Is Stochastic Volatility relevant for Portfolio Decisions?*

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Abstract

Literature on dynamic portfolio choice with stochastic volatility has been finding that volatility risk has low impact on portfolio decisions. For example, using long-run US data, Chacko and Viceira (2005) found that intertemporal hedging demand is empirically small even for high risk averse investors, therefore suggesting a low impact of stochastic volatility on portfolio choice. Recovering Knight (1921) uncertainty conceptualization, we want to assess if that evidence continues to be true if uncertainty about the stochastic volatility of the risky asset's return is considered in a broader perspective, by taking into account an "ambiguity dimension" alongside the standard "risk dimension". Adopting robust control and perturbation theory techniques we study the problem of long-horizon investor with recursive preferences that is ambiguous about the stochastic processes driving the investment opportunity set, including the stochastic volatility process. Using the same calibration of Chacko and Viceira (2005), we find that such ambiguity has a relevant empirical dimension. We therefore conclude that stochastic volatility can be very relevant for portfolio choice, not because of the volatility risk, but because of investor's ambiguity about its stochastic process.

Keywords: Asset Allocation, Stochastic Volatility, Ambiguity, Perturbation Theory, Robust Control

JEL Classification: C61 · D81 · E21 · G11.

1 Introduction

We study optimal dynamic consumption and portfolio choice under a stochastic investment opportunity set, of an investor that is averse both to risk and ambiguity. In our setting, the stochastic variance of the risky asset's return is simultaneously the source of the risk and the ambiguity that are perceived by the investor.[1]

There is a large literature on portfolio choice (see, for e.g., Kogan and Uppal (2000) and Campbell and Viceira (2002) for a survey), but few works study optimal dynamic portfolio choice with stochastic

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[1] Throughout this paper by the term "volatility" we mean variance. Moreover, for mathematical convenience, we work with precision, the reciprocal of variance.

variance of the risky asset's return. Some examples are Kim and Omberg (1996), Chacko and Viceira (2005) and Liu (2007) under incomplete markets, and Schroder and Skiadas (1999) under complete markets. Schroder and Skiadas (2003) contains a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. In those papers, potential adverse changes in the investment opportunity set are associated with the stochastic variance of the risky asset's return, which therefore represents a source of risk to investors. This implies, from Merton (1973), that stochastic variance is a key factor driving the investor's optimal intertemporal hedging demand. However, in those papers there is only risk, and no ambiguity.

Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and therefore for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a large body of empirical studies (surveyed, for e.g., in Camerer and Weber (1992) and Epstein and Schneider (2010)).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years ignored ambiguity, being based on the expected utility theory of von Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

However, gradually, ambiguity is being incorporated in decision theory. Two main approaches are being used: (i) the multiple priors (MP) approach, where the single probability measure of the standard expected and subjective utility models (precise beliefs) is replaced by a set of probabilities or priors (imprecise beliefs); (ii) the robust control (RC) approach, associated to an assumption of model uncertainty. The relationship between MP and RC approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006).

Ahn et al. (2011) have found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also conclude that ambiguity aversion can be observed in competitive markets and that it matters for portfolio choices and for asset prices.

In studies of portfolio choice with ambiguity, Gollier (2006) and Garlappi et al. (2007) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.[2] The same conclusion was reached in a dynamic MP setting (e.g. Chen et al. (2011)) and in a dynamic RC model (e.g. Maenhout (2004, 2006) and Xu et al. (2011)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, with exception of Xu et al. (2011), the source of ambiguity is exclusively the expected risky asset's return or the risky asset's return process.

[2] Although the result in Gollier (2006) requires some restrictions on the set of priors and on the investor's attitude towards risk.

In this paper, we extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice with stochastic variance, by introducing ambiguity about the data generating process of the stochastic investment opportunity set. Motivation for this is twice. On one side, it is provided by Chacko and Viceira themselves:

"*An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.*"

On the other side, literature on dynamic portfolio choice with stochastic variance has been finding that variance risk has low impact on portfolio decisions (e.g. Chacko and Viceira (2005) and Liu (2007)). Recovering Knight (1921) uncertainty conceptualization, we want to understand if that evidence continues to be true if uncertainty about the stochastic variance of the risky asset's return is considered in a broader perspective, by taking into account an "ambiguity dimension" alongside the standard "risk dimension".

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2011)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity also about the variance process of the risky asset's return because (i) there is no "a priori" reason to assume that investors are not ambiguous about it and (ii) we are able to find an asymptotic analytical solution and test it empirically.

In Faria et al. (2009), the setting of Chacko and Viceira (2005) is extended by considering a representative investor that is ambiguous about one specific parameter of the stochastic variance process (the expected value). It is adopted a MP approach and the conclusion is that ambiguity does not impact the instantaneous optimal portfolio rule. The ambiguity effect only exists, and is found to be empirically relevant, when the investor does not update continuously his portfolio. In Faria and Correia-da Silva (2010) we derive the optimal portfolio rule under a dynamic setting, with stochastic variance and ambiguity about its process. It is assumed that the representative investor derives utility exclusively through terminal wealth, implying that the intertemporal consumption-savings decision is ignored, and ambiguity is treated through a RC approach. Analytical expression for the optimal portfolio rule is derived, with ambiguity aversion having an additive impact vs. risk aversion. No empirical test in performed.

The closest paper to the present work is that of Xu et al. (2011), where preferences of the representative investor are given by the SDU function introduced by Duffie and Epstein (1992b) and ambiguity about the data generating process with stochastic variance is also considered and studied through a RC approach. Compared with Xu et al. (2011) contribution, the present paper brings two major novelties. The first results from the fact that we adopt a different RC methodology: the "constrained preferences" instead of the "multiplier preferences" approach (also applied in Faria and Correia-da Silva

(2010)).[3] The key differences between methodologies will be briefly exposed, but a key idea is that under "constrained preferences" ambiguity impacts the optimal portfolio through a non-uniform way in the state variables domain. By other words, the ambiguity aversion impact is more than simply an enhanced risk aversion level, bringing "first order" effects to optimal portfolio decisions. Moreover, in order to derive optimal policies under ambiguity, we make use of perturbation theory methods under RC settings, as for e.g. in Trojani and Vanini (2002, 2004). The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): "... *formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems"*. In our case, as in Trojani and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold for neighborhoods of a model with no ambiguity aversion.

The second major difference versus Xu et al. (2011), is that we test empirically our model, measuring the empirical dimension of the ambiguity impact on optimal portfolio decision. This is crucial as, ultimately, we are addressing the question if stochastic variance is relevant for portfolio decisions.

We find that investor's ambiguity aversion about stochastic variance process reduces the demand for the risky asset through a non-linear way. Using long-run US data, our simulation shows that the ambiguity impact on the allocation to the risky asset is empirically relevant. Stochastic variance may therefore have a significantly higher impact on the portfolio choice than what is suggested in the existing literature, which excludes ambiguity considerations (e.g. Chacko and Viceira (2005) and Liu (2007)). Our results therefore suggest that the relevant stochastic variance dimension to the dynamic optimal portfolio decision is the "ambiguity" and not the "risk" dimension.

The paper is organized as follows. In section 2, we state the problem to be solved. In section 3, we present the analytical solution to the problem and the key results. In section 4, we show simulation outputs. In section 5, we conclude the paper with some remarks.

2 Consumption and Portfolio Choice Problem

In section 2.1, the investment opportunity set is described. In section 2.2, representative investor preferences are presented. In section 2.3, the dynamic optimization problem to be solved is disclosed.

2.1 Investment Opportunity Set

It is assumed that all wealth is allocated between a riskless asset with price B_t and a risky asset with price S_t. The instantaneous return of the riskless asset is described by:

$$\frac{dB_t}{B_t} = r dt \,, \tag{1}$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} \left(\rho dW_y + \sqrt{1-\rho^2} dW_\varepsilon \right) \,, \tag{2}$$

[3] We adopt this terminology from Hansen and Sargent (2006).

where μ is the expected return of the risky asset and y_t is the instantaneous precision of the risky asset's return process (the instantaneous variance is $v_t = \frac{1}{y_t}$). From (2), the expected excess return of the risky asset versus the riskless asset, $\mu - r$, is constant over time.

The precision, y_t, follows a mean-reverting, square-root process as used by Cox et al. (1985):

$$dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t} dW_y \, , \tag{3}$$

where the expected value of the precision is $E[y_t] = \theta$, the reversion parameter is $\kappa > 0$, and, thus, $Var[y_t] = \frac{\sigma^2 \theta}{2\kappa}$. To guarantee standard integrability conditions, it is assumed that $2\kappa\theta > \sigma^2$, as in Cox et al. (1985). W_ε and W_y are two independent standard Brownian motions.

Applying Itô's Lemma to (3), a mean-reverting, square-root process for proportional changes in variance is obtained:

$$\frac{dv_t}{v_t} = \kappa_v (\theta_v - v_t) dt - \sigma \sqrt{v_t} dW_y \, , \tag{4}$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \kappa \left(\theta - \frac{\sigma^2}{\kappa}\right) = \frac{\kappa}{\theta_v}$.

Taking expectations of the second-order Taylor expansion of v_t around θ, the approximate unconditional mean of instantaneous variance is:

$$E[v_t] \approx \frac{1}{\theta} + \frac{1}{2} \frac{\sigma^2}{\theta^2 \kappa} = \frac{1}{\theta} + \frac{Var(y_t)}{\theta^3} \, . \tag{5}$$

As the expected return of the risky asset, μ, is assumed to be constant, (5) is also the unconditional variance of the risky asset's return. Chacko and Viceira (2005) perform a Monte Carlo simulation that validates this statement and the accuracy of the approximation (5), concluding that this approximation understates the true variance by 0.27%.

It is assumed that shocks to precision (W_y) are correlated with shocks to the return on the risky asset with correlation given by $\rho > 0$. From (4), this implies that the instantaneous correlation between proportional changes in variance and the risky asset's return is given by:

$$Corr_t \left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t} \right) = -Corr_t \left(dy_t, \frac{dS_t}{S_t} \right) = -\rho dt \, . \tag{6}$$

This investment opportunity setting incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect" property (given by the negative correlation between return and its variance), and the clustering property (as proportional changes in variance are higher when variance is high).

2.2 Investor Preferences

It is assumed that the representative investor is not totally sure about the data-generating processes (2)-(3) that characterize the investment opportunity set dynamics. By other words, the uncertainty faced by the representative investor has two dimensions: risk and ambiguity.

Additionally, it is assumed that the representative investor preferences are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory in Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution by not constraining to be reciprocals of one another (as in standard additive intertemporal utility function).

The utility process that defines the SDU function is represented by:

$$J = E_t \left[\int_t^{\infty} f\left(C_s, J_s\right) ds \right], \tag{7}$$

where C_s represents current consumption and J_s is the continuation utility for the consumption flow C, at time $t = s$, with infinite time horizon. In our setting, the function $f\left(C_s, J_s\right)$ is the normalized aggregator that generates J, defining a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available. The normalized aggregator $f\left(C, J\right)$ is given by (Appendix 6.1):

$$f\left(C, J\right) = \frac{\beta}{1 - \frac{1}{\psi}} \left(1 - \gamma\right) J \left[\left(\frac{C}{\left(\left(1 - \gamma\right) J\right)^{\frac{1}{1-\gamma}}} \right)^{1 - \frac{1}{\psi}} - 1 \right], \tag{8}$$

where $\gamma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the rate of time preference. With $\gamma = \frac{1}{\psi}$, (8) becomes the standard power utility representation.

When $\psi = 1$, the normalized aggregator $f\left(C, J\right)$ takes the form:

$$f(C, J) = \beta \left(1 - \gamma\right) J \left\{ ln\left(C\right) - \frac{1}{1-\gamma} ln \left[\left(1 - \gamma\right) J\right] \right\}. \tag{9}$$

A remark regarding the preference for the timing of the resolution of risk. With the preference structure in Kreps and Porteus (1978), investors can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertemporal utility function implies that investors are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between ψ and γ: if $\gamma > \frac{1}{\psi}$ $(<, =)$ investors have preference for early (late, indifferent) resolution of risk. Our specification (7) from Duffie and Epstein (1992a), being the continuous-time limit of Epstein and Zin (1989), inherits this property. However, on the contrary of other streams of literature with Epstein and Zin (1989) preferences, for e.g. the "long-run risk" literature (from the seminal work of Bansal and Yaron (2004)), we do not restrict the investor to have preference for early resolution of risk. Two main reasons support this decision: (i) as our model evolves in a long-run setting, the possibility of the "cost" becoming higher than the "benefit" of planning advantages brought by the early resolution of risk (Arai (1997)) should not be excluded and (ii) there is evidence that investors may have preference for late resolution of risk (Epstein and Zin (1991)).

2.3 Dynamic Optimization Problem

Ambiguity about the investment opportunity set is studied with robust control (RC) techniques, firstly introduced in economics by Hansen and Sargent (1995). The representative investor considers a base model as his reference belief but, being ambiguous about it, considers a family of alternative models around it. Those alternative models are statistically difficult to distinguish from the benchmark model.

Under the RC approach, two main formulations have been used in the ambiguity related literature: the "constraint preferences" and the "multiplier preferences". Under the "constraint preferences" (e.g. in Hansen et al. (2006)), there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Conversely, under the "multiplier preferences" type of model (e.g. in Maenhout (2006)), preferences for robustness are constructed by penalizing deviations from the benchmark model. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different.

One difference is that under the "constraint preferences" RC approach, the specification of the ambiguity aversion can be based on a rectangular set of priors, which guarantees a dynamically consistent preference ordering. In those cases, preferences can be represented by the recursive multiple priors utility (RMPU) specification (Chen and Epstein (2002) and Epstein and Schneider (2003)).[4] Note however that, on the contrary of the RMPU setting, RC methodology is focused in formulating optimal dynamic rules where rectangularity is not a crucial property "per se" for the formulation of the set of priors of the model (see, for e.g., Trojani and Vanini (2004)).

Additionally, the "constraint preferences" approach allows ambiguity to have "first-order" effects on portfolio choice (as pointed out by Epstein and Schneider (2010)), that is, it enables the expansion of the range of qualitative behavior that can be rationalized versus the standard expected utility theory. This contrasts with the "multiplier preferences" RC approach, which is observationally equivalent to the expected utility theory: it enables reinterpretations of some results in the expected utility theory, that can be quantitatively more appealing,[5] but does not enlarge the spectrum of qualitative behavior that can be rationalized. In this sense, the "multiplier preferences" RC approach only brings "second-order" effects on portfolio choice, with the ambiguity aversion being in practice translated in an enhanced risk aversion level (as concluded for optimal dynamic portfolio choice in Faria and Correia-da Silva (2010); Maenhout (2004, 2006) and Xu et al. (2011)).

In this paper we adopt a "constraint preferences" RC approach. The ambiguous agent considers contaminations (alternative models), P^h, around his reference belief, P under which processes (2)-(3) evolve. The contaminations are assumed to be absolutely continuous with respect to P, and, therefore, are equivalently described by contaminating drift processes, h. In each of the alternative models, P^h, the Brownian motion becomes $W^h(t) = W(t) + \int_0^t h(s)\,ds$.[6]

[4] Rectangularity is the property that allows updating every prior under the recursive multiple priors utility through a Bayes rule. See Epstein and Schneider (2003) for details about this property. In Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.

[5] For example, as ambiguity aversion translates into a higher effective risk aversion, it is a contribution for the explanation of the equity premium puzzle.

[6] For tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between P and P^h guarantees the equivalence property between the probability measures and, consequently, that the Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

An upper bound is imposed on the contaminating drift processes, h:

$$h^{\top}h \leqslant 2\eta, \tag{10}$$

where $\eta \geqslant 0$ is a parameter that can be interpreted as the level of ambiguity.

This bound (10) should be such that alternative models are statistically close to the "reference belief" model: otherwise the agent would easily distinguish among them and, consequently, would not face ambiguity. That is, η should be small. Moreover, the bound (10) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, P, and any admissible contaminated belief, P^h. Trojani and Vanini (2004) explain that the set $\left\{h : h^{\top}h \in [0\,,\,2\eta]\;,\,\forall t \geqslant 0\right\}$ defines a rectangular set of priors because any process h (and therefore any probability measure P^h) in this set corresponds to a selection of transition densities from t to $t+dt$, $t \geqslant 0$, such that $h^{\top}h \in [0\,,\,2\eta]$.[7]

Considering the reference system (2)-(3), ambiguity is therefore introduced through contaminating drift processes restricted to be $h = \begin{bmatrix} h^y & h^{\varepsilon} \end{bmatrix}^{\top}$. The class of admissible Markovian drift contaminations satisfying this restriction and the entropy bound (10) is denoted by $\mathcal{H}$.

Under an admissible contamination, P^h, the investment opportunity set is therefore described by:

$$\begin{cases} \frac{dS_t}{S_t} & = \left(\mu + \sqrt{\frac{1}{y_t}}\rho h^y + \sqrt{\frac{1}{y_t}}\sqrt{(1-\rho^2)}h^{\varepsilon}\right)dt + \sqrt{\frac{1}{y_t}}\rho dW_y + \sqrt{\frac{1}{y_t}}\sqrt{(1-\rho^2)}dW_{\varepsilon} \\ \\ dy_t & = \left(\kappa\left(\theta - y_t\right) + \sigma\sqrt{y_t}h^y\right)dt + \sigma\sqrt{y_t}dW_y \end{cases} \tag{11}$$

Note that in the "contaminated" system (11) describing the investment opportunity set, the diffusion component continues to be driven by the same vector of independent Brownian motions as in (2)-(3).

The corresponding intertemporal budget constraint faced by the ambiguous representative investor is given by:

$$dX_t = \left[\pi_t\left(\mu + \sqrt{\frac{1}{y_t}}\rho h^y + \sqrt{\frac{1}{y_t}}\sqrt{(1-\rho^2)}h^{\varepsilon} - r\right)X_t + rX_t - C_t\right]dt + \pi_t\sqrt{\frac{1}{y_t}}X_t\left(\rho dW_y + \sqrt{1-\rho^2}dW_{\varepsilon}\right), \tag{12}$$

where C_t, X_t and π_t represent instantaneous consumption, wealth and fraction of wealth invested in the risky asset, respectively.

The intertemporal optimization problem has a max-min structure: (i) Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative investor chooses from the set of alternative models, the one that corresponds to the worst case scenario, i.e., the one associated with the lowest expected utility. This brings the min component; (ii) The investor has to choose the consumption function, $C : [t_0, +\infty[\rightarrow \mathbb{R}_+$, and the fraction of wealth invested in the

[7] In Trojani and Vanini (2004), p.289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (10), and how this rectangular set of priors can be defined in the *k-ignorance* model of Chen and Epstein (2002).

risky asset throughout time, $\pi : [t_0, +\infty[\to \mathbb{R}$, which maximize his expected utility (7):

$$\underset{\pi,C}{sup}\, \underset{h\in\mathcal{H}}{inf} E_t^h \left[\int_t^{\infty} f(C_s, J_s)\, ds \right] \quad , \tag{13}$$

subject to the contaminated precision and wealth processes in (11) and (12), respectively.

The Bellman equation of this problem is:

$$\begin{aligned} 0 \;=\; & \underset{\pi,C}{sup}\, \underset{h^y,h^\varepsilon}{inf} \left\{ f(C,J) + \left[\pi_t \left(\mu + \sqrt{\frac{1}{y_t}} \rho h^y + \sqrt{\frac{1}{y_t}} \sqrt{(1-\rho^2)} h^\varepsilon - r \right) X_t + rX_t - C_t \right] J_X + \right. \\ & \left. + \left[\kappa (\theta - y_t) + \sigma \sqrt{y_t} h^y \right] J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} \right\} , \end{aligned} \tag{14}$$

where $f(C,J)$ is the normalized aggregator given in (8) and (9) for general values of ψ and for $\psi = 1$, respectively, and J_X, J_y, J_{XX}, J_{yy} and J_{Xy} are partial derivatives of value function J.

Solving for the optimal vector (h^y, h^ε), i.e. for the optimal contamination, and placing that result into (14), the Bellman equation of the problem becomes (Appendix 6.2):

$$\begin{aligned} 0 \;=\; & \underset{\pi,C}{sup} \left\{ f(C,J) + \pi_t (\mu - r) X_t J_X + rX_t J_X - C_t J_X + \kappa (\theta - y_t) J_y + \frac{1}{2} \pi_t^2 \frac{1}{y_t} X_t^2 J_{XX} \right. \\ & \left. + \frac{1}{2} \sigma^2 y_t J_{yy} + \pi_t X_t \rho \sigma J_{Xy} - \sqrt{2\eta \left(\sigma^2 y_t J_y^2 + 2\sigma\rho\pi_t X_t J_y J_X + \pi_t^2 \frac{1}{y_t} X_t^2 J_X^2 \right)} \right\} . \end{aligned} \tag{15}$$

3 Optimal Consumption and Portfolio Rules

In general, obtaining closed form solutions under stochastic investment opportunity sets is difficult. This difficulty is further enhanced by the consideration of ambiguity. In this paper, we follow perturbation theory methodology under robust control (e.g. Trojani and Vanini (2002)) to describe the solution of the problem under study. As in Trojani and Vanini (2004), we extend the asymptotic methods in Kogan and Uppal (2000) from models based on standard expected utility to models with ambiguity considerations. This is allowed by the homotheticity nature of the robust control problem (13) - (11) - (12). This homothetic property will become clear by the structure of the value function that solves the problem and the corresponding optimal consumption-wealth and portfolio policies, which are wealth scale-invariant.[8]

The rationale behind the perturbation (asymptotic) method is well described by Trojani and Vanini (2004): "*... formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*". In our case, as in Trojani

[8]As explained in Trojani and Vanini (2002), studying non-homothetic robust control settings with perturbation methods is more difficult. Moreover, Maenhout (2004) points out some reasons to support the homotheticity assumption: *"Although economics exhibit growth, rates of return are stationary. Second, when the scale of the state variables matters, natural unit invariance of optimal decisions disappears and calibrations have to take this into account. Finally, homotheticity facilitates aggregation and the construction of a representative agent"*. As it is stated in Maenhout (2004), preserving this homotheticity property guarantees that *"...robustness will no longer wear off as wealth rises"*.

and Vanini (2004), the asymptotic solution of the problem under ambiguity will hold for neighborhoods of a model with no ambiguity aversion.

The first step is therefore to identify a set of parameters that parametrize the problem under study and specific parameters values for which the value function solution is known explicitly. In our case, there are two critical parameters for this step: the intertemporal elasticity of substitution ψ and the level of ambiguity η. In Chacko and Viceira (2005) the intertemporal optimization problem in section 2 is studied but without ambiguity considerations ($\eta = 0$): it is found an exact solution of the problem for $\psi = 1$. Following the rational above described, we will therefore perturb the equilibrium of a benchmark economy with $\psi = 1$ and $\eta = 0$.[9]

The value function J that solves (15), for $\psi = 1$ and $\eta = 0$, is given by:

$$J\left(X_t, y_t\right) = exp\left\{g\left(y_t\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma}\,, \tag{16}$$

where $g(y_t) = Ay_t + B$, with A and B being given by

$$A = \frac{\gamma\left(1-\gamma\right)\left[\left(\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right) \pm \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}}\right]}{\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}\,, \tag{17}$$

$$B = \frac{\left(1-\gamma\right)\left(\beta ln\beta + r - \beta\right)}{\beta} + \frac{\kappa\theta}{\beta}A\,. \tag{18}$$

The sign of the square-root in A is "+" for $\gamma > 1$ and "-" for $\gamma \leqslant 1$ (Appendix 6.3).

In order to obtain the asymptotic expansions of the optimal policies of the problem under study, function $g(y_t)$ in (16) is expanded in $\left(\sqrt{\eta}\right)$ to first order:

$$g\left(y_t\right) \;=\; g_0\left(y_t\right) + g_1\left(y_t\right)\sqrt{2\eta} + \mathcal{O}^2\left(\sqrt{\eta}\right)\,, \tag{19}$$

where $\mathcal{O}^2\left(\sqrt{\eta}\right)$ is a symbol representing terms of second order in $\left(\sqrt{\eta}\right)$. As it is immediate from (19), g_0 is the specification of function $g\left(y_t\right)$ for the scenario when there is no ambiguity ($\eta = 0$). And, as it will be proved, g_0 is enough to characterize first order optimal policies under this optimization problem with ambiguity aversion.

Proposition *Asymptotic optimal consumption and portfolio policies under ambiguity, when* $\psi = 1$

[9] In Chacko and Viceira (2005) it is also found an approximate solution for the general case of $\psi \neq 1$ that converges to the exact solution when $\psi = 1$. However, we only present the scenario with $\psi = 1$ because key qualitative and empirical results do not change under $\psi \neq 1$: (i) optimal consumption rule is not affect by ambiguity aversion; (ii) optimal portfolio rule is impacted by the consideration of ambiguity aversion through the same non-linear structure, and we verify through empirical simulation that differences between $\psi \neq 1$ and $\psi = 1$ scenarios are immaterial.

and $\gamma \geqslant \omega$ *, where* $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$*, are:*[10]

$$C_t = \beta X_t + \mathcal{O}^2\left(\sqrt{\eta}\right), \tag{20}$$

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G_0(y_t)}}}\left[(\mu - r)\, y_t + \left(1 - \frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0\left(y_t\right)}}\right)\sigma\rho A y_t\right] + \mathcal{O}^2\left(\sqrt{\eta}\right), \tag{21}$$

with $G_0\left(y_t\right) = \left[\left(\frac{(\mu-r)+\rho\sigma A}{\gamma}\right)^2 + \left(\frac{\sigma A}{1-\gamma}\right)^2 + \frac{2\sigma\rho A((\mu-r)+\rho\sigma A)}{(1-\gamma)\gamma}\right] y_t$ *and* A *given by (17).*

Proof. *Appendix 6.4.*

The first comment on the above Proposition, is that the domain of the problem solution depends on the combination of the level of investor's risk aversion and on the characterization of the investment opportunity set dynamics (represented by ω). Note also that regarding the investor's preferences for the temporal resolution of risk, the domain of analysis $\gamma \geqslant \omega$ includes scenarios where the investor: has preference for late resolution of risk ($\omega \leqslant \gamma < 1$); has preference for early resolution of risk ($\gamma > 1$); is indifferent to that timing ($\gamma = 1$). Only scenarios where the investor has a strong preference for late resolution of risk ($\gamma < \omega$) are excluded.

From (20) and (21), it is immediate to conclude that when there is no ambiguity, $\eta = 0$, optimal consumption and portfolio rules are as follows:

$$C_t = \beta X_t\,, \tag{22}$$

$$\pi_t = \frac{1}{\gamma}\left(\mu - r\right) y_t + \frac{\sigma\rho}{\gamma} A y_t\,, \tag{23}$$

which are the results in Chacko and Viceira (2005).

Comparing (20) and (22) it is clear that the asymptotic optimal consumption rule under ambiguity continues to be to consume the same constant fraction β of wealth throughout time. In this setting, the ambiguity impact on consumption must be of order higher than one. Additionally, the fact of the optimal rule being a constant fraction of wealth means that the income and substitution effects on consumption that result from a change in the investment opportunity set are always exactly canceled out.

Conversely, from (21) and (23), it is immediate to concluded that ambiguity aversion has an impact on the optimal dynamic portfolio decision. And there are novelties regarding existing results in the literature. First, the optimal allocation to the risky asset is instantaneously impacted by investor's ambiguity (parameter η), which contrasts with results in Faria et al. (2009). Secondly, the optimal portfolio rule is a non-linear function of y_t, which differs from the linear relationship when there exists no ambiguity (e.g. Chacko and Viceira (2005)) or when ambiguity is considered through multiplier preferences RC techniques (e.g. Maenhout (2004, 2006); Faria and Correia-da Silva (2010) and Xu et al. (2011)).

[10] For $\gamma < \omega$, A in (17) is a complex number. From (16) this implies a complex value function J. As J is the optimized intertemporal utility, and utility functions are defined in the real space, the domain of the problem is restricted to values of parameters such that $\gamma \geqslant \omega$.

The structure of the optimal portfolio rule under ambiguity (21) continues to be the sum of two well know components from Merton (1973): (i) myopic portfolio demand, in this setting given by $\frac{(\mu-r)}{\gamma+\sqrt{\frac{2\eta}{G_0(y_t)}}}y_t$; and (ii) intertemporal hedging demand, given by $\frac{\left(1-\frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(y_t)}}\right)\sigma\rho A}{\gamma+\sqrt{\frac{2\eta}{G_0(y_t)}}}y_t$. Comparing with the optimal portfolio rule without ambiguity (23), it continues to be true that the intertemporal hedging demand is zero (and therefore the portfolio myopic demand is optimal) when: the investor has unit coefficient relative risk aversion ($\gamma = 1$); investment opportunities are constant ($\sigma = 0$) or, being time-varying, it is not possible to use the risky asset to hedge against those changes ($\rho = 0$). However, the ratio between the myopic and intertemporal hedging demand components depends on y_t, on the contrary of what happens when there is no ambiguity ($\eta = 0$).

Additionally, without ambiguity an investor with $\gamma > 1$ has a negative intertemporal hedging demand, and the opposite when $\omega \leqslant \gamma < 1$, which is consistent with the findings in Chacko and Viceira (2005). When risk aversion is low ($\omega \leqslant \gamma < 1$), the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that $\rho > 0$). An investor with high risk-aversion ($\gamma > 1$) is not willing to accept this trade-off. The introduction of ambiguity makes this relation not so trivial: for low risk averse investors ($\gamma < 1$) but sufficiently highly ambiguous (high η) the intertemporal hedging also becomes negative.[11]

Note that the myopic and intertemporal hedging demand components under ambiguity aversion ($\pi_t^M(\eta)$ and $\pi_t^H(\eta)$) can be rewritten:

$$\pi_t^M(\eta) = \pi_t^M(0)\frac{\gamma}{\gamma+\sqrt{\frac{2\eta}{G_0(y_t)}}}, \tag{24}$$

$$\pi_t^H(\eta) = \pi_t^H(0)\frac{\gamma\left(1-\frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(y_t)}}\right)}{\gamma+\sqrt{\frac{2\eta}{G_0(y_t)}}}, \tag{25}$$

where $\pi_t^M(0)$ and $\pi_t^H(0)$ represent the myopic and intertemporal hedging demand components without ambiguity aversion. It is immediate to conclude that ratios $\pi_t^M(\eta)/\pi_t^M(0)$ and $\pi_t^H(\eta)/\pi_t^H(0)$ are y_t dependent functions (due to $G_0(y_t)$), and that ambiguity affects both components through a non-uniform way in the state variable y_t domain.

At last, it is easy to demonstrate that $G_0(y_t) > 0$ which, from (24) and (25), implies that the reduction in the optimal risky asset demand is a positive function of the level of ambiguity of the representative investor (higher η). The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a setting where stochastic precision is one of the sources of ambiguity, and ambiguity is treated through a "constraint preferences" RC setting.

4 Simulation

Chacko and Viceira (2005) found that, calibrating their model to long-run US data, the optimal intertemporal hedging demand is empirically small. The same conclusion is reached in Liu (2007).

[11]For this to happen under the $\omega \leqslant \gamma < 1$ domain, it is necessary that $\gamma > 1 - \sqrt{\frac{2\eta}{G_0(y_t)}}$.

This suggests that the "risk dimension" of stochastic variance is empirically not very relevant to the dynamic optimal portfolio decision. However, Chacko and Viceira (2005) in their concluding remarks, acknowledged that an important caveat of their analysis is that they have counterfactually assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have set up a model to account for ambiguity about the precision stochastic process under the same stochastic investment opportunity set of Chacko and Viceira (2005). As a result, the "myopic demand" and the "intertemporal hedging demand" become "ambiguity" adjusted.

Using the calibration of Chacko and Viceira (2005), our simulation suggests that the ambiguity impact on the allocation to the risky asset has a relevant empirical dimension, essentially through the adjusted myopic demand. Stochastic variance may therefore have a significantly higher impact on the portfolio choice than what is suggested by the results in the literature.

The reference parameter values used in the simulation are those estimated in Chacko and Viceira (2005) based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill rate from January 1926 through December 2000:

$$\begin{aligned} \mu - r &= 0.0811, \\ \kappa &= 0.3374, \\ \theta &= 27.9345, \\ \sigma &= 0.6503, \\ \rho &= 0.5241, \\ r &= 0.015, \\ \beta &= 0.06. \end{aligned} \tag{26}$$

From (5), the expected standard deviation of returns is 19.1314%.

Implications on the optimal allocation to the risky asset from the consideration of ambiguity are exemplified in Table 1. The first column presents results for the scenario without ambiguity. The other three columns represent scenarios with three arbitrary levels of ambiguity: $\eta = 0.005$, 0.01, 0.02. Recall that alternative models have to be statistically close so that the investor is ambiguous about the reference model. This implies small η values. In Trojani and Vanini (2004) two values for η are used $(0.005, 0.01)$ while the value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.0136.

Simulations are run for different levels of risk aversion ($\gamma = 0.75$, 1, 2, 4, 20, 40) with unit elasticity of intertemporal consumption ($\psi = 1$). In panel A it is showed the mean allocation to the risky asset (percentage). In panel B, the intertemporal hedging demand is shown as a percentage of the myopic demand. In panels C and D, the ambiguity effect is explicitly calculated as a percentage of total risky asset demand and of myopic demand.

Table 1: Ambiguity impact on optimal risky asset demand.

	η			
	0	0.005	0.01	0.02
	A - Mean allocation to risky asset (%)			
γ				
0.75	305,66	245,80	227,16	204,98
1.00	226,55	180,98	166,91	150,24
2.00	111,37	89,59	82,80	74,73
4.00	55,24	44,43	41,07	37,07
20.00	10,98	8,83	8,16	7,37
40.00	5,48	4,41	4,08	3,68
	B - Ratio of hedging demand over myopic demand (%)			
γ				
0.75	1,19	0,36	0,01	-0,48
1.00	0,00	-0,84	-1,19	-1,68
2.00	-1,68	-2,46	-2,78	-3,24
4.00	-2,47	-3,24	-3,56	-4,01
20.00	-3,09	-3,85	-4,16	-4,61
40.00	-3,17	-3,92	-4,24	-4,68
	C - Reduction in allocation to risky asset due to ambiguity (%)			
γ				
0.75	0,00	-19,58	-25,68	-32,94
1.00	0,00	-19,57	-25,67	-32,92
2.00	0,00	-19,56	-25,65	-32,90
4.00	0,00	-19,56	-25,65	-32,89
20.00	0,00	-19,55	-25,64	-32,88
40.00	0,00	-19,55	-25,64	-32,88
	D - Reduction of myopic demand due to ambiguity (%)			
γ				
0.75	0,00	-18,92	-24,81	-31,81
1.00	0,00	-18,92	-24,81	-31,81
2.00	0,00	-18,92	-24,81	-31,81
4.00	0,00	-18,92	-24,81	-31,81
20.00	0,00	-18,92	-24,81	-31,81
40.00	0,00	-18,92	-24,81	-31,81

Note 1 **- Panel A:** $E\left[\pi_t\left(y_t\right)\right] \times 100$, with $E\left(y_t\right)=\theta$; **Panel B:** $\frac{\left(1-\frac{1}{1-\gamma}\sqrt{\frac{2\eta}{G_0(E(y_t))}}\right)\sigma\rho A}{(\mu-r)} \times 100$, with $E\left(y_t\right)=\theta$; **Panel C:** $\left[\frac{(21)}{(23)}-1\right] \times 100$, with $E\left(y_t\right)=\theta$; **Panel D:** $\left[\frac{\gamma}{\gamma+\sqrt{\frac{2\eta}{G_0(E(y_t))}}}-1\right] \times 100$, with $E\left(y_t\right)=\theta$;

Note 2 - $\omega=0.14<0.75$: domain of Proposition in section 3 is guarantied.

Results presented in Table 1 are consistent with comments in section 3. Let us start with an example based on panel A. Consider a risk-averse investor, with $\gamma=2$, that is ambiguity neutral. His mean optimal allocation to the risky asset corresponds to 111.4% of his wealth. If this investor becomes ambiguous, for example with ambiguity level given by $\eta=0.01$, his mean optimal allocation to the risky asset declines to 82,8% of his wealth. Panel A in Table 1 shows that the demand for the risky asset is decreasing with the risk aversion, γ, and with the level of ambiguity, η.

Panel B reports the estimates of the intertemporal hedging demand, measured as a ratio of myopic demand. Again, results show that ambiguity reinforces the effect of risk aversion: the higher the ambiguity and risk aversion the higher the relative dimension of the intertemporal hedging demand. However, the intertemporal hedging demand is always small - even for a very high risk and ambiguity averse investor ($\gamma=40$, $\eta=0.02$). The novelty with ambiguity is that for low risk averse investors but with high ambiguity the intertemporal hedging demand becomes negative (e.g. $\gamma=0.75$, $\eta=0.02$).

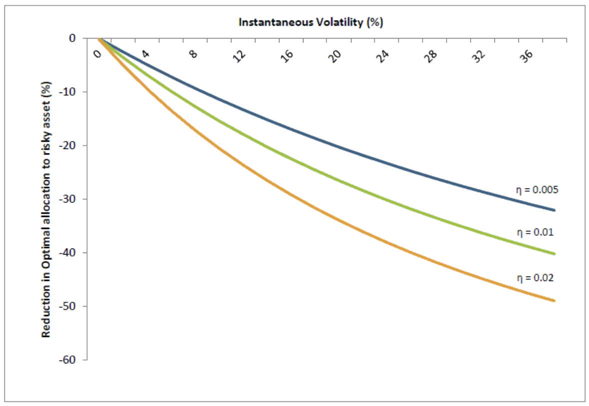
This confirms the predictions highlighted in section 3. With no ambiguity or moderate levels of ambiguity the intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma \geqslant 1$.

The fact of the intertemporal hedging demand component being empirically small, even for higher levels of ambiguity, means that ambiguity impacts optimal portfolio decision essentially through the myopic component, which is confirmed by results disclosed in Panels C and D.

Overall, results in Table 1 show that ambiguity is empirically relevant: even for a low level of ambiguity (second column in Table 1), ambiguity implies a 20% decrease of the mean optimal demand of the risky asset (panel C).

Moreover, the higher the level of ambiguity and the level of instantaneous volatility (inverse of precision) the higher is the impact from ambiguity about the volatility stochastic process. This is represented in Figure 1, where it is also evident that this ambiguity effect through the relevant support of precision or, equivalently, of variance has a non-linear nature (concave and convex, respectively).

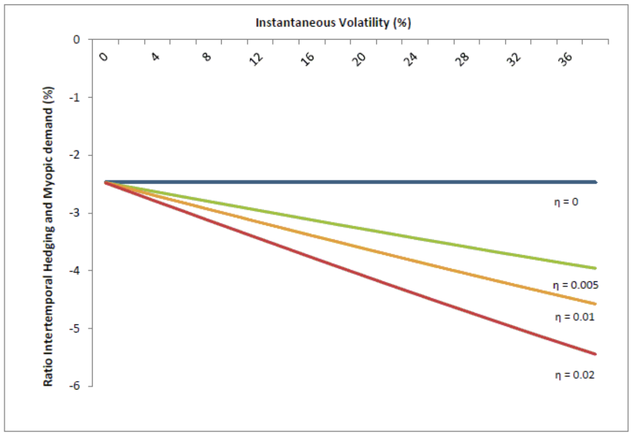
Figure 1: Ambiguity effect as a function of instantaneous volatility



Note 1 - Volatility understood as variance v_t, computed from the level of precision y_t using (5);
Note 2 - Simulation with $\gamma = 4$.

Additionally, as pointed out in section 3, the ratio between the intertemporal hedging demand and myopic demand components becomes y_t - state dependent when ambiguity is considered. This is graphically highlighted in Figure 2:

Figure 2: Hedging Demand vs Myopic Demand as a function of instantaneous volatility



Note 1 - Volatility understood as variance v_t, computed from the level of precision y_t using (5);

Note 2 - Simulation with $\gamma = 4$.

At last, in Table 2 it is possible to see that the long-term expected return on wealth, measured by $(\pi_\theta (\mu - r) + r)$, of an investor that is both risk and ambiguity averse is a decreasing function of both risk aversion and the level of ambiguity, which was expectable considering results from Table 1. This is further illustrated in Figure 3.

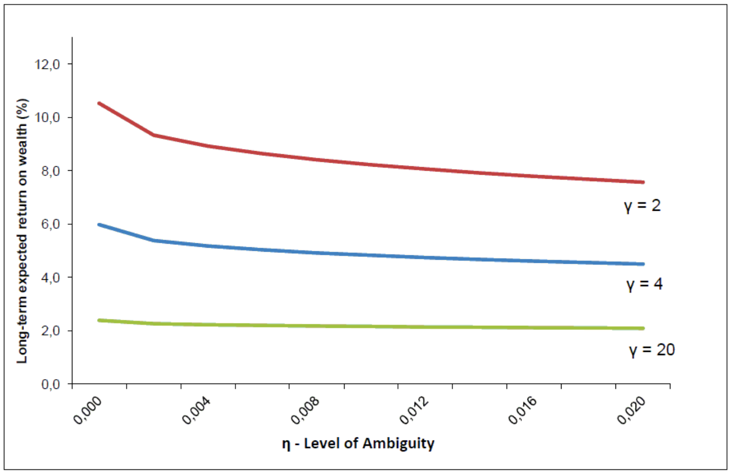
Table 2: Long-term expected return on wealth.

	η			
	0	**0.005**	**0.01**	**0.02**
	Long-term expected return on wealth (%)			
γ				
0.75	26,29	21,43	19,92	18,12
1.00	19,87	16,18	15,04	13,68
2.00	10,53	8,77	8,22	7,56
4.00	5,98	5,10	4,83	4,51
20.00	2,39	2,22	2,16	2,10
40.00	1,94	1,86	1,83	1,80

Note 1 - $(\pi_\theta (\mu - r) + r) \times 100$.

Note 2 - $\omega = 0.14 < 0.75$: domain of Proposition in section 3 is guarantied.

Figure 3: Long-term expected return on wealth as a function of ambiguity and risk aversion.



5 Concluding Remarks

We study optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the data generating processes driving the stochastic investment opportunity set. In our setting, precision of the risky asset return is therefore simultaneously the source of risk and ambiguity perceived by the risk and ambiguity averse investor.

Long-horizon investors with recursive preferences, as defined by Duffie and Epstein (1992b) with Kreps and Porteus' (1978) specification, have two assets to invest in, a risk-free asset and a risky asset. The precision of the risky asset return is stochastic and the investor is ambiguous about its process. The investor considers a reference model but, not being totally sure about it, takes into account a set of statistically close models (with the relative entropy between models being bounded). Ambiguity aversion in the spirit of Gilboa and Schmeidler's (1989) implies that investor will consider the worst possible alternative model, i.e., the one associated with the lowest expected utility. Optimal dynamic policies under ambiguity are deducted by making use of perturbation theory techniques for robust control problems.

The main conclusions of this paper concern the impact on optimal dynamic policies from ambiguity about the data generating processes and, particularly, about the stochastic variance process. Regarding the optimal consumption policy, it does not depend on the level of precision, being a constant fraction of wealth. Moreover, ambiguity does not impact the optimal consumption-wealth ratio, at least until a first order approximation with respect to the level of ambiguity.

Conversely, ambiguity reduces the optimal demand for the risky asset, with that effect being non-uniform in the precision domain. It is also found that the ratio of intertemporal hedging demand vs. myopic demand depends on the level of precision, and that for low risk averse investors that are highly ambiguous the intertemporal hedging demand becomes negative.

At last, making use of long-run US data, we conclude that ambiguity about the stochastic processes driving the investment opportunity set is empirically relevant for portfolio decisions. Chacko and Viceira (2005) and Liu (2007), concluded that the variance of the risky asset's return generates a small intertemporal hedging demand, suggesting low relevance of the variance "risk dimension" in the dynamic portfolio decision. Using the same calibration of Chacko and Viceira (2005), our simulation suggests that the ambiguity effect on the risky asset demand is very relevant, acting mainly through the myopic component. Recovering the Knightian conceptual dichotomy of ambiguity versus risk (Knight (1921)), we conclude that the "ambiguity dimension" of variance is significantly more relevant than its "risk dimension" for dynamic optimal portfolio decisions implying that stochastic variance has a much higher impact on investors portfolio decisions than it has been found in the literature.

6 Appendices

6.1 Stochastic Differential Utility

Consumption processes C are chosen from the space D of square-integrable progressively measurable processes valued in $\mathcal{C} = \mathbb{R}_+$ (see Duffie and Epstein (1992a) footnote 2 for further details). Duffie and Epstein (1992a) define the stochastic differential utility (SDU) function $U : D \to \mathbb{R}$ by two primitive functions, $\bar{f} : \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ and $\bar{A} : \mathbb{R} \to \mathbb{R}$. The function $\bar{A}(\cdot)$ is called variance multiplier, as it applies a penalty (or reward) as a multiple of the utility "volatility". For deterministic consumption process, $\bar{A}(\cdot)$ is therefore irrelevant and only $\bar{f}(\cdot)$ matters.

As stated in Duffie and Epstein (1992a) if, for each consumption process C, there exists a well-defined utility process $\bar{J}$, then the SDU function U is defined by $U(C) = \bar{J}_0$, the initial value of that utility process. The pair $(\bar{f}, \bar{A})$ generating $\bar{J}$ is called an aggregator.

Two aggregators $(\bar{f}, \bar{A})$ and (f, A) are said to be ordinally equivalent if they generate ordinally equivalent utility functions, i.e. represent the same preference ordering of consumption processes. In Duffie and Epstein (1992a) it is presented a method through which for any aggregator $(\bar{f}, \bar{A})$ an ordinally equivalent aggregator (f, A), with variance multiplier $A(\cdot)$ equal to zero, is obtained. (f, A) is called the normalized aggregator of $(\bar{f}, \bar{A})$ and generates the utility process J given by:

$$J_t^T = E_t \left[\int_t^T f(C_s, J_s)\, ds \right]. \tag{27}$$

Note how (27) is close to (7). The difference between the expressions is the infinite time horizon in the later which is also considered in Duffie and Epstein (1992a) with $J_t \equiv \lim_{T\to\infty} J_t^T$.

The method to obtain (f, A) from $(\bar{f}, \bar{A})$, consists in a change of variables through a twice continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and with $\varphi(0) = 0$. φ satisfies the differential equation $\varphi''(x) = \bar{A}(x)\varphi'(x)$, which implies:

$$\varphi(J) = \delta_2 + \delta_1 \int_{J_0}^{J} exp \left[\int_{J_0}^{u} \bar{A}(x)\, dx \right] du, \tag{28}$$

where J_0 is arbitrary, δ_2 and δ_1 are constants, with $\delta_1 > 0$, defined so that $\varphi(0) = 0$. Using φ, the relationship between the two ordinally equivalent aggregators $\left(\bar{f}, \bar{A}\right)$ and (f, A) is given by:

$$\begin{aligned} \bar{f}(C, z) &= \frac{f(C, \varphi(z))}{\varphi'(z)}, \quad (C, z) \in \mathcal{C} \times \mathbb{R}, \\ \bar{A}(x) &= \varphi'(x) A[\varphi(x)] + \frac{\varphi''(x)}{\varphi'(x)}. \end{aligned} \tag{29}$$

In this paper, the utility process J generated by the normalized aggregator $f(C_s, J_s)$ is set to define a SDU function ordinally equivalent to Kreps and Porteus (1978) utility specification. The aggregator $\left(\bar{f}, \bar{A}\right)$ for this particular utility function is defined in Duffie and Epstein (1992a) as:

$$\bar{f}(C, J) = \frac{\beta}{\xi} \frac{C^{\xi} - J^{\xi}}{J^{\xi-1}}, \qquad \bar{A}(J) = \frac{\alpha - 1}{J}, \tag{30}$$

with $\mathcal{C} = \mathbb{R}_{+}$, $0 \neq \xi \leq 1$, $0 \leq \beta$, $0 \neq \alpha \leq 1$, and $J > 0$.

Deduction of expression (8)

Replacing (30), $\bar{A}(x) = \frac{\alpha-1}{x}$, in (28):

$$\begin{aligned} \varphi(J) &= \delta_2 + \delta_1 \int_{J_0}^{J} exp\left[\int_{J_0}^{u} \frac{\alpha - 1}{x} dx\right] du \\ &= \delta_2 + \frac{\delta_1}{J_0} \int_{J_0}^{J} u^{\alpha-1} du \\ &= \delta_2 + \delta_1 \left(\frac{J^{\alpha}}{\alpha}\right). \end{aligned} \tag{31}$$

To have $\varphi(0) = 0$, we set $\delta_2 = 0$. Assuming $\delta_1 = 1$, possible as in Duffie and Epstein (1992a) the only requirement is $\delta_1 > 0$, expression (31) yields $\varphi(J) = \frac{J^{\alpha}}{\alpha}$ and:

$$\varphi'(J) = J^{\alpha-1}. \tag{32}$$

Introducing (30) and (32) in (29), gives:

$$\frac{\beta}{\xi} \frac{C^{\xi} - J^{\xi}}{J^{(\xi-\alpha)}} = f(C, \varphi(J)). \tag{33}$$

From $\varphi(J) = \frac{J^{\alpha}}{\alpha}$ we get $(\alpha\varphi(J))^{\frac{1}{\alpha}} = J$.

Introducing this result in (33):

$$
\begin{aligned}
\frac{\beta}{\xi}\frac{C^{\xi}-(\alpha J)^{\frac{\xi}{\alpha}}}{(\alpha J)^{\frac{\xi-\alpha}{\alpha}}} &= f(C,J) \\
\frac{\beta}{\xi}\alpha J\left(\left(\frac{C}{(\alpha J)^{\frac{1}{\alpha}}}\right)^{\xi}-1\right) &= f(C,J)\,. \qquad (34)
\end{aligned}
$$

Expression (8) follows simply by changing notation: $\xi = 1-\frac{1}{\psi}$ and $\alpha = 1-\gamma$.

□

6.2 Bellman Equation (15)

Starting to define Λ as the diffusion matrix of state variables y_t and X_t processes in (11) and (12):

$$
\Lambda = \begin{bmatrix} \sigma\sqrt{y_t} & 0 \\ \rho\pi_t\sqrt{\frac{1}{y_t}}X_t & \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2} \end{bmatrix}. \qquad (35)
$$

The minimization of (14) with respect to the vector h gives (see, for e.g., Anderson et al. (1998) p. 22):

$$
\begin{bmatrix} h^y \\ h^{\varepsilon} \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{\begin{bmatrix} J_y & J_X \end{bmatrix}\Lambda\Lambda^{\top}\begin{bmatrix} J_y & J_X \end{bmatrix}^{\top}}}\Lambda^{\top}\begin{bmatrix} J_y \\ J_X \end{bmatrix}. \qquad (36)
$$

Replacing (35) in (36), the vector of optimal contaminating drifts is obtained and given by:

$$
\begin{bmatrix} h^y \\ h^{\varepsilon} \end{bmatrix} = -\frac{\sqrt{2\eta}}{\sqrt{J_y^2\sigma^2 y_t + 2J_yJ_X\sigma\rho\pi_t X_t + J_X^2\pi_t^2\frac{1}{y_t}X_t^2}}\begin{bmatrix} \sigma\sqrt{y_t}J_y + \rho\pi_t\sqrt{\frac{1}{y_t}}X_tJ_X \\ \pi_t\sqrt{\frac{1}{y_t}}X_t\sqrt{1-\rho^2}J_X \end{bmatrix}.
$$

Substituting this result into (14), after some algebra, gives (15).

6.3 Sign of the square-root in (17)

With $\psi = 1$, as $\gamma \to 1$, the utility representation (9) converges to the log-utility representation. The optimal portfolio rule without ambiguity ($\eta = 0$), given by (23), in the special case of log-utility ($\gamma = \psi = 1$) is well-known (Merton (1969, 1971, 1973)):

$$
\pi_t = (\mu - r)\,y_t\,,
$$

i.e., the intertemporal hedging demand component disappears (if $\psi = \gamma = 1$, then $A = B = 0$). It is therefore necessary to guarantee that with $\psi = 1$, $\lim\limits_{\gamma\to 1} A = 0$[12] as the limit of (23) as $\gamma \to 1$ is given by:

$$\lim_{\gamma\to 1}\pi_t = (\mu - r)\, y_t + \left(\lim_{\gamma\to 1} A\right) \rho\sigma y_t\,.$$

From (17), $\lim\limits_{\gamma\to 1} A$ is:

$$\lim_{\gamma\to 1} A = \frac{(\beta+\kappa) \pm \lim\limits_{\gamma\to 1} (1-\gamma)\gamma\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}}}{\sigma^2}. \tag{37}$$

If $\gamma \to 1^+$, i.e., $\gamma > 1$, then $(1-\gamma) < 0$ and the discriminant of the square root in (37) is always > 0. By assumption, $(\beta + \kappa) > 0$, which implies that, in order to have $\lim\limits_{\gamma\to 1^+} A = 0$, the " + " sign must be considered.

The same rational implies that when $\gamma < 1$, the " $-$ " sign of the square root guarantees $\lim\limits_{\gamma\to 1^-} A = 0$ (it can be easily shown that the discriminant of the square root in (37) is positive as γ approaches 1 from below).

When $\gamma = 1$, only the "-" sign of the square root gives $A = 0$, as:

$$A\mid_{\gamma=1} = \frac{(\beta+\kappa) \pm (\beta+\kappa)}{\sigma^2}.$$

□

6.4 Optimal Consumption and Portfolio rules

6.4.1 Domain $\gamma \geqslant \omega$

The domain of analysis is set so that A in (17) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 \quad \geqslant \quad \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}. \tag{38}$$

For $\gamma > 1$ it is straightforward to conclude that $\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 > \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}$, and therefore (38) is always true.

[12]From (18) $\lim\limits_{\gamma\to 1} A = 0 \Longrightarrow \lim\limits_{\gamma\to 1} B = 0$.

For $\gamma < 1$, (38) is true as long as:

$$\begin{aligned}
\frac{\gamma}{1-\gamma} &\geqslant \frac{\sigma^2 (\mu - r)^2}{(\beta+\kappa)^2} + \frac{2\rho\sigma(\mu - r)}{(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \frac{\sigma^2 (\mu - r)^2 + 2\rho\sigma(\mu - r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2 (\mu - r)^2 + 2\rho\sigma(\mu - r)(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \omega,
\end{aligned}$$

by making $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}$. Note that $\omega < 1$ as $(\beta+\kappa)^2 > 0$.

The domain of analysis is therefore $\gamma \geqslant \omega$.

6.4.2 Optimal rules (20) and (21)

Considering the Bellman equation (15) and a value function with a structure as that of (16), the FOC with respect to π_t gives:

$$\pi_t = \frac{1}{\gamma + \sqrt{\frac{2\eta}{G(\pi)}}} \left[(\mu - r) y_t + \left(1 - \frac{1}{1-\gamma} \sqrt{\frac{2\eta}{G(\pi)}} \right) \sigma\rho y_t \frac{\partial g(y)}{\partial y} \right], \tag{39}$$

where:

$$G(\pi_t) = \frac{\pi_t^2}{y_t} + \frac{\sigma^2 y_t J_y^2}{X_t^2 J_X^2} + \frac{2\sigma\rho\pi_t J_y}{X_t J_X}, \tag{40}$$

i.e., optimal portfolio rule under ambiguity is the solution of an implicit function in π_t. In order to solve this optimization problem under ambiguity, following the perturbation theory under robust control problems, functions $g(y_t)$ and $\pi(y_t)$ are expanded in $(\sqrt{\eta})$ to first order. Expansion of function $g(y_t)$ is given in (19) and expansion of function $\pi_t(y_t)$ is given by:

$$\pi(y_t) = \pi_0(y_t) + \pi_1(y_t)\sqrt{2\eta} + \mathcal{O}^2(\sqrt{\eta}), \tag{41}$$

where $\mathcal{O}^2(\sqrt{\eta})$ is a symbol representing terms of second order in $(\sqrt{\eta})$. $\pi_0(y_t)$ represents the solution when there is no ambiguity ($\eta = 0$), with expression known from (23). The next step is to expand function $G(\pi_t)$.

Making use of expanded functions $g(y_t)$ and $\pi(y_t)$, the expanded $G(\pi_t)$ function is obtained:

$$G(\pi_t(y_t)) = G_0(y_t) + G_1(y_t)\sqrt{2\eta} + \mathcal{O}^2(\sqrt{\eta}),$$

where following the same rational as before, $G_0(y_t)$ represents the solution without ambiguity. So the last step is to deduct the expression of $G_0(y_t)$. This is done by going back to (40) and consider functions $g_0(y_t)$ ($= Ay_t + B$ with A and B given by (17) and (18)) and $\pi_0(y_t)$, given by (23), which results in:

$$G_0(y_t) = \left[\left(\frac{(\mu-r)+\rho\sigma A}{\gamma}\right)^2 + \left(\frac{\sigma A}{1-\gamma}\right)^2 + \frac{2\sigma\rho A\left((\mu-r)+\rho\sigma A\right)}{(1-\gamma)\gamma}\right] y_t.$$

With this result and going back to (39), the asymptotic expansion for the optimal rule under ambiguity (21) is immediately obtained.

Regarding the optimal consumption rule (20), computations are more straightforward. Considering the Bellman (15) and the aggregator (9), the FOC with respect to variable C_t is simply:

$$f_C = J_X,$$

where f_C is the gradient of the aggregator (9) with respect to consumption. The optimal portfolio rule is given $C_t = \beta X_t$, not depending on the ambiguity parameter η. Its expansion in $\left(\sqrt{\eta}\right)$ is therefore simply given by:

$$C_t = \beta X_t + \mathcal{O}^2\left(\sqrt{\eta}\right),$$

which is (20).

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