Static vs Dynamic Auctions with Ambiguity Averse Bidders

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March 12, 2012

Abstract

This paper presents the outcome of a dynamic price-descending auction when the distribution of the private values is uncertain and bidders exhibit ambiguity aversion. In contrast to sealed-bid auctions, in open auctions the bidders get information about the other bidders' private values and may therefore update their beliefs on the distribution of the values. The bidders have smooth ambiguity preferences and update their priors using consequentialist Bayesian updating.

It is shown that ambiguity aversion usually affects bidding behavior the same way risk aversion does, but the main result is that this is not the case for continuous price descending auctions. This is new among a few theoretical cases where ambiguity aversion does not reinforce the risk aversion implications.

I would like to thank Dolf Talman, Beatrice Pataracchia, Marta Serra-Garcia and Adam Dominiak

JEL Classification: D03, D44, D89
Keywords: Ambiguity, Auctions, Revenue Equivalence

1 Introduction

A strong but unavoidable assumption done in Auction Theory states that bidders know the distribution from which the private values are drawn. If this distribution is uncertain, a subjective distribution of possible distributions is still needed for modeling purposes, which can then be reduced to a single distribution.

This is not the case under ambiguity aversion, the case where a decision maker is averse to uncertainty about the risk. Ambiguity aversion is portrayed by the seminal experiment in Ellsberg (1961), where decision makers prefer to bet on lotteries with known probabilities, instead of unknown, even if a priori their expected payoff is the same.

This paper studies the consequences of relaxing the assumption of knowledge of the distribution of private values, on equilibrium bidding behavior. Ambiguity averse preferences are modeled using the smooth ambiguity model developed in Klibanoff, Marinacci, and Mukerji (2005). In the first-price sealed-bid auctions, ambiguity aversion leads to higher bids even if bidders are risk neutral, whereas ambiguity has no consequence on dynamic auctions, either price ascending or descending, if the price changes continuously. This latter result is independent of the risk attitude of bidders, and it is a new qualitative result on ambiguity aversion.

There results have implications on the use of first-price sealed-bid auctions as a sale procedure. On one side the expected revenue is higher for the auctioneer if the distribution of the private values is ambiguous to the bidders. On the other side there might be some crowding out of bidders if one considers costs of participation, given that the expected utility of the participating is lower.

This paper is structured as following. Section 2 describes the evolution of the literature and some of its issues, Section 3 presents and explains the basics, Section 4 discusses static auctions under ambiguity aversion, Section 5 goes through a dynamic auction, and Section 6 concludes.

2 Literature

Knight (1921) makes apparently the first distinction between risk and ambiguity, calling the latter uncertainty, reason for which the terms *ambiguity*, *uncertainty* and *Knightian uncertainty* are used interchangeably in the literature. Knight refers to "measurable uncertainty" as *risk*, whereas *uncertainty* should be restricted to cases not "susceptible of measurement". Ellsberg (1961) provides on the other hand the first formal definition of ambiguity, through some experiments that violate Savage's Subjective Expected Utility Axioms. In these experiments, later called the Ellsberg paradox, subjects tend to prefer unambiguous lotteries in a way that cannot be reproduced by risk aversion.

The Ellsberg's paradox consists in an experiment with an urn with 30 red balls and 60 being either black or yellow with unknown distribution. Define lotteries as the vector (r_R, r_B, r_Y) which pays r_i, $i \in \{R, B, Y\}$, if a ball of color i is drawn. Subjects are to make two choices, first between lottery $(1, 0, 0)$ and lottery $(0, 1, 0)$, second between lottery $(1, 0, 1)$ and lottery $(0, 1, 1)$. Typically subjects prefer $(1, 0, 0)$ over $(0, 1, 0)$ implying that their subjective probability for red is higher than that for black. However they tend to prefer lottery $(0, 1, 1)$ over $(1, 0, 1)$ which implies the opposite, their subjective probability for red is lower than that for black. This paradox is independent of the risk aversion of the subjects and thus cannot be explained by it. Intuitively subjects have a preference towards known risks, i.e. unambiguous lotteries. Ellsberg's results have been replicated by other experiments, see Camerer and Weber (1992) for a survey.

Schmeidler (1989) suggests that individuals act as if their subjective probability for ambiguous events were lower than for objective equivalent ones. That is, the subjective probability attached to black in the experiment, is lower than that for red. This leads to non-additive probabilities, i.e. subjective probabilities that do not add up to one. Taking this to calculate the expected utility using the usual Riemann Integral with a probability measure leads to inconsistencies like discontinuities in the integrand and violation of monotonicity (see Chapter 16 in Gilboa (2009)). Schmeidler (1989) uses therefore capacities, generalized probabilities. The expected utility of an act using capacities is given by the Choquet Integral, from which this model derives its name, Choquet Expected Utility. Taking v to be the capacity (probability), the Choquet Expected Utility of a given act (a mapping from the states of nature to outcomes) f, with $f(\omega) \geq 0$ for all $\omega \in \Omega$, is given by

$$V(f) = (C) \int_{\Omega} f dv \equiv \int_{0}^{\infty} v(f \geq t) dt,$$

where $(C) \int$ stands for the Choquet Integral and Ω is the state space. If the capacity of event A, $v(A)$, is interpreted as the worth of coalition A in a Transferable Utility Cooperative Game, the Choquet Integral can be written in a more intuitive way. Given the non-additivity of $v(\cdot)$ and its ambiguity aversion interpretation given above, $v(\cdot)$ should be convex - some authors take this convexity as the definition of ambiguity aversion (for a discussion on the formal definition of Ambiguity Aversion see Epstein (1999)). Schmeidler (1986) shows that in this case, the above Choquet integral can be written as

$$(C) \int_{\Omega} f dv = \min_{p \in Core(v)} \int_{\Omega} f dp. \tag{1}$$

As in the core of a transferable utility game where the allocation of a player should not only be checked against its value alone but also against all coalitions she may belong to, the probability of a state should not enter directly but checked over all the capacities of subsets of the states of nature to which it belongs.

The Multiple Priors or Maxmin Expected Utility model proposed by Gilboa and Schmeidler (1989), while derived from independent axioms, has an intuition which is related to expression (1). It assumes that the individual acts as if she had multiple (additive) priors for the subjective probability. The expected utility of an act is the minimum expected utility across the priors. The individuals then proceed to maximize across these minima, therefore the name Maxmin Utility. Utility of act f over the set of priors $\mathcal{P}$ is given by

$$V(f) = \min_{p \in \mathcal{P}} E_p[f].$$

This model coincides with the Choquet Expected Utility if the set of priors $\mathcal{P}$ equals the core of some capacity v.

As Gilboa (2009) points out the set of priors should not be interpreted as the set of all possible (given the available information) probability distributions, which would be too broad, but as implicit subjective probabilities in line with Savage's Subjective Probability Framework.

Bewley (2002) (originally from 1986) proposes another multiple priors model, where act f is preferred over act g if its expected utility is higher for all priors.

Ghirardato, Maccheroni, and Marinacci (2004) suggest that ambiguity, i.e. uncertainty on the probabilities of the states of nature, and ambiguity attitude, i.e. the way agents react to ambiguity, should be separated in the utility functionals. They propose axiomatically the α-Maxmin Expected Utility where the utility of act f is given by

$$V(f) = \alpha \min_{p \in \mathcal{P}} E_p[f] + (1 - \alpha) \max_{p \in \mathcal{P}} E_p[f],$$

where α is a parameter that captures the ambiguity attitude of the agent. For $\alpha = 1$ the agent will be ambiguity averse as in the Maxmin model.

Variational Preferences were proposed by Maccheroni, Marinacci, and Rustichini (2006), inspired on the Multiplier Preferences from Hansen and Sargent (2001) which draws from Robust Control, where different priors p are weighted through an *ambiguity index* $c(p)$, whose value increases with the ambiguity level of the prior,

$$V(f) = \min_{p \in \Delta(\Omega)} \left(\int_{\Omega} u(f) dp + c(p) \right),$$

$u(\cdot)$ being the usual Bernoulli utility function and $\Delta(\Omega)$ being the set of distributions over the state space Ω. Notice that the minimization is carried out over all possible priors.

A further set of models also weights priors, in a similar way that outcomes are weighted with their probability of occurring in the Expected Utility Model. This class is called Recursive Expected Utility or Second Order Beliefs, because each prior is assigned a (second-order) probability of being the correct one, these second-order being distributed with probability measure μ. Usually priors are indexed through some parameter $\theta \in \Theta$ and p_θ is the probability distribution for prior θ. The utility of an act will be calculated by aggregating the certainty equivalent of each probability prior, across all priors. The most common model in this category, the Smooth Ambiguity Preferences, is proposed in Klibanoff, Marinacci, and Mukerji (2005) where this aggregation is a concave (convex for ambiguity loving preferences) functional $\phi(\cdot)$ which could be interpreted as a second order Bernoulli function. Its concavity represents the aversion to uncertainty on the correct prior. The utility of act f is defined as

$$V(f) = \int_{\Theta} \phi \left(\int_{\Omega} u(f) dp_\theta \right) d\mu.$$

While clearly routed in the multiple priors model, the smooth ambiguity preferences have a straightforward intuition. In terms of attitude towards risk, a concave Bernoulli utility function performs the task of assigning lower weight to high outcomes and higher weight to low ones when adding the outcomes up, so that a risk averse individual focuses more on the bad results. With ambiguity, an ambiguity averse individual with a concave $\phi(\cdot)$ will analogously stress those priors, i.e. those possible probability distributions, that yield the worst scenarios in terms of expected outcome.

Ambiguity aversion and dynamics, i.e. preference updates as new information is gathered, have been two concepts difficult to be reconciled. The main issue can be discussed using a dynamic version of the Ellsberg paradox proposed by Epstein and Schneider (2003). Consider the same experiment but with an additional step after the ball is taken from the urn, where the individual gets to know whether the ball is yellow or not. Initially an ambiguity averse individual prefers lottery $(0, 1, 1)$ over $(1, 0, 1)$. After the ball is drawn, she will have $(0, 1, 1) \sim (1, 0, 1)$ if the ball is yellow. In the other case, if she bayesianly updates the priors for the remaining balls, she shall have $(1, 0, 1) \succ (0, 1, 1)$. Take for instance the Maxmin Expected Utility model with the following set of priors $\mathcal{P} = \{(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})\}$. Conditional on not being yellow these priors become $\{(\frac{2}{5}, \frac{3}{5}, 0), (\frac{2}{3}, \frac{1}{3}, 0)\}$ using Bayes rule. So the maxmin expected utility for $(0,1,1)$ is initially $\frac{2}{3}$ and then $\frac{1}{3}$, while for $(1,0,1)$ it decreases only from $\frac{1}{2}$ to $\frac{2}{5}$. Thus, the individual does not keep his preferences in none of the intermediate states, that is the preferences do not satisfy dynamic consistency. In this context dynamic consistency may be loosely defined as the non-reversal of preferences from period t to $t+1$ between two acts which are equal until t, but one is preferred for every possible prior in $t+1$.

Different solutions have been proposed in the literature. One enforces dynamic consistency through the choice of the time aggregating functional (as in Klibanoff, Marinacci, and Mukerji (2009) for the Smooth Ambiguity Model), backward induction like the sophisticated agents in Pollak (1968) (as in Siniscalchi (2010)), the imposition of consistency conditions on the priors (as in Epstein and Schneider (2003)), or discretionary priors update rules which depend on the preferences, the events and the choice problem (as in Klibanoff and Hanany (2007) and Hanany and Klibanoff (2009)). In the above example, a dynamically consistent ambiguity averse individual would then compulsory prefer $(1,0,1)$ over $(0,1,1)$ in the beginning.

Another approach is to discard dynamic consistency, accept the above apparent preference change and impose consequentialism, which states that the decision maker is indifferent between two acts which yield the same payoffs for all priors, irrespectively of what happened in the previous periods. To understand the implication of this assumption in the above example, consider the preference on $(1,0,0)$ vs. $(0,1,0)$. The intermediate step bares no change, either the ball is yellow and payoff is zero in both or it is not yellow and the two available lotteries are still between red and black, so an ambiguity averse individual should prefer the first over the second in both periods. If now compared to the preferences $(0,1,1) \succ (1,0,1)$ in the first period, consequentialism then states that the ambiguity averse decision maker should switch its preference in the intermediate step if the ball is not yellow, because $(1,0,1)$ and $(0,1,1)$ coincide with $(1,0,0)$ and $(0,1,0)$, respectively, in the remaining nodes. Consequentialism is satisfied if the decision maker follows a Bayesian update rule for the priors.

Consequentialist priors update rules have been axiomatized according to different requirements. Gilboa and Schmeidler (1993) axiomatize the Dempster-Shafer update rule for the Multiple Priors Model. As new information becomes available for the decision maker, she picks those priors that assign maximum likelihood to the information and updates them with Bayes rule. They also show this coincides with Bayesian updating for capacities, provided that the Choquet and Maxmin preferences coincide. Pires (2002) axiomatizes a different Bayesian update rule where all priors are kept and all are updated according to Bayes rule.

Ozdenoren and Peck (2008) further suggests that dynamic inconsistent behavior of ambiguity averse individuals can be interpreted as consistent subgame perfect equilibrium strategies in a game against nature, which influences ambiguous outcomes.

There is also a rich empirical, applied and experimental literature on Ambiguity Aversion.

Hey, Lotito, and Maffioletti (2007) use an inventive device to simulate ambiguity in the lab. Subjects can see a bingo blower and estimate the number of balls with different colors. Not only do the authors confirm a widespread existence of ambiguity averse preferences but conclude, through a series of binary tests, that Choquet Expected Utility fits the data the best, but also claim that the decisions vary a lot across individuals.

In a portfolio choice application, Dow and Werlang (1992) show that an agent with Maxmin Expected Utility has a price range for which she chooses not to buy and not to sell an asset, a result unexplainable by standard preferences. This behavior is not due to some status quo bias (as in the Bewley (2002) model) but as a safe allocation consideration.

Epstein and Schneider (2003) claim that ambiguity aversion may explain the home bias that investors exhibit. Ju and Miao (2009) use ambiguity aversion in an asset pricing model to show that it can explain the equity premium and its volatility.

This is not to say that this literature is consensual. For instance, the experiments in Halevy (2007) show that there is a significant positive correlation between displaying ambiguity aversion and violating the reduction of compound objective lotteries. See Al-Najjar and Weinstein (2009) for further criticism.

Dominiak, Dürsch, and Lefort (2009) test the dynamic version of the Ellsberg experiment and find that most subjects tend to follow consequentialism, meaning that they are not acting in a dynamically consistent way.

Liu and Colman (2009) compare decisions between single-choice and repeated-choice Ellsberg urn choices. In the latter, decision makers tend to pick the ambiguous option more frequently.

For more comprehensive reviews on the literature see Etner, Jeleva, and Tallon (2009).

Few work has been put forward assessing the impact of ambiguity aversion on auctions. Using Choquet Expected Utility, Salo and Weber (1995) show that ambiguity aversion may explain the (beyond risk aversion) overbidding in first-price sealed-bid auctions when the distribution of private values or the number of bidders is ambiguous. Lo (1998) examines first and second-price sealed-bid auctions when both the bidders' and the auctioneer's preferences follow the Maxmin model, indicating that the effects of ambiguity attitudes are similar, but not equal, to those of risk in terms of bidding and revenue. Bose, Ozdenoren, and Pape (2006) study the optimal static auction mechanism with ambiguity. Chen, Katuscak, and Ozdenoren (2007) compare experimentally the bidding behavior of bidders that know the distribution of the opponents and bidders that have to learn it through repeated auctions. They find that the latter have lower bids. Bose and Daripa (2009) is the first analyzing dynamic auctions with ambiguity (bidders choose strategies from backward induction), but from the optimal auction point of view. They show that with ambiguity, modeled with Maxmin preferences, the auctioneer can extract almost all surplus, in contrast to the unambiguous case.

The experiments in Armantier and Treich (2009) indicate that probabilistic bias are the main drive of overbidding in first-price sealed-bid auctions. Some experimental literature use compound lotteries to simulate ambiguity. While theoretically they are very different concepts, most ambiguity aversion models can also have a bad reduction of compound lotteries interpretation. Kocher and Trautmann (2011) run an experiment where subjects can choose to participate in a risky or in an ambiguous first-price sealed-bid auction. While the equilibrium price is the same in both, bidders tend to avoid the ambiguous auction.

3 Framework

In conventional Auction Theory the bidders (and the auctioneer) have limited information of each other. They are not aware of the value that the auctioned object represents for the other players and therefore they do not know the other players' payoffs. For any results to be established one must clearly make quantitative assumptions, so it is assumed that the probabilistic distribution of these values is common knowledge. While the assumption of perfect information on the probabilistic distribution may be too strong, any more elaborate assumptions end up being equivalent to it through compound lottery reduction.

It is known that, risk aversion aside, individuals display aversion to risky choices when the probability distribution of the outcomes is not perfectly known, i.e. they display Ambiguity Aversion. A popular method to generalize Expected Utility Theory to allow for these preferences to be included, is the Smooth Ambiguity Model from Klibanoff, Marinacci, and Mukerji (2005). Instead of using a single distribution of the unknown parameters, ambiguity is introduced through multiple possible distributions.

Formally there are multiple prior probability measures π_θ, where $\theta \in \Theta$ indexes the priors, over the possible states of nature ω, with $\omega \in \Omega$. Particular to this ambiguity model is the assumption of a probability measure over the different priors, represented by μ defined from 2^Θ to $[0, 1]$. Ambiguity Aversion is then modeled in a similar way as Risk Aversion, that is, using a concave function $\phi(\cdot)$ to aggregate the (certainty equivalent of the) outcomes of act f over all priors with μ, that is aggregating $\int_\Omega u(f(\omega))d\pi_\theta$ over θ. Act f maps a state of nature $\omega \in \Omega$ to an outcome $f(\omega)$ yielding utility $u(f(\omega))$, where the utility function $u(\cdot)$ is taken to be (weakly) concave to represent risk aversion. The utility of f in the smooth ambiguity model is given by

$$U(f) = \int_\Theta \phi\left(\int_\Omega u(f(\omega))d\pi_\theta\right) d\mu. \tag{2}$$

This model is chosen for several reasons. It is a smooth model, meaning that differentiable functionals may be used so that the utility itself is differentiable, in opposition to most Ambiguity Aversion models. Moreover the model allows to distinguish between the consequences of different levels of ambiguity, given by the spread of the prior, and those of idiosyncratic ambiguity aversion, given by the shape of $\phi(\cdot)$. A further reason is related to dynamic decisions under ambiguity, namely the update of priors as new information is received. Having a probability measure on the priors allows to put more weight on priors that seem to be more credible with the new information[1], whereas other models discard some priors altogether.

In all the basic auctions being considered here an indivisible good is being auctioned. The private values of the good to the n bidders are randomly drawn from distribution F_θ with support $[0, 1]$, with $\theta \in \Theta$. Private values are assumed to be independently drawn across agents. The probability of each possible distribution F_θ is given by the measure μ on 2^Θ.

[1] For a critic on the Smooth Ambiguity Model see Epstein (2010) and a reply to it Klibanoff, Marinacci, and Mukerji (2011).

To enable a comparison with the unambiguous case, an equivalent subjective probability distribution F_U will be defined, satisfying

$$\int F_\theta^{n-1}(x)d\mu = F_U^{n-1}(x),\ \forall x \in [0,1]. \tag{3}$$

F_U can be interpreted as the reduced probability distribution that an ambiguity neutral bidder considers. Let $G_\theta(x) = F_\theta^{n-1}(x)$ and similarly for $G_U(x)$,

$$\int G_\theta(x)d\mu = G_U(x),\ \forall x \in [0,1].$$

Notice that this implies

$$\int \frac{d}{dx}G_\theta(x)d\mu = \frac{d}{dx}G_U(x),\ \forall x \in [0,1].$$

Moreover it is assumed that all priors θ, $\theta \in \Theta$, are such that an auction with F_θ as the value distribution has a unique monotonic equilibrium pricing strategy.

It should be underlined that these priors are the same across all bidders and they represent the beliefs that the bidders have after learning their own value. Otherwise, given their own value, they would update their second order beliefs μ according to it.

4 Static ambiguous auctions

The two most common types of static auctions are considered, the first-price sealed-bid auction and the second-price sealed-bid auction.

4.1 First-price sealed-bid auction

In the first-price sealed-bid auction, all bidders submit one bid at the same time. The good is then given to the bidder with the highest bid, for which she pays the offered price.

Ambiguity neutrality

Consider the case of ambiguity neutral bidders with F_θ for priors and μ the measure on the priors. The first-price sealed-bid auction will be equivalent to the unambiguous case where values follow the F_U distribution defined in equation (3). This follows directly from the usual reduction of compound lotteries, or mathematically as the combination of the two integrals in (2) to a single measure. With ambiguity neutrality, that is with $\phi(y) = y$, any expectation becomes simply

$$\begin{aligned} U(f) &= \int_\Theta \phi\left(\int_\Omega f(\omega)dF_\theta\right)d\mu \\ &= \int_\Theta \int_\Omega f(\omega)dF_\theta d\mu \\ &= \int_\Omega f(\omega)dF_U, \end{aligned}$$

which is the ambiguity neutrality case.

Ambiguity aversion

If bidders have ambiguity aversion modeled as in (2), the priors cannot be reduced to a single distribution. Consider a given increasing differentiable strategy for the first-price sealed-bid auction $\beta_1(\cdot)$, where the index 1 stands for first-price, followed by the $n-1$ opponents. A bidder with value v who chooses to bid as if she had value z, will win the auction with probability $G_\theta(z)$, yielding in that case a utility of $u(v-\beta_1(z))$, according to prior $\theta \in \Theta$. The certainty equivalent of this choice is then, still according to prior θ, $G_\theta(z)u(v-\beta_1(z))$. To compute the expected utility one has to aggregate over all priors, which leads to the expected utility

$$\int \phi\left(G_\theta(z)u(v-\beta_1(z))\right) d\mu.$$

The best response for the strategy $\beta_1(\cdot)$ will therefore solve

$$\max_z \int \phi\left(G_\theta(z)u(v-\beta_1(z))\right) d\mu.$$

First order condition yields

$$\begin{aligned} \int \phi'\left(G_\theta(z)u(v-\beta_1(z))\right) \times \\ \left[G'_\theta(z)u(v-\beta_1(z)) - G_\theta(z)u'(v-\beta_1(z))\beta'_1(z)\right] d\mu = 0. \end{aligned} \tag{4}$$

The term in the second bracket is the optimum condition for the unambiguous case for each prior θ. The equilibrium can be seen as a weighted mean, the $\phi'(\cdot)$ terms being the weights. Introducing ambiguity aversion renders $\phi'(\cdot)$ decreasing, stressing those terms in the integral where $G_\theta(z)$ is lower.

In equilibrium the bidders bid according to their value, i.e. $z=v$, hence the above equation may be rewritten as

$$\beta'_1(v) = \frac{\int \phi'\left(G_\theta(v)u(v-\beta_1(v))\right) G'_\theta(v) d\mu}{\int \phi'\left(G_\theta(v)u(v-\beta_1(v))\right) G_\theta(v) d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}.$$

Assume for this section that $\phi(\cdot)$ is such that $\phi'(ab)=\phi'(a)\phi'(b)$, for example of the usual exponential form, $\phi(h)=\frac{1}{\alpha}h^\alpha$, for some $\alpha \in (0,1)$, this simplifies to

$$\beta'_1(v) = \frac{\int \phi'\left(G_\theta(v)\right) G'_\theta(v) d\mu}{\int \phi'\left(G_\theta(v)\right) G_\theta(v) d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}. \tag{5}$$

Suppose now that all priors are such that they can be ordered in the following way, $F_{\theta_1}(x) < F_{\theta_2}(x)$ for any $x>0$ if $\theta_1 < \theta_2$. This implies that $G_{\theta_1}(x) < G_{\theta_2}(x)$ for any $x>0$. Thus for higher θ, the term $\phi'(G_\theta(v))$ will be lower for the same $v>0$. Following this assumption on the ordering of the cumulative distribution functions, it is also assumed[2] that for the hazard rate

$$\frac{F'_{\theta_1}(x)}{F_{\theta_1}(x)} > \frac{F'_{\theta_2}(x)}{F_{\theta_2}(x)} \quad \forall x>0, \text{ if } \theta_1 < \theta_2.$$

[2] The second assumption while independent from the first, is not a strong one. To see this notice that the numerators are ordered in an increasing way.

Following the definition of $G_\theta(\cdot)$, its derivative $G'_\theta(x)$ equals $(n-1)F_\theta^{n-2}(x)F'_\theta(x)$ so that

$$\frac{G'_\theta(x)}{G_\theta(x)} = (n-1)\frac{F'_\theta(x)}{F_\theta(x)}.$$

Using the last assumption this implies that

$$\frac{G'_{\theta_1}(x)}{G_{\theta_1}(x)} > \frac{G'_{\theta_2}(x)}{G_{\theta_2}(x)}.$$

See below for some examples.

Now, it is easy to see that the expression $\frac{a_{-i}+ca_i}{b_{-i}+cb_i}$ moves monotonously from $\frac{a_{-i}}{b_{-i}}$ to $\frac{a_i}{b_i}$ as c goes from 0 to ∞. Think of c as θ and $\frac{a_i}{b_i}$ as $\frac{G'_\theta(x)}{G_\theta(x)}$ in the integrals of the first fraction of expression (5). The terms of priors with lower θs will thus have a higher weight as ambiguity aversion increases. Given that lower θs have a higher $\frac{G'_\theta(x)}{G_\theta(x)}$ ratio, the first fraction in (5) will be higher for higher ambiguity aversion. Therefore the concavity of $\phi(\cdot)$ implies

$$\frac{\int \phi'\left(G_\theta(v)\right) G'_\theta(v)d\mu}{\int \phi'\left(G_\theta(v)\right) G_\theta(v)d\mu} > \frac{\int G'_\theta(v)d\mu}{\int G_\theta(v)d\mu}, \tag{6}$$

and the ratio on the left-hand side is decreasing with the ambiguity aversion parameter α, i.e. increasing with ambiguity aversion. The ratio in the right-hand side is the ratio that appears in the differential equation defining the ambiguity neutral bidding equilibrium strategy, $\beta_{1,N}(\cdot)$, where the index N stands for Neutrality, that is the one in case of linear $\phi(\cdot)$,

$$\beta'_{1,N}(v) = \frac{\int G'_\theta(v)d\mu}{\int G_\theta(v)d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))}.$$

Now if $\beta_1(v) < \beta_{1,N}(v)$ then $\frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))} > \frac{u(v-\beta_{1,N}(v))}{u'(v-\beta_{1,N}(v))}$, and given (6) one gets $\beta'_1(v) > \beta'_{1,N}(v)$. But at $v=0$ it is easy to see that $\beta_1(0) = \beta_{1,N}(0) = 0$. One can therefore not have $\beta_1(v) < \beta_{1,N}(v)$ for any $v > 0$ because that would imply $\beta'_1(v) > \beta'_{1,N}(v)$, a contradiction. Thus it must be that $\beta'_1(v)$ is higher than $\beta'_{1,N}(v)$ for any $v > 0$. This implies the following result.

Lemma 1 *In the First-Price Sealed-Bid Auction with Smooth Ambiguity the equilibrium bid increases as ambiguity aversion arises.*

The following examples illustrate the lemma.

4.1.1 Ambiguous order with linear priors

Consider a set of priors in $[0,1]$ where values are drawn from distributions with the following probability density functions $F'_\theta(x) = (1+\theta) - 2\theta x$, with $\theta \in [-1,1]$. For $\theta_1 < \theta_2$ it holds that $F_{\theta_1}(x) < F_{\theta_2}(x)$ and

$$\frac{F'_{\theta_1}(x)}{F_{\theta_1}(x)} > \frac{F'_{\theta_2}(x)}{F_{\theta_2}(x)},$$

because $\frac{F'_\theta(x)}{F_\theta(x)} = \frac{1}{x} - \frac{1}{1/\theta+1-x}$ for any x.

Recall that the ambiguity aversion term $\phi'(F_\theta(x))$ stresses those priors with lower $F_\theta(x)$, i.e. those with lower θ. Take for instance $\theta = -1$. According to this prior, the value of the opponent will be drawn from $F_{-1}(x) = x^2$, meaning that there is higher probability of confronting a bidder with a higher value, in comparison to the other extreme case $\theta = 1$, when $F_1(x) = 2x - x^2$ for example. There is so to say ambiguity regarding the ordering of the values of the opponents. The ambiguity averse bidder will therefore choose to place a higher bid in equilibrium.

Ambiguous order with exponential priors

Consider the priors $F_\theta(x) = x^\theta$ for $0 \leq x \leq 1$ with $\theta > 0$. The hazard rate will be

$$\frac{F'_\theta(x)}{F_\theta(x)} = \frac{\theta}{x}.$$

The assumptions are clearly satisfied (in reverse order though), i.e. $F_{\theta_1}(x) > F_{\theta_2}(x)$ and $\frac{F'_{\theta_1}(x)}{F_{\theta_1}(x)} < \frac{F'_{\theta_2}(x)}{F_{\theta_2}(x)}$ for any x if $\theta_1 < \theta_2$.

Ambiguous mean

Consider the case with two equally likely priors $\theta = 1, 2$ with uniform distribution of length $a < 1$, whose total support is $[0, 1]$. These priors create the following conceptual problem to a bidder whose private value v is not included in the support of all priors, for instance if $v = 0.1$ and there are two priors with support $[0, 0.8]$ and $[0.2, 1]$. This bidder will reject the second prior from the start, so that the ambiguity is not the same across bidders.

It is therefore assumed that the prior distributions are of the following type for some $0 < \epsilon < \frac{1}{1-a}$,

$$F'_\theta(x) = \begin{cases} a^{-1} - \frac{1-a}{a}\epsilon & \text{if } x \in [0, a] \text{ for } \theta = 1 \text{ or if } x \in [1-a, 1] \text{ for } \theta = 2, \\ \epsilon & \text{otherwise.} \end{cases}$$

As $\epsilon \to 0$, some of the fractions $\frac{F'_\theta(x)}{F_\theta(x)}$ become undetermined. Using $\phi(h) = \frac{1}{\alpha}h^\alpha$, $\alpha \in (0, 1)$, it can still be proved that

$$\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)}$$

weakly decreases with α. See the appendix.

4.1.2 Closed-form solutions

One can get an explicit solution for the equilibrium bidding strategies if the priors are chosen appropriately. Take n risk neutral bidders and a finite set of priors $\mathcal{P} = \{F_1, \ldots, F_m\}$, all equally probable (i.e., $\mu_i = \frac{1}{m}$ for all $i = 1, \ldots, m$), such that $\frac{1}{m}\sum_{i=1}^m F'_i(x) = 1$ for all $x \in [0, 1]$. Such set of priors satisfies $\frac{1}{m}\sum_{i=1}^m F_i(x) = x$, meaning that for an ambiguous neutral bidder with only one opponent ($n = 2$), these priors correspond to a uniform distribution. For $n > 2$ and $x \in (0, 1)$ one has that $\frac{1}{m}\sum_{i=1}^m F_i^{n-1}(x) \geq x^{n-1}$ or $F_U(x)^{n-1} \geq x^{n-1}$, with strict inequality if there are at least two priors with different values.

In words, with this set of priors $\mathcal{P}$ the reduced cumulative distribution of the opponents, F_U, has a higher value for any value x than a uniform distribution with $n-1$ opponents would have. That is for any value v that the bidder may have, there is here a lower probability of having opponents with higher values than it would happen with a uniform distribution. In an auction with ambiguous neutral bidders, the equilibrium bidding strategy would therefore assign lower bids for each value than the corresponding bid in an auction with uniform distribution.

Choosing the ambiguity aversion parameter $\alpha = \frac{1}{n-1}$, simplifies the equilibrium conditions considerably,

$$\begin{aligned}
\beta_1'(v) &= \frac{\int \phi'\left(G_i(v)\right) G_i'(v) d\mu}{\int \phi'\left(G_i(v)\right) G_i(v) d\mu} \times \frac{u(v-\beta_1(v))}{u'(v-\beta_1(v))} \\
&= (n-1)\frac{\sum_{i=1}^m F_i(v)^{(\alpha-1)(n-1)} F_i(v)^{n-2} F_i'(v)}{\sum_{i=1}^m F_i(v)^{(\alpha-1)(n-1)} F_i(v)^{n-1}}(v-\beta_1(v)) \\
&= (n-1)\frac{\sum_{i=1}^m F_i(v)^{\alpha(n-1)-1} F_i'(v)}{\sum_{i=1}^m F_i(v)^{\alpha(n-1)}}(v-\beta_1(v)) \\
&= (n-1)\frac{\sum_{i=1}^m F_i'(v)}{\sum_{i=1}^m F_i(v)}(v-\beta_1(v)) \\
&= \frac{n-1}{n}(v-\beta_1(v)).
\end{aligned}$$

The equilibrium bid is thus the same as the basic non-ambiguous with uniformly distributed values, $\beta_1(v) = \frac{n-1}{n}v$, even if there are less opponents with higher values. Like risk aversion, aversion to ambiguity pushes the bidders to play a safer strategy which increases their chance to win at the expense of lower payoffs.

Take for instance the set of equally probable priors $\mathcal{P} = \{F_1, F_2\}$ with $F_1(x) = x^a$ and $F_2(x) = 2x - x^a$, where $0 \le x \le 1$, $a \in [1,2]$ and $n = 3$. At $a = 1$ the priors are both the uniform distribution so there is no ambiguity and the usual equilibrium arises. At $a > 1$, however, the reduced distribution with which an ambiguous neutral bidder ($\alpha = 1$) calculates her expected payoff is different. For $a = 2$ it will be $F_U^2(x) = \frac{1}{2}(x^4 + (2x - x^2)^2) = x^2(1 + (1-x)^2) > x^2$ for any $x > 0$. Now for $\alpha = \frac{1}{2}$ and for any $a \in [1,2]$, the ambiguity averse bidders have as equilibrium strategy the usual $\beta_1(v) = \frac{2}{3}v$. Notice that increasing the parameter a increases the probability of a low value of opponents but increases the ambiguity, and has no effect in this solution because the two effects cancel out.

4.2 Second-price sealed-bid auction

Lemma 2 *In the ambiguous Second-Price Sealed-Bid Auction with ambiguity averse bidders with smooth ambiguity preferences, bidding their own value, i.e.* $\beta_2(v) = v$, *is an equilibrium.*

Proof. The proof is straightforward as in the ambiguity neutral and risk neutral case. Provided that other bidders play according to $\beta_2(v)$, bidding less than v decreases the probability of winning the auction without yielding higher payments, and bidding more than v increases the number of chances in which the auction is won, but all of which will yield negative payoffs. ■

This result is confirmed experimentally in Chen, Katuscak, and Ozdenoren (2007).

5 Dynamic ambiguous auction

Dynamics and Ambiguity Aversion have been difficult to stitch together in the literature, as it was remarked in Section 2. Different approaches yield quite different forecasts. In this section a consequentialist Bayesian update[3] rule is adopted for various reasons. First, the only empirical evidence available indicates that subjects follow consequentialist update rules in the simple dynamic Ellsberg experiment, see Dominiak, Dürsch, and Lefort (2009). Second, models with dynamically consistent preferences use recursive update rules. In a price-descending auction where the price decreases continuously it is not clear how this recursive rule should be applied. And if a discrete process is considered, the size of the price decrease in each period would have an important impact on the outcome of these models[4].

The setting in an open price descending auction bidders is much richer than in a static auction, since bidders can collect information as the auction runs. When the distributions are not ambiguous, as the auction price descends and no bid is placed, there is only one type of information that bidders learn, namely they learn that there are no opponents with values above some given threshold.

But that is not the case with ambiguity. Consider the case where bidders have two priors on the distribution of the opponents. One indicates a higher probability of higher values, and the other of lower values. As the price descends and bidders exclude the possibility of having opponents with the highest possible values, the first prior starts to look less likely than in the beginning, since the first prior decrees that there is a stronger possibility of the auction ending with a high bid. As the auction goes on, bidders take the second prior to be more believable and evaluate their strategies according to this update believe. Conditional on the fact that no bidder stopped the auction until price p, the prior beliefs, both F_θ, $\theta \in \Theta$, and μ, will be 'updated'.

Let the conditional Bayesian beliefs, conditional on the fact that $x \leq y$ for some given y, $0 \leq y \leq 1$, be represented by $F_{\theta,y}(x)$, i.e.,

$$F_{\theta,y}(x) = \frac{F_\theta(x)}{F_\theta(y)}, \quad x \leq y, \theta \in \Theta.$$

The probability measure on the priors is also updated to μ_y. For given y,

[3] Updating is arguably not the best term given that strictly speaking there is no new information. Put differently, in the beginning of the auction bidders can infer what will be their beliefs at some future point, provided that that point is reached.

[4] It could still be argued that forward looking decision makers could recognize their changing preferences and choose suboptimal bidding strategies, i.e. stopping earlier, to prevent the predicted outcome if that would maximize their expected payoff, in line with Siniscalchi (2010). While proving that that cannot be the case is beyond the scope of this paper, all numerical simulations that were conducted show that at no point the bidders prefer to bid at the current price instead of the equilibrium one - except obviously for the equilibrium price bid. More on this issue later in the section.

$0 \leq y \leq 1$, it is defined by

$$\mu_y(A) = \frac{\int_A F_\theta^{n-1}(y)d\mu}{\int_\Theta F_\theta^{n-1}(y)d\mu} = \frac{\int_A G_\theta(y)d\mu}{\int_\Theta G_\theta(y)d\mu},\ A \in 2^\Theta.$$

5.0.1 Ambiguity neutrality

When individuals are ambiguity neutral, the existence of ambiguity should not affect the equilibrium, even if their probability measure μ is updated. In this section it is shown that indeed ambiguity does not affect the equilibrium outcome.

Take $\beta_{D,N}(v)$ to be the monotonous equilibrium bidding strategy for a bidder with value v, D standing for Dutch auctioneer. Suppose the $n-1$ opponents are playing this strategy and the descending price reaches level p, implying that the values of the opponents are smaller than $\beta_{D,N}^{-1}(p)$. For a given own private value v, the bidder may bid the good at p receiving

$$\int v - pd\mu_z = v - p = v - \beta_{D,N}(z),$$

where z is the private value for which p is the optimal bid, $z = \beta_{D,N}^{-1}(p)$. The bidder may consider to bid as a lower type $y < z$, whose bid wins with probability (according to the updated priors) $G_{\theta,z}(y) = F_{\theta,z}^{n-1}(y)$, receiving

$$\int G_{\theta,z}(y)(v - \beta_{D,N}(y))d\mu_z.$$

Let y be marginally smaller than z, $y = z - \Delta$, and let Δ go to zero. The marginal gain from Δ will be

$$\begin{aligned}
& \int G_{\theta,z}(z)\,(v - \beta_{D,N}(z)) - \\
& \Delta\left(G'_{\theta,z}(z)(v - \beta_{D,N}(z)) - G_{\theta,z}(z)\beta'_{D,N}(z)\right)d\mu_z - (v - \beta_{D,N}(z)) \\
= & \int (v - \beta_{D,N}(z)) - \\
& \Delta\left(G'_{\theta,z}(z)(v - \beta_{D,N}(z)) - \beta'_{D,N}(z)\right)d\mu_z - (v - \beta_{D,N}(z)) \\
= & \int -\Delta\left(G'_{\theta,z}(z)(v - \beta_{D,N}(z)) - \beta'_{D,N}(z)\right)d\mu_z,
\end{aligned}$$

where $G_{\theta,z}(z) = 1$ for any θ is used. In equilibrium the optimal response has $v = z$ such that the marginal gain is zero,

$$\begin{aligned}
\beta'_{D,N}(v) - (v - \beta_{D,N}(v))\int G'_{\theta,v}(v)d\mu_v &= 0, \\
\beta'_{D,N}(v) - (v - \beta_{D,N}(v))\int \frac{G'_\theta(v)}{G_\theta(v)}\frac{G_\theta(v)}{\int G_\vartheta(v)d\mu}d\mu &= 0, \\
\beta'_{D,N}(v) - (v - \beta_{D,N}(v))\int \frac{G'_\theta(v)}{G_U(v)}d\mu &= 0, \\
\beta'_{D,N}(v) = (v - \beta_{D,N}(v))\frac{G'_U(v)}{G_U(v)}&.
\end{aligned}$$

The best response satisfies the same condition as the optimal bid in the static auction. The equilibrium conditions for both auctions are therefore equivalent.

5.0.2 Ambiguity aversion

Let $\beta_D(v)$ be the equilibrium bid in an Open Price Descending Auction. The gains from delaying Δ are now

$$\begin{aligned}
& \int \phi\left(G_{\theta,z}(z-\Delta)u(v-\beta_D(z-\Delta))\right) d\mu_z - \phi\left(u(v-\beta_D(z))\right) \\
\approx & \int \phi\left(G_{\theta,z}(z)u(v-\beta_D(z))\right) - \Delta\phi'\left(G_{\theta,z}(z)u(v-\beta_D(z))\right) \\
& \left(G'_{\theta,z}(z)u(v-\beta_D(z)) - G_{\theta,z}(z)u'(v-\beta_D(z))\beta'_D(z)\right) \\
& -\phi\left(u(v-\beta_D(z))\right) d\mu_z \\
= & -\Delta\phi'\left(u(v-\beta_D(z))\right)\int G'_{\theta,z}(z)u(v-\beta_D(z)) - u'(v-\beta_D(z))\beta'_D(z)d\mu_z.
\end{aligned}$$

As $\Delta \to 0$, in equilibrium the marginal gain should be zero at $z = v$,

$$\begin{aligned}
\phi'\left(u(v-\beta_D(v))\right)\left[u'(v-\beta_D(v))\beta'_D(v) - \int G'_{\theta,v}(z)u(v-\beta_D(v))d\mu_v\right] &= 0, \\
u'(v-\beta_D(v))\beta'_D(v) - u(v-\beta_D(v))\int G'_{\theta,z}(v)d\mu_z &= 0, \\
u'(v-\beta_D(v))\beta'_D(v) - u(v-\beta_D(v))\int \frac{G'_\theta(v)}{G_\theta(v)}\frac{G_\theta(v)}{\int G_\vartheta(v)d\mu}d\mu &= 0, \\
u'(v-\beta_D(v))\beta'_D(v) - u(v-\beta_D(v))\int \frac{G'_\theta(v)}{G_U(v)}d\mu &= 0, \\
\beta'_D(v) &= \frac{u(v-\beta_D(v))}{u'(v-\beta_D(v))}\frac{G'_U(v)}{G_U(v)}. \quad (7)
\end{aligned}$$

This result holds for any differentiable $\phi(\cdot)$, implying that in the dynamic auction, the optimal strategy does not depend on the ambiguity aversion level of the bidders.

Lemma 3 *In a Dutch Auction with Smooth Ambiguity the equilibrium bidding strategy is independent of the Ambiguity Attitude of the bidders, i.e.* $\beta_D = \beta_{D,N}$.

Proof. Above. ■

Lemma 4 *Expected utility, given by smooth ambiguity preferences, from an ambiguous Dutch auction is lower than that of the equivalent unambiguous one.*

Proof Given the concavity of ϕ, it follows that

$$\begin{aligned}
\int \phi\left(G_\theta(v)u(v-\beta_D(v))\right) d\mu &< \phi\left(\int G_\theta(v)u(v-\beta_D(v))d\mu\right) \\
&= \phi\left(\int G_\theta(v)u(v-\beta_{D,N}(v))d\mu\right). \ ■
\end{aligned}$$

One important corollary follows from the previous results.

Corollary 1 *If there is any participation cost in the Dutch Auction, less bidders will choose to participate in an ambiguous auction than in the equivalent unambiguous one.*

These results also show that first-price sealed-bid auctions are not equivalent to open price descending auctions when ambiguity and ambiguity aversion are present. Karni (1988) points out that that equivalence only holds necessarily with expected utility maximizing agents. Moreover, this bidding difference which cannot be explained by risk aversion, is in agreement with the experimental literature (see e.g. Kagel and Roth (1995)) which shows that first-price sealed-bid auctions have bids and revenues which are higher than those of the risk neutral Nash Equilibrium and of the Dutch auctions.

5.0.3 Anticipating consequentialism

As discussed in the introduction of this section, it is not clear how dynamic ambiguity should be modeled. It is possible however to see that even if the bidder anticipates his consequentialist and therefore possibly dynamic inconsistent beliefs, she still chooses to play the same equilibrium bidding strategy - provided that the others do the same.

A bidder who evaluates her equilibrium strategy before the bidding price arrives, that is with previous priors, may find the equilibrium strategy to be suboptimal. That is the case at the beginning, where the bidder would rather behave as in the first-price sealed-bid auction. Given that there is no a priori way of setting the bid in a dynamic auction, the bidder can only choose to bid immediately instead of bidding at the equilibrium strategy. So one should compare the certain payoff at a higher bid b with the expected payoff of waiting until the equilibrium, using for this the priors updated until then.

Given that closed form solutions are needed to make this comparison it is impossible to establish a general result, but some examples indicate that the bidders opt for playing the equilibrium strategy defined above. Take for instance the set of equally probable ($\mu_1 = \mu_2 = \frac{1}{2}$) priors $\mathcal{P} = \{F_1, F_2\}$ with $F_1(x) = x^{\frac{m}{n-1}}$ and $F_2(x) = (2x^{n-1} - x^m)^{\frac{1}{n-1}}$, with m chosen appropriately (guaranteeing that F_1 and F_2 are non-decreasing and with codomain $[0, 1]$), risk neutrality and $\phi(h) = \frac{1}{\alpha}h^\alpha$. The reduced distribution will be $G_U(x) = x^{n-1}$ so that the equilibrium strategy is $\beta_D(v) = \frac{n-1}{n}v$. The updated priors conditional on the maximum value of bidders having values lower than y, $0 \leq y \leq 1$, will be

$$F_{i,y}(x) = \frac{F_i(x)}{F_i(y)}, \qquad \mu_i(y) = \frac{\frac{1}{2}F_i^{n-1}(y)}{\frac{1}{2}\left(F_1^{n-1}(y) + F_2^{n-1}(y)\right)} = \frac{F_i^{n-1}(y)}{y^{n-1}}, \quad i = 1, 2.$$

At bid b the bidder with value v compares the payoff of stopping, $\frac{1}{\alpha}(v - b)^\alpha$ with that of waiting until the equilibrium bid $\beta_D(v)$,

$$\sum_{j=1,2} \mu_j(y) \left[(F_{j,y}(v))^{n-1} \left(v - \frac{n-1}{n}v \right) \right]^\alpha,$$

where $y = \min\{1, \frac{n}{n-1}b\}$. Notice that for $b > \frac{n-1}{n}$ and assuming that all bidders play the equilibrium strategy, there is still no value that can be discarded

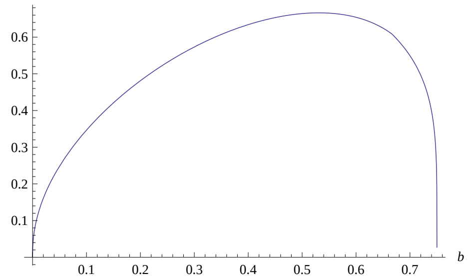

Figure 1: Expected utility as anticipated at the beginning of the auction, as a function of the bid b, for a bidder with $v = \frac{3}{4}$.

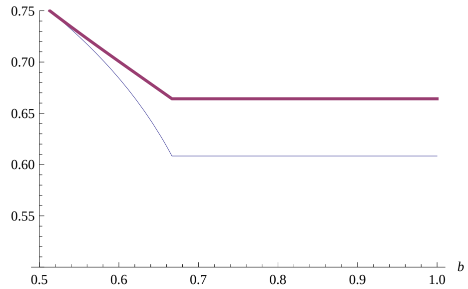

Figure 2: Expected utility as anticipated as bid b is reached, of playing the equilibrium bid strategy β_D (thick) and of accepting the momentary price b, for a bidder with $v = \frac{3}{4}$.

because b is higher than any equilibrium bid. There is therefore no update of the priors.

Let $m = 4$, $n = 3$ and $\alpha = \frac{1}{2}$. At the beginning of the auction, stopping at a future b yields the expected utility displayed in Figure 1 for a bidder with value $v = \frac{3}{4}$. This is the problem that the bidder faces in the first-price sealed-bid auction, the maximum payoff occurs thus at a bid higher than the equilibrium strategy in the open price descending auction, $\frac{n-1}{n}v = \frac{1}{2}$. There is a slight kink at $b = \frac{2}{3}$, which is the equilibrium bid of the bidder with the highest value. The probability of winning has therefore a kink because it goes below 1 for $b < \frac{2}{3}$.

Figure 2 represents the expected utility of two possible strategies, equilibrium strategy β_D and stopping at the current bid b, at different timings of the descending auction, more precisely at bid b. Contrary to Figure 1, here only the probability of winning is changing with b. For $b \in [\frac{2}{3}, 1]$ there is no type of opponent that can be discarded, there is thus no update of the priors and the probabilities are fixed. The important aspect of this graph is to show that the equilibrium strategy β_D (even if not being the optimal bid for any point in time with $b > \beta_D$) always outperforms the only possibility that the bidder at ongoing bid b has, to stop at b. At each point the bidder that anticipates his changing preferences, cannot do better than wait and play β_D.

6 Conclusion

In Auction Theory one of the basic assumptions is that of common knowledge of the distribution of the private values of the bidders, that is each bidder knows the distribution from which the values of her opponents are drawn. This paper relaxes this assumption in the spirit of the literature in Ambiguity Aversion with multiple priors and derives the equilibrium bids in basic single-good auctions.

It is shown that ambiguity aversion increases the bid in the first-price sealed-bid auction, but ambiguity has no impact in open price descending auctions. While the first result is intuitive, the second result follows from the fact that as the auction occurs and the price descends, the bidders learn about the distribution of the values of their opponents, eroding thus the ambiguity that was present in the beginning.

This entails two important results. The first concerns Auction Theory, it indicates that first-price sealed-bid auctions and open price descending need not to be theoretically equivalent. This implies that, in the presence of ambiguity, there is no revenue equivalence between those auctions.

The second is a significant result in Ambiguity Aversion, because the paper provides a new example where ambiguity aversion and risk aversion do not have the same qualitative effect on the outcomes of a model. Gollier (2009) in a portfolio choice model, shows that ambiguity aversion may lead to an increased demand of a risky or ambiguous asset. The present paper sustains that ambiguity aversion has the same qualitative consequence on static auctions as risk aversion, but that it is not the case for dynamic auctions.

7 Appendix

7.1 Ambiguous mean

Here it will be proven that for the example with ambiguous mean, the ratio

$$\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)}$$

weakly decreases with α. For any v with $v \leq 1-a$, the fraction is

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= \frac{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v) \cdot (n-1) F_\theta^{n-2}(v) F'_\theta(v)}{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v) \cdot F_\theta^{n-1}(v)} \\
&= (n-1) \frac{\frac{1}{a}\left(\frac{v}{a}\right)^{\alpha(n-1)-1} + 0}{\left(\frac{v}{a}\right)^{\alpha(n-1)} + 0} \\
&= (n-1)\frac{1}{v},
\end{aligned}$$

which is independent of α.

For any v with $1-a < v \leq a$, the fraction is

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= \frac{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v) \cdot (n-1) F_\theta^{n-2}(v) F'_\theta(v)}{\sum_\theta F_\theta^{(n-1)(\alpha-1)}(v) \cdot F_\theta^{n-1}(v)} \\
&= (n-1) \frac{\frac{1}{a}\left(\frac{v}{a}\right)^{\alpha(n-1)-1} + \frac{1}{a}\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)-1}}{\left(\frac{v}{a}\right)^{\alpha(n-1)} + \left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)}} \\
&= (n-1) \frac{1 + \left(\frac{v-(1-a)}{v}\right)^{\alpha(n-1)-1}}{v + \left(\frac{v-(1-a)}{v}\right)^{\alpha(n-1)}} \\
&= \frac{n-1}{v} \frac{1 + \left(1 - \frac{1-a}{v}\right)^{\alpha(n-1)-1}}{1 + \left(1 - \frac{1-a}{v}\right)^{\alpha(n-1)}}. \qquad (8)
\end{aligned}$$

The following derivative

$$\frac{\partial}{\partial q} \frac{1 + y^{q-1}}{1 + y^q} = \frac{(1-y) y^{q-1} \ln y}{(1+y^q)^2},$$

with $y \in (0,1)$ and $q > 0$ is negative. Substituting $y = 1 - \frac{1-a}{v}$ and $q = \alpha(n-1)$, it is concluded that (8) is decreasing in α for any $v \in (1-a, a]$.

For any v with $v > a$,

$$\begin{aligned}
\frac{\sum_\theta \phi'\left(G_\theta(v)\right) G'_\theta(v)}{\sum_\theta \phi'\left(G_\theta(v)\right) G_\theta(v)} &= (n-1) \frac{0 + \frac{1}{a}\left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)-1}}{1 + \left(\frac{v-(1-a)}{a}\right)^{\alpha(n-1)}} \\
&= (n-1) \frac{1}{a\left(\frac{v-(1-a)}{a}\right)^{1-\alpha(n-1)} + (v-(1-a))}.
\end{aligned}$$

Given that $a < v \leq 1$ it follows $0 < \frac{v-(1-a)}{a} \leq 1$, so the above fraction is decreasing in α.

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