

Fiscal policy, entry and capital accumulation: hump-shaped responses.

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Abstract

In this paper we consider the entry and exit of firms in a Ramsey model with capital and an endogenous labour supply. At the firm level, there is a fixed cost combined with increasing marginal cost, which gives a standard U-shaped cost curve with optimal firm size. The costs of entry (exit) are quadratic in the flow of new firms. The number of firms becomes a second state variable and the entry dynamics gives rise to a richer set of dynamics than in the standard case: in particular, there is likely to be a hump shaped response of output to a fiscal shock with maximum impact after impact and before steady-state is reached. Output and capital per firm are also likely to be hump shaped.

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1 Introduction

Stephen Turnovsky developed the τ continuous time representative agent model to include an endogenous labour supply¹ in order to explore the effects of fiscal policy² (τ, τ, τ, τ). When applied to fiscal policy, this model is essentially Ricardian in nature: for every government expenditure flow there is an equivalent tax shadow in the form of current or past taxes. This is an income effect which serves to reduce consumption (crowding out) and increase the labour supply (when leisure is normal). If we look at the dynamics, we find that the impact on consumption is greater in the short-run than the long-run: consumption falls a lot on impact and then increases gradually to the new steady-state (this increasing pathway is determined by the fact that the marginal product of capital exceeds the discount rate, but the gap is diminishing as capital is accumulated). This "overshooting" of consumption is also present in the response of output: there is a big initial response, with output falling to its new higher level. If we look at empirical studies of macroeconomic time-series in the form of VARs, we find a different story³. The impulse-responses implied by VARs indicate that output follows a hump shaped response: the maximum impact on output is not on impact, but some time later (3-4 quarters) - see for example τ . This sort of response is ruled out in the standard Ramsey model, which is clearly missing some vital ingredient. In this paper, we argue that if we include a process of entry (and exit) then we can keep the basic structure of the Ramsey model and move towards understanding the processes giving rise to the sorts of behavior we find in the data as represented by empirical VARs. More specifically, we find that output can have a hump-shaped response to a fiscal shock with the peak effect being after some time. Furthermore, this is not a special case at all, but a general (although not universal) feature. Whilst we do find that consumption can have a non-monotonic path, this is more exceptional.

In this paper we analyze a continuous-time model of entry in which firms produce output with capital and labour. The creation or destruction of firms (both flows) is determined endogenously as in τ and τ : the cost of entry (exit) is determined by the flow of firms into (out of) the market, so that the equilibrium price of entry (exit) equals the net present value of incumbency. We use this to explore the response of the economy to fiscal policy in terms of a permanent unanticipated change in

¹The model with an exogenous labour supply is often known as the Cass-Koopmans model after τ and τ - see τ for a discussion and exposition.

²This mirrored what was also being done in the discrete time real business cycle approach, although fiscal policy was not incorporated until τ .

³Of course, the VAR methodology is inherently reduced form and in that sense it has proven very problematic to uncover and differentiate between exogenous and endogenous fiscal shocks as well as across shocks.

government expenditure. We find that the presence of an endogenous labour supply allows for a variety of local dynamics. In particular, the stable eigenvalues can both be real or complex. With real eigenvalues, the adjustment path of either or both capital and the number of firms can be hump-shaped. With complex eigenvalues, the adjustment path of both capital and the number of firms are oscillatory. However, when we analyze the effects of fiscal policy, we find that only the number of firms can be non-monotonic. This is because the "initial position" also matters, and there is a linear relation between the number of firms and the capital stock as Government expenditure varies which restricts the dynamics to be monotonic for capital. However, whilst the dynamics of consumption and capital are monotonic, we find that there can be a non-monotonic "hump shaped" response of output and employment to fiscal policy, taking the form an initial jump response, followed by a smooth hump shaped overshooting of the new steady-state. Moreover, we find that in addition the impact effect of fiscal policy can be greater than or less than the long-run effect. Hence the introduction of entry into the Ramsey model enables a theoretical understanding of output and consumption responses that are more in line with the empirical evidence than the classic Ramsey model.

In the classic Ramsey model, all that matters for determining aggregate output and productivity is the aggregate labour and capital: how it is organized at the firm level does not matter. This approach is justified if there is constant returns to scale at the firm level. However, in this paper we assume an explicit firm-level technology which gives rise to the text-book U-shaped average cost with strictly increasing marginal cost. In the long-run with zero-profits, the aggregate economy will display constant returns in capital and labour⁴. However, in the short-run as we move towards steady-state, output at the firm level will deviate from the efficient minimum AC output. This is important, because it means that capacity⁵ utilization will vary and hence the marginal products of labour and capital: for a given level of capital, more firms means less capital per-firm which increases the marginal product of capital. With entry, the dynamics of the number of firms can act to counteract the effect of capital accumulation on reducing the marginal product of capital. It is this crucial difference which enables the model in this paper to display behavior which is more in line with the broad outline of the empirical data.

The model of entry we employ can be contrasted with the ? model used in trade models. In the Melitz model, the cost of entry is constant and does not depend on

⁴Since zero-profits in the long-run means that the number of firms rises proportionately with capital and labour so that firms remain at efficient scale at the point where average and marginal cost are equal

⁵Note that we use capacity utilisation in its standard sense to mean the level of actual output relative to the reference level of efficient output. Some authors have used this to mean capital utilisation, which has to do with the intensity of how capital is used.

the flow of entry. However, there is post-entry heterogeneity in the productivity of firms, unlike in the Ramsey framework we adopt where all firms have access to the same technology. Another important difference is that there are increasing returns to scale at the firm level in the Melitz model, not the U-shaped average cost curves in this paper. In our steady-state, all firms are efficient irrespective of the number of firms: inefficiency occurs out of steady-state as (all) firms operate at an output above or below efficient capacity. In the Melitz model, the steady state is not optimal in efficiency terms: there is excess entry due to the monopolistic post-entry equilibrium, and the model is incompatible with perfect competition. The entry model also differs from ?, where there are no sunk entry costs and there is a zero profit condition that flow operating profits cover the flow overheads. As in ? and ?, the elasticity of demand varies due to the effect of market share on firm elasticity. Again, this is an essentially monopolistic framework which is not generally efficient in equilibrium: in the absence of love of variety, there will be excess entry and each firm has an increasing returns to scale technology.

It should be stressed that we are providing a *theoretical* framework and not an empirical model: we present numerical calibrations merely to illustrate the general results. If we look at empirical DGSE models that are currently popular, they have many ingredients that are absent in our theoretical model. In ?, ?, ? there are a combination of factors determining the dynamic response a fiscal policy: habit formation in consumption, capital adjustment costs, capital utilization as well as new Keynesian price and wage nominal rigidity. Indeed, as ? points out, the reaction to monetary policy may be a crucial factor, something which is totally absent in our real model. However, theory is still useful in providing a general understanding that calibrated numerical models cannot (even when they appear to "fit" the data). In particular, we are able to show in our model that a hump shaped response of output is a general (although not universal) feature that can be understood in terms of a phase diagram, and is not just the outcome of particular sets of parameter values.

The outline of the paper is as follows. In section ?? we outline the basic model of the consumer, firm and the entry process. In section ?? we determine the equilibrium in the economy in terms of the social planner's problem. In section ?? we analyze the steady-state and present a graphical analysis of the long-run fiscal multiplier. Section ?? provides a general analysis of the local dynamics. Section ?? then applies the previous section to the dynamics of the economy in response to fiscal policy and in particular non-monotonic trajectories in state and non-state variables. Section ?? concludes.

2 The model

In this paper we model a perfectly competitive dynamic general equilibrium economy with endogenous entry/exit of firms and accumulation of capital as a Social Planning optimum. We will first lay out the model in terms of household preferences, firm level and aggregate technology, and the costs of entry and exit. We will then derive the optimality conditions for the first-best optimum.

In each instant, households consume and supply labour $(C(t), L(t))$, with the intertemporal utility function defined by the flow utility and discount rate $\rho > 0$:

$$\bar{U} = \int_0^\infty U(C(t), L(t))e^{-\rho t} dt$$

Assumption 1. Representative Household Preferences. *The instantaneous utility function: is increasing in C , is decreasing in L , is additively separable and concave in (C, L) ($U_C > 0$, $U_L > 0$, $U_{CC} < 0$, $U_{LL} < 0$, and $U_{CL} = 0$); Inada conditions on consumption and labour supply hold: $\lim_{C \rightarrow 0^+} U_C = +\infty$, $\lim_{C \rightarrow +\infty} U_C = 0$, $\lim_{L \rightarrow \infty} U_L = -\infty$, and $\lim_{L \rightarrow 0^+} U_L = 0$.*

Output is produced by capital and labour which is distributed across a continuum of firms $i \in R_+$. At instant t , there is a measure $n(t)$ such that $i \in [0, n(t)]$ are active (incumbent) and $i > n(t)$ are inactive (potential entrants). An active firm $i \in [0, n(t)]$ incurs a fixed overhead cost and produces output $y(i)$ according to the following technology:

$$y(i, t) = AF(k(i, t), l(i, t)) - \phi$$

Assumption 2. Firm level technology. *The active firm's production function: is increasing in k and l and is concave in (k, l) ($F_k > 0$, $F_l > 0$, $F_{kk} < 0$, $F_{ll} < 0$, $F_{kl} \geq 0$, and $F_{kk}F_{ll} - F_{lk}^2 > 0$); it is homogeneous of degree ν , where $\nu \in (0, 1)$; and standard Inada conditions hold. There is a (flow) overhead cost $\phi > 0$.*

Firms which are *inactive* produce no output, employ no labour and capital and incur no overhead. Note that even if an active firm produces no output, it will still incur the overhead⁶. Under assumption ??, the firm level technology corresponds

⁶We do not impose the restriction that $y(i, t) \geq 0$; for example, if the firm hires no labour or capital, it will have to pay the overhead ϕ , which is a negative output corresponding to negative profits. The only way it can avoid the overhead is by exiting the market.

to the Marshallian U -shaped average cost curve with increasing marginal cost. As outlined in ?, the corresponding *efficient* level of production is:

$$y^e = \frac{\phi\nu}{1 - \nu} \quad (1)$$

which is increasing in ϕ and ν , and independent of A .

Now, for a given measure of active firms $n(t)$, production is maximized if aggregate capital $K(t)$ and labour $L(t)$ are divided equally between the active firms, so that we have

$$k(i, t) = \frac{K(t)}{n(t)}, \quad l(i, t) = \frac{L(t)}{n(t)}.$$

Hence the aggregate output produced by active firms (dropping the time subscript):

$$Y = \int_0^n y(i) di = n \left[AF \left(\frac{K}{n}, \frac{L}{n} \right) - \phi \right],$$

where $Y = Y(K, L, n, A, \phi)$ is the aggregate production function. Y is homogenous of degree 1 in (K, L, n) and we have the following marginal products:

$$\begin{aligned} r &= AF_k(K, L, n) = AF_k \left(\frac{K}{n}, \frac{L}{n} \right) \\ w &= AF_l(K, L, n) = AF_l \left(\frac{K}{n}, \frac{L}{n} \right) \\ \pi &= \pi(K, L, n, A, \phi) = (1 - \nu) AF \left(\frac{K}{n}, \frac{L}{n} \right) - \phi \end{aligned}$$

The derivatives are as in ?. Because capital and labour are distributed equally across firms, the marginal products of labour and capital at the aggregate level are equal to the marginal products at the firm level.

The marginal product of n corresponds to the profit or surplus per firm: total production less the cost of capital and labour when they are valued at their marginal products, and can be positive or negative. An additional firm means that capital and labour per firm are lower which raises the marginal products; however, an additional overhead is also incurred. The marginal product of n is zero when output per firm is equal to the efficient level (??). Hence zero profits corresponds to an efficient scale of production at the firm level, y^e . Profits are positive when firms produce more than y^e and negative when below. Using this we can define the "efficient" production function:

$$Y^e(K, L, A, \phi) = \max_n Y(K, L, n, A, \phi) \quad (2)$$

Assumption 3. Entry: *There is a quadratic adjustment cost in the flow of entry (exit).*

The total cost of entry $Z(t)$ is the integral

$$Z(t) = \gamma \int_0^{e(t)} i \, di = \frac{\gamma}{2} e(t)^2.$$

The model of entry is represented in the decentralized form in ? and is based on ? and ?. Here we present it in the equivalent form for the Social Planner. There is a (marginal) cost $q(t)$ to setting up a new firm which is a linear function of the flow of entry $e(t) = \dot{n}(t)$. Likewise, there is a cost of $-q(t)$ to dismantle an existing firm. The marginal cost of entry depends on the flow of entry

$$q(t) = \gamma e(t) \quad (3)$$

There is a congestion effect or a fixed factor involved in the process of setting up or dismantling a firm.

The output of firms is used for consumption, government spending and the setting up new firms or dismantling existing ones and investment. We assume for simplicity that there is no depreciation of capital $I(t) = \dot{K}(t)$. Hence:

$$Y(t) = C(t) + I(t) + Z(t) + G(t)$$

Hence from (??) we get the capital accumulation equation:

$$\dot{K} = n \left[AF \left(\frac{K}{n}, \frac{L}{n} \right) - \phi \right] - C - \gamma \frac{e^2}{2} - G \quad (4)$$

equivalently as an equilibrium condition in the product market

We assume that the government demands output G which is financed by a Lump-sum tax T . Since there is Ricardian equivalence in this model, we assume the budget is balanced instant by instant:

$$T(t) = G(t).$$

3 The social planner's problem

In ? we analyzed the market (decentralized) equilibrium and showed that it is equivalent to the social planner's problem. In this paper we adopt the social planner's problem of maximizing the utility of the representative household:

$$\max_{C,L,e} \int_0^{+\infty} U(C, L) e^{-\rho t} dt$$

subject to equation (??) and $\dot{n} = e$, given $K(0) = K_0$ and $n(0) = n_0$. There are two state variables (n, K), there are three control (jump) variables (C, L, e) and the usual boundedness properties for the state variables are assumed to hold. We treat A and G as exogenous parameters: later on we will derive how the system responds in the long and short run to variations in G .

As both the utility function and the constraints of the problem are concave functions of the controls, then (if the transversality conditions hold) the Pontryagin maximum principle will give us necessary and sufficient conditions for optimality. The current value Hamiltonian is,

$$H \equiv U(C, L) + p_K \left\{ n \left[AF \left(\frac{K}{n}, \frac{L}{n} \right) - \phi \right] - C - \gamma \frac{e^2}{2} - G + qe \right\}$$

where p_K is the co-state variable associated to the aggregate capital stock and q is the relative cost associated to the number of firms. The static arbitrage condition for consumption, labour supply and entry (C, L, e) are:

$$U_L(L) + U_C(C)AF_L \left(\frac{K}{n}, \frac{L}{n} \right) = 0 \quad (5)$$

From equation (??) the optimal choice of labour can be represented as a function of (C, K, n, A)⁷

$$L = \hat{L}_{-,+}(\underline{C}, \underline{K}, \underline{n}, \underline{A}). \quad (6)$$

Note that G does not affect L independently of C : for a given wage w , there is an Income Expansion path (*IEP*) which is strictly monotonic and hence there is a 1 – 1 relation of C to L . Changes in G simply move the household up and down the *IEP* and hence given C there is a unique L .

Using (??), we can represent the marginal product of capital r and the profit per firm π and gross output Y as "reduced form" functions of (C, K, n, A, ϕ) by substituting out the labour supply $L = \hat{L}$:

$$\begin{aligned} r(\underline{C}, \underline{K}, \underline{n}, \underline{A}) &= AF_k \left(\frac{\underline{K}}{\underline{n}}, \frac{\hat{L}}{\underline{n}} \right) \\ \pi(\underline{C}, \underline{K}, \underline{n}, \underline{A}, \phi) &= (1 - \nu)AF \left(\frac{\underline{K}}{\underline{n}}, \frac{\hat{L}}{\underline{n}} \right) \\ Y(\underline{C}, \underline{K}, \underline{n}, \underline{A}, \phi) &= n \left[AF \left(\frac{\underline{K}}{\underline{n}}, \frac{\hat{L}}{\underline{n}} \right) - \phi \right] \end{aligned}$$

⁷By using the implicit function theorem we have: $\hat{L}_C = -wU_{CC}/H_{LL} < 0$, $\hat{L}_K = -w_K U_C/H_{LL} > 0$, $\hat{L}_n = -w_n U_C/H_{LL} > 0$ and $\hat{L}_A = -w_A U_C/H_{LL} > 0$. where $H_{LL} = U_{LL} + U_C w_L$.

Note that whilst the effect of n on output is in general ambiguous, around the steady-state it is unambiguously (strictly) positive: that is because the direct effect of n on Y is zero ($Y_n = \pi = 0$), but there is a strictly positive indirect effect on the marginal product of labour which leads to more labour being supplied⁸. This indirect effect would not happen if F was homogenous to degree 1 ($\nu = 1$), since then the marginal product would depend only on the ratio of capital to labour at the firm level which would not vary as n varies. This indirect effect of n on the labour supply is always positive, whilst the direct effect captured by $\pi(n)$ can be negative (or positive) depending on whether n is above (or below) its efficient level given K and L .

The optimality condition for entry is

$$\gamma e = q. \tag{7}$$

If we define the inverse of the elasticity of intertemporal substitution $\sigma(C) = -U_{CC}(C)C/U_C(C)$, we get the Euler equation for consumption

$$\dot{C} = \frac{C}{\sigma(C)}(r(C, K, n, \cdot) - \rho). \tag{8}$$

The relative co-state variable associated to the number of firms, verifies the differential equation in q :

$$\dot{q} - r(C, K, n, \cdot)q = -\pi(C, K, n, \cdot). \tag{9}$$

Integrating this gives us:

$$q(t) = \int_t^\infty \pi(s) e^{-\int_t^s r(\tau) d\tau} ds,$$

so that along the optimal path the marginal cost of entry (exit) is equated to the net present value of profits. Furthermore, (??) is also an *arbitrage equation*, since it can be written as:

$$r(C, K, n, \cdot) = \frac{\pi(C, K, n, \cdot)}{q} - \frac{\dot{q}}{q}.$$

Investing one unit of the consumption good in capital yields the marginal product of capital r . Investing one unit in a new firm yields a share q^{-1} of the profit flow π of the new firm and a capital gain/loss \dot{q} . Along the optimal path, these two will be equated.

⁸Formally: $Y_n = \pi + w\hat{L}_n$.

From equation (??), then $\dot{e}\gamma = \dot{q}$, and we obtain

$$\dot{e} = r(C, K, n, \cdot)e - \frac{\pi(C, K, n, \cdot)}{\gamma}. \quad (10)$$

Hence the dynamic general equilibrium is represented by the paths (C, e, K, n) which solve the system (suppressing the parameters (A, ϕ)) (??) -(??) and

$$\dot{K} = Y(C, K, n, \cdot) - C - \frac{\gamma}{2}e^2 - G, \quad (11)$$

$$\dot{n} = e \quad (12)$$

together with the initial conditions $K(0) = K_0$ and $n(0) = n_0$, and the transversality conditions

$$\lim_{t \rightarrow \infty} e^{-\rho t} U_C(C(t))K(t) = \lim_{t \rightarrow \infty} e^{-\rho t} U_C(C(t))e(t)n(t) = 0. \quad (13)$$

We have the two equations (??) and (??) governing the state variables (n, K) , and the two optimality conditions for the controls (C, e) (??) and (??).

Perfect foresight equilibrium trajectories converge to the steady state of the system composed by equations (??), (??), (??) and (??). Therefore, the stable manifold of this system is the set of all equilibrium trajectories. In the next section ?? we determine the steady state and discuss how it changes in fiscal policy, and in section ?? we characterize the stable manifold by presenting a method for dealing with short-run dynamics in the present four-dimensional system. In particular, we establish the conditions under which we will have non-monotonic equilibrium paths.

4 The Steady-State and long-run effects of fiscal policy.

For a given level of Government expenditure, the steady state, (C^*, e^*, K^*, n^*) , exists and is unique and verifies ⁹:

$$r^* \equiv r(C^*, K^*, n^*, A) = \rho, \quad (14)$$

$$\pi^* \equiv \pi(C^*, K^*, n^*, A, \phi) = 0, \quad (15)$$

$$e^* = 0, \quad (16)$$

$$C^* + G = Y^* \equiv \hat{Y}(C^*, K^*, n^*, A, \phi), \quad (17)$$

⁹A simple proof of the uniqueness of the steady state results from the fact that the partial derivatives of the functions r and π as regards C, K, n are globally non-zero and the application of the inverse function theorem. Equations (??) and (??) can be solved for K and n to get K^* and n^* as a function of C . Then substituting in equation (??) we get C^* .

and

$$n^* = \left(\frac{\phi\nu}{1-\nu} \right) Y^*. \quad (18)$$

Therefore, in the steady state profits are zero, there is no net entry, the real rate of return is equated to the rate of time preference and output is only used for (private and government) consumption purposes. Since profits are zero, output per firm is at the efficient level and hence from (??) the number of firms is proportional to output. In order to see how the steady-state operates, it is useful to provide a simple illustration (which we will use in later sections):

Example: benchmark case We consider a Cobb-Douglas technology and a separable and isoelastic instantaneous utility functions:

$$\begin{aligned} F(K, L) &= K^\alpha L^\beta, \quad 0 < \alpha, \beta < 1, \\ U(C, L) &= \frac{C^{1-\sigma} - 1}{1-\sigma} - \xi \frac{L^{1+\eta}}{1+\eta}, \end{aligned}$$

where $\sigma > 0$, $\xi > 0$, $\eta \geq 0$ and $\nu = \alpha + \beta \in (0, 1)$.

The unique steady state for the benchmark case is given by $K^* = k^*n^*$, $L^* = l^*n^*$ where the firm level stock of capital k^* and labour input l^* are dependent on the technological parameters

$$k^* = \frac{\alpha\phi}{\rho(1-\nu)}, \quad l^* = \left(\left(\frac{\rho}{\alpha} \right)^\alpha \left(\frac{\phi}{1-\nu} \right)^{1-\alpha} \right)^{1/\beta}, \quad y^* = \frac{\nu\phi}{1-\nu}.$$

We determine consumption as a function of n ,

$$C(n) = \left(\frac{\beta}{\xi} y^* (l^*)^{-(1+\eta)} (n)^{-\eta} \right)^{1/\sigma}.$$

Then steady state consumption is $C^* = C(n^*)$ where n^* cannot be determined in closed form, but is implicitly given by:

$$n^* = \{n : ny^* = C(n) + G\},$$

If $G = 0$ then we can determine C^* and n^* explicitly:

$$C^* = \left(\frac{\xi\nu}{\beta l^*} (y^*)^{1+\eta} \right)^{1/(\eta+\sigma)}$$

and $n^* = C^*/y^*$.

□

The purpose of this paper is to focus on the dynamic effects of fiscal policy in terms of the effects over the aggregate variables, in particular C, K, n . However, as a useful preliminary, we will first develop a simple graphical analysis of the long-run (steady-state) effects of changes in G on these variables. The dynamic system has four dimensions with two state variables (n, K) and two controls (e, C) . We can look at this graphically by projecting the system onto two-dimensional subspaces. In our analysis of the dynamics we will focus on (n, K) . However, for the purposes of focussing on the steady-state, we also look at consumption-labour space (C, L) . The formal derivation of the long-run multipliers is given in Proposition ?? below.

Turning first to (n, K) space, we can take the two equations (??)- (??) and define the two locii:

$$\begin{aligned} r(K, L^*(G), n, A) &= \rho \\ \pi(K, L^*(G), n, A, \phi) &= 0 \end{aligned}$$

The first locus $r = \rho$ defines the combinations of (K, n) which equate the marginal product of capital with ρ given steady-state labor supply $L^*(G)$ and technology parameter A . This is upward sloping, since by diminishing marginal productivity at the firm level, an increase in K (given n) *reduces* r , whilst an increase in n *increases* it. The second locus $\pi = 0$ is the combinations of (K, n) which yield zero-profits given $(L^*(G), A, \phi)$. As we will see later, this locus may be positively or negatively sloped in the (K, n) -space when the labour supply is allowed to vary, but since we are fixing it at $L^*(G)$ it is always positively sloped. However, the following relationship it is always less steep than $r = \rho$:

$$\left. \frac{dn}{dK} \right|_{r^*=\rho} > \left. \frac{dn}{dK} \right|_{\pi^*=0} .$$

Steady-state occurs where the two locii intersect. See Figure ?? ¹⁰.

Secondly, we look at the steady-state in consumption-labour space to see how labour supply $L(G)$ and consumption $C(G)$ are determined given G . First, note that the steady-state equilibrium conditions (??), (??) define the steady-state real wage w^* (marginal product of labour). This is because output per firm is at the efficient level (??), and the marginal product of capital is equal to ρ . Since the function F is *Hod1* in (K, L) , (??) determines the capital-labour ratio and hence the marginal product of labour in steady-state. In (C, L) space, we can trace out the steady-state

¹⁰See the derivations of the curves in Appendix ?? for the benchmark case.

Income Expansion Path (IEP) for consumption and labour given w^* at different levels of non-labour income¹¹. The *IEP* is *downward sloping* (since consumption and the absence of work are normal goods). The convex slope of the *IEP* results from diminishing marginal utility of consumption and increasing disutility of labour with additive separability. The Inada conditions (Assumption 1(b)) ensure that C goes to infinity as L reaches 0 and vice versa.

Furthermore, given the steady-state capital labour ratio, we also know output-per-unit labour in steady-state Y^*/L^* . Hence, we can depict the aggregate trade-off between consumption and leisure in steady-state, or the *Euler Frontier*¹² (*EF*) (see ??): (??) become $C + G = L.(Y^*/L^*)$. The *EF* is an upward sloping line, with $C = -G$ when $L = 0$. The economy is in steady-state where the *IEP* and *EF* intersect, as depicted in Figure ??.

[Figure ?? here]

We can now trace the effects of a permanent increase in G : the initial equilibrium is A , and the equilibrium after the increase is B . Turning first to consumption-leisure space, the *EF* shifts down by a vertical distance equal to the increase in government expenditure. As in the new Keynesian analysis of fiscal policy (?, ?, ?), this income effect leads to move down the *IEP* with consumption falling and labour supply (and output) increasing. Hence the fiscal multiplier is strictly positive but less than 1 due to the crowding out of consumption. The function $C^*(G)$ traces out the steady-state relation between C and G , and is strictly decreasing with a derivative less than -1 . Turning next to (n, K) space, since we know that the increase in G has decreased C^* , and for a given (n, K) the steady-state labour supply $L^*(G)$ has increased. This results in a rightward shift in the $r^* = \rho$ locus and an upward shift in the $\pi^* = 0$ locus. The new equilibrium is at point B , to the north-west A . Indeed, since the steady-state equations define the capital-labour ratio, the steady-state ratio $(n/K)^*$ is also determined. As we vary G , in (n, K) the steady-state moves along a ray through the origin $n = K./k^*$, the dotted line passing through A and B . We denote this linear steady-state locus "the dG line"

With an exogenous labour supply, fiscal policy is very simple to analyze. In effect, the labour supply function (??) becomes fixed at some level $\hat{L} = \bar{L}$, and all of

¹¹Note that in standard microeconomics, the *IEP* is defined in terms of leisure and consumption. The format is slightly different here, because we have labour supply as a bad in the utility function without any explicit reference to leisure.

¹²We use the term "Euler Frontier" because it means that for a given labour supply L , with zero-profits (efficient production), capital has accumulated to equate the marginal product of capital with ρ . The resultant output can be divided between C and G . Note that the real wage in steady state is *less* than the slope of the *EF*, since wages form only part of total income (the rest being capital's share). Hence the *EF* is *not* tangential to the indifference curve at equilibrium.

the derivatives of $\hat{L} = 0$. In terms of consumption-labour space, the *IEP* becomes vertical at \bar{L} . There is 100% crowding out: so the function $C^*(G)$ takes the simple linear form $C^*(G) = C^*(0) - G$ where $C^*(0)$ is the level of consumption when there is no government expenditure. In terms of (n, K) space, the two loci $r = \rho$ and $\pi = 0$ become fixed if there is an exogenous labour supply, since it was the change in steady-state $L^*(G)$ which shifts them in the endogenous labour case depicted in Figure ??.

5 Local Dynamics

We now analyze the behavior of the perfect foresight equilibrium paths away from the steady state, by characterizing the local stable manifold of system (??), (??), (??) and (??) around the steady-state (corresponding to a particular value of G) given by equations (??), (??), (??) and (??). Besides having a high dimensionality, the system does not have a closed form solution (even in the case in which have specific utility and production functions). Next, we will go as far as possible in characterizing the dynamics arbitrarily close to the steady state.

The introduction of the number of firms as an additional state variable while keeping the saddle-path dynamics increases its dimension to two. The stable manifold is two-dimensional. We proved, for the exogenous labour supply case, in ?, that this could be a source of non-monotonic transitional dynamics. In that paper we found that the Jacobian had two negative real eigenvalues, and if the initial values of the state variables, $K(0)$ and $n(0)$, verify certain conditions then a particular hump-shaped transition path can occur, although just for one state variable. The extension to endogenous labour supply allows not only for a hump-shaped transition for a single variable, but also for other different types of non-monotonic transitional dynamics: non-monotonic hump-shaped transitions or oscillatory behavior for *both* state variables. The first case occurs when both eigenvalues of the Jacobian are real and the second occurs when they are complex conjugate. In order to systematically determine the conditions under which all the previous types of non-monotonicity may occur we use and extend the geometrical method for dealing with the mechanics of non-monotonicity within our four-dimensional dynamic system which was introduced in ?. Essentially, the method consists in projecting the transition paths in the space of the state variables (K, n) , and partitioning that space according to the initial points that generate several different types of the transition paths. In section ?? we show that this method simplifies a lot the characterization time-response of the model to fiscal policy shocks.

5.1 The local stable manifold

The local stable manifold, W^s , is defined as the set of (C, e, K, n) such that, if the economy starts from there, there is asymptotic convergence to the steady state (C^*, e^*, K^*, n^*) . It can be approximated locally by the stable eigenspace, E^s , which is the linear space that is tangent to, W^s , at the steady state. In spite of the dimension of the system we conclude next that the stable manifold is always two-dimensional, which means that the dynamics can be projected in the two-dimensional space of the pre-determined variables (K, n) , and we can prove general results concerning the slopes of the equilibrium trajectories in its neighborhood.

The linearized system has the Jacobian

$$J := \begin{pmatrix} C^* r_C^* / \sigma & 0 & C^* r_K^* / \sigma & C^* r_n^* / \sigma \\ -\pi_C^* / \gamma & \rho & -\pi_K^* / \gamma & -\pi_n^* / \gamma \\ Y_C^* - 1 & 0 & Y_K^* & Y_n^* \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (19)$$

where the partial derivatives refer to the capital return, profit and aggregate output optimal functions, i.e, which consider the optimal response of labour supply, evaluated at the steady state¹³

The eigenvalues of Jacobian J are (see Lemma ?? in Appendix ??)

$$\lambda_1^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2} \right)^2 - \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \lambda_2^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2} \right)^2 - \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (20)$$

are function of

$$\mathcal{T} = \mathcal{Q} + \frac{\pi_n^*}{\gamma}, \quad \mathcal{D} = \frac{\pi_n^*}{\gamma} \mathcal{Q} - \frac{\mathcal{O}}{\gamma}$$

and have the "inner" discriminant

$$\Delta = \left(\frac{\mathcal{T}}{2} \right)^2 - \mathcal{D} = \left[\frac{1}{2} \left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} \right) \right]^2 + \frac{\mathcal{O}}{\gamma}. \quad (21)$$

Because $\pi_n^* < 0$, and

$$\mathcal{O} \equiv \frac{C^*}{\sigma} (r_n^* \pi_K^* + Y_n^* (r_C^* \pi_K^* - r_K^* \pi_C^*)), \quad (22)$$

$$\mathcal{Q} \equiv \frac{C^*}{\sigma} (r_C^* Y_K^* - r_K^* (Y_C^* - 1)) < 0 \quad (23)$$

¹³For example for the derivatives for the interest rate function we set $r_j^* = \partial \hat{r}_j(C, K, n, \cdot) / \partial j$ for $j = K, n, C$, evaluated at the steady state values $K = K^*$, $n = n^*$ and $C = C^*$.

we have $\mathcal{T} < 0$ and $\mathcal{D} > 0$. However, as \mathcal{O} is ambiguous then the sign of the discriminant Δ is also ambiguous. This was not the case in the exogenous labour case (see ?) where the discriminant was always positive.

Next we prove that the stable manifold is always two dimensional:

Proposition 1. *Characterization of the eigenvalues.* *Let assumptions ??, ?? and ?? hold. Then the Jacobian J has always two eigenvalues with negative real parts. In particular: (a) if $\mathcal{O} < 0$ and $\Delta < 0$, then it has one pair of complex conjugate eigenvalues with negative real part, $Re(\lambda_2^s) = Re(\lambda_1^s) < 0$; (b) if $\mathcal{O} = 0$ and $\Delta = 0$ or $\mathcal{O} < 0$, and $\Delta < 0$, then it has two negative multiple real eigenvalues, $\lambda_2^s = \lambda_1^s < 0$; (c) if $\mathcal{O} > 0$, or if $\mathcal{O} < 0$ and $\Delta > 0$, then it has two real and distinct negative eigenvalues, $\lambda_2^s < \lambda_1^s < 0$.*

See Appendix ?? for the proofs.

This means that in a properly chosen projection space the model behaves like a sink (if the eigenvalues are real) or a stable node (if they are complex conjugate).

The result on the dimension of the stable manifold is explained by the presence of stabilizing forces acting independently over the two state variables, the stock of capital and the number of firms. We may identify, in the expression for \mathcal{T} the three main channels: (1) the negative effect of the number of firms on profits, (2) the negative effect of capital accumulation on the real interest rate, and (3) the interaction between the indirect effects of consumption on labour supply and both the direct and indirect effects of the capital stock over the real interest rate and aggregate output. The first effect acts over the number of firms and the other effects will dampen shocks exerted on the stock of capital. In a model with exogenous labour only the direct components of the first two will be present.

From Proposition ?? we can note that a *necessary* condition for the existence of complex eigenvalues is $\mathcal{O} < 0$, and a *sufficient* condition for the existence of real distinct real eigenvalues is $\mathcal{O} > 0$. If we look inside (??) we see there are two parts. The first effect is $r_n^* \pi_K^*$ captures the stabilizing effects of capital on entry and entry on capital as discussed before: both terms are positive. The second terms $Y_n^*(r_C^* \pi_K^* - r_K^* \pi_C^*)$ explicitly capture the effects of consumption on the labour supply, and hence on profits and the return on capital (more labour means an increase in π and r). From (??), more consumption means less labour. Hence the sign of the second terms in brackets is ambiguous: it can be negative and so can result in complex eigenvalues. A sufficient condition for the non-existence of complex roots is that \mathcal{O} is positive, when the first effect dominates the second. In an exogenous labour model the second effect is switched off $r_C^* = \pi_C^* = 0$: hence $\mathcal{O} > 0$, $\Delta > 0$ and there can be no complex roots.

We can now understand how an endogenous labour supply can lead to complex dynamics (non-monotonicity in both variables along the equilibrium path). Consumption influences the labour supply, which has an effect of the same sign on both the entry and the return on capital. This can push both (n, K) in the same direction which can overcome (for some time) the natural tendency towards the steady-state. However, as consumption itself converges to the steady-state, its effect will diminish leading the two state-variables to reverse direction and fall back to steady-state. The possibility of complex eigenvalues with an endogenous labour supply arises from this effect which is absent with an exogenous labour supply (where eigenvalues are only real ?).

As there are no eigenvalues with zero real part, the equilibrium dynamics belonging to the stable manifold associated to the unique stationary equilibrium point $(C^*, e^*; K^*, n^*)$, W^s , can be qualitatively approximated by the eigenspace E^s which is tangent to stable manifold.

The trajectories of the variables in the model belonging to space E^s are given by the following equation:

$$\begin{pmatrix} C(t) - C^* \\ e(t) - 0 \\ K(t) - K^* \\ n(t) - n^* \end{pmatrix} = w_1^s \begin{pmatrix} v_{1,1}^s \\ \lambda_1^s \\ v_{3,1}^s \\ 1 \end{pmatrix} e^{\lambda_1^s t} + w_2^s \begin{pmatrix} v_{1,2}^s \\ \lambda_2^s \\ v_{3,2}^s \\ 1 \end{pmatrix} e^{\lambda_2^s t}, \quad t \geq 0 \quad (24)$$

where (see Lemma ?? in Appendix ??)

$$v_{1,1}^s \equiv \frac{\frac{C^*}{\sigma} (r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_1^s)}{l_2 - \frac{\pi_n^*}{\gamma}}, \quad v_{3,1}^s \equiv \frac{\frac{C^*}{\sigma} ((Y_C^* - 1)r_n^* - Y_n^* r_C^*) + Y_n^* \lambda_1^s}{l_2 - \frac{\pi_n^*}{\gamma}} \quad (25)$$

$$v_{1,2}^s \equiv \frac{\frac{C^*}{\sigma} (r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_2^s)}{l_1 - \frac{\pi_n^*}{\gamma}}, \quad v_{3,2}^s \equiv \frac{\frac{C^*}{\sigma} ((Y_C^* - 1)r_n^* - Y_n^* r_C^*) + Y_n^* \lambda_2^s}{l_1 - \frac{\pi_n^*}{\gamma}}, \quad (26)$$

and

$$w_1^s \equiv \frac{(K(0) - K^*) - v_{3,2}^s (n(0) - n^*)}{v_{3,1}^s - v_{3,2}^s}, \quad w_2^s \equiv -\frac{(K(0) - K^*) - v_{3,1}^s (n(0) - n^*)}{v_{3,1}^s - v_{3,2}^s}.$$

We denote $l_1 = \lambda_1^s \lambda_1^u$ and $l_2 = \lambda_2^s \lambda_2^u$. Equation (??) is formally valid for any type of eigenvalue with negative real part, real or complex. However, If the eigenvalues are complex then $l_1 = \mathcal{T}/2 + |\Delta|^{1/2}i$ and $l_2 = \mathcal{T}/2 - |\Delta|^{1/2}i$ are complex, and the exponential is a complex function with coefficients dependent on time, because $e^{\lambda_1^s t} = e^{(\alpha_s + \beta i)t} = e^{\alpha_s t} (\cos(\beta t) + \sin(\beta t)i)$ and $e^{\lambda_2^s t} = e^{(\alpha_s - \beta i)t} = e^{\alpha_s t} (\cos(\beta t) - \sin(\beta t)i)$.

Every variable in equation (??) is a weighted sum of the two time-dependent exponential factors, $e^{\lambda_1^s t}$ and $e^{\lambda_2^s t}$. The weighting factors are given by the eigenspaces

E_1^s and E_2^s and by two components which ensure that the transversality conditions w_1^s and w_2^s hold. The first component governs the co-movement of the variables and the second can be used to describe the dynamics by projecting it into the state space (K, n) . The eigenspaces (which can be obtained from the solutions of (??) for $w_2^s = 0$ and $w_1^s = 0$),

$$E_1^s := \{(C, e, K, n) : C - C^* = v_{1,1}^s(n - n^*), e = \lambda_1^s(n - n^*), (K - K^*) = v_{3,1}^s(n - n^*)\}$$

and

$$E_2^s := \{(C, e, K, n) : C - C^* = v_{1,2}^s(n - n^*), e = \lambda_2^s(n - n^*), (K - K^*) = v_{3,2}^s(n - n^*)\}.$$

span the stable eigenspace E^s , which is tangent to the stable sub-manifold. As in (?, p. 345) equilibrium trajectories starting at an initial position far away from the steady state, will initially be parallel to E_2^s) and will converge asymptotically to E_1^s . Therefore, the slope of the two eigenspaces will give a qualitative characterization of the correlation between the variables far away and close to the steady state.

Proposition 2. *Let the eigenvalues, λ_1^s and λ_2^s , be real numbers. Then K and n are asymptotically positively correlated if $\mathcal{O} > 0$ or if $\mathcal{O} < 0$ and $\pi_n^* > \gamma\mathcal{Q}$, and they are asymptotically negatively correlated if $\mathcal{O} < 0$ and $\pi_n^* < \gamma\mathcal{Q}$. Far away from the steady state, K and n are negatively correlated if $\mathcal{O} > 0$ or if $\mathcal{O} < 0$ and $\pi_n^* < \gamma\mathcal{Q}$ and are positively correlated if $\mathcal{O} < 0$ and $\pi_n^* > \gamma\mathcal{Q}$.*

5.2 A taxonomy for transitional dynamics

From Propositions ?? and ?? we derive a taxonomy for the main types of transition dynamics. First, we distinguish case **C**, when $\Delta < 0$ and the eigenvalues are complex, from cases **R**, when $\Delta \geq 0$ and the eigenvalues are real. Second, we can split **R** into three different cases: case **R**₁, if $\mathcal{O} > 0$, in which the the state variables K and n are negatively correlated far away from the steady state and positively correlated asymptotically; case **R**₂, if $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* > \gamma\mathcal{Q}$, in which the state variables are positively correlated both far away and close to the steady state; and **R**₃, if $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* < \gamma\mathcal{Q}$, in which they are negatively correlated both away and close to the steady state. In the exogenous labour case in ? only case **R**₁ exists.

The sign of the discriminant is an important element in characterizing the different types of transition, by separating complex-eigenvalue generated, non-monotonic, dynamics from real-eigenvalue, potentially non-monotonic, dynamics. Furthermore, as $\Delta < 0$ only if $\mathcal{O} < 0$ then, in some sense, case **C** separates **R**₂ from **R**₃.

The discriminant can be written as a polynomial over $1/\gamma$,

$$\Delta = \frac{1}{(2\gamma)^2} \{ \mathcal{Q}^2 \gamma^2 + 2(2\mathcal{O} - \pi_n^* \mathcal{Q})\gamma + (\pi_n^*)^2 \}$$

We find the following roots of the polynomial equation $\Delta(\gamma) = 0$: if there is only one root

$$\tilde{\gamma} \equiv \left(\frac{\sigma}{C^*} \right) \left(\frac{\pi_n^*}{r_K^*(1 - Y_C^*) + r_C^* Y_K^*} \right) > 0 \quad (27)$$

and, if there are two roots

$$\gamma_{1,2} = (\pi_n^*)^2 \left\{ \pi_n^* \mathcal{Q} - 2\mathcal{O} \pm 2 [\mathcal{O}(\mathcal{O} - \pi_n^* \mathcal{Q})]^{1/2} \right\}^{-1}. \quad (28)$$

Then from (??),(??) and (??): $\Delta < 0$ if and only if $\gamma \in (\gamma_1, \gamma_2)$, $\Delta = 0$ if and only if $\gamma = \tilde{\gamma}$ and $\Delta > 0$ if and only if $\gamma \notin (\gamma_1, \gamma_2)$.

As \mathcal{O} independent from γ , in order to conduct a complete bifurcation analysis we need to choose another parameter or a combination of parameters. A natural choice is σ (the inverse of the elasticity of intertemporal substitution).

Example: benchmark model Let us consider how the parameters (γ, σ) determine the dynamics in the case of the *benchmark model* of the previous section. In the benchmark model, the explicit formula for \mathcal{O} is:

$$\mathcal{O} = \frac{\rho^2(1 - \nu)\phi}{(1 - \beta + \eta)^2 n^*} \left(\frac{\nu(1 + \eta)^2}{\sigma} \frac{C^*(G)}{C^*(G) + G} - \frac{\phi\beta^2}{\alpha} \right).$$

In order to derive critical conditions for the existence of fluctuations, let us take σ as a first critical parameter and call $\tilde{\sigma}$ to the value of σ such that $\mathcal{O}(\sigma) = 0$, that is

$$\tilde{\sigma} \equiv \left\{ \sigma : \sigma \left(\frac{C^*(G, \sigma) + G}{C^*(G, \sigma)} \right) \leq \frac{\nu}{\alpha} \left(\frac{1 + \eta}{\beta} \right)^2 \right\}$$

If $G = 0$ then we can determine explicitly

$$\tilde{\sigma} = \alpha\nu \left(\frac{1 + \eta}{\beta} \right)^2$$

Then $\mathcal{O} \leq 0$ is equivalent to $\sigma \geq \tilde{\sigma}$, which is a necessary condition for the existence of complex eigenvalues.

The following expressions,

$$\mathcal{Q} = -\frac{\rho^2}{(1 - \beta + \eta)\alpha} \left(\frac{C^*(1 - \nu)(1 - \nu + \eta(1 - \alpha))}{n^* \sigma \phi} + \beta \right) < 0$$

$$\pi_n^* = \frac{\phi}{n^*} \left(\frac{\beta - \nu(1 + \eta)}{1 - \beta + \eta} \right) < 0$$

and

$$2\mathcal{O} - \pi_n^* \mathcal{Q} = - \left(\frac{\rho^2}{\alpha n^* (1 - \beta + \eta)^2} \right) \{ \beta \phi [2\beta(1 - \nu) + \alpha + \nu\eta] + \frac{(1 - \nu)[\alpha(1 + \eta)^2(\nu - 1) + \beta\eta(1 - \beta + \eta)C^*]}{\sigma n^*} \}.$$

enter the definition of the critical values for γ such that $\Delta = 0$ in equation (??) .

Figure ?? depicts the partitions in the domain of (γ, σ) , corresponding to the four cases \mathbf{R}_1 , \mathbf{R}_2 , \mathbf{R}_3 and \mathbf{C} , which are separated by schedules $\mathcal{O} = 0$ and $\Delta = 0$, for the case in which $G = 0$. We have $\mathcal{O} > 0$ below the first curve and $\Delta < 0$ inside the parabola defined by the second curve. The figure also presents the loci where $\mathcal{Q} - \pi^*/\gamma = 0$ and the combinations of parameter values such that it is negative (above) and positive (below).

[Figure ?? here]

In Table ?? we pick parameter values that lead to the dynamics being in each of the four regions, by fixing the technology and preference parameters (α, β, ρ) and by varying all the other parameters $(\sigma, \rho, \xi, \eta, \phi)$.

Table 1: Cases: Cobb-Douglas isoelastic case

	α	β	ρ	σ	ϕ	γ	ξ	η	\mathcal{O}	Δ	$\mathcal{Q} - \pi^*/\gamma$
\mathbf{R}_1	0.3	0.5	0.025	1	0.3	3	0.01	0.5	+	+	
\mathbf{R}_2	0.3	0.5	0.025	2.2	0.4	50	0.02	0.01	-	+	-
\mathbf{R}_3	0.3	0.5	0.025	4	0.1	1	0.01	0.3	-	+	+
\mathbf{C}	0.3	0.5	0.025	3	0.2	15	2	0.02	-	-	

Case \mathbf{R}_1 occurs for relatively high elasticity of intertemporal substitution, for any value of γ , given the value of the other parameters. For lower levels of the elasticity of intertemporal substitution we will tend to have all the other cases. Negative asymptotic correlations between the aggregate stock of capital and the number of firms tend to occur for low levels of the adjustment costs and of the elasticity of intertemporal substitution. γ . If $\xi \rightarrow 0$ there is convergence to the exogenous labour case, which corresponds to case \mathbf{R}_1 . If the disutility of labour ξ increases the complex-eigenvalue case \mathbf{C} becomes likelier. \square

5.3 The geometry of non-monotonicity

As the stable manifold is a two-dimensional surface in the domain of (C, e, K, n) we may consider its projection into the space (K, n) through the mapping $(C, e) = S(K, n)$. This mapping cannot be determined explicitly. However, we can approximate it locally in the neighborhood of the steady state by

$$\begin{pmatrix} C - C^* \\ e - e^* \end{pmatrix} = \begin{pmatrix} S_{CK} & S_{Cn} \\ S_{eK} & S_{en} \end{pmatrix} \begin{pmatrix} K - K^* \\ n - n^* \end{pmatrix}. \quad (29)$$

Matrix S is a real matrix and its coefficients satisfy: $S_{CK} > 0$, $S_{en} < 0$, for any value of the parameters, and $S_{Cn} < 0$ and $S_{eK} > 0$ if $\mathcal{O} > 0$, or $S_{Cn} > 0$ and $S_{eK} < 0$ if $\mathcal{O} < 0$. (see Lemma ?? in the Appendix ??). Then in case \mathbf{R}_1 the coefficients verify $S_{CK} > 0$, $S_{en} < 0$, $S_{Cn} < 0$, and $S_{eK} > 0$ and in cases \mathbf{R}_2 , \mathbf{R}_3 and \mathbf{C} we have $S_{CK} > 0$, $S_{en} < 0$, $S_{Cn} > 0$, and $S_{eK} < 0$.

The equilibrium dynamics along the eigenspace can be represented by the local projection in the space (K, n) described by the solutions of

$$\begin{aligned} \dot{K} &= \dot{K}(K, n) = \dot{K}(S_C(K, n), K, n) \\ \dot{n} &= \dot{n}(K, n) = S_e(K, n). \end{aligned}$$

The projection of the phase diagrams in space (K, n) have generic properties and correspond to sinks for cases \mathbf{R}_1 , \mathbf{R}_2 and \mathbf{R}_3 , and to a stable node for case \mathbf{C} . This is proved in Lemma ?? in Appendix ?? and is illustrated in Figure ??.

In figure 3 we have the following curves: (1) the stable eigenspaces E_1^s , to which trajectories are tangent asymptotically, and E_2^s , to which trajectories are parallel far away from the steady state (see Proposition ??) ; (2) the projections of the tangent to the isoclines close to the steady state, which are determined as, for instance for isocline $\dot{K} = 0$

$$d\dot{K} = (Y_K^* + (Y_C^* - 1)S_{CK}) dK + (Y_n^* + (Y_C^* - 1)S_{Cn}) dn = 0,$$

and likewise for $\dot{n} = 0$; (3) the projections of the isoclines $\dot{C} = 0$ (or $r(K, n) = \rho$) and $\dot{e} = 0$ (or $\pi(K, n) = 0$); and (4) the projection for the locus corresponding to $\dot{Y} = 0$, determined as

$$\begin{aligned} d\dot{Y} &= ((Y_K^* + Y_C^* S_{CK})(Y_K^* + (Y_C^* - 1)S_{CK}) + (Y_n^* + Y_C^* S_{Cn})S_{eK}) dK + \\ & ((Y_K^* + Y_C^* S_{CK})(Y_n^* + (Y_C^* - 1)S_{Cn}) + (Y_n^* + Y_C^* S_{Cn})S_{en}) dn = 0. \end{aligned}$$

The slopes of the isoclines for K are always positive and the slope for n is positive for case \mathbf{R}_1 and is negative for the others. The slope for the isocline $\dot{Y} = 0$ has the opposite slope. Given any initial value for the two state variables $(K(0), n(0))$,

the transition path is initially parallel to E_2^s and converges asymptotically to line E_1^s . If the projection of a trajectory "hits" one of the isoclines, the corresponding variable varies non-monotonically. In case **C**, depending on how far the initial point is from the steady state, there are several "hits" along the way, for every isocline. and therefore the trajectory is oscillatory for all variables. However, as shown in ? non-monotonic trajectories may also occur for cases **R** associated to sinks. However, in the last cases there can only be one "hit" at one or more isoclines, which produces a hump-shaped trajectory, but only under certain conditions.

In order to characterize non-monotonicity, we use the same method as in ?. First, we define the following subsets of the domain of (K, n) : $\mathbf{M}_K = \mathbf{N}_K \cup \mathbf{S}_K$ and $\mathbf{M}_n = \mathbf{E}_n \cup \mathbf{W}_n$ such that

$$\begin{aligned} \mathbf{N}_K &\equiv \left\{ (K, n) : n - n^* > \max \left\{ \frac{1}{v_{32}^s}, \frac{\lambda_2^s v_{32}^s - \lambda_1^s v_{31}^s}{(\lambda_2^s - \lambda_1^s) v_{31}^s v_{32}^s} \right\} \cdot (K - K^*) \right\} \\ \mathbf{S}_K &\equiv \left\{ (K, n) : n - n^* < \min \left\{ \frac{1}{v_{32}^s}, \frac{\lambda_2^s v_{32}^s - \lambda_1^s v_{31}^s}{(\lambda_2^s - \lambda_1^s) v_{31}^s v_{32}^s} \right\} \cdot (K - K^*) \right\} \\ \mathbf{E}_n &\equiv \left\{ (K, n) : K - K^* > \max \left\{ v_{32}^s, \frac{\lambda_2^s v_{31}^s - \lambda_1^s v_{32}^s}{\lambda_2^s - \lambda_1^s} \right\} \cdot (n - n^*) \right\} \\ \mathbf{W}_n &\equiv \left\{ (K, n) : K - K^* < \min \left\{ v_{32}^s, \frac{\lambda_2^s v_{31}^s - \lambda_1^s v_{32}^s}{\lambda_2^s - \lambda_1^s} \right\} \cdot (n - n^*) \right\}. \end{aligned}$$

From Lemma ?? we can see that: if $\mathcal{O} > 0$ (case **R**₁) then sets \mathbf{M}_K and \mathbf{M}_n are disjoint; if $\mathcal{O} < 0$ and $\mathcal{Q} - \pi_n^*/\gamma < 0$ (case **R**₂) then there are non-empty intersections between \mathbf{N}_K and \mathbf{W}_n and between \mathbf{S}_K and \mathbf{E}_n ; and if $\mathcal{O} < 0$ and $\mathcal{Q} - \pi_n^*/\gamma > 0$ (case **R**₃) then there are non-empty intersections between \mathbf{N}_K and \mathbf{E}_n and between \mathbf{S}_K and \mathbf{W}_n .

Then the fundamental result on non-monotonic dynamics follows:

Proposition 3. *Non-monotonic dynamics over the over space (K, n) . Let $(K(0), n(0)) \neq (K^*, n^*)$.*

1. *If the eigenvalues are complex then the trajectories are always oscillatory and rotate clockwise (because the coefficient of the complex part is positive $\beta > 0$) for any $(K(0), n(0))$.*
2. *If the eigenvalues are real then three types of trajectories exist: (1) $(K(0), n(0))$ belongs to \mathbf{M}_K the trajectory for K is hump-shaped; (2) if $(K(0), n(0))$ belongs to \mathbf{M}_n the trajectory for n is hump-shaped; (3) if $(K(0), n(0))$ belongs to $\mathbf{M}_K \cap \mathbf{M}_n$ then trajectories for both K and n are hump-shaped; (4) if $(K(0), n(0))$ does not belong to $\mathbf{M}_K \cup \mathbf{M}_n$ then trajectories for both K and n are monotonic.*

This has the following general consequence on the type of non-monotonic adjustment that may exist when there are real eigenvalues. If $\mathcal{O} > 0$ then, in the transition to the steady state, just one variable may have a humped-shaped trajectory and will change direction only once. This can be seen in the phase diagram for case \mathbf{R}_1 depicted in panel NW of Figure ??, which displays the non-empty and disjoint sets \mathbf{N}_K and \mathbf{S}_K , between curves E_2^s and $\dot{K} = 0$, and \mathbf{E}_n and \mathbf{W}_n , between curves E_2^s and $\dot{n} = 0$, and the rest of the domain in which no monotonic trajectories are associated. Phase diagrams associated to $\mathcal{O} < 0$, also shown in the NE and SW panels of Figure ?? for cases \mathbf{R}_2 and \mathbf{R}_3 , also have the same subsets. However, differently from case \mathbf{R}_1 there are non-empty intersections between the previous sets: in the phase diagram for \mathbf{R}_2 the intersections between \mathbf{N}_K and \mathbf{E}_n and between \mathbf{S}_K and \mathbf{W}_n lie between curves E_2^s and $\dot{n} = 0$, and in the phase diagram for \mathbf{R}_3 the intersections between \mathbf{S}_K and \mathbf{W}_n and between \mathbf{N}_K and \mathbf{E}_n lie between curves E_2^s and $\dot{K} = 0$.

[Figure ?? here]

Some representative trajectories are shown in the phase diagrams in Figure ??.

Phase diagram \mathbf{R}_3 is in the of SW panel Figure ??. We can see that all trajectories originating in \mathbf{N}_K and \mathbf{S}_K have both a hump-shaped response first for K and later on for n as well. Along the way they also cross $\dot{Y} = 0$. Take the first case. As both curves E_1^s and E_2^s are negatively sloped, K grows initially while n diminishes, then the capital stock and output will eventually diminish, and finally close to the steady state there will be net entry and the number of firms will increase. Note that as the $\dot{Y} = 0$ is less steep than the $\dot{K} = 0$ the hump in capital peaks before the hump in output, or there is no hump for capital if the dynamics originates in \mathbf{E}_n or \mathbf{W}_n . In any of this cases the trajectories cross the $\dot{n} = 0$ isocline, indicating an increase in entry close to the steady state. Phase diagram \mathbf{C} , associated with the existence of complex eigenvalues, is depicted in Figure ??, SE panel and is qualitatively similar to \mathbf{R}_3 , however the number of crossings depends upon the distance of the initial point from the steady state.

Phase diagram \mathbf{R}_1 is depicted in Figure ??, NW panel. Its main features are the following. First, asymptotically the aggregate capital stock and the number of firms are positively related in the neighborhood of the steady state, and will be tangent asymptotically to E_1^s . However, since E_2^s is negatively sloped the initial co-movement is negative. If we take trajectories originating in \mathbf{N}_K a sequence of crossings first of $\dot{Y} = 0$ and then of $\dot{K} = 0$ is possible, meaning that initially output and the capital stock increase and the number of firms decrease, then output crossed the hump down followed by the capital stock. Asymptotically, the trajectories of n , K , and Y all move together down (this is the same as the exogenous labour case ?). The number

of firms do not have a hump, differently from case \mathbf{R}_3 . Phase diagram \mathbf{R}_2 is depicted in Figure ??, NE panel. It shares some similar properties with the phase diagram \mathbf{R}_1 , in particular, the positive co-movement of the two state variables and Y in the neighborhood of the steady state. However, the timing of the crossings of $\dot{K} = 0$ and $\dot{Y} = 0$ are the opposite. Far away from the steady state an initial positive correlation is also possible, when the origin is in \mathbf{W}_n , as E_2^s has a positive slope.

6 Fiscal policy.

Now we consider unanticipated and permanent variations of public expenditures G which perturbs an economy which is at a steady state.

We consider the locally projected space (K, n) and its perturbation by G

$$\dot{K} = \dot{K}(S_C(K, n), K, n, G) \quad (30)$$

$$\dot{n} = S_e(K, n) \quad (31)$$

Then we can determine if there are hump-shaped or general non-monotonic responses to a permanent unanticipated shock, dG , if (K, n) happens to be initially placed in the subsets \mathbf{M}_k or \mathbf{M}_n , relative to the new steady state. We can trace out the the location of the initial and by considering the steady state projection

$$dn = \left. \frac{dn}{dK} \right|_{dG} dK = \frac{\partial n / \partial G}{\partial K / \partial G} dK$$

determined from the long run multipliers of the projected system (??)-(??). First we need to determine the long run and impact multipliers.

6.1 Long run and impact responses.

Proposition 4. Fiscal policy multipliers

Long run multipliers: they are negative for consumption, positive for the stock of capital and the number of firms. Impact multipliers: they are always positive for entry. For consumption they are negative if $\mathcal{O} < 0$ and undershoot the long run variation, and they are ambiguous for $\mathcal{O} > 0$.

Long run multipliers for G (a permanent shock) were described in section : their

explicit expression is given by:

$$\begin{aligned}\frac{dC^*}{dG} &= -\frac{(r_K^*\pi_n^* - r_n^*\pi_K^*)C^*}{\sigma\gamma\mathcal{D}} < 0 \\ \frac{de^*}{dG} &= 0 \\ \frac{dK^*}{dG} &= \frac{(r_C^*\pi_n^* - r_n^*\pi_C^*)C^*}{\sigma\gamma\mathcal{D}} > 0 \\ \frac{dn^*}{dG} &= -\frac{(r_C^*\pi_K^* - r_K^*\pi_C^*)C^*}{\sigma\gamma\mathcal{D}} > 0\end{aligned}$$

This implies the long-run employment and output multipliers:

$$\begin{aligned}\frac{dL^*}{dG} &= L_C\frac{dC^*}{dG} + L_K\frac{dK^*}{dG} + L_n\frac{dn^*}{dG} > 0, \\ \frac{dY^*}{dG} &= Y_K^*\frac{dK^*}{dG} + Y_n^*\frac{dn^*}{dG} > 0\end{aligned}$$

which means that $dC^* + dG > 0$ ¹⁴.

The Impact multipliers for consumption and entry are

$$\frac{dC(0)}{dG} = \frac{dC^*}{dG} - \left(S_{CK}\frac{dK^*}{dG} + S_{Cn}\frac{dn^*}{dG} \right) \quad (32)$$

$$\frac{de(0)}{dG} = -\left(S_{eK}\frac{dK^*}{dG} + S_{en}\frac{dn^*}{dG} \right) > 0 \quad (33)$$

If $\mathcal{O} < 0$, including the case when we have complex eigenvalues, then from Lemma ?? both coefficients S_{CK} and S_{Cn} associated for the consumption multiplier are positive, whilst S_{eK} and S_{en} associated with entry are negative: therefore we have $\frac{dC(0)}{dG} < \frac{dC^*}{dG} < 0$ and $\frac{de(0)}{dG} > 0$.

If $\mathcal{O} > 0$ and we have real eigenvalues, the coefficients do not have the same sign, and therefore the impact multipliers are potentially ambiguous and the impact effect can be lower or higher than the long-run effect. This possibility is novel and results from the fact that both state variables K and n effect both of the control variables (C, e) . Turning first to consumption, the term S_{CK} captures the effect of capital on consumption with the traditional "negative" relationship: as capital moves closer to equilibrium, the gap between the marginal product and the discount rate gets smaller,

¹⁴If we take the benchmark model, the long run multipliers are: $\frac{dC^*}{dG} = -\frac{\eta C^*}{\sigma Y^* + \eta C^*} < 0$, $\frac{dK^*}{dG} = \frac{\sigma K^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dn^*}{dG} = \frac{\sigma n^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dL^*}{dG} = \frac{\sigma L^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dY^*}{dG} = \frac{\sigma Y^*}{\sigma Y^* + \eta C^*} > 0$, and $\frac{dy^*}{dG} = \frac{dk^*}{dG} = \frac{dl^*}{dG} = 0$.

hence consumption continues to grow but at a slower rate, implying the monotonic trajectory of consumption rising alongside capital accumulation. However, the term S_{Cn} captures the effect of entry on the Euler equation for consumption. More firms can counteract the increase in capital leading to a non-monotonic trajectory in the marginal product of capital (which depends on capital per firm k , not aggregate capital) and hence also non-monotonicity in consumption. This leads to an ambiguous relation between initial consumption and its long-run value if the entry flow is large initially so that capital per firm decreases and the marginal product of capital increases. This sort of *consumption undershooting* response is impossible in the classic Ramsey model where the marginal product depends only on the aggregate capital stock or capital /labour ratio and the impact response of consumption always *overshoots* the long-run response¹⁵.

6.2 Hump-shaped responses.

The short run dynamics is similar to the one generated by equation (??). The existence of non-monotonic adjustment trajectories depend both on the parameters and on the type magnitude of the shock. If we take the after-shock levels of capital and number of firms as K^* and n^* and the pre-shock values as initial values, we can compare the variation with the expressions for the expressions in Proposition ??.

[Figure ?? here]

Proposition 5. *Assume there is a permanent fiscal policy shock. If $\Delta > 0$ then the adjustment for K is always monotonic and the adjustment for n is hump-shaped for case \mathbf{R}_3 and monotonic for cases \mathbf{R}_1 and \mathbf{R}_2 . If $\Delta < 0$ the adjustment is oscillatory for both variables.*

In Figure ?? we have already shown the four possible cases of short-run dynamics. We see that there is a monotonic adjustment of both (n, K) for cases \mathbf{R}_1 and \mathbf{R}_2 , and a non-monotonic adjustment for cases \mathbf{C} and \mathbf{R}_3 . Since we are looking at the response of changes to fiscal policy the initial position will lie on the dG line: hence. Figure ?? superposes the dG line. We can see that if we started from an initial steady-state and it is perturbed by dG , the initial steady-state will belong to to sets \mathbf{W}_n and \mathbf{E}_n of the new steady-state. and only the number of firms can adjust non-monotonously. This is an "initial condition effect", and would not hold if we were to consider technology shocks. Whilst for all cases $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ and \mathbf{C} capital will adjust monotonically in response to fiscal policy, capital stock per firm may be non-monotonic in all cases. If

¹⁵In the classic Ramsey model in (C, K) , the saddlepath is upwards sloping so (C, K) move together with an endogenous labour supply.

we turn to output, in case \mathbf{R}_1 the $\dot{Y} = 0$ isocline is downward sloping, which implies that output always responds monotonically to fiscal policy. However, the $\dot{Y} = 0$ isocline is upward sloping and flatter than the dG line in the other cases: output will always be non-monotonic in cases \mathbf{R}_3 and \mathbf{C} and may be in case \mathbf{R}_2 .

Phase diagram \mathbf{R}_3 is depicted in Figure ??, SW panel. In this case both curves E_1^s and E_2^s are negatively sloped. Since $E_1^s < 0$ there is a *negative co-movement* between the two state variables (K, n) as we approach the steady-state. Furthermore, since dG lies in between the $\dot{K} = 0$ and $\dot{n} = 0$ isoclines, the initial co-movement is *positive*. The trajectories also cross the $\dot{n} = 0$ isocline, indicating an overshooting hump shape for n . The other point to note is that the $\dot{Y} = 0$ line is positively sloped. This means that any trajectory originating from the dG line must pass through $\dot{Y} = 0$: *hence output must have a hump shape response*. Phase diagram \mathbf{C} , associated with the existence of complex eigenvalues, is depicted in Figure ??, SE panel and is qualitatively similar to \mathbf{R}_3 : the reason for this is that all trajectories starting on dG must start off with positive co-movement (in order to get negative co-movement initially, we would need to start from a point the "other side" of the $\dot{K} = 0$ isocline which is off the dG line).

Phase diagram \mathbf{R}_1 is depicted in Figure ??, NW panel. Its main features are the following. First, asymptotically the aggregate capital stock and the number of firms are positively related in the neighborhood of the steady state, and will be tangent asymptotically to E_1^s . Since the initial position lies on dG , the initial co-movement will also be positive. Since the trajectory cannot cross any of the $\dot{n}, \dot{K} = \bar{Y} = 0$ lines, along the trajectory of n, K , and Y all move together (this is the same as the exogenous labour case ?). Phase diagram \mathbf{R}_2 is depicted in Figure ??, NE panel. It shares some similar properties with the phase diagram \mathbf{R}_1 , in particular, the positive co-movement of the two state variables in the neighborhood of the steady state. However, it differs in that trajectories originating from dG may cross the $\dot{Y} = 0$ line. Hence there may be a hump shaped response of output to fiscal policy.

Another way of looking at the dynamics is through the impulse-response functions for the main variables to a fiscal shock in this model, (C, Y, K, L, n, e, k, I) where $k = K/n$ is capital per firm and I is investment in new firms and capital (see Figures ?? to ??). The impulse-response functions for all cases show that capital-per firm k is non-monotonic. However, we also observe that whilst consumption and capital in all cases follow monotonic trajectories (in this benchmark case), this is not so for output Y and employment L . In the case of phase diagram \mathbf{C} , where there are two complex eigenvalues and n adjusts non-monotonically, we can see that both output and employment have a hump-shaped response function: output and employment "jump" up, but continue upwards for a time overshooting the long-run effect, peaking and then converging to the new steady-state.

[Figures ?? to ?? here]

Whilst we have not found an example of a non-monotonic trajectory for consumption in the benchmark model, we can determine what it would look like if it were to happen. Since the marginal product of capital will fall as we approach close to the steady-state, we know that consumption and capital will then have a positive co-movement. In the classical Ramsey model, with overshooting, the positive co-movement occurs all along the path. However, if there is undershooting on impact, then consumption will at first fall to below (in the case of a fiscal expansion) the new steady-state, so that it can then increase with capital. There will thus be an initial phase of negative co-movement resulting in a hump shape.

7 Conclusion

In this paper we have analyzed Ricardian fiscal policy in the context of the classic Ramsey model extended to include an endogenous labour supply and a real time entry and exit process. Both of these extensions have been done on their own: entry was developed in ? and an endogenous labour supply in (?, ch.9). We find that with both extensions together we are able to obtain a much richer set of possible dynamic responses to fiscal policy. Whilst the long-run dynamics are similar to standard models, the short-run can be very different. The dynamics allows for complex eigenvalues so that both state variables (capital and the number of firms) can be non-monotonic.

However, in the analysis of fiscal policy there is an "initial condition" effect that implies that only the number of firms will be non-monotonic as we move from the initial steady-state to the new one. Whilst the response of capital to fiscal policy will be monotonic, we find that there can be a hump shaped response of output in a wide range of cases. This is analyzed in terms of a phase diagram in the state-space of capital and the number of firms, where we are able to define the isocline for output (the combinations of capital and the number of firms at which the time derivative of output is zero). We are able to show that in certain well defined cases the trajectory of the economy in response to a fiscal shock will pass through this isocline and thus exhibit a hump shaped response.

Much of macroeconomics today is thought of in terms of calibrated or estimated numerical models. Whilst theory has its limitations in being limited to shedding light on relatively simple models, theoretical models allow us to understand the general mechanisms underlying macroeconomic relationships in a way that numerical simulation cannot. We have extended the standard Ramsey model by introducing a real time model of entry. Whilst the analysis is complex and four dimensional, we are able to interpret the results in terms of two dimensional phase diagrams with relatively clear economic interpretation.

A Appendix: proofs and auxiliary results

Lemma 1. Let $\mathcal{D} = \det(J)$, $\mathcal{T} = M_2 - \rho^2$, where M_2 is the sum of the principal minors of order 2 of J . Then, the eigenvalues of Jacobian J , are

$$\lambda_1^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2} \right)^2 - \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \lambda_2^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2} \right)^2 - \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (34)$$

where the discriminant is $\Delta \equiv \left(\frac{\mathcal{T}}{2} \right)^2 - \mathcal{D}$.

Proof. The characteristic polynomial for a four dimensional system of ODEs is:

$$c(\lambda) = \lambda^4 - M_1\lambda^3 + M_2\lambda^2 - M_3\lambda + M_4$$

where M_j is the sum of the principal minors of order $j = 1, \dots, 4$. In infinite horizon optimal control problems with two state variables the following constraints on the value of the sums of the principal minors hold: $M_2 = 2\rho$, which is the trace, and $M_3 = \rho(M_2 - \rho^2)$. Therefore, there are only two independent coefficients involving the sums of the principal minors of even order: $\mathcal{T} \equiv M_2 - \rho^2$ and $\mathcal{D} \equiv M_4 = \det(J)$. Then the characteristic polynomial can be written as

$$c(\lambda) = \left(\frac{\rho}{2} \right)^4 \left[\omega^2 + \left(\left(\frac{\rho}{2} \right)^{-2} \mathcal{T} - 2 \right) \omega + \left(\left(\frac{\rho}{2} \right)^{-4} M_4 - \left(\frac{\rho}{2} \right)^{-2} \mathcal{T} + 1 \right) \right],$$

where $\omega \equiv \left(\lambda - \frac{\rho}{2} \right)^2 \left(\frac{\rho}{2} \right)^{-2}$. Then we get the eigenvalues as $\left(\lambda - \frac{\rho}{2} \right)_{1,2}^{s,u} = \mp \left[\left(\frac{\rho}{2} \right) \omega_{1,2} \right]^{\frac{1}{2}}$. Then, by solving the polynomial equation $c(\lambda) = 0$, we get the eigenvalues in equation (??). \square

Lemma 2. If the discriminant is non-negative, $\Delta \geq 0$, then the four eigenvalues are real and verify

$$\lambda_2^s \leq \lambda_1^s < 0 < \lambda_1^u \leq \lambda_2^u.$$

If the discriminant is negative, $\Delta < 0$, then there are two pairs of complex conjugate eigenvalues

$$\lambda_1^s = \alpha_s + \beta i, \quad \lambda_2^s = \alpha_s - \beta i, \quad \lambda_1^u = \alpha_u - \beta i, \quad \lambda_2^u = \alpha_u + \beta i$$

where $i^2 = -1$, and $\alpha_s < 0 < \alpha_u$:

$$\alpha_{s,u} = \frac{\rho}{2} \mp \left(\frac{1}{2} \right)^{1/2} \left\{ \left[\left(\frac{\rho}{2} \right)^4 - \left(\frac{\rho}{2} \right)^2 \mathcal{T} + \mathcal{D} \right]^{1/2} + \left(\frac{\rho}{2} \right)^2 - \frac{\mathcal{T}}{2} \right\}^{1/2}$$

and

$$\beta = \left(\frac{1}{2}\right)^{1/2} \left\{ \left[\left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 \mathcal{T} + \mathcal{D} \right]^{1/2} - \left(\frac{\rho}{2}\right)^2 + \frac{\mathcal{T}}{2} \right\}^{1/2} > 0.$$

Furthermore, the following relationships hold:

1. $\lambda_i^s + \lambda_i^u = \rho > 0$ (if they are complex we have $\alpha_s + \alpha_u = \rho$);
2. $\lambda_1^s \lambda_1^u = l_1$ and $\lambda_2^s \lambda_2^u = l_2$,

$$l_1 = \begin{cases} \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}}, & \text{if } \Delta \geq 0 \\ \frac{\mathcal{T}}{2} + (|\Delta|)^{\frac{1}{2}}i, & \text{if } \Delta < 0 \end{cases}, \quad l_2 = \begin{cases} \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}}, & \text{if } \Delta \geq 0 \\ \frac{\mathcal{T}}{2} - (|\Delta|)^{\frac{1}{2}}i, & \text{if } \Delta < 0 \end{cases}$$

verify $l_2 \leq l_1 < 0$ are real if $\Delta \geq 0$ and are complex conjugate with a negative real part, if $\Delta < 0$

3. $l_1 + l_2 = \mathcal{T} < 0$ and $l_1 l_2 = \mathcal{D} > 0$ are real for any type of eigenvalue, real or complex.

Proof. Proof of Lemma ?? The first part of the Lemma is obvious from Lemma ??. When $\Delta < 0$ we use the fact that given the complex number $y = a + bi$ it is known that $\sqrt{y} = \sqrt{a + bi} = 1/2 \{ \sqrt{r + a} + \text{sign}(b)\sqrt{r - ai} \}$ where $r \equiv \sqrt{a^2 + b^2}$. The rest of the proof is obvious. \square

Proof. Proof of Proposition ?? We start by proving that the conditions of Lemma ?? on the sums of the principal minors hold.

First, as $\frac{C^*}{\sigma} r_L L_C + w L_K = 0$ then $\frac{C^* r_C^*}{\sigma} + Y_K^* = \rho$ which implies that the trace is $M_1 = \rho + \frac{C^* r_C^*}{\sigma} + Y_K^* = 2\rho$. Second, the sum of the principal minors of order two and three are

$$M_2 = \rho \left(\frac{C^* r_C^*}{\sigma} + Y_K^* \right) + \frac{C^*}{\sigma} (r_C^* Y_K^* - r_K^* (Y_C^* - 1)) + \frac{\pi_n^*}{\gamma}$$

and

$$M_3 = \rho(M_2 - \rho^2) - \frac{1}{\gamma} \left(\frac{C^* r_n^*}{\sigma} \pi_C^* + \pi_K^* Y_n^* \right)$$

But as $\frac{C^* r_n^*}{\sigma} \pi_C^* + \pi_K^* Y_n^* = 0$ then $M_3 = \rho(M_2 - \rho^2)$. Then, we can use Lemma ?? to determine the eigenvalues, noting that

$$\mathcal{T} \equiv M_2 - \rho^2 = \frac{C^*}{\sigma} (r_C^* Y_K^* - r_K^* Y_C^*) + \frac{C^* r_K^*}{\sigma} + \frac{\pi_n^*}{\gamma} \quad (35)$$

and

$$\mathcal{D} \equiv M_4 = \frac{C^*}{\sigma \gamma} [(1 - Y_C^*)(\pi_n^* r_K^* - \pi_K^* r_n^*) + Y_K^*(\pi_n^* r_C^* - \pi_C^* r_n^*) - Y_n^*(\pi_K^* r_C^* - \pi_C^* r_K^*)] \quad (36)$$

In order to obtain the dimension of the stable manifold, we have to determine the signs of \mathcal{T} and \mathcal{D} . From the signs of the partial derivatives, $r_K^* < 0$, $\pi_n^* < 0$ and as $r_C^* Y_K^* - r_K^* Y_C^* = L_C(w_K r - r_K w) < 0$ then $\mathcal{T} < 0$. After tedious algebra, we can show that

$$\mathcal{D} = \frac{C^*}{\sigma\gamma} \left(\frac{L^*}{n^*} \right) \left(\frac{1}{u_C w_L + u_{LL}} \right) \left(AF w u_{CC} + u_{LL} \frac{L^*}{n^*} \right) (r_K w_L - r_L w_K) > 0 \quad (37)$$

From the fundamental theorem of algebra (i.e., on the relationship between the eigenvalues and the coefficients of the characteristic polynomial) it is easy to see that there are two eigenvalues with negative real parts: $\mathcal{D} > 0$ implies that there are no zero eigenvalues and that the number of eigenvalues with negative real part is even; but as $M_1 > 0$ they should be reduced to zero or two; and, finally as $M_3 = \rho\mathcal{T} < 0$ then we should have one pair of negative eigenvalues (M_2 is ambiguous, in general).

Now, let $z \equiv \frac{1}{\gamma}$. First, observe that the discriminant can be written as a quadratic function of z :

$$\Delta(z) = \left(\frac{\pi_n^*}{2} \right)^2 \left[z^2 + 2 \left(\frac{2\mathcal{O} - \pi_n^* \mathcal{Q}}{(\pi_n^*)^2} \right) z + \left(\frac{\mathcal{Q}}{\pi_n^*} \right)^2 \right]. \quad (38)$$

Then $\Delta(z) = 0$ if and only if $F(z) = z^2 + a_1 z + a_0 = 0$ where $a_1 \equiv 2 \left(\frac{2\mathcal{O} - \pi_n^* \mathcal{Q}}{(\pi_n^*)^2} \right)$ and $a_0 \equiv \left(\frac{\mathcal{Q}}{\pi_n^*} \right)^2$. The sign of a_1 is ambiguous but $a_0 > 0$.

Second, observe that $F(z)$ is a convex function of z and consider that $z > 0$ and is real. This polynomial has a minimum at $\tilde{z} > 0$ such that $F'(\tilde{z}) = 0$. If $F(\tilde{z}) = 0$ then if $z = \tilde{z}$ then there will be multiple real eigenvalues and if $z \neq \tilde{z}$ there will be real distinct eigenvalues. If $F(\tilde{z}) < 0$ this means that the polynomial has two roots, z_1 and z_2 such that $z_1 > z_2$. Then there are related values for γ , $\gamma_1 < \gamma_2$. This implies that $\Delta(z) < 0$ for $z_1 > z > z_2$ and the eigenvalues, λ_i^j for $j = s, u$ and $i = 1, 2$ will be complex conjugate.

Therefore, we have only to investigate if there are any positive real roots for the polynomial equation $F(z) = 0$. Its roots are

$$\begin{aligned} z_1 &= (\pi_n^*)^{-2} \left\{ \pi_n^* \mathcal{Q} - 2\mathcal{O} + 2 [\mathcal{O}(\mathcal{O} - \pi_n^* \mathcal{Q})]^{\frac{1}{2}} \right\}, \\ z_2 &= (\pi_n^*)^{-2} \left\{ \pi_n^* \mathcal{Q} - 2\mathcal{O} - 2 [\mathcal{O}(\mathcal{O} - \pi_n^* \mathcal{Q})]^{\frac{1}{2}} \right\}. \end{aligned}$$

Observe that as $a_0 > 0$ then the two roots, z_1, z_2 , have the same sign, but may be real or complex.

Let $\Delta_z \equiv \mathcal{O}(\mathcal{O} - \pi_n^* \mathcal{Q})$. Three cases may occur:

(a) $\Delta_z = 0$ if $\mathcal{O} = 0$ or $\mathcal{O} = \pi_n^* \mathcal{Q} > 0$. In each of those cases there is only one multiple solution for $F(z) = 0$. In the first case, we get $z = \tilde{z} = \mathcal{Q}/\pi_n^* > 0$, given in equation (??). In the second case we get $z = -\mathcal{Q}/\pi_n^* < 0$ which does not belong to the domain of z .

(b) $\Delta_z < 0$ if $\mathcal{O} - \pi_n^* \mathcal{Q} < 0 < \mathcal{O}$ (another formally possible case where $\mathcal{O} - \pi_n^* \mathcal{Q} > 0 > \mathcal{O}$ is not admissible because $\pi_n^* \mathcal{Q} > 0$). In this case z_1 and z_2 are complex conjugate and therefore do not belong to the domain of z .

(c) $\Delta_z > 0$ if $\mathcal{O} < 0$, implying that $\mathcal{O} - \pi_n^* \mathcal{Q} < 0$, or if $\mathcal{O} > \mathcal{O} - \pi_n^* \mathcal{Q} > 0$. In the first case, as a_1 is positive because $\pi_n \mathcal{Q} - 2\mathcal{O} > 0$, and as a_0 is positive, then the two roots for $F(z) = 0$ are real and positive. In the second case, as $\pi_n^* \mathcal{Q} - \mathcal{O} < \pi_n^* \mathcal{Q} - \mathcal{O} < 0$ then $a_1 < 0$ and the two roots are real but negative, and, therefore do not belong to the domain of z .

Summing up, the only case in which $\Delta(z) < 0$ for $z > 0$ is if $\mathcal{O} < 0$ and if $z_1 < z < z_2$. In all the other cases the eigenvalues are real.

The rest of the proof follows from Lemma ??.

□

Lemma 3. *If there is no multiplicity then is the eigenvector associated to eigenvalue λ_j^s , $j = 1, 2$ of matrix J is*

$$V_j^s = (v_{1,j}^s, \lambda_j^s, v_{3,j}^s, 1)^\top, \quad j = 1, 2 \quad (39)$$

where

$$v_{1,1}^s \equiv \frac{\frac{C^*}{\sigma} (r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_1^s)}{l_2 - \frac{\pi_n^*}{\gamma}}, \quad v_{3,1}^s \equiv \frac{\frac{C^*}{\sigma} ((Y_C^* - 1)r_n^* - Y_n^* r_C^*) + Y_n^* \lambda_1^s}{l_2 - \frac{\pi_n^*}{\gamma}} \quad (40)$$

$$v_{1,2}^s \equiv \frac{\frac{C^*}{\sigma} (r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_2^s)}{l_1 - \frac{\pi_n^*}{\gamma}}, \quad v_{3,2}^s \equiv \frac{\frac{C^*}{\sigma} ((Y_C^* - 1)r_n^* - Y_n^* r_C^*) + Y_n^* \lambda_2^s}{l_1 - \frac{\pi_n^*}{\gamma}} \quad (41)$$

Proof. Eigenvector V_j^i is obtained as the non-zero solution of the homogeneous system $(J - \lambda_j^i I_4)V_j^i = 0$,

$$V_j^i = \left[\frac{\frac{C^*}{\sigma} (r_K^* Y_n^* - r_n^* (Y_K^* - \lambda_j^i))}{\left(\frac{C^*}{\sigma} - \lambda_j^i\right)(r_K^* - \lambda_j^i) - \frac{C^*}{\sigma} r_K^* (Y_C^* - 1)}, \lambda_j^i, \frac{\frac{C^*}{\sigma} r_n^* (Y_C^* - 1) - Y_n^* \left(\frac{C^*}{\sigma} r_C^* - \lambda_j^i\right)}{\left(\frac{C^*}{\sigma} r_C^* - \lambda_j^i\right)(r_K^* - \lambda_j^i) - \frac{C^*}{\sigma} r_K^* (Y_C^* - 1)}, 1 \right],$$

for $i = s, u$ and $j = 1, 2$. But $\left(\frac{C^*}{\sigma} - \lambda_j^i\right)(r_K^* - \lambda_j^i) - \frac{C^*}{\sigma} r_K^* (Y_C^* - 1) = (\lambda_j^i)^2 - \lambda_j^i \left(\frac{C^*}{\sigma} r_C^* + Y_K^*\right) + \frac{C^*}{\sigma} r_C^* Y_K^* - \frac{C^*}{\sigma} r_K^* (Y_C^* - 1) = \lambda_j^i (\lambda_j^i - \rho) + \mathcal{T} - \frac{\pi_n^*}{\gamma} = -l_i + l_1 + l_2 - \frac{\pi_n^*}{\gamma}$, because $\mathcal{T} = l_1 + l_2$ and $\lambda_i^s + \lambda_i^u = \rho$. Then, for V_1^s and V_1^u the denominator becomes $l_2 - \frac{\pi_n^*}{\gamma}$

and for V_2^s and V_2^u the denominator becomes $l_1 - \frac{\pi_n^*}{\gamma}$. Then equation (??) results. If the eigenvalues are complex, then $l_1 = \lambda_1^s \lambda_1^u = (\alpha_s + \beta i)(\alpha_u - \beta i) = \alpha_s \alpha_u + \beta^2 - \beta(\alpha_u - \alpha_s)i = \mathcal{T}/2 + |\Delta|^{1/2}i$ and l_2 is its complex conjugate $l_2 = \mathcal{T}/2 - |\Delta|^{1/2}i$. \square

Proof. Proof of Proposition ??

We start with the determination of the slopes of the (K, n) projections of the eigenspaces in space (K, n) , for the $\Delta > 0$ case: $\frac{dn}{dK} \Big|_{E_1^s} = 1/v_{3,1}^s$ and $\frac{dn}{dK} \Big|_{E_2^s} = 1/v_{3,2}^s$. The numerators of the $v_{1,i}^s$ and $v_{3,i}^s$, for $i = 1, 2$, in equations (??)-(??), are negative. In order to determine the sign of the denominators observe that $l_1 - \frac{\pi_n^*}{\gamma} = \frac{1}{2} \left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} + \Delta^{\frac{1}{2}} \right)$ and $l_2 - \frac{\pi_n^*}{\gamma} = \frac{1}{2} \left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} - \Delta^{\frac{1}{2}} \right)$ where $\mathcal{Q} < 0$, $\pi_n^* < 0$, and $\Delta = \left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} \right)^2 + 4\frac{\mathcal{O}}{\gamma}$. As the sign of \mathcal{O} is ambiguous then the signs of Δ , l_1 and l_2 are also ambiguous. However, if we want to guarantee that $\Delta > 0$ we readily conclude that, three cases may occur: (1) if $\mathcal{O} > 0$ then the second member is larger in absolute value and therefore $l_1 - \frac{\pi_n^*}{\gamma} > 0 > l_2 - \frac{\pi_n^*}{\gamma}$. To see this note that we have $\left[\left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} \right)^2 + 4\frac{\mathcal{O}}{\gamma} \right]^{\frac{1}{2}} > \left| \mathcal{Q} - \frac{\pi_n^*}{\gamma} \right| \geq \mathcal{Q} - \frac{\pi_n^*}{\gamma}$. Then $v_{1,1}^s > 0$ and $v_{3,1}^s > 0$, and $v_{1,2}^s < 0$ and $v_{3,2}^s < 0$. If $\mathcal{O} < 0$ and $\Delta > 0$, the first member is larger in absolute value and therefore the sign of the denominators are equal and are the same as the sign of $\mathcal{Q} - \frac{\pi_n^*}{\gamma}$. Therefore, we have two cases: (2) if $\mathcal{O} < 0$, and $\mathcal{Q} - \frac{\pi_n^*}{\gamma} < 0$, then $v_{1,1}^s > 0$, $v_{3,1}^s > 0$, $v_{1,2}^s > 0$ and $v_{3,2}^s > 0$; or (3) if $\mathcal{O} < 0$, and $\mathcal{Q} - \frac{\pi_n^*}{\gamma} > 0$ then $v_{1,1}^s < 0$, $v_{3,1}^s < 0$, $v_{1,2}^s < 0$ and $v_{3,2}^s < 0$.

Then, the signs of $\frac{dn}{dK} \Big|_{E_1^s}$ and $\frac{dn}{dK} \Big|_{E_2^s}$ are clear. In addition, we have a relationship between those slopes

$$\frac{dn}{dK} \Big|_{E_1^s} - \frac{dn}{dK} \Big|_{E_2^s} = \frac{\frac{C^*}{\sigma} r_n^* (l_1 - l_2) + Y_n^* (\lambda_2^s - \lambda_1^s) (\mathcal{Q} - \lambda_1^u \lambda_2^u)}{\left(\frac{C^*}{\sigma} r_n^* + Y_n^* \lambda_1^u \right) \left(\frac{C^*}{\sigma} r_n^* + Y_n^* \lambda_2^u \right)} > 0$$

for any case of the three previous cases. Then: 1. If $\mathcal{O} > 0$ then $\frac{dn}{dK} \Big|_{E_1^s} > 0$ and $\frac{dn}{dK} \Big|_{E_2^s} < 0$; 2. If $\mathcal{O} < 0$ and $\pi_n^* > \gamma \mathcal{Q}$ then $\frac{dn}{dK} \Big|_{E_1^s} > \frac{dn}{dK} \Big|_{E_2^s} > 0$; 3. If $\mathcal{O} < 0$ and $\pi_n^* < \gamma \mathcal{Q}$ then $0 > \frac{dn}{dK} \Big|_{E_1^s} > \frac{dn}{dK} \Big|_{E_2^s}$. \square

Lemma 4. (Stable eigenspace projection). The linear space which is tangent to the stable manifold is given by

$$E^s \equiv \{ (C, e, K, n) : (C - C^*, e)^\top = S(K - K^*, n - n^*)^\top \}$$

$$S_{CK} = \frac{\frac{C^*}{\sigma}(r_K^* Y_n^* - r_n^* Y_K^*)(l_1 - l_2) + \frac{C^*}{\sigma} r_n^* (\lambda_1^s - \lambda_2^s)(\mathcal{Q} - \lambda_1^s \lambda_2^s)}{\frac{C^*}{\sigma} r_n^* (l_2 - l_1) + Y_n^* (\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} \quad (42)$$

$$S_{Cn} = \frac{\frac{C^*}{\sigma} r_n^* (\lambda_1^s - \lambda_2^s) \mathcal{O} / \gamma}{\frac{\pi_K^*}{\gamma} \left[\frac{C^*}{\sigma} r_n^* (l_2 - l_1) + Y_n^* (\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u) \right]} \quad (43)$$

$$S_{eK} = \frac{(\lambda_2^s - \lambda_1^s) \mathcal{O} / \gamma}{\frac{C^*}{\sigma} r_n^* (l_2 - l_1) + Y_n^* (\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} \quad (44)$$

$$S_{en} = \frac{\frac{C^*}{\sigma} ((Y_C^* - 1)r_n^* - Y_n^* r_C^*) (\lambda_2^s - \lambda_1^s) (\lambda_1^s \lambda_2^s - \frac{\pi_n^*}{\gamma}) + Y_n^* \lambda_1^s \lambda_2^s (l_1 - l_2)}{\frac{C^*}{\sigma} r_n^* (l_2 - l_1) + Y_n^* (\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} \quad (45)$$

where $S_{CK} > 0$, $S_{en} < 0$. Additionally $S_{Cn} < 0$ and $S_{eK} > 0$ if $\mathcal{O} > 0$, or $S_{Cn} > 0$ and $S_{eK} < 0$ if $\mathcal{O} < 0$.

Proof. Following the same method as in ? we prove this by solving the last two equations of (??) for the two constants, w_1^s and w_2^s as

$$w_1^s = \left(\frac{K(t) - K^* - v_{3,2}^s (n(t) - n^*)}{v_{3,1}^s - v_{3,2}^s} \right) e^{-\lambda_1^s t}$$

$$w_2^s = \left(\frac{-(K(t) - K^*) + v_{3,1}^s (n(t) - n^*)}{v_{3,1}^s - v_{3,2}^s} \right) e^{-\lambda_2^s t}$$

and substitute in the solutions for C and e in equation (??) to get $(C(t) - C^*, e(t) - e^*)^\top = S(K(t) - K^*, n(t) - n^*)^\top$ where

$$S \equiv \frac{1}{v_{3,1}^s - v_{3,2}^s} \begin{pmatrix} v_{1,1}^s - v_{1,2}^s & v_{1,2}^s v_{3,1}^s - v_{3,2}^s v_{1,1}^s \\ \lambda_1^s - \lambda_2^s & \lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s \end{pmatrix}.$$

After some algebra we get equations (??) to (??). In those equations we can sign $r_K^* Y_n^* - r_n^* Y_K^* < 0$ and $(Y_C^* - 1)r_n^* - Y_n^* r_C^* < 0$ from the concavity properties of the production and utility functions, $r_n^* > 0$, $Y_n^* > 0$, $\pi_K^* > 0$ and $\pi_n^* < 0$. If the eigenvalues are real then all the denominators are negative because $l_2 - l_1 < 0$, $\mathcal{Q} < 0$, and because of the relationships between the eigenvalues, $\lambda_2^s < \lambda_1^s < 0 < \lambda_1^u < \lambda_2^u$. The numerator of S_{CK} is negative and the numerator of S_{en} is positive. At last, we see that the signs of the numerators of S_{Cn} and S_{eK} are symmetric. Note that, if $\mathcal{O} > 0$ then $l_2 - \mathcal{Q} < 0 < l_1 - \mathcal{Q}$ and if $\mathcal{O} < 0$ then the signs of $l_2 - \mathcal{Q}$ and $l_1 - \mathcal{Q}$ are the same. On the other hand $(l_1 - \mathcal{Q})(l_2 - \mathcal{Q}) = -\mathcal{O} / \gamma$. Then the sign of S_{eK} is the same as \mathcal{O} and is the symmetric of S_{Cn} .

If the eigenvalues are complex we can proof that all the coefficients are of type $(ai)/(bi) = a/b$ and they are also real. The denominator is a complex number with zero real part and a negative coefficient to the complex part $l_2 - l_1 = -2|\Delta|^{1/2}i$ and

$(\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u) = (\mathcal{Q} - (\alpha_u)^2 - \beta^2)2\beta i$ have both negative coefficients. The numerator of S_{CK} is also complex number with negative coefficient to the complex part as $(\lambda_1^s - \lambda_2^s)(\mathcal{Q} - \lambda_1^s \lambda_2^s) = (\mathcal{Q} - (\alpha_s)^2 - \beta^2)2\beta i$. If we apply the same method to the other coefficients we observe that the numerator of all the other coefficients are complex numbers with zero real part and S_{Cn} has a negative coefficient to the complex part and S_{eK} and S_{en} have positive coefficients to the complex part. \square

Lemma 5. (Phase diagrams) *The geometry of the projections of the isoclines and of the eigenspaces over space (K, n) are as follows:*

$$\mathbf{R}_1 \text{ if } \mathcal{O} > 0 \text{ then } \left. \frac{dn}{dK} \right|_{\dot{K}=0} > \left. \frac{dn}{dK} \right|_{r=\rho} > \left. \frac{dn}{dK} \right|_{E_1^s} > \left. \frac{dn}{dK} \right|_{\pi=0} > \left. \frac{dn}{dK} \right|_{\dot{n}=0} > 0 > \left. \frac{dn}{dK} \right|_{E_2^s};$$

$$\mathbf{R}_2 \text{ If } \mathcal{O} < 0, \Delta > 0 \text{ and } \pi_n^* > \gamma\mathcal{Q} \text{ then } \left. \frac{dn}{dK} \right|_{r=\rho} > \left. \frac{dn}{dK} \right|_{\dot{K}=0} > \left. \frac{dn}{dK} \right|_{E_1^s} > \left. \frac{dn}{dK} \right|_{E_2^s} > 0 > \left. \frac{dn}{dK} \right|_{\dot{n}=0} > \left. \frac{dn}{dK} \right|_{\pi=0};$$

$$\mathbf{R}_3 \text{ If } \mathcal{O} < 0, \Delta > 0 \text{ and } \pi_n^* < \gamma\mathcal{Q} \text{ then } \left. \frac{dn}{dK} \right|_{r=\rho} > \left. \frac{dn}{dK} \right|_{\dot{K}=0} > 0 > \left. \frac{dn}{dK} \right|_{\dot{n}=0} > \left. \frac{dn}{dK} \right|_{\pi=0} > \left. \frac{dn}{dK} \right|_{E_1^s} > \left. \frac{dn}{dK} \right|_{E_2^s};$$

$$\mathbf{C} \text{ If } \mathcal{O} < 0 \text{ and } \Delta < 0 \text{ then } \left. \frac{dn}{dK} \right|_{r=\rho} > 0, \left. \frac{dn}{dK} \right|_{\dot{K}=0} > 0, \left. \frac{dn}{dK} \right|_{\dot{n}=0} < 0, \left. \frac{dn}{dK} \right|_{\pi=0} < 0.$$

Proof. The slopes of the (K, n) projections of the eigenspaces in space (K, n) were already determined in the proof of Proposition ???. Next, we deal with the projection of the of the isoclines in the diagram (K, n) , on the stable eigenspace, in a neighborhood of the steady state. Obviously, as they depend on (C, e, K, n) , the projection in (K, n) will shift if consumption and entry are not on their steady state values. As the stable manifold is a 2-dimensional manifold, there is a functional relationship between (C, e) and (K, n) , which is approximated linearly by the expressions defining the eigenspace E^s . Therefore, the projection of the intersection of the isoclines with the stable eigenspace on the space (K, n) have the following slopes,

$$\begin{aligned} \left. \frac{dn}{dK} \right|_{\dot{C}=0} &= -\frac{r_K^* + r_C^* S_{CK}}{r_n^* + r_C^* S_{Cn}} = \frac{\lambda_2^s v_{1,2}^s - \lambda_1^s v_{1,1}^s}{\lambda_2^s v_{3,1}^s v_{1,2}^s - \lambda_1^s v_{1,1}^s v_{3,2}^s}, \\ \left. \frac{dn}{dK} \right|_{\dot{e}=0} &= -\frac{\pi_K^* + \pi_C^* S_{CK} - \rho\gamma S_{eK}}{\pi_n^* + \pi_C^* S_{Cn} - \rho\gamma S_{eK}} = \frac{(\lambda_2^s)^2 - (\lambda_1^s)^2}{v_{3,1}^s (\lambda_2^s)^2 - v_{3,2}^s (\lambda_1^s)^2} \\ \left. \frac{dn}{dK} \right|_{\dot{K}=0} &= -\frac{Y_K^* + (Y_C^* - 1)S_{CK}}{Y_n^* + (Y_C^* - 1)S_{Cn}} = \frac{\lambda_2^s v_{3,2}^s - \lambda_1^s v_{3,1}^s}{v_{3,1}^s v_{3,2}^s (\lambda_2^s - \lambda_1^s)} \\ \left. \frac{dn}{dK} \right|_{\dot{n}=0} &= -\frac{S_{CK}}{S_{Cn}} = \frac{\lambda_2^s - \lambda_1^s}{\lambda_2^s v_{3,1}^s - v_{3,2}^s \lambda_1^s}. \end{aligned}$$

Tedious calculations, available upon request, allowed us to find unambiguous signs such as $\frac{dn}{dK} \Big|_{\dot{C}=0} > 0$ and $\frac{dn}{dK} \Big|_{\dot{K}=0} > 0$, and $\frac{dn}{dK} \Big|_{\dot{K}=0} - \frac{dn}{dK} \Big|_{\dot{n}=0} > 0$. All the other signs for the slopes and relationships for the slopes depend on the sign of $(l_1 - \mathcal{Q})(l_2 - \mathcal{Q})$ which depends on the sign of \mathcal{O} . \square

Proof. Proof of Proposition ?? The proof is similar to the one presented in ? Proposition 5. \square

Proof. Proof of Proposition ?? Observe that, in the (K, n) -graphs, the pre- and post-shock levels of those variables will be located on the linear schedule $n = K \cdot (n/K)^*$ with a positive slope:

$$\frac{dn}{dK} \Big|_{dG} = -\frac{r_C^* \pi_K^* - r_K^* \pi_C^*}{r_C^* \pi_n^* - r_n^* \pi_C^*} > 0,$$

passing through the initial and ex-post steady-states. The pre-shock initial point following a positive (negative) variation of G will be located below (above) the post-shock steady state along the dG line. This means that using the results in Proposition ?? we can have a geometric determination of the existence of non-monotonic effects of fiscal policy by comparing the dG line with the projections of the isoclines for K and n and with curve E_2^s . Interestingly we have:

$$\begin{aligned} \text{sign} \left(\frac{dn}{dK} \Big|_{\dot{K}=0} - \frac{dn}{dK} \Big|_{dG} \right) &= \text{sign} \left(1 - (Y_C^* - 1) \frac{dC(0)}{dG} \right) \\ \text{sign} \left(\frac{dn}{dK} \Big|_{\dot{n}=0} - \frac{dn}{dK} \Big|_{dG} \right) &= -\text{sign} \left(\frac{de(0)}{dG} \right) \end{aligned}$$

We find that

$$\frac{dn}{dK} \Big|_{dG} > \frac{dn}{dK} \Big|_{\dot{n}=0}, \quad \frac{dn}{dK} \Big|_{dG} < \frac{dn}{dK} \Big|_{\dot{K}=0}, \quad \frac{dn}{dK} \Big|_{dG} > \frac{dn}{dK} \Big|_{E_2^s}$$

That is: the long-run relation represented by the dG line is less steep than the projection of the isocline $\dot{K} = 0$ but is steeper than the projection of the isocline $\dot{n} = 0$ and the eigenspace E_2^s . This means that we that schedule $dn/dK|_{dG}$ can only belong to sets \mathbf{W}_n and \mathbf{E}_n , for case \mathbf{R}_3 that is we will only find the case in which the number of firms can evolves non-monotonically. Capital will always respond in a monotonic manner to fiscal policy, except for the case in which there are complex eigenvalues, case \mathbf{C} . \square

B Benchmark model: Euler frontier etc.

In Figure ?? we trace out two curves. In this section we derive these. The long run income expansion path

$$IEP \equiv \{(L, C) : U_L(L) = U_C(C)w(K, L, n), r(K, n, L) = \rho, \pi(K, n, L) = 0\}$$

is given by the schedule

$$C = \mu_C L^{-\eta/\sigma}$$

where

$$\mu_C = \left\{ \frac{\beta A}{\xi} \mu_K^\alpha \mu_n^{1-\alpha-\beta} \right\}^{1/\sigma}$$

and

$$\mu_K = \left[\frac{\phi}{A(1-\alpha-\beta)} \mu_n^{\alpha+\beta} \right]^{1/\alpha}$$

and

$$\mu_n = \left[\left(\frac{\phi}{A(1-\alpha-\beta)} \right)^{1/\alpha} \left(\frac{\rho}{\alpha A} \right)^{1/(1-\alpha)} \right]^{\alpha(\alpha-1)/\beta}$$

The short run income expansion path is defined for a given pair (C, L) as

$$IEP(K, n) \equiv \{(L, C) : U_L(L) = U_C(C)w(K, L, n)\}$$

is parameterized by (K, n) and is

$$C = \left(\frac{\beta A}{\xi} K^\alpha n^{1-\nu} \right)^{1/\sigma} L^{-(1+\eta-\beta)/\sigma}$$

The Euler frontier is

$$EF(G) = \{(L, C) : U_L(L) = U_C(C)w(K, L, n), r(K, n, L) = \rho, Y(L, K, n) = C + G\}$$

becomes

$$C = A \left(\frac{\alpha A}{\rho} \right)^{1/(1-\alpha)} L - G$$

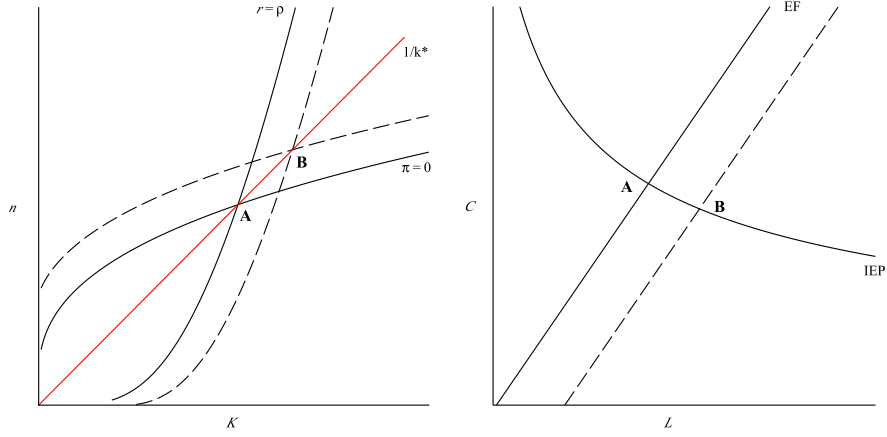


Figure 1: Steady state in (K, n) and (L, C) before (**A**) and after (**B**) a permanent increase in G for the case in which the isocline $\pi = 0$ has a positive slope at the steady state.

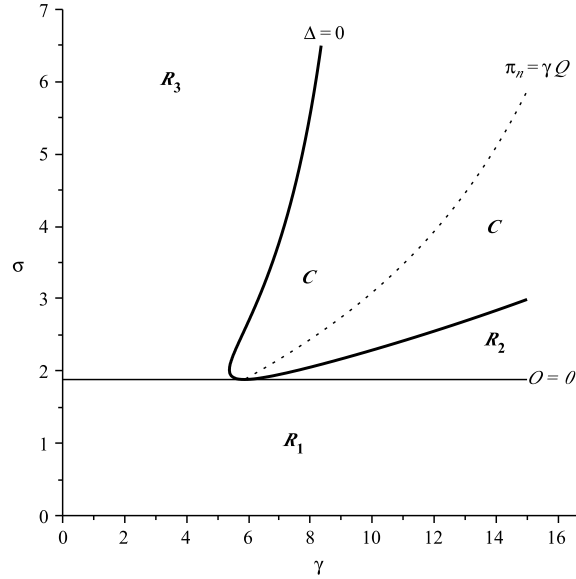


Figure 2: Bifurcation diagram on the (γ, σ) -space for $\alpha = 0.3$, $\beta = 0.5$, $\rho = 0.025$, $\phi = 0.1$, $\xi = 0.01$, $\eta = 0.4$ and $G = 0$, displaying the values for the parameters associated with the four main types of phase diagrams: phase diagram \mathbf{R}_1 , for $\mathcal{O} > 0$, phase diagram \mathbf{R}_2 for $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* > \gamma\mathcal{Q}$, phase diagram \mathbf{R}_3 for $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* < \gamma\mathcal{Q}$ and \mathbf{C} for $\Delta < 0$.

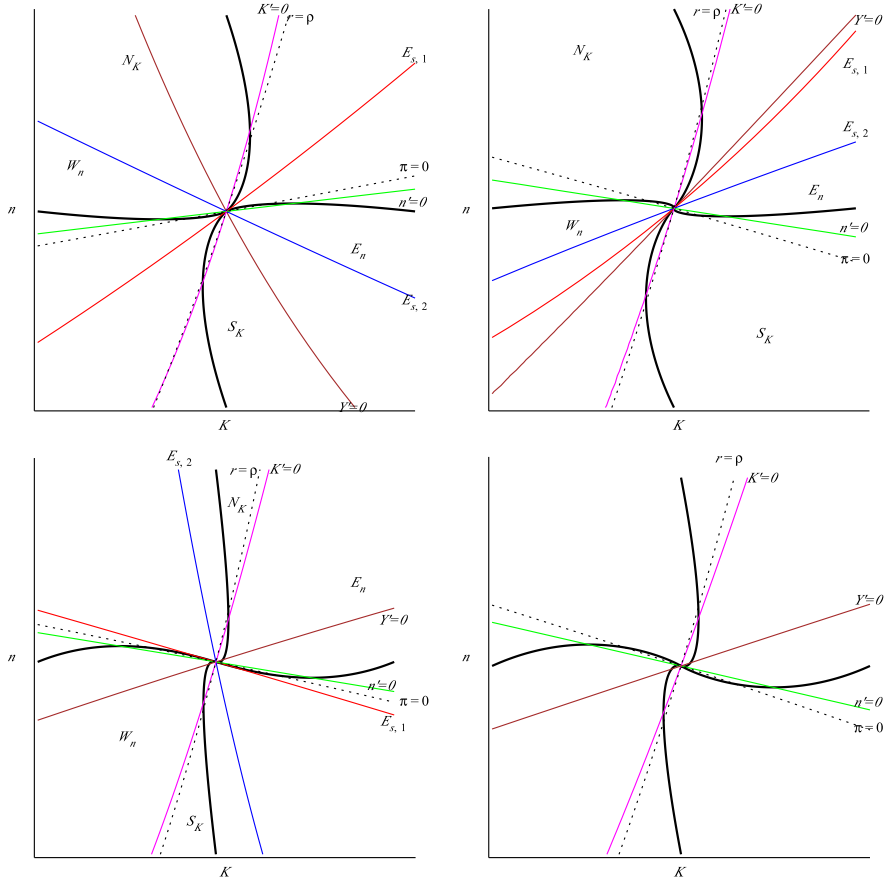


Figure 3: Two-dimensional projections of the phase diagrams in (K, n) -space, for cases \mathbf{R}_1 , \mathbf{R}_2 in the upper panels, \mathbf{R}_3 and \mathbf{C} , in the lower panel. Each panel displays the isoclines $\dot{K} = 0$, $\dot{n} = 0$ for the state variables and the isoclines $\dot{C} = 0$ ($r = \rho$) and $\dot{e} = 0$ ($\pi = 0$), the projections of the two eigenspaces E_1^s and E_2^s and the four areas \mathbf{N}_k and \mathbf{S}_k , between isocline for K and E_2^s and \mathbf{W}_n and \mathbf{E}_n , between isocline for n and E_2^s .

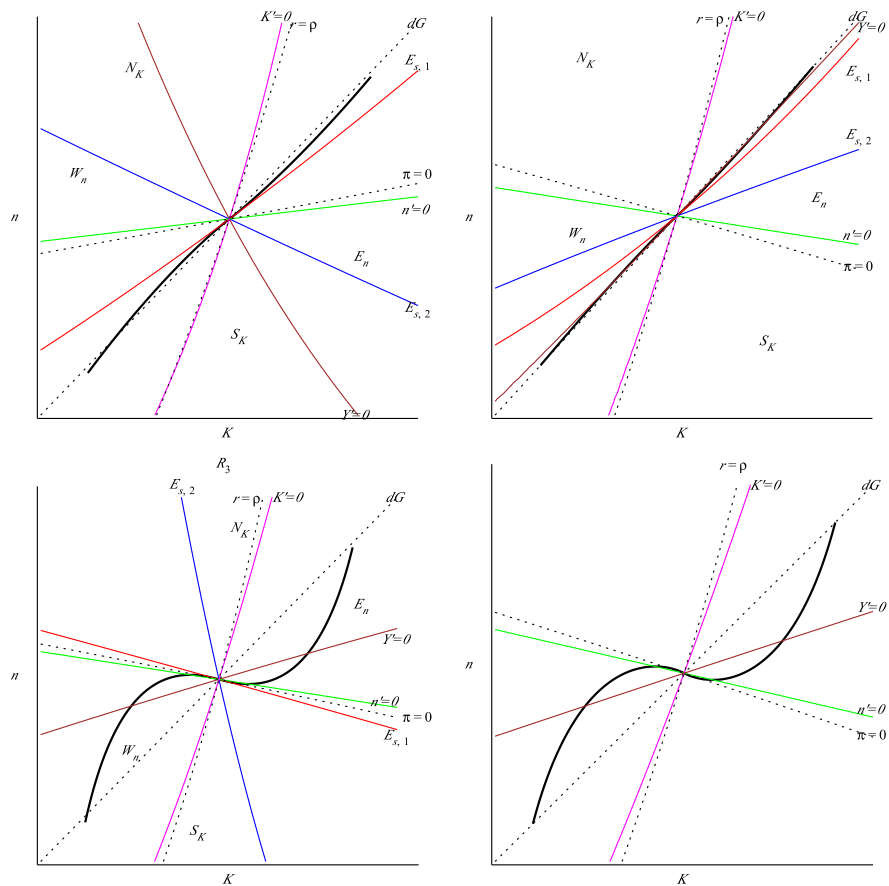


Figure 4: Two-dimensional projections of the phase diagrams in (K, n) -space and the effects of a fiscal shock: monotonous adjustment for cases \mathbf{R}_1 and \mathbf{R}_2 in the upper panels, hump-shaped adjustment for n in case \mathbf{R}_3 and hump-shaped adjustment for K and n in case \mathbf{C} .

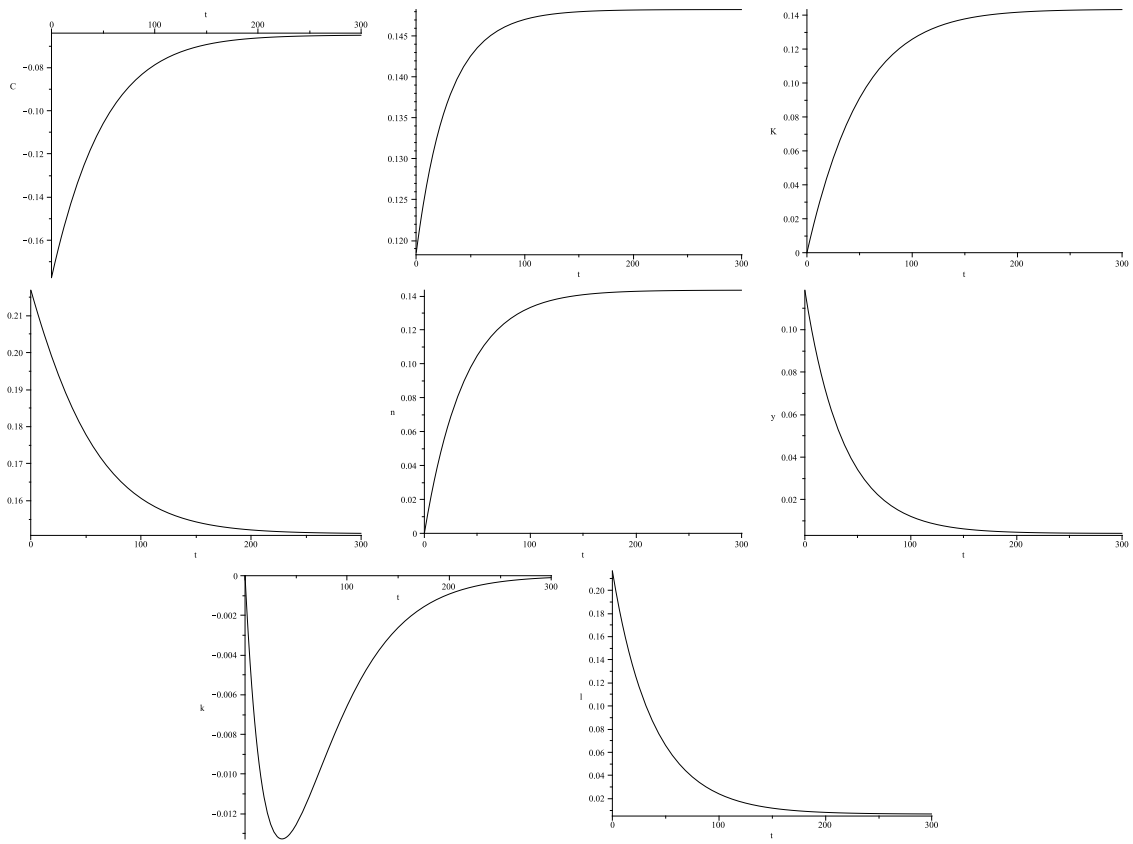


Figure 5: Impulse response for fiscal policy, case R_1

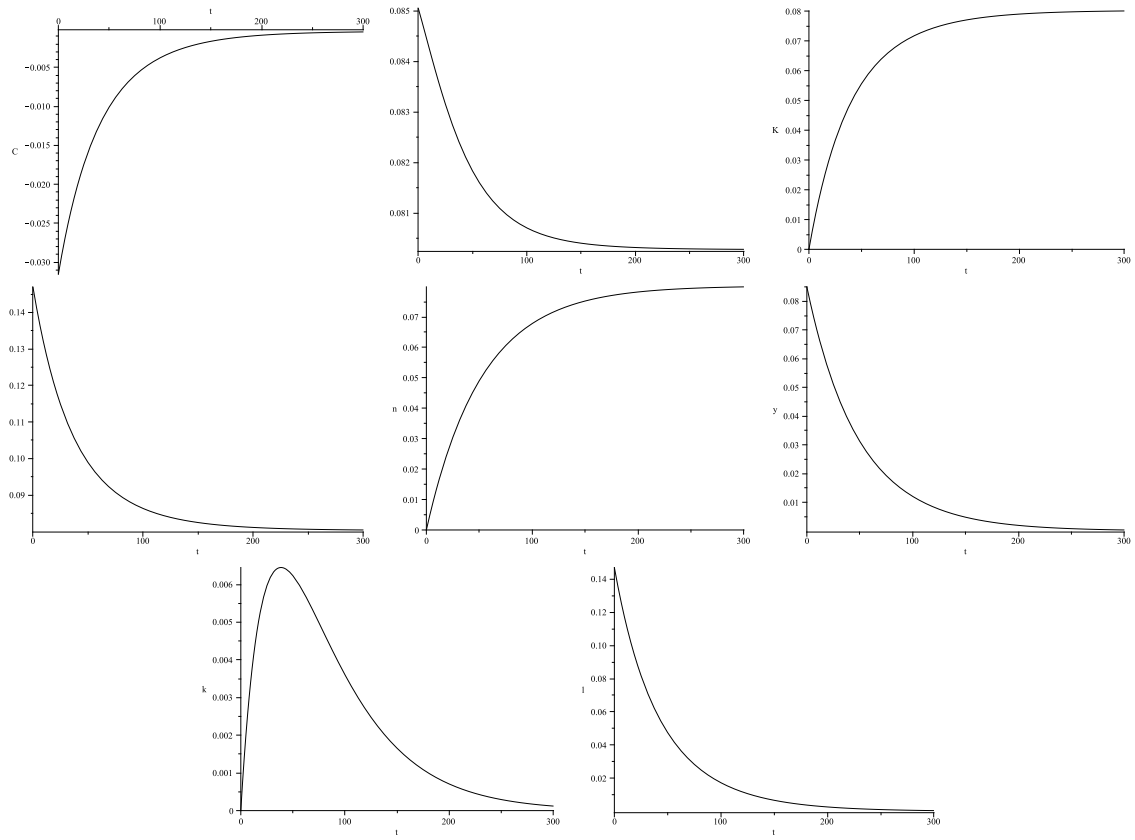


Figure 6: Impulse response for fiscal policy, case \mathbf{R}_2

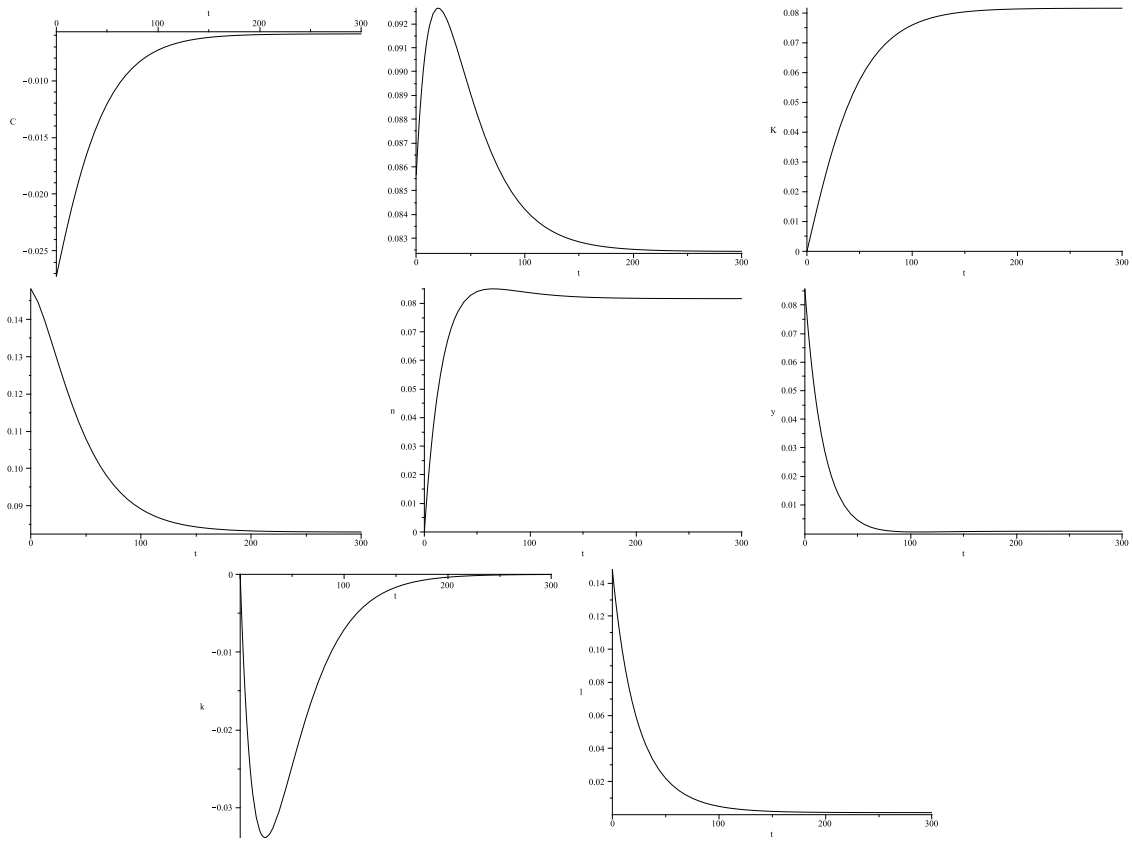


Figure 7: Impulse response for fiscal policy, case R_3

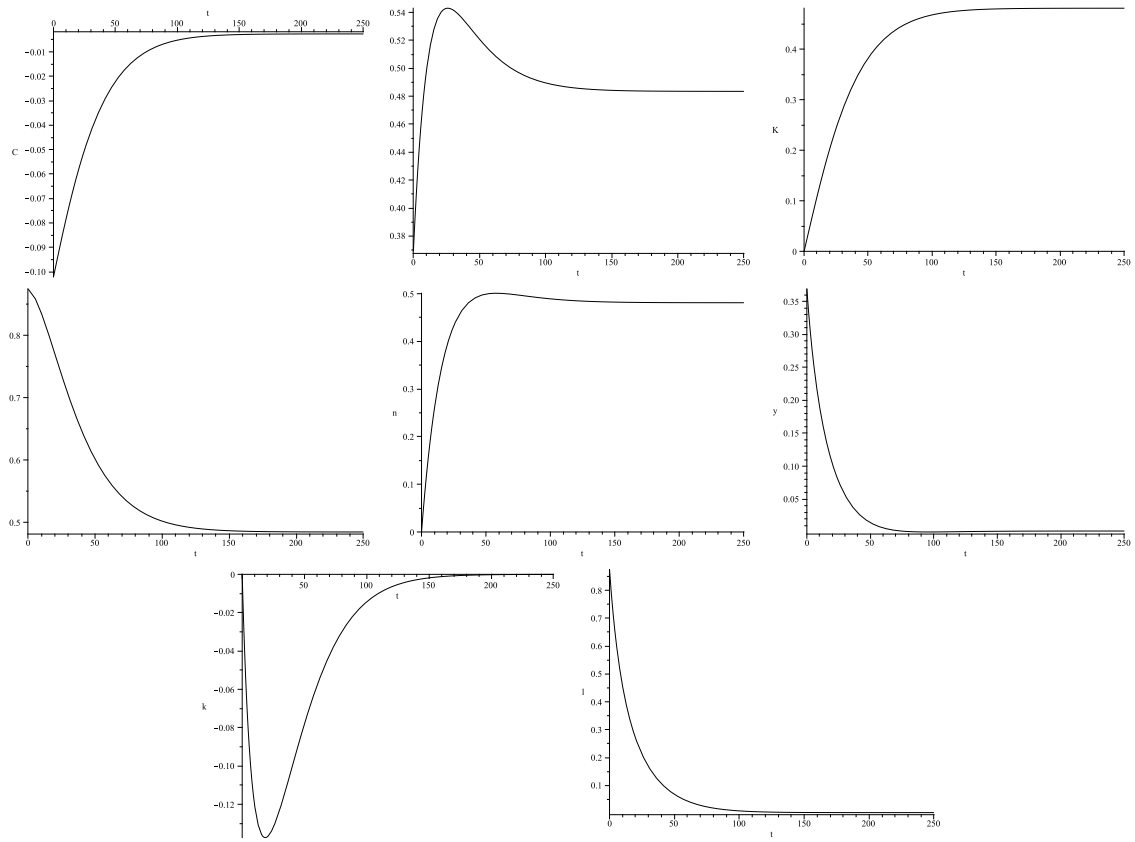


Figure 8: Impulse response for fiscal policy, case C