C} = 1$, assume, without loss of generality, that $Y \in \mathcal{C}$ and $Z \notin \mathcal{C}$. Then $E(Y, Z) = \{Y\}$, which implies that $n_Y \geq Q$ and hence $Y \notin \mathcal{E}$.

Proof of Lemma 8:

(a) Without loss of generality, assume $X \in \mathcal{C}, Y \notin \mathcal{C}, Z \notin \mathcal{C}, X \in \mathcal{E}, Y \in \mathcal{E}$ and $Z \notin \mathcal{E}$. We need to prove $E(X, Y) = \{X\}$. Since $X \in \mathcal{C}$, all we need to rule out is $E(X, Y) = \{X, Y\}$. Suppose indeed that $E(X, Y) = \{X, Y\}$. Then, since $Y \notin \mathcal{C}$ and $Z \notin \mathcal{C}$, $E(X, Z) = \{X\}$ and $E(Y, Z) = \{Z\}$. The latter implies $Z \in \mathcal{E}$, a contradiction.

(b) Without loss of generality, assume $X \in \mathcal{C}, Y \notin \mathcal{C}, Z \in \mathcal{C}, X \notin \mathcal{E}, Y \in \mathcal{E}$ and $Z \in \mathcal{E}$. We need to prove $E(Y, Z) = \{Z\}$. Because $X \in \mathcal{C}$ and $X \notin \mathcal{E}$, $E(X, Y) = \{X, Y\}$ and $E(X, Z) = \{X, Z\}$. This implies, by $Y \notin \mathcal{C}$, $E(Y, Z) = \{Z\}$, which was to be proven.

Proof of Lemma 9:

Assume that both $D_{XY}$ and $D_{XZ}$ are strong. Then

$$n_{ZX} + n_{ZY} + n_{XZ} \geq Q$$
$$n_{YX} + n_{YZ} + n_{XY} \geq Q$$

which sums to $n \geq 2Q$, contradicting $Q > \frac{n+1}{2}$.

Proof of Lemma 10:

Take the pair of factions in the lemma to be, without loss of generality, $n_{XY}$ and $n_{YX}$. If both of these factions are unnecessary, then

$$n_{XZ} + n_{YX} + n_{ZX} \geq Q$$
$$n_{YZ} + n_{XY} + n_{ZY} \geq Q$$

which sums to $n \geq 2Q$, or $Q \leq \frac{n}{2}$, a contradiction to $Q \geq \frac{n+1}{2}$. If both of these factions are
necessary, then

\[ n_{XZ} + n_{YX} + n_{ZX} < Q \]
\[ n_{YZ} + n_{XY} + n_{ZY} < Q \]

which rewrites, because \( Q \) is an integer, as

\[ n_{XZ} + n_{YX} + n_{ZX} \leq Q - 1 \]
\[ n_{YZ} + n_{XY} + n_{ZY} \leq Q - 1 \]

and sums to \( n \leq 2Q - 2 \). When \( Q = \frac{n+1}{2} \) this equation becomes \( n \leq n - 1 \), a contradiction.

**Proof of Lemma 11:**

When \( \#C = 0 \) then each binary choice between two alternatives has to have unique outcome and we can, without loss of generality, assume that \( E(X, Y) = \{Y\} \), \( E(X, Z) = \{X\} \) and \( E(Y, Z) = \{Z\} \). The set of strong factions is \( \{n_{XZ}, n_{YX}, n_{ZY}\} \) and the set of weak factions is \( \{n_{XY}, n_{YZ}, n_{ZX}\} \). Because each of the strong factions elects its second alternative when it concedes,

\[ n_{XY} + n_{XZ} + n_{YX} \geq Q \]
\[ n_{YX} + n_{YZ} + n_{ZY} \geq Q \]
\[ n_{ZX} + n_{ZY} + n_{XZ} \geq Q \]

which sums to \( n + n_s \geq 3Q \). If each of the weak factions is necessary for electability of its own party

\[ n_{XZ} + n_{YX} + n_{ZX} < Q \]
\[ n_{YX} + n_{XY} + n_{ZY} < Q \]
\[ n_{ZY} + n_{XZ} + n_{YZ} < Q \]

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which sums to $n + n_s < 3Q$.

**Proof of Lemma 12:**

Assume $Q = \frac{n+1}{2}$, $\#E = 3$, $\#C = 1$. Since $Q = \frac{n+1}{2}$, every election has unique outcome, which implies that there are three non-empty strong factions. Without loss of generality assume both that $X \in C$, which implies $E(X, Y) = \{X\}$ and $E(X, Z) = \{X\}$, and that $E(Y, Z) = \{Z\}$. This implies that $n_{XZ}$, $n_{YX}$ and $n_{ZX}$ are strong (and non-empty) and $n_{XY}$, $n_{YZ}$ an $n_{ZY}$ are weak. Now assume that the number of weak non-empty factions is zero or one. Because $Y$ is electable we have $n_{YX} + n_{YZ} + n_{XY} + n_{ZY} \geq Q$, where only the first faction is strong. If the number of non-empty weak factions is zero, the equation rewrites as $n_{YX} \geq Q$ so that $Y$ wins the election immediately, a contradiction. If the number of non-empty weak factions is one, then the equation rewrites either as $n_{YX} + n_{YZ} \geq Q$, a contradiction or as $n_{YX} + n_{XY} \geq Q$ so that $n_{XY}$ is strong, a contradiction, or as $n_{YX} + n_{ZY} \geq Q$ so that $n_{ZY}$ is strong, again a contradiction.

To prove the claimed structure when the number of non-empty weak factions is two, without loss of generality, assume that $E(X, Y) = \{X\}$, $E(X, Z) = \{X\}$ and $E(Y, Z) = \{Y\}$, so that $X \in C$. Clearly, the set of strong factions is $\{n_{XY}, n_{YX}, n_{ZX}\}$. By electability of $Z$, $n_{ZX} + n_{ZY} + n_{XZ} + n_{YZ} \geq Q$, where only the first faction is strong. If $n_{XZ} = 0$ the equation rewrites as $n_{ZX} + n_{ZY} + n_{YZ} \geq Q$, so that $n_{YZ}$ is strong, a contradiction. If $n_{YZ} = 0$ the equation rewrites as $n_{ZX} + n_{ZY} + n_{XZ} \geq Q$, so that $n_{XZ}$ is strong, a contradiction. Hence $n_{YZ} = 0$. To show that $n_{YZ}$ is necessary for electability of $Y$, assume, towards contradiction, that $n_{YZ}$ is not necessary. Along with electability of $Z$ this gives

$$n_{YX} + n_{XY} + n_{ZY} \geq Q$$

$$n_{ZX} + n_{ZY} + n_{XZ} + n_{YZ} \geq Q$$

which sums to $n \geq 2Q = n + 1$ since $n_{ZY} = 0$, a contradiction.

To prove the claimed structure when the number of non-empty weak factions is three, without loss of generality, assume that $E(X, Y) = \{X\}$, $E(X, Z) = \{X\}$ and $E(Y, Z) = \{Y\}$, so that $X \in C$. The set of strong factions is $\{n_{XY}, n_{YX}, n_{ZX}\}$ and the set of weak factions is $\{n_{XZ}, n_{YZ}, n_{ZY}\}$. To see that $n_{YZ}$ and $n_{ZY}$ cannot be both unnecessary, assume, towards contradictions, that both are.
$n_{YZ}$ and $n_{ZY}$ both unnecessary rewrites as

\[ n_{YX} + n_{XY} + n_{ZY} \geq Q \]
\[ n_{ZX} + n_{XZ} + n_{YZ} \geq Q \]

and sums to $n \geq 2Q = n + 1$, a contradiction. To see that $n_{YZ}$ and $n_{ZY}$ cannot be both necessary, assume, towards contradiction, that both are. $n_{YZ}$ and $n_{ZY}$ both necessary rewrites as

\[ n_{YX} + n_{XY} + n_{ZY} < Q \]
\[ n_{ZX} + n_{XZ} + n_{YZ} < Q \]

or, because $Q$ is an integer,

\[ n_{YX} + n_{XY} + n_{ZY} \leq Q - 1 \]
\[ n_{ZX} + n_{XZ} + n_{YZ} \leq Q - 1 \]

which sums to $n \leq 2Q - 2 = n - 1$, a contradiction.

Proof of Lemma 13:

By Proposition 4 electorates must have 3 strong factions and 3 non-empty necessary weak factions are impossible.

Proof. $C = \{Z\}$ implies $E(Y, Z) = Z$, $E(X, Z) = Z$, and let $E(X, Y) = Y$ without loss of generality. Hence the following inequalities hold

\[ n_Y + n_{ZY} \geq \frac{n + 1}{2} > n_X + n_{ZX} \]
\[ n_Z + n_{XZ} \geq \frac{n + 1}{2} > n_Y + n_{XY} \]
\[ n_Z + n_{YZ} \geq \frac{n + 1}{2} > n_X + n_{YX} \]

so that $D_{ZY}, D_{XZ},$ and $D_{YZ}$ are the strong factions and $D_{ZX}, D_{XY},$ and $D_{YX}$ are weak; that is we are in an electorate $v = (w, s, w, s, w, s)$. By Lemma 10 weak factions $D_{XY}$ and $D_{YX}$ cannot be
both necessary.

**Proof of Lemma 14:**

\( \mathcal{C} = \emptyset \), implies that \( E(X, Y) = X, E(Y, Z) = Y, E(X, Z) = Z \). Now \( E(X, Y) = X \) is equivalent to \( n_X + n_{ZX} \geq Q > n_Y + n_{ZY} \) which implies that \( D_{ZX} \) is strong and necessary. Similarly \( D_{XY} \) and \( D_{YZ} \) are strong and necessary. So we must have an electorate \( \mathbf{v} = ((sn, w), (w, sn), (sn, w)) \).

By Lemma 11 if there are three non-empty weak factions, at least one of them is unnecessary for electability of its own party.

To complete the proof we provide examples for all the scenarios:

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</table>

Examples for Proposition

**Example**

\( n = 7, Q = 4, ((0, 1), (2, 1), (1, 2)) \)
\( n = 7, Q = 4, ((0, 1), (2, 1), (2, 1)) \)
\( n = 7, Q = 4, ((1, 1), (1, 1), (1, 2)) \)
\( n = 9, Q = 5, ((1, 1), (2, 1), (1, 3)) \)
\( n = 9, Q = 5, ((1, 1), (2, 1), (2, 2)) \)