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A More Reasonable Model of Insurance Demand: Clarifications and Further Results

by

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Abstract

Much of the traditional economic theory of insurance is based on the assumption that the risk against which insurance is to be purchased is entirely exogenous. This is usually modelled by simply allowing the individual to include insurance as a mechanism of covering risk, without any real analysis of how this insurance is paid for. However, in almost all real-life consumer insurance, the size of the risk is itself a choice variable (the type of car to purchase, the type of employment to take, the amount to invest in an insurable asset, etc.), and decisions are made taking budget constraints explicitly into account. While an enormous number of interesting theorems can be derived in the standard model, these results are typically not robust to the extention of making risk an endogenous choice variable and the explicit inclusion of a budget constraint. A small, but growing literature (led by Prof. J. Meyer) exists that studies the simultaneous choice of risk and insurance, but the current models are somewhat restricted due to the desire to introduce a very general setting. Here, we use a simple two state model of the demand for insurance in which the insurable risk itself is a choice variable. In the model, we confirm the comparative statics results found in the existing literature, although we can now be far more explicit about them, and we also study some further comparative statics results.

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1 Introduction

Traditionally, the economic theory of insurance is based on the study of preferences over wealth lotteries, using the indirect utility function to describe preferences. In the standard model, two particularly important assumptions are often made. Firstly, that the individual is always able to pay for his insurance contract out of that part of wealth that is not subject to loss. In particular, the implicit assumption is that first the individual uses the non-random part of her wealth to finance the purchase of insurance coverage, and then dedicates whatever terminal wealth ensues for consumption purposes. Secondly, the standard model assumes that the insurable risk itself is fully endowed and non-alterable. This is, of course, a question that is related to the first. If we do allow the individual to make a simultaneous choice of insurance and other goods under a budget constraint, then one of the goods that she could contemplate purchasing is precisely that which pertains to the underlying risk. In this case, decisions on both the amount of risk that impinges upon wealth, and the degree to which that risk is insured, would be made simultaneously (and simultaneously with the purchase of other goods) making the amount of risk an endogenous variable.

The basic motivation for models in which the amount of risk and the amount of insurance are chosen simultaneously is the following. It is well known that the standard model returns ambiguous comparative statics of the demand for coverage. Specfically, if the total wealth of the individual is increased and the premium is such that the demand for coverage is partial, then if (as is very reasonable to assume) the individual's preferences are characterized by decreasing absolute risk aversion she will respond by purchasing less coverage.[1] Thus insurance is an inferior good in the standard model under DARA. For more general utility functions, the result is ambiguous. On the other hand, in the standard portfolio selection model, where no insurance per se is contemplated, it is easy to show that, under the same assumption of decreasing absolute risk aversion, risk itself is a normal good – that is, a strictly positive part of any increase in wealth will be dedicated to increasing the risky element of the portfolio. Thus, exactly what happens when we allow the

[1] The increase in wealth implies that the individual is less risk averse, and so will be less willing to insure any given risk.

individual the option of setting endogenously both the amount of risk and the amount of insurance is unclear. We would expect that an increase in wealth will result in an increase in risk, but that this increased risk position will be insured relatively less, and so it is not clear whether the total purchase of insurance will increase or decrease.

Recently, in a paper that follows on from a series of earlier papers (see Meyer and Ormiston, 1995, Eeckhoudt, Meyer and Ormiston, 1997, and Meyer and Meyer, 1999 all of which consider the same basic problem of simultaneous choice of risk and insurance, in a variety of settings), Meyer and Meyer (2004) show that (among other results) when the amount of risk that is undertaken is chosen together with the amount of insurance that is purchased, then insurance can still be a normal good under some DARA utility functions (specifically, those that display non-decreasing relative risk aversion), and that it will be ordinary whenever relative risk aversion is not greater than 1.[2] In order to derive this result, Meyer and Meyer (2004) note that it is relevant to consider the purchase of risky and non-risky goods as a composite commodity, and so the relative amounts of each that are purchased cannot be varied. This simple, but powerful, assumption leads directly to a model that is comparable in nature to any other model of consumer choice, and thus allows the normality of insurance to be properly addressed.

In this paper we analyse the demand for insurance in a simple two-state version of Meyer and Meyer's (2004) model, with the objective of providing both clarifications of some of the results found in the earlier model, and also adding some further results to those already existing. Our emphasis is on providing a point of comparison, both with the Meyer and Meyer (2004) paper, and the comparative statics of the traditional model of insurance (Mossin 1968). More specifically, we do not claim that our results on normality and ordinarity of insurance are in general superior to, or extensions of, those already noted by Meyer and Meyer (among others). Simply, we find that the analysis in the more complex settings clouds much of the underlying intuition to the extent that the results thus obtained are of limited practical use. We contend that the use of the two-state model, while clearly less general, is interesting in its own right for at least three reasons;

[2] Actually, these results appear in some of the earlier papers also.

1) by retaining the familiar and simple graphical environment of many text-books,[3] it allows the intuition behind the results to be understood more deeply, 2) it allows the existing results on normality and ordinarity to be stated in a far more clear-cut and, hopefully useful, manner, and 3) it allows a greater array of comparative static results to be achieved.

The paper proceeds in the following way; in the next section our model of insurance is set out and we establish the optimal solution. Then, in section 3, we consider the comparative statics that emerge from the model, using traditional demand theory. Section 4 concludes and offers some suggestions for future research.

2 The model

Consider an individual who is endowed with a certain amount of wealth, w, which includes all borrowing capacity, which she can distribute between the purchase of a composite commodity consisting of risky and non-risky goods, and insurance of this composite commodity. Let x be the amount of composite commodity purchased, and let v be the price per unit of this good. The risky portion of the composite good, which following Meyer and Meyer (2004) is held fixed, implies that, *ex-ante*, there are two states of nature; in the no-loss state the total amount purchased of the composite good becomes available for consumption, and in the loss state a proportion a of the amount purchased disappears. The loss state occurs with known probability p, where $0 < p < 1$. The assumption is that the risk is one-off, that is, once the accident resulting in loss has or has not occurred, there is no further risk to the composite good. Thus, *ex-post*, there is no risky portion of the composite commodity. The other good, insurance coverage, can be purchased to provide indemnity against the loss state at an amount c at unit price of π, i.e. in the loss state, the individual receives an amount of the composite commodity (now riskless) equal to c.[4]

We restrict the choice of indemnity to be not greater than the proportion of the composite good that may be lost, i.e. $c \leq ax$. In addition, we impose $x \geq 0$ and $c \geq 0$; there exists no market

[3] See, for example, the analysis of chapter 3 of the well known text by Hirshleifer and Riley (1992). There, the comparative statics as wealth changes of the risk bearing optimum in a two-state environment is analysed graphically, although the formal expressions are not given.

[4] Since the model is restricted to only two states of nature, there is no formal difference between coverage being based on a deductible, or on co-insurance. Either of these options can be accommodated by the model. Also, the indemnity could be set at that monetary amount that is sufficient to purchase c units of the composite good.

for short-selling of the composite good and the insured cannot sell insurance. The problem of the individual is to maximize the expected (direct) utility that she gets from the choice of x and c subject to her budget constraint. The general problem is

$$\max_{x,c} \; U(x,c) \equiv (1-p)u(x) + pu((1-a)x + c)$$

$$s.t. \quad w \geq vx + \pi c \; \text{ and } \; c \leq ax$$

We will indicate the solution to this problem by the vector (x^*, c^*).

Note that we consider a simple optimisation problem for an individual in a single period. In this period, insurance and the composite good are available to the individual at prices π and v respectively. We do not justify their existence, nor the possible existence of other risky goods in the economy. Our objective is to focus on the individual demand for insurance for a single insurable good.[5]

2.1 The optimal solution

It turns out that the comparative statics of this model are easier to work through if we introduce a simple change of variable. Let the no-accident state (occurs with probability $1-p$) be state 1, and the accident state (occurs with probability p) be state 2. Let us denote state 2 consumption by y, so that

$$y \equiv (1-a)x + c$$

In this case, we have $c = y - (1-a)x$, and so the budget constraint becomes $w \geq vx + \pi(y - (1-a)x) = (v - \pi(1-a))x + \pi y$. We interpret $v - \pi(1-a)$ to be the "net" price of state 1 consumption, and we denote it by $v - \pi(1-a) \equiv q$. Naturally, we assume that this price is always strictly positive ($q = v - \pi(1-a) > 0$). The problem is now to maximise $(1-p)u(x) + pu(y)$ under the conditions that $w \geq qx + \pi y$ and $y \leq x$. We will indicate the solution to this problem by the vector (x^*, y^*).

Since the objective function is concave, and the restrictions are linear, we know that there

[5] To avoid confusions, we stress that our model is an optimisation model, **not** a general equilibrium model. We do **not** assume a two-state economy, only a two state decision scenario for our individual.

exists a unique optimum for the problem. The Lagrangian is:

$$L(x, y, \delta) = (1-p)u(x) + pu(y) + \delta_1(w - qx - \pi y) + \delta_2(x - y)$$

where δ_1 is the multiplier corresponding to the budget constraint, and δ_2 is the multiplier corresponding to the coverage restriction (naturally, we restrict $\delta_i \geq 0$ for $i = 1, 2$). The first order conditions for the optimum are

$$\frac{\partial L}{\partial x} = (1-p)u'(x^*) - q\delta_1 + \delta_2 = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = pu'(y^*) - \pi\delta_1 - \delta_2 = 0 \tag{2}$$

and the complementary slackness conditions are

$$\delta_1(w - qx^* - \pi y^*) = 0 \text{ and } \delta_2(x^* - y^*) = 0 \tag{3}$$

Firstly, note that from (1) it must be true that $\delta_1 > 0$, so the budget constraint must always saturate:

$$w = qx^* + \pi y^* \tag{4}$$

Secondly, assuming an interior solution ($\delta_2 = 0$), from (1) and (2) we get the tangency condition in state contingent space:[6]

$$\frac{u'(x^*)}{u'(y^*)} = \frac{pq}{(1-p)\pi} \tag{5}$$

We begin by studying the relationship between full/partial coverage, and the insurance premium.

Proposition 1 *In the optimal solution,*

$$\begin{array}{rclcl} \pi & \leq & \dfrac{pv}{1-pa} & \Longrightarrow & c^* = ax^* \\ \pi & > & \dfrac{pv}{1-pa} & \Longrightarrow & c^* < ax^* \end{array}$$

Proof. For an interior solution, the optimum is given by the simultaneous solution to (4) and (5). Firstly, consider the case $(1-p)\pi \leq pq$. In this case the right-hand-side of (5) is greater than or equal to 1. This would indicate that $y^* \geq x^*$, but since the second constraint does not allow

[6] Of course, this is nothing more than the usual condition of equality between the marginal rate of substitution and the ratio of state-contingent prices.

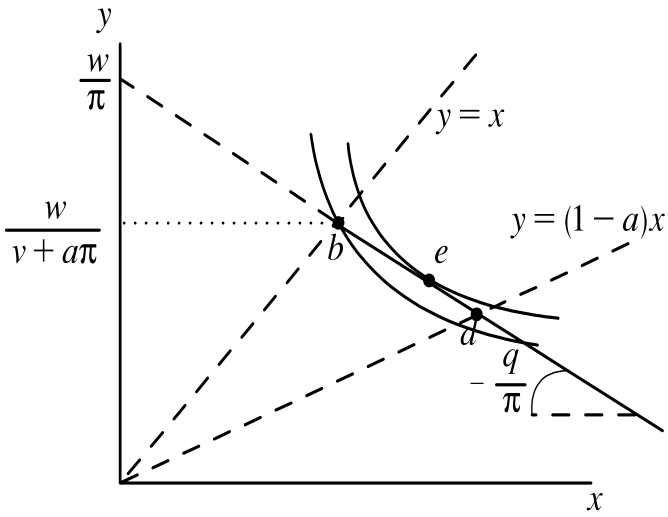

Figure 1:

y to go above x, we get the corner solution in which $y^* = x^*$, that is $c^* = ax^*$. Recalling that $q = v - \pi(1-a)$, it turns out that $(1-p)\pi \leq pq$ reorders directly to $\pi \leq \frac{pv}{1-pa}$, the first statement of the proposition is proved.

Secondly, consider $(1-p)\pi > pq$, that is $\pi > \frac{pv}{1-pa}$. In this case the right-hand-side of (5) is less than 1, which can only imply $y^* < x^*$, that is $c^* < ax^*$. ■

Proposition 1 is illustrated in figure 1 in state contingent consumption space. The straight line with slope $-\frac{q}{\pi}$ is the budget constraint, and the indifference curves have slope $-\frac{(1-p)u'(x)}{pu'(y)}$. The initial endowment point is labelled as point d, at the intersection of the budget constraint and the ray from the origin indicating the fact that state 2 consumption with no insurance is a predetermined fraction, $1-a$, of state 1 consumption. The equilibrium point e, at tangency between the budget constraint and the indifference map, is the optimal solution, where $y < x$: partial insurance is purchased. This is preferred to point b on the certainty line, where $x = y = \frac{w}{v+a\pi}$.

3 Comparative statics

Of particular interest to us are the comparative statics of the solution vector as the parameters of the system are altered. Perhaps the most interesting of these comparative statics exercises is the effect of changes in w on the optimal values of x and c, but we shall also consider changes in other parameters, concretely the effect of a change in the unit price of insurance (π), in the fraction of risky asset that is at risk (a), in the probability of the loss state[7] (p), and a change in risk aversion (captured by the curvature of the utility function, u). For this entire section, we assume that the solution is interior, that is, we assume $\pi > \frac{pv}{1-pa}$.

3.1 Normality of insurance

We begin by considering the normality of the goods in our model. Recall, that in the traditional model (where the amount of risky good is held constant), under DARA insurance is inferior. Insurance is also inferior under DARA in the models of Meyer and Ormiston (1995) and Eeckhoudt, Meyer and Ormiston (1997).[8] Meyer and Meyer (2004) find that insurance is normal, even under DARA, if relative risk aversion is non-decreasing. In the two-state model we get an even more clear-cut result:

Proposition 2 $\frac{\partial c^*}{\partial w} \gtreqless 0$ *as* $R_a(x^*) \gtreqless (1-a)R_a(y^*)$, *and* $\frac{\partial x^*}{\partial w} > 0$ *always.*

Proof. Differentiating the budget constraint (4) with respect to w conditional on being at a solution gives

$$q\frac{\partial x^*}{\partial w} = 1 - \pi\frac{\partial y^*}{\partial w} \tag{6}$$

Secondly, differentiating the tangency condition (5) with respect to w conditional on being at a solution, and simplifying gives

$$R_a(y^*)\frac{\partial y^*}{\partial w} = R_a(x^*)\frac{\partial x^*}{\partial w} \tag{7}$$

where $R_a(x)$ is the Arrow-Pratt measure of absolute risk aversion. Combining these two equations, we get

$$\frac{\partial x^*}{\partial w} = \frac{R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \tag{8}$$

$$\frac{\partial y^*}{\partial w} = \frac{R_a(x^*)}{qR_a(y^*) + \pi R_a(x^*)} \tag{9}$$

[7] In our model, first-order stochastic dominance (FSD) changes in the risk can be split into changes in either the proportion of the composite commodity that may be lost, or changes in the probability of the loss state. This contrasts to earlier models with continuous loss densities, in which FSD changes can only be studied by shifts in the loss density.

[8] Actually, in Eeckhoudt, Meyer and Ormiston (1997), DARA preferences leads to the conclusion that it is very likely that insurance is inferior (concretely, either insurance is inferior, or the risky asset is normal, or both).

Clearly, from (8), it always happens that the optimal purchase of the risky good is increasing in wealth in this model.

Finally, since $\frac{\partial c^*}{\partial w} = \frac{\partial y^*}{\partial w} - (1-a)\frac{\partial x^*}{\partial w}$, we have

$$\frac{\partial c^*}{\partial w} = \frac{R_a(x^*) - (1-a)R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \tag{10}$$

Thus,

$$\frac{\partial c^*}{\partial w} \gtreqless 0 \quad \text{as} \quad R_a(x^*) \gtreqless (1-a)R_a(y^*) \tag{11}$$

■

The necessary and sufficient condition for normality/inferiority of insurance indicated in this proposition, $R_a(x^*) \gtreqless (1-a)R_a(y^*)$, can be written in several different interesting ways. These can in turn be used to provide for other sufficient conditions. Firstly, simple but direct re-ordering of (11) reveals that it is equivalent to

$$\frac{\partial c^*}{\partial w} \gtreqless 0 \quad \text{as} \quad a \gtreqless \frac{R_a(y^*) - R_a(x^*)}{R_a(y^*)} \tag{12}$$

That is, insurance is normal if the fraction of the composite good that is at risk is greater than the relative change in absolute risk aversion from state of nature 2 to state of nature 1.

Secondly, note that if we multiply condition (11) by x^*, and then sum $c^*R_a(y^*)$ to each side, it reads

$$\frac{\partial c^*}{\partial w} \gtreqless 0 \quad \text{as} \quad R_r(x^*) + c^*R_a(y^*) \gtreqless R_r(y^*)$$

where $R_r(\cdot)$ is the Arrow-Pratt measure of relative risk aversion. This reorders to[9]

$$\frac{\partial c^*}{\partial w} \gtreqless 0 \quad \text{as} \quad c^*R_a(y^*) \gtreqless R_r(y^*) - R_r(x^*) \tag{13}$$

Thirdly, directly from (13), dividing both sides by $y^*R_a(y^*)$ we get

$$\frac{\partial c^*}{\partial w} \gtreqless 0 \quad \text{as} \quad \frac{c^*}{y^*} \gtreqless \frac{R_r(y^*) - R_r(x^*)}{R_r(y^*)} \tag{14}$$

where the right-hand-side is just the relative change in relative risk aversion over the two states of nature.

[9] $cR_a(y)$ is related to, but not equal to, the measure of partial risk aversion introduced by Menezes and Hanson (1970). Their measure, defined as $tR_a(w+t)$, is based on the case where an exogenous risk t is added to riskless initial wealth w.

Sufficient, but not necessary, conditions for normality of insurance can be obtained directly from any of these equivalent expressions. Three such conditions are:

Corollary 1 *If $a = 1$, then insurance is always a normal good.*

Proof. The proof is immediate from (12), the right-hand-side of which is strictly less than 1. Also, directly from (10) the same result can be found. ■

This result is to be expected – when the loss state has a total loss, state 2 consumption is simply the insurance indemnity, and it is well known that under separable utility, all state contingent consumptions are normal.

Corollary 2 *Insurance is a normal good if the utility function displays non-decreasing absolute risk aversion.*

Proof. Since (by assumption) we are at an internal solution, we have $y^* < x^*$. Thus, under non-decreasing absolute risk aversion, we have $R_a(x^*) \geq R_a(y^*)$, and so (with $a > 0$) it is always true that $R_a(x^*) > (1-a)R_a(y^*)$. ■

This result is also to be expected, it also holds in the traditional model of insurance.

Corollary 3 *Insurance is a normal good if the utility function displays DARA and non-decreasing relative risk aversion (assuming some insurance is purchased).*

Proof. Directly from (13), since the left-hand-side is strictly positive when some insurance is purchased, we get the result that a sufficient condition for insurance to be a normal good is that the right-hand-side is negative, which holds if $R_r' \geq 0$ (since at an internal solution $y^* < x^*$). ■

This third corollary mirrors the result of Meyer and Meyer (2004), however our model stresses the fact that normality of insurance occurs for a wider range of utility types than those displaying non-decreasing relative risk aversion. *If absolute risk aversion is high enough, there is room for the true condition, (13), to be satisfied, even under decreasing relative risk aversion.*

It is interesting to note that out of all the models mentioned, only in Meyer and Meyer (2004) and here is non-decreasing relative risk aversion a sufficient condition for normality of insurance. This may seem strange since the normality of insurance is determined by changes in the purchased *amount* of insurance, when wealth changes. It is naturally linked to absolute risk aversion, which governs the *amount* of risky goods in the overall consumption bundle. Thus, in the standard model (and in Meyer and Ormiston (1995)), a wealth increase means an exogenous increase in safe wealth, the risky portion of wealth being given. Under DARA, this will necessarily lead to a counter-balancing reduction in the optimal amount of insurance to yield an increase in risk held. In contrast, relative risk aversion governs the optimal *proportion* of riskless goods to be purchased. It is not directly linked to the normality of insurance.

However, Meyer and Meyer (2004) and the current paper present models that allow for three goods to be present (risky good, risk-free good, and insurance), but that hold the relative proportions of risky and risk-free goods constant. Although our model has been described as simply one in which the loss state is partial, clearly such an assumption is exactly equivalent to the model of Meyer and Meyer, as that part of the risky good that is not lost is clearly a riskless part of the consumption bundle. The reason why relative risk aversion drives the normality of insurance in these models is that there is a fixed *proportional* relationship between the risky (uninsured) part and the riskless part of the consumption bundle. In the IARA and CARA cases, leading to increasing relative risk aversion (IRRA), the amount of risky good purchased does not increase when wealth increases. This implies that the amount allocated to the riskless good cannot increase either. All of the wealth increase (more than that in the IARA case) must then be allocated to the insured risky good: the demand for insurance increases. Under DARA, some additional wealth is allocated to the risky (uninsured) good. But then the constant proportionality with the riskless good implies that some of the additional wealth must be allocated to the riskless good. Under IRRA and CRRA, this implies further that the demand for insurance must increase (as the share of insured goods purchased must increase or remain constant). Under decreasing relative risk aversion (DRRA), this is not necessarily the case. To obtain the desired more risky allocation, given that the relationship between risky and riskless goods in the composite commodity has remained the same, it may be necessary to decrease the amount invested in insured goods.

3.2 Ordinarity of insurance

As any normal good is ordinary, if the condition for normality of insurance is satisfied, then insurance is ordinary. For example, in our model if we assume $a = 1$, clearly insurance is a normal good from Corollary 1. However, even if insurance is inferior, it still may turn out to be ordinary. Also, from Corollary 3, it turns out that insurance is an ordinary good if relative risk aversion is non-decreasing, *whatever its value.* We now go on to consider the ordinarity of insurance.

Proposition 3 $\frac{\partial c^*}{\partial \pi} \gtreqless 0$ *as* $c^* \left[(1-a)R_a(y^*) - R_a(x^*)\right] \gtreqless \frac{v^2}{\pi q}$.

Proof. The proposition can be proved directly from the Slutsky equation;

$$\frac{\partial c^*}{\partial \pi} = \frac{\partial c^h}{\partial \pi} - c^* \frac{\partial c^*}{\partial w} \tag{15}$$

where c^h is the Hicksian demand for insurance (that is, the demand that eventuates when the cost of (c, x) is minimised subject to utility being no less than some minimum).

Consider the Hicksian problem defined in the space (x, y):

$$\min_{x,y} \; qx + \pi y \quad \text{subject to} \;\; pu(y) + (1-p)u(x) \geq \overline{u}$$

The solution, (x^h, y^h) is given by the simultaneous solution to the equation

$$pu(y) + (1-p)u(x) = \overline{u} \tag{16}$$

and the corresponding tangency condition

$$\frac{u'(x^h)}{u'(y^h)} = \frac{pq}{(1-p)\pi} \tag{17}$$

Differentiating the utility constraint with respect to π yields

$$pu'(y^h)\frac{\partial y^h}{\partial \pi} + (1-p)u'(x^h)\frac{\partial x^h}{\partial \pi} = 0$$

Dividing this by $u'(y^h)$, and using (17) give us the result that at the Hicksian solution we have

$$p\frac{\partial y^h}{\partial \pi} + (1-p)\left(\frac{pq}{(1-p)\pi}\right)\frac{\partial x^h}{\partial \pi} = 0$$

that is

$$\frac{\partial y^h}{\partial \pi} = -\frac{q}{\pi}\frac{\partial x^h}{\partial \pi} \tag{18}$$

Secondly, differentiating the tangency condition (17) with respect to π, and recalling that $q = v - \pi(1-a)$ we get

$$\begin{aligned}
\frac{u''(x^h)u'(y^h)\frac{\partial x^h}{\partial \pi} - u'(x^h)u''(y^h)\frac{\partial y^h}{\partial \pi}}{u'(y^h)^2} &= \frac{-p(1-a)(1-p)\pi - p(v-\pi(1-a))(1-p)}{\pi^2(1-p)^2} \\
&= \frac{-pv}{\pi^2(1-p)}
\end{aligned}$$

However, multiplying by the marginal utility of state 2 consumption and simplifying, we get

$$u''(x^h)\frac{\partial x^h}{\partial \pi} + u'(x^h)R_a(y^h)\frac{\partial y^h}{\partial \pi} = -\frac{u'(y^h)vp}{\pi^2(1-p)}$$

Using the tangency condition itself, we have

$$\begin{aligned} u''(x^h)\frac{\partial x^h}{\partial \pi} + u'(x^h)R_a(y^h)\frac{\partial y^h}{\partial \pi} &= -\frac{u'(x^h)(1-p)\pi}{pq}\frac{vp}{\pi^2(1-p)} \\ &= -u'(x^h)\frac{v}{\pi q} \end{aligned}$$

And then dividing by the negative of marginal utility in state 1 yields

$$R_a(x^h)\frac{\partial x^h}{\partial \pi} - R_a(y^h)\frac{\partial y^h}{\partial \pi} = \frac{v}{\pi q} \tag{19}$$

The simultaneous solution to (18) and (19) is given by

$$\begin{aligned} \frac{\partial x^h}{\partial \pi} &= \frac{1}{D^h}\frac{v}{q} \\ \frac{\partial y^h}{\partial \pi} &= -\frac{1}{D^h}\frac{v}{\pi} \end{aligned}$$

where $D^h \equiv \pi R_a(x^h) + qR_a(y^h)$. Finally, from the equation that defines y, we have $\frac{\partial c^h}{\partial \pi} = \frac{\partial y^*}{\partial \pi} - (1-a)\frac{\partial x^h}{\partial \pi}$. Thus

$$\frac{\partial c^h}{\partial \pi} = \frac{1}{D^h}\left(-\frac{v}{\pi} - (1-a)\frac{v}{q}\right)$$

Using the fact that $q = v - \pi(1-a)$, this simplifies to

$$\frac{\partial c^h}{\partial \pi} = -\frac{1}{D^h}\frac{v^2}{\pi q}$$

Substituting this into the Slutsky equation, and using (10) we get

$$\begin{aligned} \frac{\partial c^*}{\partial \pi} &= -\frac{1}{D^h}\frac{v^2}{\pi q} - c^*\frac{R_a(x^*) - (1-a)R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \\ &= \frac{1}{D}\left[c^*\left((1-a)R_a(y^*) - R_a(x^*)\right) - \frac{v^2}{\pi q}\right] \end{aligned} \tag{20}$$

where $qR_a(y^*) + \pi R_a(x^*) \equiv D$, and $D = D^h$ since the solution to the Hicksian problem and the solution to the utility maximisation problem coincide for the appropriate level of initial wealth. Since $D > 0$, the proposition follows directly from (20). ■

Proposition 3 provides a link between the necessary and sufficient conditions for inferiority of insurance (Proposition 2) and the necessary and sufficient conditions to get insurance as a Giffen good. The proposition confirms that the value of $(1-a)R_a(y^*)$ is critical. If it is larger than $R_a(x^*)$, insurance is inferior. If it is even larger (larger than $R_a(x^*) + \frac{v^2}{q\pi c^*}$), insurance is a Giffen good. As a corollary, when the proportion of the composite commodity that is subject to risk, a, is high enough, insurance is a normal good.

Again, other sufficient (but not necessary) conditions can be obtained. We shall now mention two possibilities:

Corollary 4 *If insurance is an inferior good, it is still ordinary if* $\frac{v}{q} \geq R_r(y^*)$.

Proof. The full sufficient and necessary condition for ordinarity of insurance is (from Propostion 3)

$$c^*\left((1-a)R_a(y^*) - R_a(x^*)\right) < \frac{v^2}{\pi q} = \frac{v}{q}\frac{v}{\pi}$$

However, since $c^* R_a(x^*) > 0$, it is sufficient that

$$c^*(1-a)R_a(y^*) < \frac{v}{q}\frac{v}{\pi}$$

Dividing by $(1-a)$, gives the condition

$$c^* R_a(y^*) < \frac{v}{q}\frac{v}{(1-a)\pi} \tag{21}$$

Since $v - \pi(1-a) > 0$, we have $\frac{v}{\pi(1-a)} > 1$, and so $\frac{v}{q}\frac{v}{(1-a)\pi} > \frac{v}{q}$. And finally, since $y^* > c^*$, we have $R_r(y^*) = y^* R_a(y^*) > c^* R_a(y^*)$. Thus, over all, so long as $R_r(y^*) \leq \frac{v}{q}$, we have

$$c^* R_a(y^*) < R_r(y^*) \leq \frac{v}{q} < \frac{v}{q}\frac{v}{(1-a)\pi}$$

■

It is interesting to note that this particular sufficient condition for ordinarity of insurance also conditions $\frac{\partial x^*}{\partial \pi}$ to be positive. To see this, simply note that the Slutsky equation for the effect of an increase in π on x^* is given by

$$\frac{\partial x^*}{\partial \pi} = \frac{\partial x^h}{\partial \pi} - c^* \frac{\partial x^*}{\partial w}$$

Using the fact that (noted in the proof of Proposition 3) $\frac{\partial x^h}{\partial \pi} = \frac{1}{D^h}\frac{v}{q}$, equation (8), and the definition of D, we have

$$\frac{\partial x^*}{\partial \pi} = \frac{1}{D}\left[\frac{v}{q} - c^* R_a(y^*)\right]$$

Thus, $R_r(y^*) \leq \frac{v}{q}$ implies that $\frac{\partial x^*}{\partial \pi} > 0$ and $\frac{\partial c^*}{\partial \pi} < 0$ simultaneously. In terms of classical demand theory, insurance and the insurable risky asset would be "net substitutes". This is worth noting, as it is often thought that insurance, and the item being insured are complementary.[10]

The sufficient condition in Corollary 4 is also interesting since it clearly shows that insurance can be ordinary when relative risk aversion is greater than 1. Since the term $\frac{v}{q} = \frac{v}{v-\pi(1-a)}$ is clearly greater than 1, there is room for relative risk aversion to be greater than 1 while still allowing insurance to be ordinary. This is important, since much of the existing literature (see Meyer and Ormiston (1995), Eeckhoudt, Meyer and Ormiston (1997), Meyer and Meyer (1999) and Meyer and Meyer (2004)) provide sufficient conditions for insurance to be ordinary that rely on relative risk aversion being less than 1, something that is not particularly likely given empirical estimates.[11]

It is interesting to consider exactly how high relative risk aversion can get before insurance becomes Giffen. Take the following example; say the probability of the loss state and the proportion of the asset at risk are both one half, that is $p = a = 0.5$, and assume $\pi = 0.75$. Now the price of the risky good must satisfy two conditions: for an internal solution $\pi > \frac{pv}{1-pa}$ which now reads $v < 1.125$, and for a positive "net" price of state 1 consumption $v > \pi(1-a) = 0.375$. Thus, we can only consider values of v that satisfy $0.375 < v \leq 1.125$. Now, since $\frac{v}{v-\pi(1-a)}$ is decreasing in v, it reaches its greatest value close to $v = 0.375$, and its lowest value at $v = 1.125$. That is, the condition is satisfied by all cases in which relative risk aversion is no greater than the value of $\frac{1.125}{1.125-0.75\times0.5} \approx 1.0345$. However, if we use $v = 0.4$, the limit value on relative risk aversion is 16, while at $v = 0.5$, relative risk aversion can go as high as 4 and still satisfy the condition. And we really should recall that we are only discussing a sufficient condition here; the greater is the difference between c^* and y^* (that is, the greater is x^* and the smaller is a), relative risk aversion

[10] Meyer and Meyer (2004) justify making the amount of risk endogenous by stating "In a standard consumer setting, making this assumption [that the risk is exogenous] is similar to determining the demand for hotdog buns, allowing the quantity of soda to vary, but holding the number of hotdogs fixed." The connotation that perhaps insurance and the good insured are complementary is clear. It is a fact in this model that insurance would not exist if no risky asset was purchased. In this sense, insurance and the risky asset are complements. In spite of this, clearly they may react in opposite direction to an increase in the price of insurance, and thus be "net substitutes" in the sense of demand theory.

[11] There is a large (and growing) literature dedicated to estimating the true size of relative risk aversion. In almost all cases, we get numbers that are almost constant, and located between about 1 and up to about 10 (see, for example, Mehra and Prescott (1985), Szpiro (1986), and Levy (1994), all of whom find that relative risk aversion is approximately constant, and is usually valued above 1).

can go even higher and we may still satisfy the condition expressed in Proposition 3.

Nevertheless, it is also very easy to see that our model also satisfies the sufficient condition that is so frequently mentioned in the earlier literature:

Corollary 5 *If insurance is an inferior good, it is still ordinary if relative risk aversion is not greater than 1.*

Proof. Clearly, if relative risk aversion is uniformly less than 1, then $R_r(y^*) < 1$, which necessarily satisfies the condition in Corollary 4. ■

From Corollary 5, relative risk aversion larger than 1 is a necessary condition for insurance to be a Giffen good, a result already obtained by Hoy and Robson (1981).[12]

Indeed, and as a summary, if insurance is inferior, $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*)$, but it turns out that $c^* R_a(y^*) < \frac{v}{q}$, which from (21) is a weaker (but with a less intuitive meaning) sufficient condition for ordinarity than that stated in Corrollary 4, and taking 1 and $\frac{v}{q}$ as benchmarks for the value of relative risk aversion, three cases are possible:

1. Low and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*) < 1 < \frac{v}{q}$.

2. High and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*)$ and $1 < R_r(y^*) \leq \frac{v}{q}$.

3. Very high and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*)$, $c^* R_a(y^*) < \frac{v}{q}$, and $1 < \frac{v}{q} < R_r(y^*)$.

3.3 Additional comparative statics results

The rest of the propositions consider aspects of the demand for insurance that, while still interesting and important, are not so critical as the normality and ordinarity of insurance. Some of the following results have been considered previously in the literature, and most of them are perfectly in line with intution. All proofs of these results are given in the Appendix.

Firstly, if the proportion of the risky good that may be lost increases, we get the result that less of it will be purchased, and it will be more heavily insured:

12 See also Briys, Dionne and Eeckhoudt (1989).

Proposition 4 $\frac{\partial x^*}{\partial a} < 0 < \frac{\partial c^*}{\partial a}$

Secondly, we consider how the optimal solution is affected by an increase in risk aversion. Following Pratt (1964), an increase in risk aversion can be studied by substituting the utility function $u(\cdot)$ by $h(u(\cdot))$ where h is a strictly increasing and strictly concave function from $\mathbb{R}^1$ to $\mathbb{R}^1$.

Proposition 5 *An increase in risk aversion will increase the amount of coverage and decrease the amount of risky good purchased.*

The fact that an increase in risk aversion leads to both less risk being purchased and more insurance being purchased is, of course, entirely natural and in accordance with logic. It states that an increase in risk aversion will be accommodated by reducing the amount of risk undertaken, but that this reduction in risk will be split between the two possible manners in which it can be done – purchasing less risk to start out with, and insuring this risk more heavily – rather than relying on a single risk reduction method.

Next, we can consider how a change in the probability density defining the loss lottery will affect the optimal choice. Here, to begin with we do this holding the premium constant. That is, we consider the effect of an increase in the probability of loss when the insurer does not alter the premium as a result. Later, we consider the situation in which the premium is also altered endogenously with the probability.

In the traditional model of wealth lotteries with fully endowed risk, an increase in the probability of loss which is not accompanied by an increase in the premium, and where partial coverage is optimal, will always lead to an increase in the amount of coverage purchased (see Mossin (1968)). As the next proposition shows, the same is true in the current model, although here we are also able to assertain what happens to the optimal purchase of the risky good.

Proposition 6 $\left.\frac{\partial c^*}{\partial p}\right|_{\pi} > 0 > \left.\frac{\partial x^*}{\partial p}\right|_{\pi}$.

The result that insurance coverage increases and the purchase of the risky good decreases after an increase in the probability of the adverse state is very easy to see graphically (figure 1). Since an increase in p has the effect of making the indifference curves flatter at all points in (y, x) space, it will not alter the solution when the coverage constraint saturates, and it must move the solution to the left along the budget constraint when the coverage constraint does not saturate. But if y increases and x decreases, we know that c must increase.

However, as has been argued previously, it is not reasonable to assume that if the probability of the loss state increases, the premium will not increase also (Jang and Hadar (1995)). In the traditional model of insurance under such conditions the sign of the effect on insurance coverage is indeterminate, although the sign of the change does depend on the slope of absolute risk aversion. Concretely, in order to get a determinate effect, absolute risk aversion cannot be decreasing. In the present model, things get very complicated very quickly, so we shall only present an analysis pertaining to the special case of a full loss in state 2, that is $a = 1$.

Proposition 7 *Assume that $a = 1$, and that the insurer increases the premium π in response to an increase in the probability of loss, p; that is $\frac{\partial \pi}{\partial p} > 0$. Then if relative risk aversion is not less than 1, an increase in p will reduce the value of x^*. On the other hand, the value of c^* will increase, decrease or not change as $(1-p)\left(1+\left(\frac{\pi c^*}{v x^*}\right)R_r(x^*)\right)\frac{\partial \pi}{\partial p}$ is less than, greater than, or equal to $\frac{\pi}{p}$.*

One interesting point to note about this result is that, in the traditional model the sign of the effect of an increase in the probability of loss on insurance coverage depends on the slope of absolute risk aversion, but in the present model (as for some of the previous results) it depends on the size of risk aversion (in this case, of relative risk aversion). However, we can say even more about the effect of an increase in the probability of the loss state on the optimal insurance purchase.

Corollary 6 *Assume that $a = 1$ and that relative risk aversion is constant and equal to 1 (i.e. logarithmic utility), then $\frac{\partial c^*}{\partial p} \gtreqless 0$ as $\frac{\partial \pi}{\partial p} \lesseqgtr \frac{\pi}{p}$.*

In order to say more, we need to be more precise about exactly how the premium depends upon p. However, a great deal of the economics of insurance is based upon the assumption that the insurance premium is calculated as the expected value of a unit of coverage multiplied by a constant loading factor that is not less than 1. That is, the premium is set such that $\pi = kp$, where $k \geq 1$ is the loading factor. For this case, we get;

Corollary 7 *Assume that $a = 1$, and that insurance is priced according to a loading factor, $\pi = kp$, where $k \geq 1$, then an increase in the probability of loss will increase (decrease, not affect) the optimal purchase of insurance if relative risk aversion is uniformly less than (greater than, equal to) 1.*

A special, but obvious, case occurs for insurance that is sold in a perfectly competitive market, i.e. the loading factor is equal to 1, and Corollary 7 applies. Since it is far more reasonable to expect that relative risk aversion is always greater than 1, for the case of a premium based on a loading factor, when the insurer does respond to an increase in the probability of loss by increasing the premium, it turns out that the optimal purchase of insurance is likely to be decreasing in the probability of loss. This result compares with that of Jang and Hadar (1995), who consider the same problem in the traditional model (where the size of the risk is not a choice variable). In that paper, they show that for the most realistic case of decreasing absolute risk aversion, the effect of an increase in the probability of loss on the insurance decision cannot be signed. Here, the sign depends upon the size of relative risk aversion as it compares to 1, and for the most reasonable case of relative risk aversion uniformly greater than 1, the result is that the effect on insurance is the opposite to the case when the insurer does not increase the premium in response to the increase in probability. What this implies is that, when the probability of loss increases, and the premium increases as well, then the effect of the premium increase on the demand for insurance is greater than the effect of the probability increase when relative risk aversion is greater than 1.

4 Conclusions

This paper offers a simple economic model, that allows the comparative statics of insurance demand to be easily considered. The two-state setting used here mimics the application of the composite commodity theorem of Meyer and Meyer (2004), but allows for more clear-cut results to be found. The principal point of departure of the theory given here and more traditional theory (referred to here as the standard model) is that in this model we allow the decision maker to simultaneously choose the amount of risky goods and the amount of insurance coverage under a budget constraint. This very natural extension to the traditional approach yields several interesting comparative statics results.

In the model it turns out that insurance is a normal good at all wealth levels for a wide range of risk aversion characteristics. In particular, even if the utility function displays decreasing absolute risk aversion, an increase in wealth can still increase the amount of insurance. It depends on the rate at which risk aversion decreases, compared to the proportion of the consumption good that is at risk.

We have also shown that an increase in the premium will decrease the purchase of insurance for an even wider range of utility functions. It turns out that it is not unlikely that an increase in the unit price of insurance will lead to a decrease in the purchase of insurance and an increase in the purchase of the risky good, thus insurance and risk can be termed "net substitutes", in terms of traditional demand theory.

We have also shown that the intuitively logical result that an increase in risk aversion will both increase the amount of insurance coverage and decrease the amount of risk purchased also holds in the model. The need to reduce risk is met by two simultaneous movements that achieve the required end – less risk is purchased and it is more heavily insured.

Further, we have also shown that both an increase in the proportion of the risky good that is at risk and an increase in the probability of loss that is not accompanied by an increase in the premium will have the effect of decreasing the amount of risky good purchased, and increasing the amount of insurance coverage purchased. That is, these types of increase in risk will again be countered by two simultaneous risk reduction strategies – less risk will be purchased, and it will be more heavily insured. The fact that in this setting an increase in the probability of loss increases the purchase of insurance is also found in the standard model. On the other hand, for the special case of a full loss in state 2, if the insurer increases the premium after an increase in the probability of loss, conditional upon relative risk aversion being not less than 1, the optimal purchase of risky good will still decrease, and the effect on the optimal purchase of insurance coverage depends upon the value of absolute risk aversion. However, for the particularly interesting case of an insurer who sets the premium at the probability of loss multiplied by a constant loading factor, we do get a particularly clean result as far as the effect on insurance of an increase in the probability of loss. In contrast to what is found in the standard model (the effect can only be signed if the utility function displays non-decreasing absolute risk aversion, and then it is negative i.e. insurance decreases), we find here that the sign of the effect depends upon the comparison between relative risk aversion and the number 1. Concretely, for the interesting case of relative risk aversion greater than 1, it turns out that (conditional upon a competitive insurance environment) an increase in the probability of loss will decrease the optimal purchase of insurance. That is, the effect of the premium increase outweighs the effect of the increase in probability.

The model studied in the paper suggests several interesting questions. To begin with, when the individual has a need to reduce the risk of his optimal decision due to an increase in risk aversion, we know that risk will be reduced by two simultaneous strategies; x will be reduced and c will be increased. This, however, begs the question of exactly how the split will occur, that is, how much risk will be reduced by decreasing the purchase of the risky good, and how much will be reduced by increasing insurance coverage?[13]

Secondly, an interesting problem presents itself if we consider exactly how the risky asset and insurance coverage would indeed be priced if, for example, each was supplied by a monopolist. One can imagine the first order condition for an expected profit maximum by each supplier, and how each of their optimal decisions depends upon the decision of the other through the consumer's demands for each product. Using, for example, a standard Cournot type model, it would be interesting to study this price setting game, making use of the comparative statics that the current model provides.

Thirdly, it would be interesting to remove the linear nature of the indemnity without removing the discrete nature of the model by simply adding more states of nature. In such a model we could also differentiate between proportional insurance and insurance subject to a deductible.

Finally, the model has clear finance applications, which are interesting to explore. In a finance setting, the prices of the two goods (which would now be termed "assets") would have to be set taking into account a no-arbitrage condition such that there is no benefit from borrowing a unit of wealth, and then purchasing a fully insured unit of the risky asset (which also returns a unit of wealth). In such a model, at least one of the two prices (either that of insurance, or that of the risky good) becomes endogenous, and this will have particularly interesting effects when the comparative statics of such things as the proportion of the risky portfolio that may be lost, or the

[13] The same question for the case of an increase in the probability of loss is answered in the first two equations in the proof of Proposition 5.

probability of the loss state, are considered.

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Appendix

Proof of proposition 4:

The derivative of the tangency condition (5) with respect to a is:

$$\frac{u''(x^*)\frac{\partial x^*}{\partial a}u'(y^*) - u'(x^*)u''(y^*)\frac{\partial y^*}{\partial a}}{u'(y^*)} = u'(y^*)\left(\frac{p}{1-p}\right)$$

Using the tangency condition to substitute for the marginal utility of state 1 consumption on the right-hand-side, and simplifying, we get

$$R_a(y^*)\frac{\partial y^*}{\partial a} - R_a(x^*)\frac{\partial x^*}{\partial a} = \frac{\pi}{(v-\pi(1-a))} > 0$$

Thus, we have $R_a(y^*)\frac{\partial y^*}{\partial a} > R_a(x^*)\frac{\partial x^*}{\partial a}$, from which we must rule out the possibility that $\frac{\partial y^*}{\partial a} < 0 < \frac{\partial x^*}{\partial a}$.

On the other hand, differentiating the budget constraint (4) we get

$$\begin{aligned} 0 &= \frac{\partial q}{\partial a}x + q\frac{\partial x^*}{\partial a} + \pi\frac{\partial y^*}{\partial a} \\ &= \pi x^* + q\frac{\partial x^*}{\partial a} + \pi\frac{\partial y^*}{\partial a} \end{aligned}$$

that is $q\frac{\partial x^*}{\partial a} + \pi\frac{\partial y^*}{\partial a} = -\pi x^* < 0$. Thus it is not possible that $\frac{\partial x^*}{\partial a} > 0$ and $\frac{\partial y^*}{\partial a} > 0$ simultaneously; either one is negative and the other positive, or both are negative. But since we have just shown that we cannot have $\frac{\partial y^*}{\partial a} < 0 < \frac{\partial x^*}{\partial a}$, it turns out that either both are negative or it holds that $\frac{\partial x^*}{\partial a} < 0 < \frac{\partial y^*}{\partial a}$. Thus we conclude that $\frac{\partial x^*}{\partial a} < 0$. Now, if we go back to the budget constraint expressed in terms of the purchase of x and c, that is $w = vx^* + \pi c^*$, and we differentiate it with respect to a, we get $0 = v\frac{\partial x^*}{\partial a} + \pi\frac{\partial c^*}{\partial a}$. But since $\frac{\partial x^*}{\partial a} < 0$ we must have $\frac{\partial c^*}{\partial a} > 0$.

Proof of proposition 5:

At an internal solution, ($\pi > \frac{pv}{1-ap}$) the solution for the case of $v(\cdot) \equiv h(u(\cdot))$ is given by the duly adjusted versions of (1) and (2), which taking into account the internal solution assumption (i.e. $\delta_2 = 0$), and denoting the new solution by $(\widetilde{x}^*, \widetilde{y}^*)$ we have

$$(1-p)h'(u(\widetilde{x}^*))u'(\widetilde{x}^*) = (v - \pi(1-a)\delta_1$$

and

$$ph'(u(\widetilde{y}^*)u'(\widetilde{y}^*) = \pi\delta_1$$

Dividing the first of these by the second, and simplifying, we have the tangency condition in the equilibrium expressed in terms of state contingent consumption;

$$\frac{h'(u(\widetilde{x}^*))u'(\widetilde{x}^*)}{h'(u(\widetilde{y}^*)u'(\widetilde{y}^*)} = \frac{p(v-\pi(1-a))}{(1-p)\pi}$$

However, from the original problem, we also know that

$$\frac{u'(x^*)}{u'(y^*)} = \frac{p(v - \pi(1-a))}{(1-p)\pi}$$

so we have

$$\frac{h'(u(\widetilde{x}^*))u'(\widetilde{x}^*)}{h'(u(\widetilde{y}^*)u'(\widetilde{y}^*)} = \frac{u'(x^*)}{u'(y^*)}$$

Now, since we also know that in the new solution we have partial coverage (Proposition 1), then we also have a lower consumption in state 2 than in state 1, $\widetilde{x}^* > \widetilde{y}^*$, and since u is an increasing function and h is increasing and concave, it holds that $h'(u(\widetilde{y}^*)) > h'(u(\widetilde{x}^*))$, so that

$$\frac{u'(\widetilde{y}^*)}{u'(\widetilde{x}^*)} < \frac{u'(y^*)}{u'(x^*)} \tag{22}$$

Therefore we can definitely rule out the possibility that $\widetilde{x}^* = x^*$ and $\widetilde{y}^* = y^*$.

Finally, from the fact that in both solutions the budget constraint saturates, we know that $(v - \pi(1-a))\widetilde{x}^* + \pi\widetilde{y}^* = (v - \pi(1-a)x^* + \pi y^*$. Hence, it can never be true that the increase in risk aversion leads to an increase in both choice variables, or to a decrease in both - i.e. one must increase and the other must decrease. However, (22) makes it clear that we can also rule out the possibility that $\widetilde{y}^* < y^*$ and $\widetilde{x}^* > x^*$ since that would lead to a decrease in marginal utility in the no-loss state and an increase in the marginal utility in the loss state, that is an increase in the ratio of the two. Thus, we can safely conclude that the increase in risk aversion leads to a decrease in state 1 consumption (x), and an increase in state 2 consumption (y).

Finally, since $y = (1-a)x + c$, and since x decreases the only way y can increase is if c increases.

Proof of proposition 6:

Again, begin with the tangency condition in state contingent consumption space (5). Differentiating with respect to p gives

$$\frac{u''(x^*)\frac{\partial x^*}{\partial p}u'(y^*) - u'(x^*)u''(y^*)\frac{\partial y^*}{\partial p}}{u'(y^*)} = u'(y^*)\left[\frac{(v - \pi(1-a))(1-p)\pi + p(v - \pi(1-a))\pi}{(1-p)^2\pi^2}\right]$$

The term in brackets on the right-hand-side simplifies immediately so that this equation reads

$$\frac{u''(x^*)\frac{\partial x^*}{\partial p}u'(y^*) - u'(x^*)u''(y^*)\frac{\partial y^*}{\partial p}}{u'(y^*)} = u'(y^*)\left[\frac{(v - \pi(1-a))}{(1-p)^2}\right]$$

Using the tangency condition to substitute for the marginal utility of state 1 consumption on the right-hand-side and simplifying, we get

$$R_a(y^*)\frac{\partial y^*}{\partial p} - R_a(x^*)\frac{\partial x^*}{\partial p} = \frac{\pi}{(1-p)p} > 0$$

Thus we know that

$$R_a(y^*)\frac{\partial y^*}{\partial p} > R_a(x^*)\frac{\partial x^*}{\partial p} \tag{23}$$

However, from the budget constraint (4), we also know that $0 = (v - \pi(1-a)\frac{\partial x^*}{\partial p} + \pi\frac{\partial y^*}{\partial p}$. But since both of the (net) state contingent prices are positve, this indicates that the two derivatives must have opposite sign. Using (23), the only option is $\frac{\partial y^*}{\partial p} > 0 > \frac{\partial x^*}{\partial p}$. Finally, since $\frac{\partial y^*}{\partial p} = (1-a)\frac{\partial x^*}{\partial p} + \frac{\partial c^*}{\partial p}$, and since $\frac{\partial x^*}{\partial p} < 0$, we must have $\frac{\partial c^*}{\partial p} > 0$.

Proof of Proposition 7:

In the proof of proposition 6 it was shown that

$$R_a(y^*)\frac{\partial y^*}{\partial p} - R_a(x^*)\frac{\partial x^*}{\partial p} = \frac{\pi}{(1-p)p}$$

When $a = 1$, we have $y^* = c^*$, so that $R_a(c^*)\frac{\partial c^*}{\partial p} - R_a(x^*)\frac{\partial x^*}{\partial p} = \frac{\pi}{(1-p)p}$. Together with the fact that $0 = v\frac{\partial x^*}{\partial p} + \pi\frac{\partial c^*}{\partial p}$, after some simplification, it can be shown that the effects on the two optimal variables, holding the premium constant are in fact the following;

$$\begin{aligned} \left.\frac{\partial x^*}{\partial p}\right|_{\pi} &= -\frac{\pi}{p(1-p)\left[\pi R_a(x^*) + vR_a(c^*)\right]} \\ \left.\frac{\partial c^*}{\partial p}\right|_{\pi} &= \frac{v}{p(1-p)\left[\pi R_a(x^*) + vR_a(c^*)\right]} \end{aligned}$$

On the other hand, when $a = 1$, (6) reads $c^* + v\frac{\partial x^*}{\partial \pi} + \pi\frac{\partial c^*}{\partial \pi} = 0$, and (7) reads $R_a(x^*)\frac{\partial x^*}{\partial \pi} - R_a(c^*)\frac{\partial c^*}{\partial \pi} = \frac{1}{\pi}$. The simultaneous solution to these two equations, again after some simplification, yields

$$\begin{aligned} \frac{\partial x^*}{\partial \pi} &= \frac{1 - R_r(c^*)}{\pi R_a(x^*) + vR_a(c^*)} \\ \frac{\partial c^*}{\partial \pi} &= -\frac{1}{\pi}\left[\frac{v + \pi c^* R_a(x^*)}{\pi R_a(x^*) + vR_a(c^*)}\right] \end{aligned}$$

Now, the full effects of an increase in the probability of loss are

$$\begin{aligned} \frac{\partial x^*}{\partial p} &= \left.\frac{\partial x^*}{\partial p}\right|_{\pi} + \frac{\partial x^*}{\partial \pi}\frac{\partial \pi}{\partial p} \\ \frac{\partial c^*}{\partial p} &= \left.\frac{\partial c^*}{\partial p}\right|_{\pi} + \frac{\partial c^*}{\partial \pi}\frac{\partial \pi}{\partial p} \end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\partial x^*}{\partial p} &= -\frac{\pi}{p(1-p)\left[\pi R_a(x^*)+vR_a(c^*)\right]}+\frac{1-R_r(c^*)}{\left[\pi R_a(x^*)+vR_a(c^*)\right]}\frac{\partial \pi}{\partial p} \\
\frac{\partial c^*}{\partial p} &= \frac{v}{p(1-p)\left[\pi R_a(x^*)+vR_a(c^*)\right]}-\frac{1}{\pi}\frac{v+\pi c^* R_a(x^*)}{\left[\pi R_a(x^*)+vR_a(c^*)\right]}\frac{\partial \pi}{\partial p}
\end{aligned}$$

Simplifying, we get

$$\frac{\partial x^*}{\partial p} = \frac{1}{\left[\pi R_a(x^*)+vR_a(c^*)\right]}\left[-\frac{\pi}{p(1-p)}+(1-R_r(c^*))\frac{\partial \pi}{\partial p}\right] \tag{24}$$

$$\frac{\partial c^*}{\partial p} = \frac{1}{\left[\pi R_a(x^*)+vR_a(c^*)\right]}\left[\frac{v}{p(1-p)}-\left(\frac{v+\pi c^* R_a(x^*)}{\pi}\right)\frac{\partial \pi}{\partial p}\right] \tag{25}$$

Since both prices and absolute risk aversion are all positive, the signs of each of the effects are the same as the signs of the terms in parenthesis. That is, after a minimal amount of simplification, we get

$$\begin{aligned}
\frac{\partial x^*}{\partial p} &\gtreqless 0 \;\; as \;\; (1-R_r(c^*))\frac{\partial \pi}{\partial p} \gtreqless \frac{\pi}{p(1-p)} \\
\frac{\partial c^*}{\partial p} &\gtreqless 0 \;\; as \;\; \frac{\partial \pi}{\partial p} \lesseqgtr \frac{v\pi}{p(1-p)(v+\pi c^* R_a(x^*))}
\end{aligned}$$

This proves the statement concerning the effect of an increase in p upon the optimal purchase of the risky asset. To get the effect on the optimal purchase of insurance coverage, note that $\pi c^* R_a(x^*) = v\left(\frac{\pi c^*}{v x^*}\right) R_r(x^*)$. Substituting this into the previous expression, and cancelling the common term (v) and carrying out a simple reordering yields the indicated result.

Proof of Corollary 6:

Note that, from the tangency condition at the optimum we know that

$$\frac{pv}{(1-p)\pi} = \frac{u'(x^*)}{u'(c^*)} \quad \Rightarrow \quad \frac{\pi}{v} = \frac{pu'(c^*)}{(1-p)u'(x^*)}$$

so it turns out that

$$\frac{\pi c^*}{v x^*} = \left(\frac{p}{(1-p)}\right)\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right)$$

Thus the condition can be written as

$$\frac{\partial c^*}{\partial p} \gtreqless 0 \;\; as \;\; (1-p)\left(1+\left(\frac{p}{(1-p)}\right)\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right)R_r(x^*)\right)\frac{\partial \pi}{\partial p} \lesseqgtr \frac{\pi}{p}$$

that is

$$\frac{\partial c^*}{\partial p} \gtreqless 0 \;\; as \;\; \left((1-p)+p\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right)R_r(x^*)\right)\frac{\partial \pi}{\partial p} \lesseqgtr \frac{\pi}{p}$$

Note that the term in parentheses on the right-hand-side of the condition is a convex combination of the number 1 and the number $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right) R_r(x^*)$. However, consider the function $u'(z)z$. Its slope is $u''(z)z + u'(z)$, and so $u'(z)z$ is increasing (decreasing, constant) as $u''(z)z + u'(z) \gtreqless 0$, that is, as $R_r(z) \lesseqgtr 1$. For the case at hand, i.e. relative risk aversion equal to 1, it turns out that $u'(c^*)c^* = u'(x^*)x^*$, and so we get $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right) R_r(x^*) = 1$. This means that the convex combination term on the right-hand-side of the condition is exactly equal to 1, which gives the required result.

Proof of Corollary 7:

Clearly, when $\pi = kp$, we get $\frac{\partial \pi}{\partial p} = k = \frac{\pi}{p}$. Substituting this into the condition leads to

$$\frac{\partial c^*}{\partial p} \gtreqless 0 \ \ as \ \ \left((1-p) + p\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right) R_r(x^*)\right) \lesseqgtr 1$$

which in turn implies

$$\frac{\partial c^*}{\partial p} \gtreqless 0 \ \ as \ \ \left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right) R_r(x^*) \lesseqgtr 1$$

However, it was shown in the previous corollary that $u'(z)z$ is increasing (decreasing, constant) as $u''(z)z + u'(z) \gtreqless 0$, that is, as $R_r(z) \lesseqgtr 1$. Hence, since we know that for the case at hand $x^* > c^*$, it turns out that

$$\frac{u'(c^*)c^*}{u'(x^*)x^*} \lesseqgtr 1 \ \ as \ \ R_r(z) \lesseqgtr 1$$

In short then, if $R_r(z) > 1$, we get $\frac{u'(c^*)c^*}{u'(x^*)x^*} > 1$, and consequently $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*}\right) R_r(x^*) > 1$ which implies that $\frac{\partial c^*}{\partial p} < 0$. An identical reasoning suffices to prove the other two cases.