Exponential Discounting Bias*

Orlando Gomes† Alexandra Ferreira-Lopes‡ Tiago Neves Sequeira§

- March 2012 -

Abstract

This paper addresses intertemporal utility maximization under a general discount function that nests the exponential discounting and the quasi-hyperbolic discounting cases as particular specifications. The suggested framework intends to capture one important anomaly typically found when addressing the way agents discount the future, namely the evidence pointing to the prevalence of decreasing impatience. The referred anomaly can be perceived as a bias relatively to what would be a benchmark exponential discounting setting, and is modeled as such. The general discounting framework is used to address a trivial optimal growth model in discrete time. Transitional dynamics and stability properties of the corresponding dynamic setup are approached.

Keywords: Intertemporal preferences, Exponential discounting, Quasi-hyperbolic discounting, Optimal growth, Transitional dynamics.

JEL classification: C61, D91, O41.

---

*Alexandra Ferreira-Lopes and Orlando Gomes acknowledge financial support from PEst-OE/EGE/UI0315/2011. Alexandra Ferreira-Lopes and Tiago Neves Sequeira acknowledge support from FCT, project PTDC/EGE-ECO/102238/2008.

†Corresponding author. Instituto Superior de Contabilidade e Administração de Lisboa (ISCAL/IPL) & Instituto Universitário de Lisboa (ISCTE-IUL), Business Research Unit (UNIDE). Address: Instituto Superior de Contabilidade e Administração de Lisboa (ISCAL/IPL), Av. Miguel Bombarda 20, 1069-035 Lisbon, Portugal. E-mail: omgomes@iscal.ipl.pt

‡Instituto Universitário de Lisboa (ISCTE - IUL), ISCTE Business School Economics Department, BRU - IUL (BRU - Business Research Unit), and CEFAGE - UBI. E-mail: alexandra.ferreira.lopes@iscte.pt.

§Univ. Beira Interior (UBI) and CEFAGE-UBI research unit. Management and Economics Department. Universidade da Beira Interior. E-mail: sequeira@ubi.pt.

1 Introduction

Typically, the benchmark utility maximization dynamic model takes a constant rate of time discounting and, thus, intertemporal discounting is modeled as being exponential. This is an analytically convenient assumption and it is logically consistent with the idea that a constant interest rate is often used to compare the value of money over time, for instance at the level of the evaluation of investment projects. However, there are psychological effects that must be taken into account when addressing intertemporal preferences. Such effects may have a huge impact on how we perceive the behavior of the representative agent in the context of conventional economic models since they tend to generate a departure relatively to exponential discounting. In a couple of influential papers, Laibson (1997, 1998) has initiated a serious discussion on the economic implications of assuming departures from exponential intertemporal discounting. In this study, we discuss some of these implications by generalizing Laibson's basic hypothesis and by applying it to the study of a trivial discrete time neoclassical growth model.

In Xia (2011), three types of time preference anomalies that imply a deviation relatively to the standard exponential discounting framework are identified. These relate to the timing of the evaluation, to the magnitude of the reward, and to the sign of the reward. The sign effect was first highlighted by Kahneman and Tversky (1979) and basically states that gains are discounted more than losses. The magnitude effect is a matter that has received increasing attention on recent literature (see Noor (2011) and Bialaszek and Ostaszewski (2012)) and relates to the evidence that there is an inverse relation between the amount of the reward and the steepness of discounting over time, i.e., agents tend to be more patient when larger rewards are under evaluation.

The most debated issue, though, is the one concerning changes on the degree of impatience as time evolves. This is the object of Laibson's analysis and it relates essentially to the basic evidence that there is decreasing impatience over time – the rate of discount that we apply when measuring the present value of some near in time outcome is typically much larger than the discount rate applied to a distant in the future event. Putting it more clearly, human beings tend to attribute much more weight on the difference between a reward to be received today or tomorrow than on the difference between two consecutive dates in the far future. This is the same as saying that the discount rate decreases in time. This type of phenomenon is known as hyperbolic discounting and it has been widely discussed at various levels in recent years.

On one hand, hyperbolic discounting is a subject of debate on philosophical grounds. Several authors have argued that hyperbolic discounting is a rational way to form intertemporal preferences, more than exponential discounting. Prelec (2004), Dimitri (2005), Drouhin (2009), and Farmer and Geanakoplos (2009) argue that hyperbolic discounting is time consistent or rational. Decreasing impatience on a stochastic environment allows for a formal proof of such claim. Other authors are more skeptical about how hyperbolic discounting is being approached in the literature. While there is a tendency to search for analytical discount functions that may allow for an elegant treatment of economic models, one should take into account arguments as the ones by Rubinstein (2003) and Rasmussen (2008) who believe that modifying functional forms does not answer the main questions posed by the apparent lack of rationality in economic behavior. As stated by Ariel Rubinstein, a deeper understanding of intertemporal human decisions requires opening the black-box of decision making more than changing slightly the structure of the model used to address human behavior.

Other relevant contributions on the field of hyperbolic discounting relate the generalization of the concept and of the corresponding implications. In Bleichrodt, Rohde, and Wakker (2009) the commonly used discount functions are modified in order to account for other kinds of time inconsistency on the formation of preferences besides decreasing impatience. Specifically, the proposed framework accommodates the possibilities of increasing impatience and strongly decreasing impatience. Also Benhabib, Bisin, and Schotter (2010) present a general version of the discount function, that contemplates the most common specifications of exponential and hyperbolic discounting found in the literature.

The powerful notion of hyperbolic discounting, and its most common specification in economics (Laibson's quasi-hyperbolic discounting concept), have been applied to study a wide range of relevant economic issues. Just to cite a few, we highlight the contributions of Gong, Smith, and Zou (2007), concerning consumption under uncertainty, Hepburn, Duncan, and Papachristodoulou (2010) in the field of environment policy, Graham and Snower (2007) on short-run macroeconomics and inflation dynamics, and, Barro (1999) and Coury and Dave (2010) on the implications of non-exponential discounting to economic growth.

In this paper we generalize the quasi-hyperbolic discounting setting of Laibson and apply the new framework on intertemporal preferences to a trivial discrete time optimal growth problem. The setup differs from other analyses on the subject because we relate the shape of the discounting function to issues of financial literacy, following the analysis on the exponential growth bias as developed by Stango and Zinman (2009) and Almenberg and Gerdes (2011). Our argument is that in the same way people tend to underestimate future values of variables that grow at constant rates, individuals also tend to overestimate close in time values (relatively to the ones more distant in the future) when discounting them to the present. This reasoning will allow to present a discount function that is flexible enough to characterize different degrees of hyperbolic discounting and to nest the exponential discounting case as a possible limit outcome.

The proposed specification of intertemporal preferences is analytically convenient to address a discrete time optimal growth model. It allows to present explicit stability conditions and it serves to compare different degrees of deviation from the constant discount rate benchmark.

The remainder of the paper is organized as follows. Section 2 discusses in detail the notion of exponential discounting bias. In section 3 this concept is applied to compare different possibilities in terms of hyperbolic discounting. Section 4 approaches utility maximization under the general specification for intertemporal preferences. Section 5 sets up the growth model. Section 6 analyzes the dynamics of the model. Finally, section 7 concludes.

2 Anomalies in Financial Evaluation

Recently, Stango and Zinman (2009) and Almenberg and Gerdes (2011) have carefully analyzed the evidence that points to a tendency to underestimate the future value of a given variable that grows at a constant rate. This exponential growth bias clearly exists in practice, for instance in what concerns household financial decision making.

The mentioned literature emphasizes the link between the extent of the bias and the degree of financial literacy. A poor ability to perform basic calculations and the lack of familiarity with elementary financial concepts and products will, in principle, imply a wider gap between individuals' calculations and the true future values, i.e., it is observable a negative correlation between financial literacy and the exponential growth bias.

Well informed agents will be able to understand the basic notion of capitalization and to perceive the exponential path followed by any value that accumulates over time. However, many studies have been discovering serious flaws on the understanding, by the average citizen, of simple financial concepts and mechanisms. This was highlighted by Lusardi (2008) and Japelli (2010), among others. Financial literacy or, more precisely, the lack of it, can explain the kind of deficiency that consists in linearizing an exponential series in time that is the outcome of accumulating some value at a constant rate.

The important argument concerning the lack of ability on accurately addressing the value of money in time is that incorrect answers are biased. As emphasized by Almenberg and Gerdes (2011), individuals are almost twice as likely to underestimate the correct amount than to overestimate it. Thus, on the aggregate it makes sense to state that in a society where a given degree of financial illiteracy exists, the future values of a series that grows at a constant rate will be underestimated. Exponential growth bias will then be common when assessing the future value of an investment that offers a return at a given annual constant interest rate.

It is reasonable to conceive the existence of a link between the interest rate and the rate of time preference. In Farmer and Geanakoplos (2009, pages 1,2), this link is explained in simple terms,

'A natural justification for exponential discounting comes from financial economics and the opportunity cost of foregoing an investment. A dollar at time s can be placed in the bank to collect interest at rate r, and if the interest rate is constant, it will generate $\exp(r(t-s))$ *dollars at time t. A dollar at time t is therefore equivalent to* $\exp(-r(t-s))$ *dollars at time s. Letting $\tau = t - s$, this motivates the exponential discount function* $D_s(\tau) = D(\tau) = \exp(-r\tau)$, *independent of s.'*

The above sentence establishes a possible direct connection between the interest rate and the discount rate of intertemporal preferences. Nevertheless, there is a substantial difference between the two. While the interest rate is obtained as a market outcome, and might not vary if market conditions do not change, the rate of intertemporal choice is a matter of perception and preferences. Agents may want to adopt a rate of time preference that is close to the interest rate, but if they fail in understanding how future values accumulate, the lack of financial literacy eventually helps in explaining why the subjective rate of time impatience possibly departs from a constant value or, in other words, why discounting possibly deviates from the benchmark exponential discounting under a constant discount rate.

To understand how financial illiteracy might contribute to deviate aggregate preferences from exponential discounting, we just need to make the inverse path to the one that is present in the evaluation of the exponential growth bias, i.e., if individuals tend to underestimate future values when assessing them in the present, they will certainly overestimate current values when thinking about them as if they were taking decisions at some future time moment. In analytical terms, the idea of exponential growth bias is commonly presented as $FV = PV(1+r)^{(1-\theta)t}$, where FV is the future value, PV the present value, r the interest rate, t is time and $\theta \in (0,1)$ measures the magnitude of the bias. If one wants to address the present value given the future value, we just need to rearrange the previous expression and write it as $PV = FV/(1+r)^{(1-\theta)t}$.

The above relation implies decreasing impatience. Far in the future outcomes are much less valued than the ones occurring in the near future. Now the bias works on the opposite direction - near in time results are overestimated. We can call this effect **exponential discounting bias**, and we may define it as the tendency to overestimate close in time values of a variable that grows at a constant rate.

The exponential discounting bias will be as much larger as the larger is the extent of financial illiteracy and it constitutes an alternative explanation about why preferences in time tend to imply hyperbolic discounting: agents *want* to select a constant rate of time preference, namely a rate of time preference that follows the interest rate path, but their ability to undertake the proper computations is biased, in such a way that far in time values are less considered then the ones near the current period.

3 Departures from Exponential Discounting

In order to account for decreasing impatience, Loewenstein and Prelec (1992) have proposed the following hyperbolic discount function: $D_H(t) = (1+\alpha t)^{-\gamma/\alpha}$, where α and γ are two positive parameters. This discount function implies a decreasing discount rate - short-term discount rates are higher than long-term discount rates. Empirical evidence suggests that this is a much more appropriate and realistic way to approach intertemporal preferences then just considering a constant discount rate over time.

While empirically more suitable, hyperbolic discounting, considered as modeled above, is much less tractable from an analytical point of view than exponential discounting. Because of this, Laibson (1997, 1998) has proposed an approximation to hyperbolic discounting, that he dubbed quasi-hyperbolic discounting; this is straightforward to apply to the standard dynamic optimization models of economists. The discount function takes the following form: $D_{QH}(t) = \begin{cases} 1 \text{ if } t = s \\ \widehat{\beta}\widehat{\delta}^{t-s} \text{ if } t = s+1, s+2, ... \end{cases}$, with s the time period in which the future is being evaluated; $\widehat{\beta} \in (0,1)$, $\widehat{\delta} \in (0,1)$. Note that in the limit case $\widehat{\beta} = 1$ we are back at exponential discounting.

As in the hyperbolic case, the quasi-hyperbolic discount function captures the idea that discount rates decline with the passage of time. Laibson proposes, in his studies, a small exercise to compare discount rates on each of the settings. He considers exponential discounting ($\widehat{\beta} = 1; \widehat{\delta} = 0.97$), quasi-hyperbolic discounting ($\widehat{\beta} = 0.6; \widehat{\delta} = 0.99$), and hyperbolic discounting ($\alpha = 10^5; \gamma = 5 \times 10^3$) and draws a graphic where it is evident that $D_{QH}(t)$ generates a time trajectory that is considerably closer to $D_H(t)$ than the one originating on plain exponential discounting.

In the previous section, it was stated that the absence of a stable impatience level over time may be interpreted as an anomaly, something similar to the tendency that individuals have to linearize a series of values that accumulate at a constant rate (and, hence, truly exhibit an exponential path). In the proposed setting, this anomaly should be considered in the inverse way, i.e., if individuals tend to linearize exponential trajectories for the future, when discounting values to the present they will exacerbate the exponential nature of the series under analysis.

In this context, we will consider exponential discounting, $D_E(t) = \beta^{t-s}$, $\beta \in (0,1)$, but we add the possibility of existing an error of evaluation that increases short-run impatience, generating a kind of hyperbolic discounting. Let $\theta(t)$ be the anomaly term, which transforms $D_E(t)$ into a discount function with an exponential bias, i.e., $D_{EB}(t) = \beta^{(1+\theta(t))(t-s)}$. Function $\theta(t)$ will take the following form: $\theta(t) = \begin{cases} 0 \text{ if } t = s \\ \frac{\theta_1}{t-s} - \theta_0 \text{ if } t = s+1, s+2, ... \end{cases}$.

The assumption of $D_{EB}(t)$ as the discount function has two advantages. On one hand, it allows for an intuitive explanation on why we depart from exponential discounting. There is an error of evaluation by the agents; they eventually want to adopt a constant discount rate but, relatively to the periods that are closer in time they do not have the capacity to make an objective evaluation of their priorities. As time goes by, such ability evolves and, in the long-run, the error in evaluation is much smaller. On the other hand, we introduce a more general and flexible approach to time discounting than the one underlying $D_{QH}(t)$; as we will see below, the values of β, θ_0, and θ_1 can be chosen in such a way that we obtain an approximation to $D_H(t)$ that is undoubtedly better than the one provided by hyperbolic discounting.

We consider $\theta_0 \in [0,1]$ and $\theta_1 \geq 0$. Naturally, exponential discounting holds for $\theta_0 = \theta_1 = 0$, while quasi-hyperbolic discounting is also a particular case of the more general setting provided by $D_{EB}(t)$ for $\widehat{\beta} = \beta^{\theta_1}$ and $\widehat{\delta} = \beta^{(1-\theta_0)}$. Recover Laibson's example and consider the following parameter values for the exponential bias discount function: $\beta = 0.97$, $\theta_0 = 0.95$, and $\theta_1 = 23$. Figure 1 displays a graphic that is similar to the one in the original Laibson's analysis (50 periods are considered and hyperbolic and quasi-hyperbolic discount functions are displayed; pure exponential discounting is ignored in the displayed figure). To this figure, we add the exponential bias case for the parameter values that were chosen.

It is evident that the new function generates results that provide a much better fit with the hyperbolic discount function than the ones generated by quasi-hyperbolicity. After 15 periods there is almost a perfect match between $D_{EB}(t)$ and $D_H(t)$ (although, if we introduced additional periods - after 50 - we would start to see a departure of one of the series relatively to the other; nevertheless, this widening gap would never be as pronounced as the one regarding quasi-hyperbolic discounting).

Figure 2 allows for a closer look on this issue. The figure represents the distance (in percentage and in absolute value) between $D_{EB}(t)$ and $D_H(t)$ and between $D_{QH}(t)$ and $D_H(t)$. Only in three of the 50 time periods ($t = 1, t = 2$ and $t = 22$), the distance between $D_{EB}(t)$ and $D_H(t)$ exceeds the distance between $D_{QH}(t)$ and $D_H(t)$. It is notorious that the present proposal is well suited to address decreasing impatience and it is also well founded on the idea that agents lack the information, literacy, or ability to maintain a constant discount rate over time.

**** Fig. 1, 2 ****

4 Intertemporal Utility Function

In many economic settings discount functions are used to construct intertemporal utility functions. Their typical presentation is as follows,

$$U_s(c) = u(c_s) + \sum_{\tau=1}^{\infty} D(t)u(c_{s+\tau}) \tag{1}$$

Equation (1) represents the utility in the current period, $t = s$, from consuming today and in all future moments from $t = s + 1$ to an undefined future date. The term $u(c_s)$ is current consumption utility; the instantaneous utility function obeys to conventional properties of continuity, smoothness, and concavity. Future utility is taken into account for all possible time moments but discounting implies that a larger weight is put into closer in time consumption opportunities. The discount function that we will consider is the one involving the exponential bias, $D(t) = D_{EB}(t)$.

We can consider the same sequence of utility functions, but now initiating one period later. This comes,

$$U_{s+1}(c) = u(c_{s+1}) + \sum_{\tau=1}^{\infty} D(t)u(c_{s+1+\tau}) \tag{2}$$

Taking into account $U_s(c)$ and $U_{s+1}(c)$ as presented above, we can address intertemporal utility under a recursive form. The following expression is straightforward to obtain from the simultaneous consideration of (1) and (2), under exponential discounting bias (now we denote time by t instead of s, in order to reflect that the important issue is that we are considering two consecutive time periods, independently of what the first in fact is),

$$U_t(c) = u(c_t) + \beta^{1-\theta_0}\left[U_{t+1}(c) - (1-\beta^{\theta_1})u(c_{t+1})\right] \tag{3}$$

The above expression is analytically useful, because one can apply to it, directly, dynamic programming techniques, in order to obtain optimal solutions.[1] Consider a simple budget constraint according to which a representative agent accumulates financial wealth (a_t) at a constant rate (r), besides receiving a constant wage w. This constraint is

$$a_{t+1} = w + (1+r)a_t - c_t,\ a_0 \text{ given.} \tag{4}$$

Maximizing utility subject to (4) requires defining a function $V(a_t)$ such that

$$V(a_t) = \max_c \left\{u(c_t) + \beta^{1-\theta_0}\left[V(a_{t+1}) - (1-\beta^{\theta_1})u(c_{t+1})\right]\right\} \tag{5}$$

The first order conditions come,

$$\begin{aligned} &u'(c_t) + \beta^{1-\theta_0}\left[V(a_{t+1})\right]' \frac{\partial V(a_{t+1})}{\partial c_t} - \beta^{1-\theta_0}(1-\beta^{\theta_1})u'(c_{t+1}) = 0 \\ &\Rightarrow \beta^{1-\theta_0}\left[V(a_{t+1})\right]' = u'(c_t) - \beta^{1-\theta_0}(1-\beta^{\theta_1})u'(c_{t+1}) \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\left[V(a_t)\right]' = \beta^{1-\theta_0}\left[V(a_t)\right]' \frac{\partial V(a_{t+1})}{\partial a_t} \\ &\Rightarrow \left[V(a_t)\right]' = (1+r)\beta^{1-\theta_0}\left[V(a_t)\right]' \end{aligned} \tag{7}$$

We must also take into account the transversality condition

$$\lim_{t\to\infty} a_t D_{EB}(t) V(a_t) = 0$$

Combining the two optimality conditions, (6) and (7), one obtains an equation of motion for consumption. In order to simplify the analysis, take a logarithmic utility function, $u(c) = \ln c$. For this functional form, the following difference equation is computed,

$$c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})(1+r)c_t c_{t-1}}{(2-\beta^{\theta_1}+r)c_{t-1} - \frac{1}{\beta^{1-\theta_0}}c_t} \tag{8}$$

The above expression is now rewritten for the ratio $\psi_t := c_t/c_{t-1}$,

[1] The dynamic programming procedure used to solve the model was adapted from Walde (2011), and it is standard in terms of the analysis of deterministic discrete time optimization models.

$$\psi_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})(1+r)}{(2-\beta^{\theta_1}+r) - \frac{1}{\beta^{1-\theta_0}}\psi_t} \tag{9}$$

From equation (9), we can determine the steady-state value of the ratio between two consecutive values of consumption. Two solutions exist; these equilibria come: $\psi^* = \beta^{1-\theta_0}(1+r) \vee \psi^* = \beta^{1-\theta_0}(1-\beta^{\theta_1})$. The found values have direct correspondence in the exponential case with the solutions $\psi^* = \beta(1+r) \vee \psi^* = 0$.

Note that the two solutions have a different nature: the first one is unstable and the second one is stable:

$\left.\frac{d\psi_{t+1}}{d\psi_t}\right|_{\psi^*=\beta^{1-\theta_0}(1+r)} = \frac{1+r}{1-\beta^{\theta_1}} > 1$;

$\left.\frac{d\psi_{t+1}}{d\psi_t}\right|_{\psi^*=\beta^{1-\theta_0}(1-\beta^{\theta_1})} = \frac{1-\beta^{\theta_1}}{1+r} \in (0,1)$.

5 The Growth Model

We now characterize the dynamics of a neoclassical growth model under exponential discounting bias. The maximization problem is the same as in the previous section. However, the constraint differs. We take capital accumulation and a production function involving decreasing marginal returns. Let k_t represent the capital stock and assume the following parameters: $A > 0$ (technology index), $\delta \in (0,1)$ (depreciation rate), $\eta \in (0,1)$ (output-capital elasticity). The resource constraint takes the form

$$k_{t+1} = Ak_t^{\eta} - c_t + (1-\delta)k_t,\ k_0 \text{ given} \tag{10}$$

Again, we compute first order conditions to encounter an optimal dynamic relation for consumption. If $\eta = 1$ (endogenous growth model with an AK production function), we end up with exactly the same dynamics one has characterized in the previous section (with $r = A - \delta$). Under decreasing marginal returns, we will be able to find constant steady-state values for both the state and the control variable, i.e., k_t and c_t. Repeating the same procedure of calculus to find optimality conditions, we arrive to the difference equation for consumption

$$c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})\left[1+\eta A k_t^{-(1-\eta)} - \delta\right] c_t c_{t-1}}{\left[2-\beta^{\theta_1}+\eta A k_t^{-(1-\eta)} - \delta\right] c_{t-1} - \frac{1}{\beta^{1-\theta_0}} c_t} \tag{11}$$

Because one cannot address consumption dynamics independently of capital accumulation on the present setting, we end up with a system of three difference equations to analyze; the system is,

$$\begin{cases} k_{t+1} = Ak_t^{\eta} - c_t + (1-\delta)k_t \\ c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})\left[1+\eta Ak_t^{-(1-\eta)}-\delta\right]c_t z_t}{\left[2-\beta^{\theta_1}+\eta Ak_t^{-(1-\eta)}-\delta\right]z_t - \frac{1}{\beta^{1-\theta_0}}c_t} \\ z_{t+1} = c_t \end{cases} \tag{12}$$

Next, we proceed to the full characterization of the dynamics of system (12). This requires finding the steady-state and to look at local dynamics.

Proposition 1 *The steady-state of the neoclassical optimal growth problem under exponential discounting bias corresponds to a unique equilibrium point:*

$$(k^*; c^*) = \left(\left[\frac{\eta A}{1/\beta^{1-\theta_0} - (1-\delta)} \right]^{1/(1-\eta)} ; A\,(k^*)^{\eta} - \delta k^* \right)$$

Proof. The steady-state is defined as the pair of values $(k^*; c^*)$ such that $k_{t+1} = k_t$ and $c_{t+1} = c_t = c_{t-1}$. Applying these conditions to system (12), it is straightforward to determine the values in the proposition ■

The steady-state value of the capital stock increases with the output-capital elasticity and with the value of the technology index. It falls with a larger depreciation rate. It is also straightforward to observe that a higher θ_0 (stronger deviation relatively to exponential discounting) implies a larger long-run value for the capital stock; the same is true for the value of β. Relatively to parameter θ_1, this has no influence over the steady-state values of the endogenous variables.

We illustrate the results with a small numerical example. Let $\eta = 1/3$, $\delta = 0.05$. The steady-state for capital and consumption is explored for values of θ_0 in the range $0 - 1$ and values of β in the range $0.75 - 1$. Figure 3 shows how the amounts of capital and consumption vary with different values of the parameters that define intertemporal impatience. The higher is θ_0, the larger are the equilibrium values of the two variables; the same occurs for a higher β. The horizontal axis respects to β, the depth axis represents different values of θ_0 and the vertical axis gives the values of each of the variables for the selected parameter values. Although discounting is important in terms of the dynamics of the growth model, we conclude that it has little impact on the steady-state - parameter θ_1 does not have any influence in long-run equilibrium; a change in θ_0 has impact on the steady-state - when it rises, the discount factor increases as well. A larger discount factor is synonymous of increased patience, what benefits the economy in terms of long-run accumulated capital and consumption levels.

*** *Fig. 3* ***

6 Local Dynamics

In order to address stability properties, one needs to linearize the system in the vicinity of the steady-state point. Computation leads to

$$\begin{bmatrix} k_{t+1}-k^* \\ c_{t+1}-c^* \\ z_{t+1}-c^* \end{bmatrix} = \begin{bmatrix} 1/\beta^{1-\theta_0} & -1 & 0 \\ j & 1+\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} & -\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_t-k^* \\ c_t-c^* \\ z_t-c^* \end{bmatrix} \tag{13}$$

with $j = \frac{1}{\beta^{1-\theta_0}}\left[\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})}-1\right]\frac{1-\eta}{\eta}\left[1-(1-\delta)\beta^{1-\theta_0}\right]\left[1-(1-(1-\eta)\delta)\,\beta^{1-\theta_0}\right]>0$.

A general stability result is achieved,

Proposition 2 *The system is saddle-path stable. There exists one stable dimension, in the three dimensional space of the model.*

Proof. The existence of one stable dimension implies that one of the eigenvalues of the Jacobian matrix locates inside the unit circle, while the other two fall outside the unit circle. Let the eigenvalues be $\lambda_1, \lambda_2, \lambda_3$. We want to prove that $|\lambda_1| < 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$.

We start by presenting trace, Tr, determinant, Det, and sum of principal minors, ΣM, of the Jacobian matrix; these are:

$Tr = \lambda_1 + \lambda_2 + \lambda_3 = 1/\beta^{1-\theta_0} + 1 + \frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})}$;

$Det = \lambda_1\lambda_2\lambda_3 = \frac{1}{\left(\beta^{1-\theta_0}\right)^2(1-\beta^{\theta_1})}$

$\Sigma M = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = Tr + Det + j - 1$

It is straightforward to observe that $Tr > 3$, $Det > 1$ and $\Sigma M > 3$. The constraint on the determinant implies that the eingenvalues are all positive or that $\lambda_1 > 0, \lambda_2, \lambda_3 < 0$. In this second scenario, the conditions involving the trace and the sum of principal minors imply $\lambda_1 > 3 - (\lambda_2 + \lambda_3)$ and $\lambda_1 < \frac{3-\lambda_2\lambda_3}{\lambda_2+\lambda_3}$; these inequalities cannot be simultaneously satisfied for the constraints on the values of the eigenvalues. Thus, the only feasible possibility is the one under which the eigenvalues are all positive: $\lambda_1, \lambda_2, \lambda_3 > 0$. If all the eigenvalues are larger than zero, then the constraints involving the trace and the determinant allow to perceive that full stability (all eigenvalues below one) is not a possible outcome. At least one eigenvalue must be larger than 1.

Next, we resort to Brooks (2004) to identify how many eigenvalues fall inside the unit circle. According to the mentioned author, an evaluation of the characteristic polynomial allows to state that: if condition

$$-(1+\Sigma M) < Tr + Det < 1 + \Sigma M$$

is met, there exists one real eigenvalue λ_1 of magnitude less than 1 and either:

- a pair of complex conjugate eigenvalues $\lambda_2, \lambda_3 = a \pm ib$, with $|a \pm ib| < 1$;

- two more real eigenvalues of magnitude less than 1; or
- a pair of real eigenvalues of magnitude greater than 1 and having the same sign.

Since we have remarked that at least one of the eigenvalues is larger than 1, the only possibility that can hold from the three above is the last one. Thus, if the displayed double inequality is satisfied, we confirm that $0 < \lambda_1 < 1$ and $\lambda_2, \lambda_3 > 1$. It is straightforward to verify the validity of the condition since it is equivalent to

$$-(Tr + Det + j) < Tr + Det < Tr + Det + j$$

Therefore, we confirm the existence of a single stable dimension, in the three dimensional space of the assumed system ■

The above result can be illustrated through a numerical example. Recover the benchmark values for the exponential discounting bias case, i.e., $\beta = 0.97, \theta_0 = 0.95, \theta_1 = 23$. For these, one computes the eigenvalues of the Jacobian matrix for various possibilities in terms of the values of $\eta \in (0, 1)$ and $\delta \in (0, 1)$, namely, we consider η between 1/6 and 5/6 and δ between 0.025 and 0.125. The values that the eigenvalues possess for each combination of parameters are displayed in Figure 4. The figure involves three panels, one for each eigenvalue. In the horizontal axis, we have the values of δ, in the depth axis the value of η and in the vertical axis the values assumed by each eigenvalue. Panel *a* respects to λ_1 and the other two to the eigenvalues that fall outside the unit circle, independently of the values of the parameters.

*** *Fig. 4* ***

With saddle path stability, we have a result that is qualitatively similar to the one of the original Ramsey model with a constant discount rate. There is convergence towards the unique steady-state point, that follows the one-dimensional stable trajectory; this trajectory is followed because the representative agent has the possibility of adapting its initial consumption level in order to place the system over the stable path, since consumption is a control variable. The expression of the stable trajectory can be presented in general terms.

Proposition 3 *Consider a point (k_0, c_0) in the vicinity of the steady-state $(k^*; c^*)$. In the convergence from the first to the second point, contemporaneous values of consumption and capital evolve following the stable path*

$$c_t - c^* = \left(1/\beta^{1-\theta_0} - \lambda_1\right)(k_t - k^*)$$

Proof. The saddle-path stable trajectory can be obtained by computing the eigenvector associated to the eigenvalue inside the unit circle. Thus, we can solve the system

$$\begin{bmatrix} 1/\beta^{1-\theta_0}-\lambda_1 & -1 & 0 \\ j & 1+\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})}-\lambda_1 & -\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} \\ 0 & 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Letting $p_1 = 1$, the eigenvector might be written as

$$P = \begin{bmatrix} 1 \\ 1/\beta^{1-\theta_0}-\lambda_1 \\ 1/(\lambda_1\beta^{1-\theta_0})-1 \end{bmatrix}$$

The slope of the contemporaneous relation between consumption and capital is given by the ratio p_2/p_1, i.e., $c_t - c^* = (p_2/p_1)(k_t - k^*)$, which corresponds to the expression in the proposition ■

As in the original Ramsey model, the convergence relation between capital and consumption is of positive sign; thus, both variables will simultaneously increase towards their long-term values. Note that this is a generic stable path expression that can be applied to specific forms of the discount function, namely the quasi-hyperbolic case and also the pure exponential discounting case. Observe, as well, that another stable trajectory emerges from the analysis - one can also relate consumption at $t-1$ to the capital stock at t. This convergence relation is also of positive sign.

Let us return to the numerical example. Take $\eta = 1/3$ and $\delta = 0.05$ and recover the remaining parameter values of the benchmark exponential discounting bias exercise. The eigenvalue inside the unit circle is, in this case, $\lambda_1 = 0.944\,14$. The stable trajectory is, then, $c_t - c^* = 5.738\,4 \times 10^{-2}(k_t - k^*)$. For the assumed parameter values, in the convergence towards the steady-state, when the capital stock increases in one unit, consumption will increase $5.738\,4 \times 10^{-2}$ units. This rate of convergence can be compared to the one of the quasi-hyperbolic discounting setting. Recall that hyperbolic discounting was discussed for $\widehat{\beta} = 0.6; \widehat{\delta} = 0.99$, with $\widehat{\beta} = \beta^{\theta_1}$ and $\widehat{\delta} = \beta^{(1-\theta_0)}$.

Continue to consider $\beta = 0.97$; to be in the conditions of the QHD case, we must now take $\theta_0 = 0.67004$ and $\theta_1 = 16.771$. For these parameter values, the lower than 1 eigenvalue comes $\lambda_1 = 0.936\,68$ and the stable trajectory is now $c_t - c^* = 7.342\,1 \times 10^{-2}(k_t - k^*)$. In the QHD case, the saddle path is steeper than in the EDB case. If we take EDB as a closer approximation to pure hyperbolic discounting, one possible error in using QHD consists in taking a larger change in consumption as the capital stock evolves than the one that should, in fact, be taken into account.

Next, we consider the constant discount rate case. This is the case for which θ_0 and θ_1 are zero. As θ_1 approaches zero, the eigenvalue lower than 1 approaches $0.917\,99$. Thus, the stable trajectory is $c_t - c^* = 0.11294(k_t - k^*)$. This case departs even more from the hyperbolic discounting case and thus the relation between k and c is even more steeper.

The above results point to the conclusion that the more apart we are from the exponential discounting case the less consumption will vary, in the convergence towards the steady-state, as the capital stock evolves. To emphasize this outcome, let us present the slope of the stable trajectory for different values of θ_0 and θ_1. Consider, again, $\eta = 1/3$, $\delta = 0.05$, $\beta = 0.97$. Figure 5 takes θ_0 in the horizontal axis and θ_1 in the depth axis, in order to quantify the slope of the stable arm in the vertical axis. It is evident from the figure that a larger bias relatively to exponential discounting (measured by higher values of θ_0 and θ_1) implies a smaller change in c relatively to k in the process of adjustment towards the steady-state.

**** Fig. 5 ****

7 Conclusion

People do not evaluate future outcomes as if they were computers or calculators. Measuring the future value of some current event or the present value of some future event is many times an intuitive process in which individuals engage on. In the same way there is evidence of an exponential growth bias, according to which individual agents tend to linearize the sequence of accumulated future outcomes, we can conceive a kind of exponential discounting bias, according to which we can explain the evidence that points to decreasing impatience and that is analytically translated in the concept of hyperbolic discounting.

The notion of exponential discounting bias is more general than the one commonly used by economists to characterize observed intertemporal preferences, i.e., the notion of quasi-hyperbolic discounting. This allows for a flexible analysis, where we can shape the trajectory of the discount factor the way we want in order to be the closest as possible to what evidence reveals.

Furthermore, when assessing the dynamics of an intertemporal representative consumer growth model in discrete time, the new specification has appealing features from an analytical tractability point of view. It allows for constructing a three dimensional dynamic system, from which it is straightforward to analyze steady-state properties and transitional dynamics. The analysis turns possible a thorough characterization of how different intertemporal preferences may shape the optimal relation between capital accumulation and consumption.

References

[1] Almenberg, J. and C. Gerdes (2011). 'Exponential Growth Bias and Financial Literacy.' IZA discussion paper n^o 5814.

[2] Barro, R.J. (1999). 'Ramsey Meets Laibson in the Neoclassical Growth Model.' *Quarterly Journal of Economics*, vol. 114, pp. 1125-1152.

[3] Benhabib, J.; A. Bisin and A. Schotter (2010). 'Present-Bias, Quasi-Hyperbolic Discounting, and Fixed Costs.' *Games and Economic Behavior*, vol. 69, pp. 205-223.

[4] Bialaszek, W. and P. Ostaszewski (2012). 'Discounting of Sequences of Delayed Rewards of Different Amounts.' *Behavioural Processes*, vol. 89, pp. 39-43.

[5] Bleichrodt, H.; K.I.M. Rohde and P.P. Wakker (2009). 'Non-Hyperbolic Time Inconsistency.' *Games and Economic Behavior*, vol. 66, pp. 27-38.

[6] Brooks, B.P. (2004). "Linear Stability Conditions for a First-Order Three-Dimensional Discrete Dynamic." *Applied Mathematics Letters*, vol. 17, pp. 463-466.

[7] Coury, T. and C. Dave (2010). "Hyperbolic' Discounting: a Recursive Formulation and an Application to Economic Growth.' *Economics Letters*, vol. 109, pp. 193-196.

[8] Dimitri, N. (2005). 'Hyperbolic Discounting can Represent Consistent Preferences.' University of Siena, Department of Economics working paper n^o 466.

[9] Drouhin, N. (2009). 'Hyperbolic Discounting may be Time Consistent.' *Economics Bulletin*, vol. 29, pp. 2549-2555.

[10] Farmer, J.D. and J. Geanakoplos (2009). 'Hyperbolic Discounting is Rational: Valuing the Far Future with Uncertain Discount Rates.' Cowles Foundation Discussion Papers n^o 1719.

[11] Gong, L.; W. Smith and H. Zou (2007). 'Consumption and Risk with Hyperbolic Discounting.' *Economics Letters*, vol. 96, pp. 153-160.

[12] Graham, L. D.J. Snower (2007). 'Hyperbolic Discounting and the Phillips Curve.' Kiel working paper collection n^o 2.

[13] Hepburn, C.; S. Duncan and A. Papachristodoulou (2010). 'Behavioural Economics, Hyperbolic Discounting and Environmental Policy.' *Environmental and Resource Economics*, vol. 46, pp. 189-206.

[14] Japelli, T. (2010). 'Economic Literacy: an International Comparison.' *Economic Journal*, vol. 120, pp. F429-F451.

[15] Kahneman, D. and A. Tversky (1979). 'Prospect Theory: an Analysis of Decision under Risk.' *Econometrica*, vol. 47, pp. 263-292.

[16] Laibson, D. (1997). 'Golden Eggs and Hyperbolic Discounting.' *Quarterly Journal of Economics*, vol. 112, pp. 443-478.

[17] Laibson, D. (1998). 'Life-Cycle Consumption and Hyperbolic Discount Functions.' *European Economic Review*, vol. 42, pp. 861-871.

[18] Loewenstein, G. and D. Prelec (1992). 'Anomalies in Intertemporal Choice: Evidence and an Interpretation.' *Quarterly Journal of Economics*, vol. 57, pp. 573-598.

[19] Lusardi, A. (2008). 'Financial Literacy: an Essential Tool for Informed Consumer Choice?' NBER working paper n^o 14084.

[20] Noor, J. (2011). 'Intertemporal Choice and the Magnitude Effect.' *Games and Economic Behavior*, vol. 72, pp. 255-270.

[21] Prelec, D. (2004). 'Decreasing Impatience: a Criterion for non-Stationary Time Preference and 'Hyperbolic' Discounting.' *Scandinavian Journal of Economics*, vol. 106, pp. 511-532.

[22] Rasmussen, E.B. (2008). 'Some Common Confusions about Hyperbolic Discounting.' Indiana University, Kelley School of Business, working paper of the department of Business Economics and Public Policy n^o 2008-11.

[23] Rubinstein, A. (2003). 'Economics and Psychology? The Case of Hyperbolic Discounting.' *International Economic Review*, vol. 44, pp. 1207-1216.

[24] Stango, V. and J. Zinman (2009). 'Exponential Growth Bias and Household Finance.' *Journal of Finance*, vol. 64, pp. 2807-2849.

[25] Walde, K. (2011). *Applied Intertemporal Optimization*. Mainz, Germany: Mainz University Gutenberg Press.

[26] Xia, B. (2011). 'A Simple Explanation of Some Key Time Preference Anomalies.' *Canadian Journal of Economics*, vol. 44, pp. 695-708.

Figures:

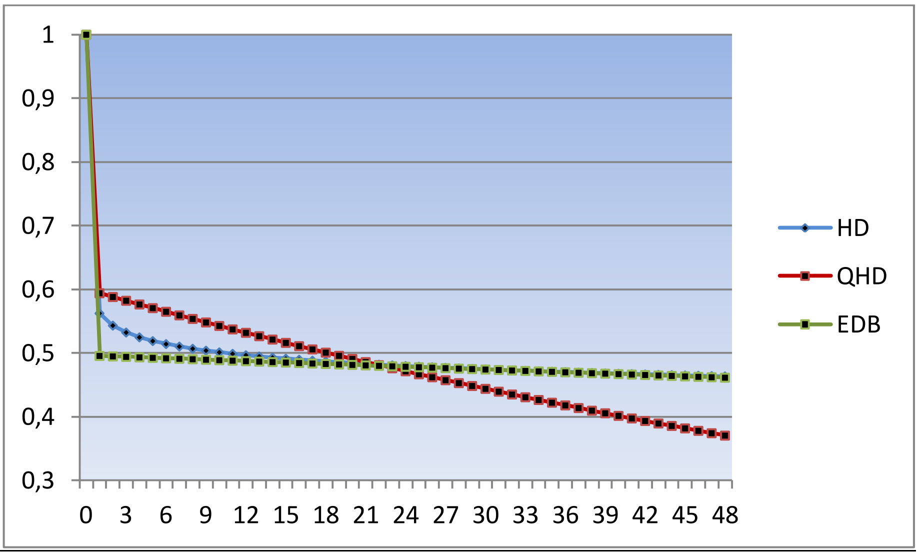


Fig.1 – Discount factors for hyperbolic discounting, quasi-hyperbolic discounting and exponential discounting bias.

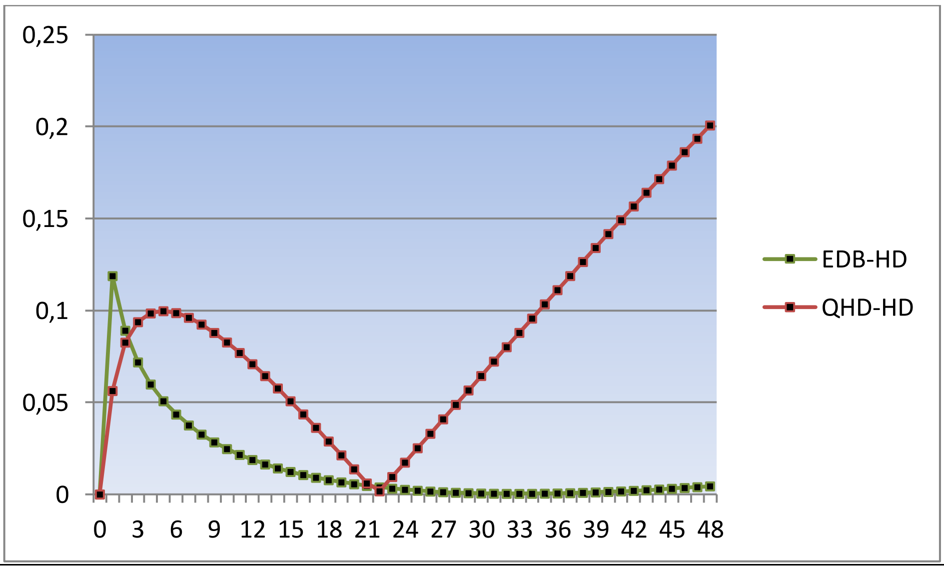
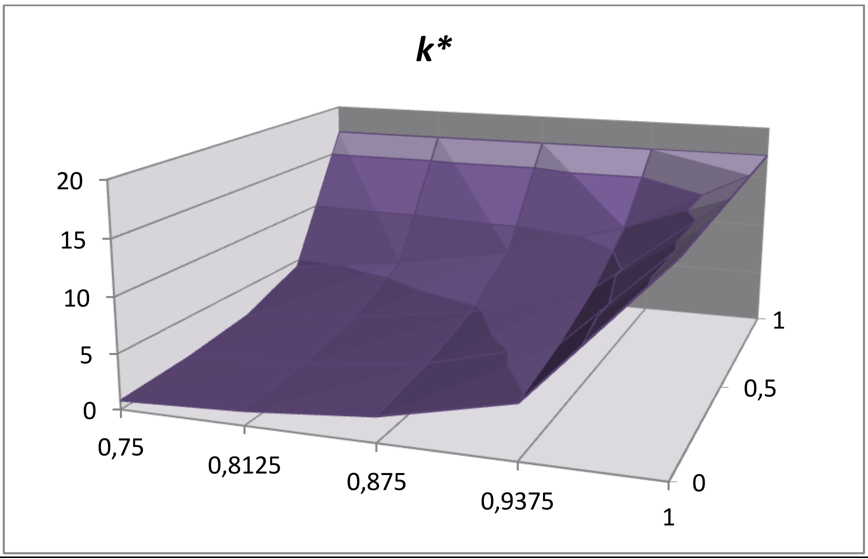
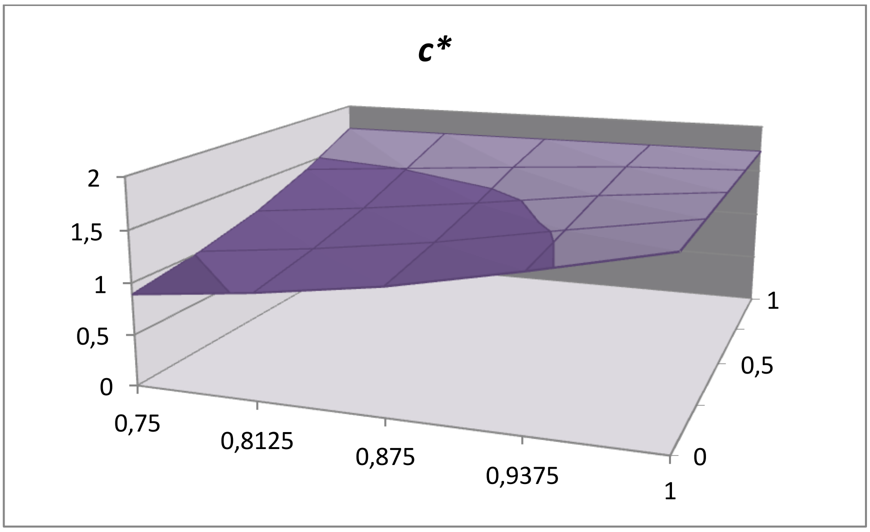


Fig.2 – Comparison between hyperbolic discounting and the two approximations (quasi-hyperbolic discounting and exponential discounting bias).

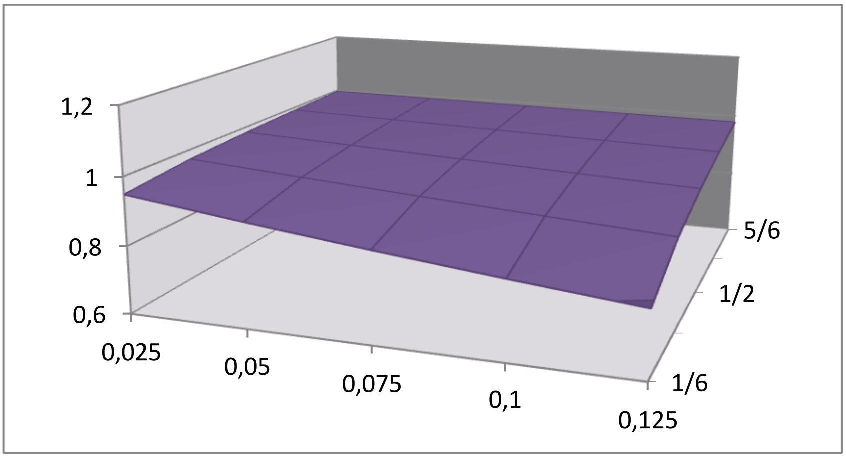


Panel a – Capital stock steady-state value.

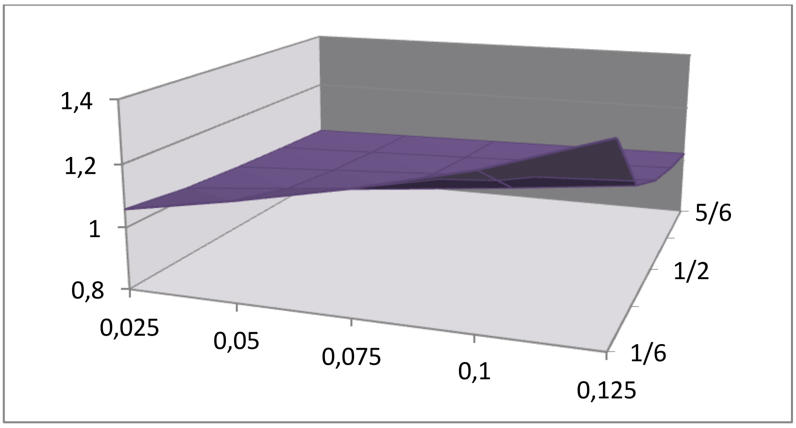


Panel b – Consumption steady-state value.

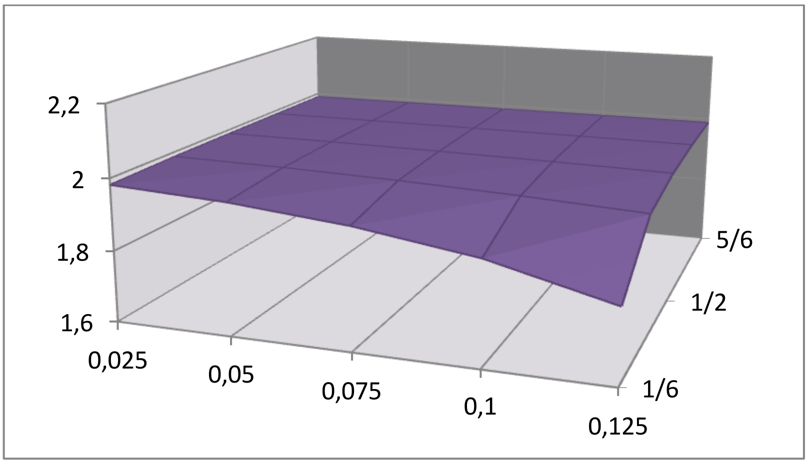
Fig. 3- Steady-state values of capital and consumption for different values of parameters β and θ_0.



Panel $a - \lambda_1$



Panel $b - \lambda_2$



Panel $c - \lambda_3$

Fig.4 – Eigenvalues for different values of parameters η and δ (exponential discounting bias case).

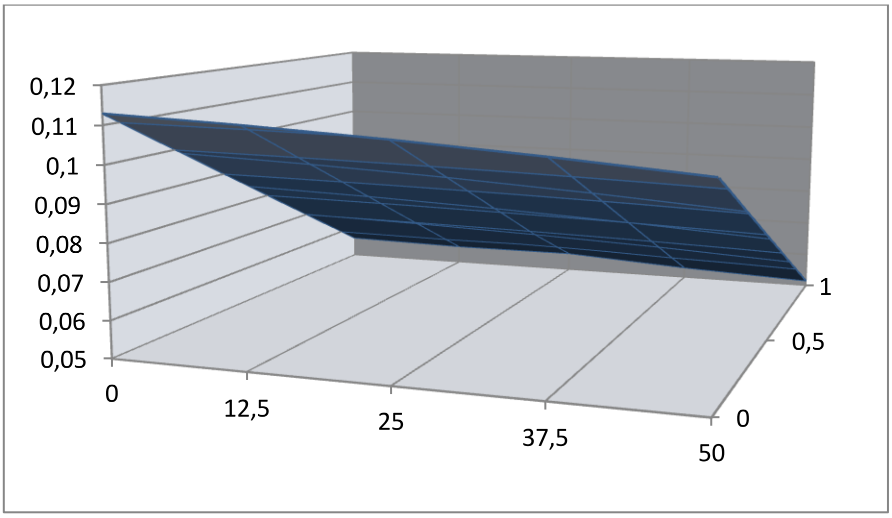


Fig. 5- Slope of the stable trajectory for different values of parameters θ_0 and θ_1.