Optimal Redistributive Policy in Debt Constrained Economies

Monica Tran Xuan

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Abstract

This paper studies optimal taxation, as a redistributive policy, in an open economy facing long-run binding debt constraints and inequality. The debt constraints arise endogenously from the government’s limited commitment, and become relevant in the long run due to the impatience of domestic agents. The limited borrowing distorts the intratemporal substitution between consumption and leisure, induces under-accumulation of capital, and discourages domestic borrowing. The redistribution concern affects the tax levels via the trade-off between equity and efficiency, and the tax dynamics via tightening/relaxing the debt constraints. The optimal labor tax is constant with unconstrained borrowing, but eventually converges to a real limit that depends on inequality and distributional preference. The efficient contract features front-loading redistribution and back-loading efficiency, allowing the economy to accumulate a large external debt position, and increase its borrowing capacity when the debt constraints bind.

Keywords: Optimal taxation; Redistribution; Limited commitment; Sovereign debt

JEL Classifications: F34; F38; H21; H23; H63

*Department of Economics, University of Minnesota and Federal Reserve Bank of Minneapolis. Email: tranx888@umn.edu. The author is indebted to Manuel Amador and V.V. Chari for their invaluable advice and guidance. The author is grateful for comments from Anmol Bhandari, Larry Jones, Chris Phelan, and David Rahman. The author also benefits from discussions with Fernando Arce, the seminar participants at University of Minnesota, the Federal Reserve Bank of Minneapolis, the Federal Reserve Bank of Richmond, and editing work by John Arechavala. The author also acknowledges the support from the 2017 AEA Summer Economics Fellowship. The views express therein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis and the Federal Reserve System.
1 Introduction

The recent debt crises featured a rapid accumulation of external debt in the periods leading up to the crises. At the same time, highly-indebted countries, such as Greece, Portugal, and Spain, also experienced high levels of income inequality. According to the European Union’s Survey of Income and Living Conditions, the Gini coefficients and income quintile share S80/S20 ratios in these countries were higher than the average across the EU-27 countries. Several papers, including Berg and Sachs (1988); Aizenman and Jinjarak (2012); Ferriere (2014); Jeon et al. (2014), have documented the connection between inequality and sovereign debt\textsuperscript{1}.

The running up of debt led to those countries facing high default risk such that they could not roll over their debt. An explanation for such situations was the lack of commitment of the government. Since international lenders and private agents could not perfectly enforce the contracts, the government had incentive to renge on its external obligations and deviate from the promised tax plans, especially when borrowing was limited. In fact, there were large increases in tax rates, reductions in welfare transfers and government spending, during such periods.

On the other hand, high inequality in the economy might motivate the government to care about redistribution. Redistributive policies such as distortionary taxes and transfers are commonly used to achieve a social distributional preference. For example, with skill inequality, a positive marginal tax rate on labor income means that the more productive, richer the agent, the heavier their tax burden. Therefore, governments that have high preferences towards the less productive, poorer agents might find it optimal to keep marginal tax rates on labor high.

Redistribution and international borrowing are connected from the perspective of setting optimal fiscal policies. Taxes and transfers determine not only the government’s budget and debt management, but also the distributions of consumption, labor, and utility across households. Hence, the desire for redistribution can have a significant impact on the government’s ability to raise revenue, which in turn impacts its ability to repay debt. Reversely, when borrowing is constrained, it hinders a country’s ability to redistribute.

Motivated by these observations, this paper aims to answer the following normative question. What are the optimal characteristics of redistributive policies, in terms of taxes and

\textsuperscript{1}Berg and Sachs (1988) showed that income inequality was a key predictor of a country’s probability of rescheduling debt and the bond spread in secondary markets. Aizenman and Jinjarak (2012) described a negative correlation between income inequality and the tax base and a positive correlation with sovereign debt. Recently, Ferriere (2014); Jeon et al. (2014) also provided empirical evidence of rising inequality significantly increases sovereign default risk.
transfers, when the government faces binding sovereign debt constraints and inequality? The approach is an environment of a small open economy with impatient domestic agents that are differentiated by labor productivity, and the government has lack of commitment. Impatience implies the need to accumulate debt over time, while the lack of commitment imposes endogenous debt constraints which will be binding in the long run. The heterogeneity in labor productivity creates income inequality and social incentive to redistribute. The paper also explicitly models distributional preference by social weights across individual utilities, and allows for a rich tax structure with lump-sum transfers.

The model uses impatience, a standard assumption in the sovereign debt literature, as an explanation why countries accumulate debt over time. Impatience means that the intertemporal discounting rate of future utilities is lower than the international intertemporal price of resources. As it is more costly to save over time, the countries prefer to increase their borrowing, front load their consumption, and back load their labor supply.

The endogenous debt constraints arise from a sovereign game between the private agents, international lenders, and the government. When the government deviates from its policy, it triggers punishment to financial autarky, where the government cannot access to both domestic and international lending markets. In each period, the sub-game perfect equilibrium that reverts to autarky can be characterized by the enforcement (sustainability) constraints, similar to the approach in Chari and Kehoe (1990, 1993). These sustainability constraints are such that the future discounted welfare of the whole economy must be higher or equal to the deviation utility in each period. They act as endogenous aggregate debt constraints that the government faces when choosing its policies.

The paper focuses on the dynamics and levels of tax policies, with a stronger emphasis on the labor income tax. The trade-off between equity and efficiency determines the level of distortions across time. At the same time, binding debt constraints generate an additional distortion on future intertemporal and intratemporal conditions. The impatient agents would like to front load consumption and leisure, but eventually are constrained to do so. As a result, the optimal labor tax is constant at first, but eventually converges to a real limit. More interestingly, the limit varies with respect to the skill distribution and social distributional preference. These results hold under the assumptions of separable and isoelastic preference, interior allocation, and a benevolent government.

The implication on the optimal path of labor tax arises because, at first, when the sustainability constraints do not bind, there are high benefit of redistribution and no cost of borrowing. The consumption-labor distortion only comes from inequality induced by the skill distribution, which is constant over time. It is then optimal for the government to use debt and lump-sum transfers to smooth finances over time and keep the distortions equal
to the level that maximizes the redistribution benefit. When the sustainability constraints bind, borrowing becomes costly. The government wants to adjust the tax rates to lower the efficiency cost of delivering the promised utility, relax the constraints, and increase the debt capacity of the whole economy. In the long run, the efficient allocation minimizes the cost of delivering the deviation utility, and maximizes the net payment to the international financial markets. The labor tax converges to a limit that associates with the maximal sustainable debt level of the whole economy. The skill distribution and distributional preference influence both sides of the debt constraints, so they affect the endogenous level of the maximum aggregate debt, which in turn determines the tax limit. In this way, primitives determine the long-run steady-steady of the economy under optimal taxation.

Generally, a higher redistribution towards the lower-skilled, poorer workers or higher inequality is associated with higher labor tax rates during the unconstrained borrowing periods, and potentially either higher or lower labor tax limits in the long run. It is because the marginal benefit of redistribution is high in both cases. When borrowing is unlimited, the government finds it optimal to equate the marginal benefit of redistribution and the marginal cost of distortion, and hence it has an incentive to keep high marginal tax rates to redistribute more. However, higher distortion also implies lower output and lower debt capacity. When the economy is debt constrained, the government has an additional motive to lower the efficiency cost. Starting at a low inequality level, the redistribution benefit can dominate the efficiency cost in the long run, so labor tax limit is increasing with respect to inequality. For high a enough inequality, the efficiency cost can eventually be higher than the redistribution benefit, so it is optimal to decrease the labor tax limit to encourage more output and increase the economy’s debt capacity.

The paper extends the sub-market analysis in Werning (2007), which studied the optimal redistribution in a closed economy, to a small open economy with limited commitment. The sub-market solution is incorporated in the debt constraints. Since all agents face the same intratemporal and intertemporal distortions in each period, the competitive equilibrium generates an efficient assignment of individual allocation, which is captured by a set of market weights that determines individual utility shares. This analysis also provides a key insight of the redistribution effect on the optimal tax limit. Specifically, when the market distribution, represented by the market weights, is identical to social distribution, implied by the exogenous social weights, there is no consumption–labor distortion in the long run. As a result, the optimal labor tax converges to zero, as in Aguiar and Amador (2016). The long-run intratemporal distortion only arises from the differences between market and social distributions.

Furthermore, in each period, the government can deviate and expropriate all capital stock
to operate on its own. To discourage the government to do so, there exists a positive tax rate on capital such that the efficient level of capital is lower than the level when there are no debt constraints. Thus, the optimal capital tax does not converge to zero in the long run. As the debt constraints bind, there is an implicit tax on domestic borrowing to discourage domestic debt accumulation.

The paper also shows a numerical exercise of a simple environment with a parametric separable isoelastic preference, an utilitarian government, and no capital. There is an exogenous autarkic deviation value as a lower bound on weighted future utility. Once the endogenous debt constraint binds, it will bind afterwards. When there is inequality, the optimal labor tax is positive when the debt constraint does not bind, implying the redistribution motive. As the debt constraint starts binding, the labor tax declines to lower the efficiency cost. There is a potential of subsidizing labor in the limit. The economy rapidly accumulates debt before hitting the constraints, and slowly increases its debt capacity when the constraints bind up to the maximum level in steady states. In addition, the paper provides a comparative static analysis of the tax rates and external debt with respect to inequality, measured as the ratio of individual labor productivities. Increasing in relative inequality implies rising unconstrained-debt levels of labor tax and a hump-shaped labor tax limits. The labor tax limit is initially increasing with respect to inequality, reflecting that the gain from redistribution is higher than the cost of long-run distortion. Nevertheless, labor tax limit starts declining when inequality is high, implying that the efficiency cost is dominating. When the debt constraints start binding, the return on domestic savings is higher the higher inequality is, but eventually converge to the steady-state non-zero rates. The external debt levels also change with respect to inequality, in which for the higher the inequality, the longer the periods of unconstrained borrowing, and the higher external debt positions in the long run. The reason is that the highly-unequal economy has a high motive for redistribution and is willing to have more periods of unconstrained borrowing to redistribute. The cost of a lot of redistribution now is the high debt accumulation later on.

When extending the analysis to separable preferences, few results change. First, the labor tax is not smooth even under unconstrained borrowing. This is due to the time-varying elasticities. The long-run convergence property of the labor tax is similar to the case of constant elasticities. If the economy converges to a steady state, the optimal labor tax also converges to a real constant. However, if steady-state allocation does not exist, the long-run optimal labor tax fluctuates in a bounded region. Similar to the case of separable isoelastic preference, this region corresponds to the region of the maximum debt capacity of the economy.
Literature Review. This research contributes to the literature of public finance and international macroeconomics.

First, the research is related to works that analyze government’s fiscal and debt policy over time (Chari and Kehoe (1999); Lucas and Stokey (1983); Aiyagari and McGrattan (1998); Aiyagari et al. (2002), and many other papers). This paper finds optimal policy by characterizing the best allocation of any tax-distorted debt constrained equilibrium, i.e. the primal approach with debt constraints. The argument for tax smoothing is consistent with findings from Werning (2007) for a deterministic model, and is related to results developed by Barro (1979); Lucas and Stokey (1983); Chari et al. (1994). The positive capital and borrowing taxes are contrast to the zero convergence of the capital tax from Judd (1985); Chamley (1986); Chari et al. (1994); Straub and Werning (2014), because here capital and borrowing have an externality in influencing the debt constraints.

The paper also relates to the sovereign debt literature. Several papers studied sovereign debt in the limited commitment environment (Eaton and Gersovitz (1981); Arellano (2008); Aguiar and Amador (2011, 2014)). There are a few recent works on optimal policies in the Eaton-Gersovitz-Arellano framework (Arellano and Bai (2016); Ferriere (2014)). This paper instead focuses on the sovereign debt model with limited commitment and self-enforcing contract as in Aguiar and Amador (2011, 2014).

The dynamic environment in this paper is an extension to one in Aguiar and Amador (2016), adding heterogeneity, distribution motive, and allowing for richer tax systems. Aguiar and Amador (2016) found that labor tax must go to zero in the long run as a result of front-loading efficient consumption and leisure allocation. In this paper, the tax limit can be any real value. More interestingly, when turning off the redistribution effect in the model, the limit of labor tax is zero, consistent with their findings. The paper shows that redistribution consideration, not heterogeneity, is the main source for the changes in optimal policies.

Work in optimal taxation with inequality and redistribution includes Werning (2007); Bhandari et al. (2016), which both found that redistribution had significant impact on optimal policies. This paper instead explores redistribution in a small open economy setting and endogenous aggregate debt constraints. Werning (2007) developed the conditions for perfect tax smoothing, while Bhandari et al. (2016) emphasized the impact of the distribution of initial asset holdings on optimal allocation. The framework and analysis in this paper are closely related to Werning (2007). The finding is that labor tax smoothing only occurs when the debt constraint does not bind. Long-run binding debt constraints then alters the dynamic of the labor tax, resulting in imperfect tax smoothing. Moreover, the initial distribution of after-tax net asset holdings matters in determining the private market shares across agents, which indirectly affect the labor tax limit.
Several recent papers addressed the trade-off between redistribution and external debt. D’Erasmo and Mendoza (2016) focused on how redistribution incentives affected defaults on domestic debt. They asserted that equilibrium with debt could be supported only when the government was politically biased towards bond holders. Ferriere (2014) showed how modifying tax progressiveness could mitigate the cost of default. Dovis et al. (2016) argued how the interaction between inequality and debt endogenized the dynamic cycles of debt, taxes, and transfers over time. Balke and Ravn (2016) studied time-consistent fiscal policy in a sovereign debt model à la Eaton and Gersovitz (1981) with inequality through unemployment. They found that austerity policies were optimal during debt crises since they reduced default premium, which was correlated with debt issuance, and increased access to international lending market. This paper instead emphasizes on the sustainability constraints arising from a self-enforcing contracting problem among the government, international lenders, and domestic agents, with the equilibrium that is triggered by financial autarky. These constraints restrict the weighted future utility, which act like endogenous debt constraints. The paper features the long-run binding debt constraints due to the private agent’s impatience. When the debt constraints bind, austerity policies might not be optimal if they generate more distortion. The analysis is consistent with any general distributional preference, instead of a particular welfare function. The redistribution motive determines both the dynamic and limiting value of taxes by affecting the debt constraints.

Outline. The paper is organized as follows. Section 2 describes the environment, the competitive equilibrium, and the lack of commitment problem. Section 3 characterizes the equilibrium. Section 4 formulates the efficiency problem, while section 5 derives the main results of the optimal policies. Section 6 analyzes the effect of inequality and redistribution on optimal taxes. Section 7 shows the numerical exercise, and Section 8 generalizes the optimal tax formulas with separable preferences. Section 9 then concludes.

2 Model

2.1 Environment

A small open economy faces exogenous world interest rates \( \{r^*_t\}_{t=0}^{\infty} \). There is a measure-one continuum of infinitely-lived agents different by labor productivity types \( (\theta^i)_{i \in I} \), which are publicly observable. The fraction of agents with productivity \( \theta^i \) is \( \pi^i \), where \( (\pi^i)_{i \in I} \) is normalized such that \( \sum_{i \in I} \pi^i \theta^i = 1 \). All agents have the same discount factor \( \beta \) and the static utility \( U(c, n) \) over consumption \( c \) and hours worked \( n \). The utility of agent with
productivity $\theta^i$ over consumption $c^i_t \geq 0$ and efficient labor $l^i_t \geq 0$ is

$$\sum_{t=0}^{\infty} \beta^t U^i(c^i_t, l^i_t)$$

(1)

where $U^i(c, l) = U(c, \frac{l}{\theta^i})$.

In addition, there is a representative firm that uses both capital and labor to produce a single output good. The production function $F(K, L)$ is constant return to scale, where $K$ and $L$ are respectively the aggregate capital and labor. The economy is subject to an exogenous sequence of government spending $\{G_t\}_{t=0}^{\infty}$. In each period, the government issues domestic and foreign bonds, imposes a lump-sum tax $T_t$, a marginal tax on labor income $\tau^n_t$, a marginal tax on capital income $\tau^K_t$, and set the return on domestic bond $r_t$. Assume that only the government can borrow abroad$^2$.

### 2.2 Equilibrium

Consider the set of prices facing by agents and firm: $w_t$ the labor wage, $r^K_t$ the return on capital, and $\delta$ the capital depreciation rate.

**Agents.** Agent of type $i \in I$ faces the sequential budget constraint in period $t$,

$$c^i_t + k^i_{t+1} + b^{d,i}_{t+1} \leq (1 - \tau^n_t) w_t l^i_t + [1 + (1 - \tau^K_t) r^K_t - \delta] k^i_t + (1 + r_t) b^{d,i}_t - T_t$$

(2)

where $c^i_t, l^i_t, k^i_t, b^{d,i}_t$ denote the consumption, effective labor, capital holding, and domestic bond holding of agent $i$ in period $t$, respectively.

Moreover, no arbitrage exists such that the after-tax return is the same when investing in capital or in domestic bonds:

$$1 + (1 - \tau^K_t) r^K_t - \delta = 1 + r_t$$

which implies $(1 - \tau^K_t) r^K_t = r_t + \delta$.

**Representative Firm.** The firm chooses the amount of capital and labor to maximize profit each period:

$$\max_{\{K_t, L_t\}} F(K_t, L_t) - w_t L_t - r^K_t K_t$$

$^2$This assumption is standard in the sovereign debt literature, based on the empirical observation that domestic households often hold a very small amount of foreign assets.
which gives the first-order conditions:

\[ r^k_t = F_K(K_t, L_t) \]

\[ w_t = F_L(K_t, L_t) \] (3)

Note that profits are zero in equilibrium because of the constant return to scale production function.

**Government.** The government needs to finance an exogenous expenditure \( \{G_t\}_{t=0}^{\infty} \). The government sells one-period bond \( B^d_t \) to domestic agents and \( B_{t+1} \) to the international lenders at a price \( Q_{t+1} \). The government’s budget constraint in each period is

\[ G_t + (1 + r_t)B^d_t + B_t \leq \tau^n_t w_t L_t + \tau^k_t r^k_t K_t + B^d_{t+1} + Q_{t+1}B_{t+1} + T_t \]

where \( L_t = \sum_{i \in I} \pi^i L^i_t \) is the aggregate labor, \( K_t = \sum_{i \in I} \pi^i k^i_t \) is the aggregate capital, \( B^d_t = \sum_{i \in I} \pi^i b^d_{t,i} \) is the aggregate domestic bond, and \( B_t \) is the amount of the government’s external debt. The government faces a no-Ponzi condition such that the present value of external debt is bounded below.

Define \( q_t \) as international price of a unit period-\( t \) consumption in terms of period-0 consumption units:

\[ q_t = \Pi_{s=0}^{t} \frac{1}{1 + r^*_{s}} \] (4)

Optimality of the risk-neutral international lenders’ problem gives \( Q_t = \frac{1}{1 + r^*_{t}} \). Using \( \{q_t\}_{t=0}^{\infty} \) as the relevant inter-temporal price, one can write the government’s present-value budget constraint as

\[ \sum_{t=0}^{\infty} q_t \{G_t - \tau^n_t w_t L_t - \tau^k_t r^k_t K_t + (1 + r_t) B^d_t - B^d_{t+1} - T_t\} \leq -B_0 \] (5)

with normalizing \( 1 + r^*_0 = 1 \) ³

**Aggregate resource constraint.** In a small open economy, markets do not have to clear in every period. However, using the agent’s budget constraints and government’s budget constraint, one can obtain an aggregate resource constraint in terms of the intertemporal

³This assumption is without loss of generality to fix the initial level of external debt.
prices and the initial external debt:

\[
\sum_{t=0}^{\infty} q_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t] - B_0 \geq 0
\]  

(6)

**Competitive equilibrium.** Given the above equations, one can define the following competitive equilibrium with taxes.

**Definition 2.1.** Given initial external debt \(B_0\) and individual wealth positions \((a_{i0})_{i \in I}\), a competitive equilibrium with taxes for an open economy, is agent’s allocation \(z^{H,i} = \{c^i_t, l^i_t, k^i_{t+1}, b_{t+1}^{d,i}\}_{t=0}^{\infty}, \forall i \in I\), the firm’s allocation \(z^F = \{K_t, L_t\}_{t=0}^{\infty}\), prices \(p = \{q_t, Q_t, w_t, r_t, r^k_t\}_{t=0}^{\infty}\), and government’s policy \(z^G = \{\tau^n_t, \tau^k_t, T_t, r_t, B_{t+1}^d, B_{t+1}^d\}_{t=0}^{\infty}\) such that (i) given \(p\) and \(z^G\), \(z^{H,i}\) solves agent’s problem that maximizes (1) subject to (2) and a no-Ponzi condition of agent’s debt value, (ii) given \(p\) and \(z^G\), \(z^F\) solves firm’s problem, which implies the first-order conditions (3), (iii) government budget constraint (5) holds, (iv) aggregate resource constraint (6) is satisfied, \(\sum_{i \in I} b_{t}^{d,i} = B_t^d\), (iv) no arbitrage condition \((1 - \tau^k_t)r^k_t = r_t + \delta\), and (v) \(p\) satisfies (4) given \(z^G\).

### 2.3 Lack of commitment

Assume that the government is benevolent in that its objective is the weighted discounted utility of all agents in the economy, given by a set of social welfare weights \(\lambda = (\lambda^i)_{i \in I}\).

The government enters into contracts with private agents and foreign creditors that specify allocation of consumption, capital, labor, domestic and foreign bonds. Nevertheless, in every period, the government can drop its external and domestic obligations, change the tax schedules, and expropriate all capital holdings. When deviating from the contracted allocation, the government faces the punishment from private agents and foreign lenders. The government then receives a deviation utility \(U_t(K_t)\) that depends on the current aggregate capital level that it can expropriate. The limited commitment implies that there exists a lower bound on future discounted aggregate utility. Specifically, for all \(t \geq 0\),

\[
\sum_{s=t}^{\infty} \beta^{s-t} \sum_{i \in I} \lambda^i \pi^i U^i (c^i_s, l^i_s) \geq U_t(K_t)
\]  

(7)

Following Chari and Kehoe (1990, 1993), this sustainability constraint is a characterization of a subgame perfect equilibrium of a dynamic sovereign game between the government.

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4 \(a_{i0} \equiv \left[1 + (1 - \tau^k_0)r^k_0 - \delta\right] k_i^0 + (1 + r_0) b_{0}^{d,i}\)

5 For the relationship between the welfare weights and the utility frontier, see Section 4.
private agents, and foreign creditors. The equilibrium sustains risk-free debt and no default on path. Appendix A provides the details of the game and the equilibrium definition. In general, the set of sustainable equilibrium payoffs of this sovereign game can be supported by trigger strategies to the equilibrium that has the worst payoff. Due to the complication in characterizing the worst equilibrium of this dynamic game, this paper uses autarky, in which there are no international and domestic financial markets, as the punishment for deviation. This assumption does not change the main results of the paper. The reverting-to-autarky equilibrium is characterized by the constraint (7) where $U_t(K_t)$ incorporates the autarky value. It is a constraint on aggregate allocation such that private agents do not directly take into account when solving their problems. The constraint imposes endogenous limits on the aggregate debt levels over time.

3 Characterizing Equilibrium

In equilibrium, because all agents have the same preference, facing the same tax rates, earn the same wage on their efficient labor units, and have the same return on savings, the intratemporal and intertemporal conditions are the same across agents, i.e. in each period $t$, for all $i$,

$$\begin{aligned}
(1 - \tau_t^n)w_t &= -\frac{U_t^i(c_t^i, l_t^i)}{U_t^i(c_t^i, l_t^i)} \\
1 + r_{t+1} &= \frac{U_t^i(c_t^i, l_t^i)}{\beta U_t^i(c_{t+1}^i, l_{t+1}^i)}
\end{aligned}$$

Given the aggregate allocation $(C_t, L_t)$ in every period, there is an efficient assignment of individual allocation $(c_t^i, l_t^i)_{i \in I}$ due to the equal marginal rates of substitution between consumption and efficient labor. Moreover, because of the equal marginal rates of substitution of future to current consumption, the efficient assignment needs to be the same across time. Any inefficiencies due to tax distortions are captured by the aggregate allocation. Werning (2007) incorporated these properties of the equilibrium allocation into first analyzing the static distortion problem, then looking at the dynamics in aggregate levels. This method allows for aggregation such that the competitive equilibrium allocation can be completely characterized in terms of aggregates and a static rule for individual allocation.

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6 Many papers have pointed out the problem in characterizing the worst equilibrium in this type of dynamic games as it might not be the repeated worst static Nash equilibrium. Instead, they made the same assumption of using autarky as the worst equilibrium (see Chari and Kehoe (1993); Dovis et al. (2016)).
Sub-market analysis. For any equilibrium, there exist market weights $\varphi = (\varphi^i)_{i \in I}$, with $\varphi^i \geq 0$ and $\sum_i \pi^i \varphi^i = 1$, such that individual assignments solve a static problem.

$$V(C, L; \varphi) = \max_{(c^i, l^i) \in I} \sum_{i \in I} \varphi^i \pi^i U^i(c^i, l^i) \quad \text{s.t.} \quad \sum_{i \in I} \pi^i c^i = C; \quad \sum_{i \in I} \pi^i l^i = L$$

The market weights capture how individual allocation are determined given any aggregate allocation. This problem gives the policy functions for each agent $i$:

$$h^i(C, L; \varphi) = (h^i_c(C, L; \varphi), h^i_l(C, L; \varphi))$$

A competitive equilibrium allocation must satisfy: $(c^i_t, l^i_t) = h^i(C_t, L_t; \varphi)$ for all $i$ and $t$. The associate competitive equilibrium prices can be computed as if the economy were populated by a fictitious representative agent with utility function $V(C, L; \varphi)$. The envelope conditions of the static problem give

$$\left(1 - r^n_t\right) w_t = -\frac{V_L[h^i(C_t, L_t; \varphi)]}{V_C[h^i(C_t, L_t; \varphi)]} \quad (8)$$

$$1 + r_{t+1} = \frac{V_C[h^i(C_{t+1}, L_{t+1}; \varphi)]}{\beta V_C[h^i(C_t, L_t; \varphi)]} \quad (9)$$

Furthermore, the present-value budget constraint for individual $i$ can be written as

$$\sum_{t=0}^{\infty} \beta^t \left[ V_C(C_t, L_t; \varphi) h^i_c(C_t, L_t; \varphi) + V_L(C_t, L_t; \varphi) h^i_l(C_t, L_t; \varphi) \right] = V_C(C_0, L_0; \varphi) \left(a^i_0 - T\right) \quad (10)$$

where $T$ is the present-value of lump-sum taxes\(^7\), and $a^i_0$ is the individual initial after-tax wealth. Equation (10) is the individual implementability constraint.

Now one has the following characterization proposition.

**Proposition 3.1.** Given initial individual wealth $\{a^i_0\}_{i \in I}$ and external debt $B_0$, an allocation $\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ can be supported as an aggregate allocation of an open economy competitive equilibrium with taxes if and only if aggregate resource constraint (6) holds, and there exist market weights $\varphi = (\varphi^i)_{i \in I}$ and lump-sum tax $T$ such that the implementability constraint (10) holds for all $i \in I$.

\(^7\) $T = \sum_{t=0}^{\infty} \beta^t \frac{V_C[h^i(C_t, L_t; \varphi)]}{V_C[h^i(C_0, L_0; \varphi)]} T_t$
4 Efficiency

This section formulates the planning problem in terms of a Ramsey problem with the additional sustainability constraints induced by the limited commitment. It follows the primal approach in public finance to characterize the best equilibrium allocation and derive the optimal policies.

4.1 Planning problem

The set of equilibrium with limited commitment can be supported as a competitive equilibrium with taxes and the sustainability constraint (7). Define the set $U$ of attainable utilities $\{u^i\}_{i \in I}$ such that $u^i = \sum_{t=0}^{\infty} \beta^t U^i (c^i_t, l^i_t)$ for any such equilibrium allocation. Given Proposition 3.1, $\{u^i\}_{i \in I}$ is the individual lifetime utilities for any allocation $\{C_t, L_t, K_t\}_{t=0}^{\infty}$ and a vector of market weights $\varphi$ such that the aggregate resource constraint and the implementability constraint all $i \in I$ are satisfied. Specifically, $u^i = \sum_{t=0}^{\infty} \beta^t U^i [h^i(C_t, L_t; \varphi)]$. An efficient allocation is defined as one that reaches the northeastern frontier of $U$, i.e. maximizing lifetime utility of one agent given that the utilities of other agents are above feasible thresholds.

Then the necessary conditions can be derived by an alternative problem of maximizing a Pareto-weighted utility, where the Pareto weights are closely related to the feasible thresholds\(^8\).

Therefore, given the Pareto weights $\lambda = \{\lambda^i\}_{i \in I}$ and exogenous international interest rates $\{r^*_t\}_{t=0}^{\infty}$, an efficient allocation maximizes the weighted utility subject to the aggregate resource constraint, the individual implementability constraints, and each-period sustainability constraint. The planning problem is formulated as

\[
(P) \equiv \max_{\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}, \varphi, T} \sum_{t=0}^{\infty} \beta^t \sum_{i \in I} \lambda^i \pi^i U^i [h^i(t; \varphi)]
\]

s.t.

\[
\sum_{t=0}^{\infty} q_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t] - B_0 \geq 0
\]

\[
\forall i, \sum_{t=0}^{\infty} \beta^t [V_C(t; \varphi)h^{i,c}(t; \varphi) + V_L(t; \varphi)h^{i,l}(t; \varphi)] \geq V_C(0; \varphi) (a^i_0 - T)
\]

\[
\forall t, \sum_{s=t}^{\infty} \sum_{i \in I} \beta^{s-t} \lambda^i \pi^i U^i [h^i(s; \varphi)] \geq U^i_t(K_t)
\]

\(^8\)As the set of attainable utilities $U$ might not be convex, an allocation that solves $(P)$ might not attain the utilities in $U$. The analysis focuses on the necessary conditions, as they are enough to develop the properties of the optimal taxes. The set of optimal taxes is a subset of the set of taxes that implement any allocation satisfying the necessary conditions for efficiency. Therefore, the optimal taxes also satisfy the attributes of taxes deriving from the necessary analysis. Park (2014); Werning (2007) made the similar argument in their work.
4.2 Characterizing efficient allocation

Let $\mu$ be the multiplier on the resource constraint, $\pi^i \eta^i$ be the multiplier on the implementability constraint for agent $i$, and $\beta^t \gamma^t$ be the multiplier on the aggregate debt constraint for period $t$. Define $\eta = (\eta^i)_{i \in I}$ and rewrite the Lagrange of the planning problem with a new pseudo-utility function that incorporates the implementability constraints:

$$\sum_{t=0}^{\infty} \beta^t W[t; \varphi, \lambda, \eta] - V_C(0; \varphi) \sum_{i \in I} \pi^i \eta^i (a^i_0 - T)$$

where

$$W[t; \varphi, \lambda, \eta] \equiv \sum_{i \in I} \lambda^i \pi^i U^i[h^i(t; \varphi)] + \sum_{i \in I} \pi^i \eta^i [V_C(t; \varphi) h^{i,c}(t; \varphi) + V_L(t; \varphi) h^{i,l}(t; \varphi)]$$

The necessary conditions to characterize the set of efficient allocation are the first-order conditions of the planning problem, the aggregate resource constraint, the sustainability constraints, and the implementability constraints.

5 Optimal Taxation

This section provides the main optimal taxation results. The work emphasizes on the case of separable isoelastic preferences. The optimal taxes are derived such that the efficient allocation can be implemented as an allocation of a competitive equilibrium with taxes. A key assumption throughout this section is the impatience of private agents with respect to the international intertemporal interest rates that the country faces when borrowing abroad, i.e.

**Assumption 1** (Impatience). There exists $M > 0$ and $T$ such that $\forall t > T$, $\beta(1 + r^*_t) < M < 1$.

Consider the following functional form of the utility:

**Assumption 2** (Separable isoelastic preference). The utility function $U : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies

$$U(c, n) = \frac{c^{1-\sigma}}{1-\sigma} - \omega \frac{n^{1+\nu}}{1+\nu}$$

for $\sigma, \omega, \nu > 0$.

Given that the preference is separable and isoelastic, individual consumption and efficient
labor supply are time-invariantly proportional to the aggregates:

\[ c^i_t = h^{i,c}(C_t, L_t; \varphi) = \psi^i_c C_t \]
\[ l^i_t = h^{i,l}(C_t, L_t; \varphi) = \psi^i_l L_t \]

where

\[ \psi^i_c = \left( \varphi^i \right)^{1/\sigma} \sum_{i \in I} \frac{\pi^i(\varphi^i)^{1/\sigma}}{\pi^i} \]
\[ \psi^i_l = \left( \theta^i \right)^{1+\nu} \left( \varphi^i \right)^{-1/\nu} \sum_{i \in I} \frac{\pi^i(\theta^i)^{1+\nu}}{\pi^i(\varphi^i)^{-1/\nu}} \]  

Then \( V, W \) inherit the separable and isoelastic properties from \( U \), i.e. \( \forall t, \)

\[ V(C_t, L_t; \varphi, \lambda, \eta) = \Phi^V_C C_t^{1-\sigma} - \Phi^V_L L_t^{1+\nu} \]
\[ W[C_t, L_t; \varphi, \lambda, \eta] = \Phi^W_C C_t^{1-\sigma} - \Phi^W_L L_t^{1+\nu} \]

and the objective is

\[ \sum_{t=0}^{\infty} \beta^t \left( \Phi^P_C C_t^{1-\sigma} - \Phi^P_L L_t^{1+\nu} \right) \]

where \( \Phi^V_C, \Phi^V_L \) depend on \( \varphi \), \( \Phi^W_C, \Phi^W_L \) depend on \( \varphi, \lambda, \eta \), and \( \Phi^P_C, \Phi^P_L \) are functions of \( \lambda \) and \( \varphi \) (see Appendix B.1).

The first-order conditions of the planning problem for any period \( t \geq 1 \) can be summarized as

\[ F_L(K_t, L_t) = \frac{\Phi^W_L + \Phi^P_L \sum_{s=0}^{t} \gamma_s}{\Phi^W_C + \Phi^P_C \sum_{s=0}^{t} \gamma_s} L_t^{1-\sigma} \]
\[ F_K(K_t, L_t) = r^*_t + \delta + \frac{\beta^t \gamma_t}{q_t} U'_t(K_t) \]

and

\[ C_t^{-\sigma} = \beta C_{t+1}^{-\sigma} \left( 1 + r^*_{t+1} \right) \left[ \frac{\Phi^W_C + \Phi^P_C \sum_{s=0}^{t+1} \gamma_s}{\Phi^W_C + \Phi^P_C \sum_{s=0}^{t} \gamma_s} \right] \]

To implement the efficient allocation, optimal taxes are derived such that the competitive equilibrium allocation satisfies the first-order conditions of the planning problem. From the competitive equilibrium characterization, the taxes on labor, capital and saving return must
satisfy

\[
(1 - \tau^n_t)F_L(K_t, L_t) = \frac{\Phi^V_L L^V_t}{\Phi^V_C C^{-\sigma}_t}
\]

\[
(1 - \tau^k_t)F_K(K_t, L_t) = r_t + \delta
\]

\[
C^{-\sigma}_t = \beta (1 + r_{t+1}) C^{-\sigma}_{t+1}
\]

5.1 Labor income tax

Dividing equation (13) by equation (16) gives

\[
\tau^n_t = 1 - \frac{\Phi^V_L}{\Phi^V_C} \left[ \frac{\Phi^W_C + \Phi^C_P \sum_{s=0}^{t} \gamma_s}{\Phi^W_L + \Phi^L_P \sum_{s=0}^{t} \gamma_s} \right]
\]

which is the equation for optimal labor income tax. Note that the time-variant component of the tax is the sum of the Lagrange multipliers on the debt constraints, reflecting how limited borrowing influences the dynamic of the taxes. When the aggregate debt constraints do not bind, i.e. \( \gamma_s = 0, \forall s \leq t \), the tax on labor income becomes

\[
\tau^n_t = 1 - \frac{\Phi^V_L \Phi^W_C}{\Phi^V_C \Phi^W_L} \equiv \tau^{n,uc}
\]

which is a time-independent constant. The labor tax is constant when the debt constraint is not relevant. The intuition is that distortionary tax is a mechanism for redistribution. The distortion reflects the trade-off between the inequality level, which is determined by the skill distribution, and the redistribution motive, which is from the social welfare weights. Because the skill distribution and welfare weights do not change, and borrowing is unconstrained, the government finds it optimal keep the intratemporal distortion constant and borrow as needed to finance expenditure. The unconstrained optimal level is formulated by (20), which indirectly depends on skill distribution and Pareto weights (see Appendix for the formulas of \( \Phi \)'s).

On the other hand, binding debt constraint limits the government’s ability to borrow and to smooth taxes over time. When the debt constraint binds, the cumulative sum of debt-constraint multipliers show up in the optimal labor tax formula. Given that private agents are impatient, the country’s aggregate debt increases over time. In the environment without debt constraints, or debt constraints never bind, the Ramsey allocation features immiseration in the long run such that the marginal utility of consumption is growing without bound. Such scenario happens when the deviation utility is unbounded below, i.e. the value of deviating is low enough such that the government will always commit to the contract.
However, if the deviation utility is bounded below, the full-commitment Ramsey allocation cannot be supported. There is no immiseration in the long run as the future utility is always bounded below. This no-immiseration result is common in many models of limited commitment.

Note that no immiseration also means that the debt constraint eventually binds, and the multiplier \( \gamma_t \) increases over time. In the long run, the cumulative sum of multipliers will diverge\(^9\). Given equation (19), it must be that

\[
\lim_{t \to \infty} \tau^n_t = 1 - \frac{\sum_{i \in I} \pi^W_i \psi^i_c}{\sum_{i \in I} \pi^L_i \psi^i_l},
\]

which is a different level from the unconstrained distortion level. The tax limit similarly depends on redistribution preference, which comes from \( \{ \lambda^i \}_{i \in I} \), and inequality, captured by \( \{ \phi^i \}_{i \in I} \) (utility shares), \( \{ \psi^i_c \}_{i \in I} \) (consumption shares), and \( \{ \psi^i_l \}_{i \in I} \) (labor shares).

Back-loading efficiency motive is a driver for the changes in the optimal tax rates. Consider the following expenditure minimization problem for each period \( t \)

\[
(EM) \equiv \min_{C_s, L_s, K_{s+1}} \sum_{s=t}^{\infty} q_s [C_s + G_s + K_{s+1} - F(K_s, L_s) - (1 - \delta)K_s]
\]

\[
s.t. \sum_{s=t}^{\infty} \beta^{s-t} \left( \Phi^P_C \frac{C_i^{1-\sigma}}{1-\sigma} - \Phi^P_L \frac{L^{1+\nu}}{1+\nu} \right) = U_t(K_i)
\]

which is the problem of minimizing the present value of the resource cost to deliver \( U_t(K_i) \) as the planner’s promised utility at period \( t \). The solution to this minimization problem can be implemented with the labor tax \( \tau^n_s = 1 - \frac{\Phi^W_C}{\Phi^P_c} \) for \( s \geq t \). Note that this formula is exactly the labor tax limit. The labor tax equation (19) incorporates the solution to the Ramsey planner when debt is unconstrained (\( \Phi^W \)’s) and as debt constraint binds, part of the solution to this expenditure minimization problem shows up (\( \Phi^P \)’s) with respect to the tightness of the constraints (\( \sum_{s \leq t} \gamma_s \)). An interpretation is that when there is unlimited borrowing, the planner should set the tax rate to achieve the most efficient allocation, which is the allocation that the planner would choose if she never runs into the debt constraints. However, when debt constraint binds, the planners want the international lenders to continue the contract by offering an allocation such that it delivers the promised utility \( U_t(K_i) \) in a less costly way. As the debt constraint binds in the long run, the optimal allocation continues to lower the delivering cost and eventually reaches the allocation with minimal cost, which is the

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\(^9\)Intuitively, optimal allocation must satisfies \( \beta^t/q_t \left( \Phi^W_C + \Phi^P_C \sum_{s=0}^{t} \gamma_s \right) C_i^{1-\sigma} = \mu \). No immiseration implies that \( C_i^{1-\sigma} \) is bounded below. Since \( \mu > 0 \), as \( \beta^t/q_t \to 0 \), it must be that \( \sum_{s=0}^{t} \gamma_s \to \infty \)
solution to \((EM)\). The labor tax starts of with the most efficient level for the country, which is \(1 - \frac{\Phi_L^V\Phi_L^C}{\Phi_C^V\Phi_C^L}\) and gradually converges to the most efficient level for the international lenders, i.e. \(1 - \frac{\Phi_L^P\Phi_L^C}{\Phi_C^P\Phi_C^L}\).

Formally, consider the following assumption on deviation utility.

**Assumption 3.** \(U_t(\cdot)\) is bounded below, i.e. there exists a finite real \(M_U\) such that \(\inf_{K_t} U_t(K_t) \geq M_U\).

Given this assumption, the consumption path is bounded below by zero in the long run, i.e.

**Lemma 5.1** (No immiseration). Suppose assumptions 2 and 3 hold, then for any efficient allocation \(\{C_t^*, L_t^*, K_t^*\}_{t=0}^{\infty}\) \(\lim_{t \to \infty} C_t^* > 0\).

The following proposition characterizes the optimal tax on labor income in an economy facing debt constraints and distributional concern.

**Proposition 5.1** (Optimal labor tax). Given assumption 2, if an efficient allocation exists and debt constraint does not bind, there is constant labor tax given by (20). Moreover, if assumptions 1 and 3 also hold, and an interior efficient allocation exists, then the optimal labor tax converges to a real constant given by (21) that depends on skill distribution and redistribution preference. These results hold with or without the lump-sum transfers.

The properties of the optimal labor tax come from the properties of the efficient allocation. The intratemporal marginal rate of substitution between consumption and leisure depends not only on the trade-off between efficiency and equality, but also the endogenous debt constraints. Initially, when the sustainability constraint never binds, impatient agents front load their consumption and leisure, accumulating debt over time. It is optimal to equate the marginal cost of distortion to the marginal benefit of redistribution. The latter does not change over time since the determinants of inequality and distributional preference are constant, implying the same for the labor tax. When the sustainability constraint binds, the discounted future utility is bounded below, which additionally distorts the consumption–leisure decision. The optimal contract wants to relax the current and future endogenous debt constraints. In the long run, because of impatience and the need to borrow, it is efficient to maximize the economy’s aggregate debt capacity, and so the efficient allocation converges to one with the maximal debt level. This property is implied by the sustainability constraints increasingly tightened, as the sum of the multipliers going to infinity.
5.2 Capital income and saving taxes

Combining equations (15) and (18), the optimal domestic return satisfies

\[ 1 + r_t = (1 + r_t^*) \frac{\Phi_W^t + \Phi_P^t \sum_{s=0}^{t} \gamma_s}{\Phi_W^t + \Phi_P^t \sum_{s=1}^{t} \gamma_s} \] (22)

Note that when the sustainability constraint is not relevant, i.e. \( \gamma_s = 0 \ \forall s \leq t \), it is optimal to set the domestic interest rates equal to the exogenous foreign interest rates. However, as the sustainability constraints start binding, equation (22) implies that \( r_t > r_t^* \), which implies a saving subsidy. As the economy reaches its debt limits, shown as the binding sustainability constraints, the government has incentive to subsidize more on saving, or tax more on borrowing, to discourage private agents to accumulate debt.

Moreover, the first-order conditions (14) and (17) give

\[ r_t^k = 1 - \frac{r_t + \delta}{r_t^* + \delta + \frac{\alpha}{\eta} \mu^U_t(K_t)} \] (23)

If \( U_t' \) is positive\(^{10}\), the above equation implies that there is a higher tax on capital income when sustainability (debt) constraints bind. A greater capital tax reflects that there is capital under-investment of the efficient allocation. Indeed, the first-order condition (14) shows that \( F_K(K_t, L_t) > r_t^* + \delta \) when \( \gamma_t > 0 \), where \( r_t^* + \delta \) is the first-best interest rate. Because the government can expropriate more capital and receives higher utility from reneging, the optimal contract discourages capital accumulation, which can be implemented by imposing more tax on capital income.

6 Inequality and Redistribution Effect

This section analyzes how inequality and redistribution motive affect the optimal labor tax. Inequality and redistribution not only influences the tax levels from the trade-off between equity and efficiency, but also the tax dynamics from the interaction with the sustainability (debt) constraints. Suppose that there is no inequality, then the problem becomes the standard representative Ramsey problem in a small open economy. If the government can lump-sum tax/transfer, there is no need for distortion, and the optimal labor tax will be zero. If the government can only impose distortionary taxes, the model collapses to the representative setting as in Aguiar and Amador (2016), where zero labor tax limit is optimal.

\(^{10}\)The higher the amount of capital the government can expropriate, the higher the deviation utility. Proposition A.1 proves a case when it is true.
An interpretation is that the optimal contract should maximize its debt capacity in the long run, which is associated with zero labor tax to achieve the highest output. Thus, it is optimal to not tax labor in the limit when there is no inequality:

**Proposition 6.1** (No inequality). Suppose that $\theta^i = \theta^j$, $\forall i, j \in I$. Then there is zero labor tax in the long run. This result holds with or without lump-sum transfers.

*Proof.* Follow from equation (21) with $\varphi^i = \psi_c^i = \psi_l^i = 1$, $\forall i \in I$. \hfill \Box

While the market weights determine how the competitive market chooses individual shares of utility, the Pareto weights regulate the social shares of utility. Any agent has an exogenous Pareto weight based on the government’s redistribution preference, and a market weight that depends on her relative skill and initial wealth. This is due to skill and initial wealth positions determine individual budget constraints, which gives individual implementability constraints in the planning problem. An interesting case is when the vector of market weights is equal to the vector of Pareto weights ($\psi = \lambda$). This implies that there is no redistribution effect, because the government, as a planner, distributes aggregate utility exactly the same way as the competitive market does. In this situation, \((21)\) turns out to be $\lim_{t \to \infty} \tau_t^n = 0$, that is, the optimal labor tax converges to zero. These results are summarized in the following proposition.

**Proposition 6.2.** There exists an efficient allocation $\{C^*_t, L^*_t, K^*_{t+1}\}_{t=0}^{\infty}, \varphi^*, T^*$ such that for all $i$, $\lambda^i = \varphi^i$. Such an allocation can be implemented with a zero labor tax in the long run.

In general, changing the welfare weights, which are a measure for redistribution motive, affects the social utility, and so the sustainability constraints\(^{11}\). Proposition 6.2 shows that redistribution motive also has effect on the levels of optimal policies, yet only when it can deviate from the distribution rising from the competitive equilibrium markets. It affects the level of tax distortions because of the trade-off between distribution and efficiency. Moreover, heterogeneity is not the source of differences in optimal policies comparing to the representative-agent setting.

To understand how redistribution and inequality affect the optimal labor tax levels,

\(^{11}\)The welfare weights determine the discounted future utility and the deviation utility, derived in the Appendix, so they influence both sides of the sustainability constraints.
consider the rewritten formulas of the labor tax:

\[
\tau^{n,uc} = 1 - \frac{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] + \operatorname{cov} \left( \psi^i_c, \frac{\lambda}{\varphi} + (1 - \sigma) \eta^i \right)}{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] + \operatorname{cov} \left( \psi^i, \frac{\lambda}{\varphi} + (1 + \nu) \eta^i \right)}
\]

\[
\lim_{t \to \infty} \tau^n = 1 - \frac{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] + \operatorname{cov} \left( \psi^i_c, \frac{\lambda}{\varphi} \right)}{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] + \operatorname{cov} \left( \psi^i, \frac{\lambda}{\varphi} \right)}
\] (24)

using the definitions \( \mathbb{E} [x^i] = \sum_i \pi^i x^i \) and \( \operatorname{cov}(x^i, y^i) = \mathbb{E} [x^i y^i] - \mathbb{E} [x^i] \mathbb{E} [y^i] \). Optimal conditions from the planning problem give \( \eta^i = \sum_j \pi^j \lambda^j / \varphi^j - \lambda^i / \varphi^i \), which allows one to write the unconstrained-borrowing tax level as

\[
\tau^{n,uc} = 1 - \frac{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] + \sigma \operatorname{cov} \left( \psi^i_c, \frac{\lambda}{\varphi} \right)}{\mathbb{E} \left[ \frac{\lambda}{\varphi} \right] - \nu \operatorname{cov} \left( \psi^i, \frac{\lambda}{\varphi} \right)}
\] (25)

Note that the optimal tax levels depend on the covariance between fractions of individual allocation to aggregates \( (\psi^i_c, \psi^i)_i \in I \) and the relative distribution ratios \( (\lambda^i / \varphi^i)_i \in I \). Changes in the redistribution preference or skill distribution affect \( \operatorname{cov}(\psi^i_c, \lambda^i / \varphi^i) \) and \( \operatorname{cov}(\psi^i, \lambda^i / \varphi^i) \), which induce the optimal changes in the labor tax. Specifically, lowering \( \operatorname{cov}(\psi^i_c, \lambda^i / \varphi^i) \) and \( \operatorname{cov}(\psi^i, \lambda^i / \varphi^i) \) will lead to a higher \( \tau^{n,uc} \), and a lower or higher \( \lim_{t \to \infty} \tau^n \).

An explanation is that while \( \operatorname{cov}(\psi^i_c, \lambda^i / \varphi^i) \) incorporates the benefit for redistribution, \( \operatorname{cov}(\psi^i, \lambda^i / \varphi^i) \) represents both the redistribution benefit and the efficiency cost. Low \( \operatorname{cov}(\psi^i_c, \lambda^i / \varphi^i) \) and \( \operatorname{cov}(\psi^i, \lambda^i / \varphi^i) \) imply a high benefit of redistribution, but also a high cost of efficiency. When borrowing in unconstrained, it is optimal to equate the marginal benefit and the marginal cost, which can be achieved by a high tax rates (as \( \tau^{n,uc} \) is high when both \( \operatorname{cov}(\psi^i_c, \lambda^i / \varphi^i) \) and \( \operatorname{cov}(\psi^i, \lambda^i / \varphi^i) \) are small). However, in the long run when debt constraints bind, marginal cost of distortion is much higher because additional distortion discourage agent’s incentive to work, hurting the government’s ability to pay back foreign debt. Due to the need to increase the economy’s debt capacity, the cost of distortion is high. At the same time, there is still benefit and the need for redistribution. It turns out that the tax limit will increase (decrease) if the marginal benefit is higher (lower) than the marginal cost.

How do changes in redistribution or inequality influence these covariance terms? Consider the case that \( \psi^i_c, \psi^i \) increase with the individual skill \( \theta^i \).

\[\text{If the initial wealth positions are equal among everyone, the more productive type is likely to have a higher effective labor and a higher consumption, as it is true in the numerical example below.}\]
cov(ψ_i^c, λ_i/ϕ_i) and cov(ψ_i^l, λ_i/ϕ_i) (more negative). Similarly, given a set of distributional weights, and the case that ψ_i^c, ψ_i^l increase with θ_i, cov(ψ_i^c, λ_i/ϕ_i) and cov(ψ_i^l, λ_i/ϕ_i) are negative\(^{13}\), and increase in the dispersion of the skill distribution imply more negative covariances.

Intuitively, given type-independent lump-sum transfers, increasing distributional preference towards the low types or increasing inequality lead to the planner’s higher redistribution motive. The planner will then want higher marginal tax rates on labor, because higher marginal tax implies that the more highly skilled, richer types will pay more taxes. When borrowing is unconstrained, it is optimal to redistribute a lot by having a higher \(\tau_{n,uc}^n\). When the borrowing is limited, the planner has an additional incentive expand the economy’s debt capacity by reducing the distortion. The trade-off between redistribution and high output determines how the labor tax limit moves.

7 Numerical Exercise

This section aims to illustrate the theoretical results by a numerical exercise with endogenous debt constraints and redistribution. There is no capital. Assume that production is linear in effective labor, i.e. \(F(L_t) = L_t\). The deviation utility is a constant finite \(U\) so that it is consistent with Assumption 3.

Lemma 7.1. If the sustainability constraint binds for some finite \(S\), then it will bind for all \(t > S\).

It must be true that the socially weighted utility, \(\sum_{i \in I} \lambda_i \pi_i U^i[h^i(C_t, L_t; \varphi)]\) is equal across all period \(t > S\) that the sustainability constraint binds. Combining this feature and the planner’s first-order conditions solves the efficient allocation at each period after the constraints bind, then the unconstrained allocation at period \(t < S\) can be derived by recursing backwards.

Consider a parametric economy consisting of two types of agents, denoted \(I = \{H, L\}\), where \(\theta^H \geq 1 \geq \theta^L\). Let \(\pi^H = \pi^L = 0.5\), which implies that \(\theta^H = 2 - \theta^L\). The two types have zero initial wealth positions, i.e. \(a_i^0 = 0, \forall i\). Let \(U(c, n) = \log c - \omega \frac{n^{1+\nu}}{1+\nu}, \omega = 1, \nu = 2\) so that the Frisch elasticity of labor supply is 0.5. Assign \(\beta = 0.94\) and \(r_t^* = r^* = 0.05\) so that \(\beta(1 + r^*) < 1\). The planner is utilitarian: \(\lambda^H = \lambda^L = 1\). The government expenditure is constant across time and set to be 20 percent of the initial external asset position, which is normalized to 1 (\(U\) is the value of autarky, which is calculated as the maximal utility

\(^{13}\)The definition of \(\psi_i^c\) gives that \(\varphi_i\) will also increase with \(\theta_i\). Fixing \(\{\lambda^i\}_{i \in I}\) implies that \(\lambda^i/\varphi_i\) decreases with \(\theta_i\).
attained from a tax-distorted competitive equilibrium of the economy with no access to international market and private savings.

Figure 1 shows the time paths of aggregate allocation, labor tax, relative domestic return with respect to the international rates, utility, and external debt when inequality is such that $\theta^H = 3\theta^L$. Time is the horizontal axis, and period $S$ is the first period at which the debt constraint binds. Panel (a) plots the aggregate consumption and efficient labor, and panel (c) plots the period utility of the planner relative to the deviation utility. Before the debt constraint binds, consumption is decreasing while labor is increasing over time, reflecting the impatience of private agents. At the same time, utility is decreasing. As the debt constraint binds for periods $S$ onward, the planner’s utility stays constant at the level of deviation utility. However, aggregate labor still goes up but at a slower speed that before, and aggregate consumption increases such that in the long run, they reach the most cost-efficient levels in steady states. Panel (b) plots the optimal labor tax and domestic return that implement the planner’s allocation. During the time periods from 0 to $S$, the labor tax starts at a positive level and constant, while optimal domestic return is equal to the international rates $r^*_t$. As the debt constraint starts binding from $S$, labor tax decreases, as the domestic return increases more than $r^*_t$. In the long run, labor tax converges to a negative level, while domestic return converges to $1 - 1/\beta$ which is higher than $r^*_\infty$ due to the impatience assumption. Panel (d) shows the path of external debt $B_t$. The economy accumulates external debt quickly in the beginning. However, when the debt constraint hits, there is a slower accumulation of debt that eventually reaches its steady state, which is the maximum debt capacity of the economy.

The high initial labor tax rate reflects the redistribution motive, and that the marginal benefit of redistribution is high. When there is no limit on borrowing, it is optimal to set the marginal distortion equal to the marginal benefit, so the tax rate is high. However, when hitting the debt constraints, the economy needs to lower the distortion cost, incentivizing more output to increase its debt capacity. It is done by lowering the labor tax rates, in return for increasing the tax on borrowing. Additionally, because the high type is very productive, the optimal contract eventually subsidizes labor to encourage more output from the high type.

Figure 2 shows a comparative static, with respect to relative inequality $\theta^H/\theta^L$, of the optimal labor tax rates in periods where borrowing is unconstrained ($\tau^{n,\text{uc}}$) and at the limit ($\lim_{t\to\infty} \tau^n_t$), for the utilitarian planner. When there is no inequality ($\theta^H = \theta^L$), the problem collapses to a Ramsey’s problem of a representative-agent small open economy. Due to the presence of lump-sum tax, it is optimal to have zero labor distortion in all periods. With inequality, equation (25) implies that $\tau^{n,\text{uc}}$ is positive and increasing with respect to higher
inequality\textsuperscript{14}. As inequality increases, the government, while debt constraint is not binding, would want to set higher tax rates for redistributive purposes. Therefore, as shown in panel (a), $\tau^{n,uc}$ increases with $\theta^H/\theta^L$.

On the other hand, setting high tax rates during unconstrained-debt periods leads to high cost of distortion when debt constraint binds. Panel (b) plots the tax limits for a smaller range of inequality. There tax limit curve is hump-shaped, where initially tax limit increases with respect to inequality, yet eventually decreases and become negative\textsuperscript{15}. For a low enough inequality, in the long run, there is higher benefit for redistribution, so $\lim_{t \to \infty} \tau^n_t$ is increasing and positive. However, when inequality is high enough, it is then optimal for the government possibly subsidizing labor in the long run, as $\lim_{t \to \infty} \tau^n_t$ declines and becomes negative. In addition, there are two interesting points in panel (b) where the labor tax limit is approximately zero. The first point (where $\theta^L = \theta^H$) corresponds to the case where $\lambda = \varphi$, which is the condition for zero distortion from Proposition 6.2. The other point is the case where the market weights are equal to the labor productivity types. The equation for labor

\textsuperscript{14}$\sigma = 1$ and $\lambda^j = 1$ imply that $\tau^{n,uc} = 1 - \frac{1}{\{E[1/\varphi^j] - \nu \text{cov}(\psi^j, 1/\varphi^j)\}}$. Note that $\text{cov}(\psi^j, 1/\varphi^j) \leq 0$, and increasing $\theta^H/\theta^L$ leads to higher $E[1/\varphi^j]$ and lower $\text{cov}(\psi^j, 1/\varphi^j)$.

\textsuperscript{15}In this case, $\lim_{t \to \infty} \tau^n_t = 1 - 1/\{E[1/\varphi^j] + \text{cov}(\psi^j, 1/\varphi^j)\}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Time paths of economic aggregates when $\theta^H = 3\theta^L$}
\end{figure}
tax limit (21) implies that \( \lim_{t \to \infty} \tau^n_t = 0 \).

Since the private agents borrow over time due to impatience, the government collects its revenue from taxing domestic borrowing. Figure 3 plots the relative return on domestic bonds to the international interest rates, \( r_t/r^* - 1 \), in periods after the debt constraint binds, with respect to inequality \(^{16}\). In the long-run, the domestic interest rate reaches \( 1 - 1/\beta \) so that the economy reaches its steady-state allocation. Higher inequality is associated with higher relative returns on savings after the debt constraint binds, implying higher subsidies on savings or higher taxes on borrowing.

Figure 4 presents the dynamics of government’s external debt \( B_t \) with respect to inequality. The dot in each line represents the time period and debt level where the debt constraint starts binding for each level of inequality. While all economies start with the same initial external debt position, an economy with higher inequality accumulates higher debt over time. Recall that a highly-unequal economy has high labor tax rate in the beginning, which helps financing a large amount of foreign debt. When the debt constraint binds, by keeping the low distortion, expected future utility is high, relaxing the borrowing constraints and increasing higher debt capacity of the economy.

\(^{16}\)Section 5.2 showed that this relative return is zero when the debt constraint is not binding.
Interestingly, higher inequality is associated with longer time of unconstrained borrowing. One explanation is that since a highly-unequal economy has more motive for redistribution. It is optimal for the government to prolong the periods that the debt constraint does not bind, in which the planner can redistribute the most. Appendix C.3 illustrates the dynamics of aggregate allocation with respect to inequality. Note that high inequality implies lower aggregate consumption and labor in the beginning of time, as the debt constraint does not bind. Both consumption and labor increase after the debt constraints bind, and potentially reach a higher level than the allocation level without inequality, allowing the country to accumulate a much higher level of external debt.
8 General Case: Separable Preference

This section extends the results of optimal labor taxation with separable preferences. As the elasticity of consumption intertemporal substitution and the elasticity of labor supply vary across time, the optimal labor tax fluctuates. In general, the labor tax is bounded in the long run. If the steady states exist, in the long run, the optimal tax goes to a real limit as in the case with separable isoelastic preferences. In either case, the redistribution preference alters the level of optimal taxes in the long run. The results rely on the following assumptions of separability and boundedness.

**Assumption 4** (Separable preference). \(U^i(c,l) = u(c) - v(l/\theta^i)\), where \(u_c(\cdot) > 0, u_{cc}(\cdot) < 0, \lim_{c \to 0} u_c(c) = \infty, \lim_{c \to \infty} u_c(c) = 0, v_l(\cdot) > 0, v_{ll}(\cdot) > 0, \text{ and } \lim_{l \to 0} v_l(l) = \infty\)

**Assumption 5** (Bounded elasticities). \(u\) and \(v\) are such that \(\forall c, n \in \mathbb{R}_+, 0 < -\frac{u'(c)}{u(c)} c < \infty\) and \(0 < \frac{v'(n)}{v(n)} n < \infty\)

Since the preference is separable between consumption and leisure, individual consumption (labor) only depends on the aggregate consumption (labor). In addition, each agent’s allocation is increasing with respect to the aggregates.

**Lemma 8.1.** Given assumption 4, for any competitive equilibrium, there exist time-invariant functions \(h^i_c(\cdot; \varphi), h^i_l(\cdot; \varphi)\), \(\forall i\) such that \(\forall i, \forall t,\)

\[
\begin{align*}
c^i_t &= h^i_c(C^i_t; \varphi) \\
l^i_t &= h^i_l(L^i_t; \varphi)
\end{align*}
\]

where \(h^i_c(\cdot; \varphi), h^i_l(\cdot; \varphi)\) are strictly increasing.

The characterization of individual allocation from Lemma 8.1 and the bounded elasticities from assumption 5 give the long-run property of efficient allocation and optimal labor tax. Specifically, the efficient allocation also features no immiseration in the long run.

**Lemma 8.2** (No immiseration). Suppose assumptions 3 and 4 hold, then for any efficient allocation \(\{C^*_t, L^*_t, K^*_t\}_{t=0}^{\infty}\), \(\lim \inf_{t \to \infty} C^*_t > 0\).

and the long-run optimal labor tax is bounded.

**Proposition 8.1** (Optimal labor tax in the long run). If assumptions 1, 3, 4, and 5 hold, and an interior efficient allocation \(\{C_t, L_t, K_t\}_{t=0}^{\infty}, \varphi, T\) exists, then there exist \(-\infty < \tau, \bar{\tau} < \infty\) such that \(\lim \inf_{t \to \infty} \tau^*_t = \tau\) and \(\lim \sup_{t \to \infty} \tau^*_t = \bar{\tau}\). Moreover, if the steady states exist, then \(\lim_{t \to \infty} \tau^*_t\) exists. These results hold with or without the lump-sum transfers.
The results rely on the fact that the tax’s long-run value relies on the marginal changes in individual allocation with respect to the aggregates in the long run. With constant elasticities, the individual allocation is linear in the aggregate allocation, as in equations (11), so the marginal change is constant over time, which means that the limit exists. However, when preferences are not isoelastic, the marginal change fluctuates over time. Therefore, the optimal labor tax does not necessarily converge to a constant. Given the bounded elasticities, the marginal changes are bounded and so is the optimal labor tax. In case the steady state allocation exists, the marginal changes will converge to the steady state values, which implies the convergence to limit of the labor tax.

9 Conclusion

This paper analyzes the optimal taxation and debt management for a small open economy with impatient agents, endogenous debt constraints, and redistribution motive. Impatient agents borrow over time, which makes the debt constraints relevant in the long run. The optimal labor tax features a constant rate when borrowing is unconstrained, yet later a gradual convergence to a limit associated with the economy’s maximal aggregate debt limit. As the debt constraints bind, it is optimal to increase the taxes on capital and borrowing.

Redistribution significantly changes the implication for the optimal fiscal policies. Specifically, it alters the long-run limit of taxes through interacting with inequality. It also changes both the inside and outside values of the contract, which indirectly determines the debt limits. Any country’s optimal levels of taxes and debt issuance crucially depends on its labor productivity distribution as well as its social distribution preference.

It is useful to explore how optimal policies response in this model with aggregate shock that leads to periods of fiscal/debt crises. The government will have an additional incentive to insure the economy against aggregate shocks. The trade-off between insurance and redistribution motive can result in interesting tax dynamics, as illustrated in Arellano and Bai (2016); Balke and Ravn (2016). In addition, enriching the tax system to non-linear taxes can help study the optimal tax progressiveness in the presence of sovereign debt.
References


A Sovereign Game

Before setting up the game, consider the general environment where the government’s policy includes the decision to default on external bond \( \{d_t\}_{t=0}^{\infty} \), where \( d_t \in \{0, 1\} \) and \( d_t = 0 \) implies default\textsuperscript{17}. The government’s budget constraint becomes

\[
G_t + (1 + r_t)B^d_t + d_t B_t \leq \tau^n_t w_t L_t + \tau^k_t r_t K_t + B^d_{t+1} + Q_{t+1}B_{t+1} + T_t
\]

As the government cannot commit to any of its policies, one can think that the government, private agents, and international lenders enter in a sovereign game where they determine their actions sequentially. In every period, the state variable for the game is \( \{B_t, (k^i_t, b^d_t)\}_{i \in I} \).

The timing of the actions is as follows.

- Government chooses \( z^G_t = (\tau^n_t, \tau^k_t, T_t, d_t, r_t, B_{t+1}, B^d_{t+1}) \in \Pi \) such that it is consistent with the government budget constraint.

- Agents choose allocation \( z^{H,i}_t = (c^i_t, l^i_t, k^i_{t+1}, b^d,i_{t+1}) \) subject to their budget constraints, the representative firm produce output by choosing \( z^F_t = (K_t, L_t) \), and the international lenders choose holdings of government’s bonds \( z^*_t = (B_{t+1}) \) given the price \( Q_{t+1} \).

Define \( h^t = (h^{t-1}, z^G_t, (z^{H,i}_t)_{i \in I}, z^F_t, z^*_t, p) \in H^t \) as the history at the end of period \( t \). Note that the history incorporates the government’s policy, allocation and prices. Define \( h^t_p = (h^{t-1}, z^G_t) \in H^t_p \) as the history after the government announce its policies at period \( t \). The government strategy is \( \sigma^G_t : H^{t-1} \to \Pi \). The individual agent’s strategy is \( \sigma^{H,i}_t : H^t_p \to \mathbb{R}^3_+ \times \mathbb{R} \). The firm has strategy \( \sigma^F_t : H^t_t \to \mathbb{R}^2_+ \), and the international lenders have strategy \( \sigma^*_t : H^t_p \to \mathbb{R}_+ \). The prices are determined by the pricing rule: \( p : H^t_p \to \mathbb{R}_+ \).

**Definition A.1** (Sustainable equilibrium). A sustainable equilibrium is \( (\sigma^G, \sigma^H, \sigma^F, \sigma^*) \) such that (i) for all \( h^{t-1} \), the policy \( z^G_t \) induced by the government strategy maximizes the socially weighted utility given \( \lambda \) subject to the government’s budget constraint (5) (ii) for all \( h^t_p \), the strategy induced policy \( \{z^G_t\}_{t=0}^{\infty} \), allocation \( \{z^{H,i}_t, z^F_t, z^*_t\}_{t=0}^{\infty} \), and prices \( \{Q_t\}_{t=0}^{\infty} \) constitute a competitive equilibrium with taxes.

The following focuses on characterizing a set of sustainable equilibrium in which deviation triggers autarky, where there is no domestic and foreign borrowing. In this case, the value of deviation includes the autarkic payoff.

\textsuperscript{17}Since the paper focuses on characterizing the no-default equilibrium, the set-up from the main text does not explicitly model the default decision and instead takes into account that the government will pay back its foreign debt \( (d_t = 1) \).
By definition, autarky is a sustainable equilibrium. Given that the domestic agents do not save/invest, the representative firm produces only with labor, and the international creditors do not lend, the government finds it optimal to default on its external debt, set saving and capital taxes such that the after-tax gross returns on domestic bonds and capital are zero, and set the labor tax such that it maximizes the socially weighted utility. Given the government defaulting and fully taxing all returns from domestic savings and capital, international creditors do not want to lend, agents do not save or invest in capital, and output is produced only by labor. Lastly, given that the government will be in autarky in the future, it is optimal in the current period for the government to also follow the autarkic strategies.

Reverting to autarky equilibrium is defined as a sustainable equilibrium of the above game such that following any government’s deviation from the promised plans, the economy reverts to autarky. One can characterize the equilibrium as follows.

**Proposition A.1** (Reverting to autarky equilibrium). An allocation and policy \( \{ (z_{H,i})_{i \in I}, z^F, z^G \} \) can be supported by reverting to autarky equilibrium if and only if (i) given \( z^G \), there exist prices \( p \) such that \( \{ (z_{H,i})_{i \in I}, z^F, z^G, p \} \) is a competitive equilibrium with taxes for an open economy, and (ii) for any \( t \), there exists \( U_t(\cdot) \) such that \( \{ (z_{H,i})_{i \in I}, z^F, z^G \} \) satisfies the constraint

\[
\sum_{s=t}^{\infty} \beta^{s-t} \sum_{i \in I} \lambda^i \pi^i U_i \left( (c^i_{s}, l^i_{s}) \right) \geq U_t(K_t) \tag{7}
\]

Furthermore, \( U_t(\cdot) \) is increasing.

**Proof.** Define \( U_t(K_t) \) as the maximum discounted weighted utility for the agents in period \( t \) when the government deviates. In period \( t \), the agents save and the government can borrow abroad. In subsequent period \( s > t \), the economy reverts to financial autarky where the agents do not save and the government is excluded from international lending. This economy resembles a neoclassical growth closed economy that starts at period \( t \) and in which distortionary taxes and savings are only in the initial period. Then it is true that the higher the initial capital stock (in this case \( K_t \)), the higher utility that the agents receive. Hence, \( U_t(\cdot) \) is increasing.

Suppose \( \{ (z_{H,i})_{i \in I}, z^F, z^G \} \) is an outcome of the reverting to autarky equilibrium. Then by the optimal problems of the government, agents, and foreign lenders, \( \{ (z_{H,i})_{i \in I}, z^F, z^G \} \) maximizes the weighted utility of the agents, satisfies government budget constraint and foreign lender’s problem at period 0. Thus, \( \{ (z_{H,i})_{i \in I}, z^F, z^G \} \) is an open-economy tax-distorted competitive equilibrium. For any period \( t \) and history \( h^{t-1} \), an equilibrium strategy that has the government deviates in period \( t \) triggers reverting to autarky in period \( s > t \). Such
strategy must deliver the weighted value at least as high as the right-hand side of (7). So \( \{(z^{H,i})_{i \in I}, z^F, z^G\} \) satisfies condition (ii).

Next, suppose \( \{(z^{H,i})_{i \in I}, z^F, z^G\} \) satisfies conditions (i) and (ii). Let \( h^{t-1} \) be any history such that there is no deviation from \( z^G \) up until period \( t \). Since \( \{(z^{H,i})_{i \in I}, z^F, z^G\} \) maximizes agents period-0 weighted utility, it is optimal for the agents if the government’s strategy continues the plan from period \( t \) onward. Consider a deviation plan \( \hat{\sigma}^G \) at period \( t \) that receives \( U^d_t(K_t) \) in period \( t \) and \( U^{aut} \) for period \( s > t \). Because the plan is constructed to maximize period-\( t \) utility at \( K_t \), the right-hand side of (7) is the maximum attainable utility under \( \hat{\sigma}^G \). Given that \( \{(z^{H,i})_{i \in I}, z^F, z^G\} \) satisfies condition (ii), the original no-deviation plan is optimal. \(\square\)

\section{Formulas and Proofs}

\subsection{Formulas for separable isoelastic preference}

Given the formulas for \( \psi^i_c \) and \( \psi^i_l \) in (12),

\[
\Phi^V_C = \left[ \sum_{i} \pi^i (\varphi^i)^{1/\sigma} \right]^\sigma, \quad \Phi^V_L = \omega \left[ \sum_{i} \pi^i (\varphi^i)^{-1/\nu} (\theta^i)^{(1+\nu)/\nu} \right]^{-\nu}
\]

\[
\Phi^W_C = \Phi^V_C \sum_{i \in I} \pi^i \psi^i_c \left[ \frac{\lambda^i}{\varphi^i} + (1-\sigma)\eta^i \right], \quad \Phi^W_L = \Phi^V_L \sum_{i \in I} \pi^i \psi^i_l \left[ \frac{\lambda^i}{\varphi^i} + (1+\nu)\eta^i \right]
\]

\[
\Phi^P_C = \Phi^V_C A_C, \quad \Phi^P_L = \Phi^V_L A_L
\]

where

\[
A_C = \sum_{i \in I} \pi^i \frac{\lambda^i}{\varphi^i} \psi^i_c, \quad A_L = \sum_{i \in I} \pi^i \frac{\lambda^i}{\varphi^i} \psi^i_l \quad (B.1)
\]

\subsection{Proof of Proposition 3.1}

Proof. \( (\Rightarrow) \) Let \( \{C_t, L_t, K_{t+1}\}_{t=0}^\infty \) be an aggregate allocation of an open economy competitive equilibrium with taxes. Then by definition, \( \{C_t, L_t, K_t\}_{t=0}^\infty \) satisfies aggregate resource constraint for every period. Moreover, given any market weights \( \varphi \), \( \{C_t, L_t, K_{t+1}\}_{t=0}^\infty \) satisfies

\[
(1 - \tau^u_t)w_t = -\frac{V_L(C_t, L_t; \varphi)}{V_C(C_t, L_t; \varphi)}
\]

\[
1 + r_{t+1} = \frac{V_C \left[ h^i(C_t, L_t; \varphi) \right]}{\beta V_C \left[ h^i(C_{t+1}, L_{t+1}; \varphi) \right]}
\]
Substituting for \( w_t \) and \( r_t \) into the budget constraint (2), and using \((c_t^i, l_t^i) = h^i(C_t, L_t; \varphi)\) gives the implementability constraint for each agent. Importantly, choose \( \varphi \) and \( T \) such that the individual implementability constraints hold with equality.

\( (\Leftarrow) \) Given \( \varphi, T \) and an allocation \( \{C_t, L_t, K_{t+1}\}_{t=0}^{\infty} \) that satisfies the aggregate resource constraint, and individual implementability constraints, construct \( \{w_t, r_t\}_{t=0}^{\infty} \) using firm’s first-order conditions (3). \( \{\tau^k_t\}_{t=0}^{\infty} \) can be calculated using the intratemporal condition (8), while one can choose \( \{r_t\}_{t=0}^{\infty} \) that satisfy the intertemporal constraint (9). The tax on capital \( \{\tau^k_t\}_{t=0}^{\infty} \) can be derived from \( (1 - \tau^k_t) r^k_t = r_t + \delta \). Define \( \{q_t\}_{t=0}^{\infty} \) by (4).

Rewriting the aggregate resource constraint using \( F(K, L) = wL + rK \) gives

\[
\sum_{t=0}^{\infty} q_t \left\{ C_t + K_{t+1} - (1 - \tau^n_t) w_t L_t - \left[ 1 + (1 - \tau^k_t) r^k_t - \delta \right] K_t + T_t \right\} \\
+\sum_{t=0}^{\infty} q_t \left[ G_t - \tau^k_t r_t K_t - \tau^n_t w_t L_t - T_t \right] \leq -\delta_0 B_0 \tag{B.2}
\]

Aggregating up the agent’s budget constraints implies

\[
C_t + K_{t+1} + B^d_{t+1} = (1 - \tau^n_t) w_t L_t + \left[ 1 + (1 - \tau^k_t) r^k_t - \delta \right] K_t + (1 + r_t) B^d_t - T_t
\]
or

\[
C_t + K_{t+1} - (1 - \tau^n_t) w_t L_t - \left[ 1 + (1 - \tau^k_t) r^k_t - \delta \right] K_t + T_t = (1 + r_t) B^d_t - B^d_{t+1}
\]

Substituting the last equation into (B.2) gives the government’s budget constraint (5). Thus, \( \{C_t, L_t, K_{t+1}\}_{t=0}^{\infty} \) is the aggregate allocation of the constructed competitive equilibrium with taxes.

\[ \square \]

**B.3 Proof of Lemma 5.1**

**Proof.** Given an efficient allocation \( \{C^*_t, L^*_t, K^*_t\}_{t=0}^{\infty} \), suppose, by contradiction that \( \lim_{t \to \infty} C^*_t \leq 0 \). Find \( \epsilon > 0 \) such that \( \forall t, \)

\[
\sum_{s=t}^{\infty} \beta^{s-t} \left\{ \Phi^P C^{1-\sigma}_s \frac{C^{1-\sigma}_s}{1-\sigma} - \Phi^P (L^s)^{1+\nu} \frac{(L^s)^{1+\nu}}{1+\nu} \right\} \leq M_U
\]

34
with \( C_s = \epsilon \) and \( C_s = C_s^* \), \( \forall s \geq t \). Such \( \epsilon \) exists since the utility function is unbounded. Because \( \lim\inf_{t \to \infty} C_t^* \leq 0 \), there exists \( t_0 \) such that \( C_{t_0}^* < \epsilon \). Then by monotonicity,

\[
\sum_{s=t_0}^{\infty} \beta^{s-t_0} \left\{ \frac{\Phi_C^P(C_s^*)^{1-\sigma}}{1-\sigma} - \frac{\Phi_L^P(L_s^*)^{1+\nu}}{1+\nu} \right\} < \sum_{s=t_0}^{\infty} \beta^{s-t_0} \left\{ \frac{\Phi_C^P(C_s^*)^{1-\sigma}}{1-\sigma} - \frac{\Phi_L^P(L_s^*)^{1+\nu}}{1+\nu} \right\} \leq M_U \leq U_t(K_t^*)
\]

which contradicts the aggregate debt constraint at \( t_0 \).

\[ \square \]

**B.4 Proof of Proposition 5.1**

**Proof.** The first statement directly follows from equations (19) and (20). Let \( \{ C_t^*, L_t^*, K_{t+1}^* \}_{t=0}^{\infty} \), \( \varphi^* \), \( T^* \) be an interior efficient allocation. Then there exist \( \lambda \) such that \( \{ C_t^*, L_t^*, K_{t+1}^* \}_{t=0}^{\infty} \), \( \varphi^* \), \( T^* \) solves the planning problem \( (P) \). Redefine

\[
A_C = \sum_{i \in I} \pi_i \frac{\lambda_i}{\varphi_{vi}^*} \psi_c^i, \quad A_L = \sum_{i \in I} \pi_i \frac{\lambda_i}{\varphi_{vi}^*} \psi_l^i \quad (B.3)
\]

where \( \psi_c^i, \psi_l^i \) are defined by equations (12) using \( \varphi^* \). First, one can show that \( A_C \) and \( A_L \) are positive and bounded:

**Lemma B.1.** *Given an interior allocation, \( 0 < A_C < \infty \) and \( 0 < A_L < \infty \)*

**Proof.** Interior allocation means that for any \( i \), \( c_t^i, l_t^i > 0, \forall t \). This implies that \( \psi_c^i, \psi_l^i > 0 \). By (12), \( \varphi_{vi}^* > 0 \).

For all \( i \), \( \pi_i > 0, \lambda_i \geq 0 \) and since \( \sum_{i \in I} \pi_i \lambda_i = 1 \), there exists at least an \( i \) such that \( \lambda_i > 0 \). Given that \( \psi_c^i, \psi_l^i > 0, \forall i \), it must be that \( A_C, A_L > 0 \).

Since \( \sum_{i \in I} \pi_i \varphi_{vi}^* = 1 < \infty \) and \( \forall i, \pi_i, \varphi_{vi}^* > 0 \), it must be that \( \varphi_{vi}^* < \infty \). So by definition, \( \psi_c^i, \psi_l^i < \infty \). Moreover, \( \varphi_{vi}^* > 0 \) implies that \( \lambda_i / \varphi_{vi}^* < \infty \). Then by definition, \( A_C, A_L < \infty \). \[ \square \]

Define \( (P_T) \) the same problem as \( (P) \) with the restriction that \( (C_t, L_t) = (C_t^*, L_t^*) \), \( \forall t > T \), \( \varphi = \varphi^* \), \( T = T^* \), and \( K_t = K_t^* \), \( \forall t \). Note that \( \{ C_t^*, L_t^*, K_{t+1}^* \}_{t=0}^{\infty} \) is a solution to \( (P_T) \), and \( (P_T) \) has a finite number of constraints. By a Lagrangian theorem in Luenberger (1969), there exists non-negative, not identically zero vector \( \{ r_T, \mu_T, \eta_{T1}, \ldots, \eta_{T1}, \gamma_0^T, \ldots, \gamma_T^T \} \) such that the first-order and complementarity conditions hold, i.e. \( \forall t \geq 1 \)
\[
\frac{\beta^t}{q_t} \left\{ r^T A_C + \sum_i \pi^i \eta^T (1 - \sigma) \psi^i + \sum_{s=0}^t \gamma^T A_C \right\} \Phi_C^V C_t^{-\sigma} = \mu^T \tag{B.4}
\]
\[
\frac{\beta^t}{q_t} \left\{ r^T A_L + \sum_i \pi^i \eta^T (1 + \nu) \psi^i + \sum_{s=0}^t \gamma^T A_L \right\} \Phi_L^V L_t^V = \mu^T F_L(K_t, L_t) \tag{B.5}
\]

Since the allocation is interior and \( A_C, A_L > 0 \), one can rewrite the first-order conditions as
\[
\frac{\beta^t}{q_t} \left\{ r^T A_C + \sum_i \pi^i \eta^T (1 - \sigma) \psi^i + \sum_{s=0}^t \gamma^T A_C \right\} \Phi_C^V C_t^{-\sigma} = \mu^T
\]
\[
\frac{\beta^t}{q_t} \left\{ r^T A_C + \sum_i \pi^i \eta^T (1 + \nu) \psi^i A_C + \sum_{s=0}^t \gamma^T A_C \right\} \Phi_C^V C_t^{-\sigma} = \mu^T \frac{A_C}{A_L} F_L(K_t, L_t) \frac{\Phi^V C_t^{-\sigma}}{\Phi^V L_t^V}
\]

Subtracting the first from the second line gives
\[
\frac{\beta^t}{q_t} \left\{ \Phi_C^V \sum_i \pi^i \eta^T \left( \frac{A_C}{A_L} (1 + \nu) \psi^i - (1 - \sigma) \psi^i_c \right) \right\} C_t^{-\sigma} = \mu^T \left[ \frac{A_C}{A_L} F_L(K_t, L_t) \frac{\Phi^V C_t^{-\sigma}}{\Phi^V L_t^V} - 1 \right] \tag{B.6}
\]

The following lemma shows that the resource constraint binds for any sub-problem \((P^T)\) and \( T \geq 1 \).

**Lemma B.2.** In any sub-problem \((P^T)\) with \( T \geq 1 \), \( \mu^T > 0 \)

**Proof.** Suppose, by contradiction, that \( \mu^T = 0 \) so the resource constraint does not bind. Consider allocation \( \{C_t, L_t, K_{t+1}\}_{t=0}^\infty \) which is the solution to \((P^T)\). Then there exists \( \epsilon > 0 \) such that
\[
\sum_{t=0}^\infty q_t [F(K_t, L_t) + (1 - \delta) K_t - K_{t+1} - C_t - G_t] - B_0 - \epsilon \geq 0
\]

Define \( \{\hat{C}_t\}_{t=0}^\infty \) such that \( \hat{C}_1 = C_1 + \epsilon/q_1 \) and \( \hat{C}_t = C_t, \forall t \neq 1 \). Note that allocation \( \{\hat{C}_t, L_t, K_{t+1}\}_{t=0}^\infty \) satisfies the resource constraint and because of strict monotonicity, \( \{\hat{C}_t, L_t, K_{t+1}\}_{t=0}^\infty \) also satisfies the implementability constraints and the aggregate debt constraints. However,
\[
\sum_{t=0}^\infty \sum_{i \in I} \beta^t \lambda^i \pi^i U^i \left[ h^i(\hat{C}_t, L_t; \phi) \right] > \sum_{t=0}^\infty \sum_{i \in I} \beta^t \lambda^i \pi^i U^i \left[ h^i(C_t, L_t; \phi) \right]
\]
which contradicts \( \{C_t, L_t, K_t\}_{t=0}^\infty \) being optimal solution for \((P^T)\). \( \square \)
By Lemma B.2 and interior allocation, we can rewrite equation (B.6) as

\[
\frac{\Phi^V}{\mu^T} \sum_i \pi^i \eta^{T,i} \left[ \frac{A_C}{A_L} (1 + \nu) \psi^i - (1 - \sigma) \psi^i \right] = \frac{q_t}{\beta t} \left[ \frac{A_C}{A_L} F_L(K_t, L_t) \frac{\Phi^V C_i^{-\sigma}}{\Phi^V L_t^\nu} - 1 \right]
\]

Specifically, for any \( T \geq 1 \),

\[
\frac{\Phi^V}{\mu^T} \sum_i \pi^i \eta^{T,i} \left[ \frac{A_C}{A_L} (1 + \nu) \psi^i - (1 - \sigma) \psi^i \right] = \frac{q_t}{\beta} \left( C_1^* \right) \frac{A_C}{A_L} F_L(1) \frac{\Phi^V (C^*_i)^{-\sigma}}{\Phi^V (L_1^*)^\nu} - 1
\]

Note that the left-hand side is a function of \((C_1^*, L_1^*, K_1^*)\), which implies that there exists a \(-\infty < \kappa < \infty\) such that \( \forall T \geq 1 \),

\[
\frac{\Phi^V}{\mu^T} \sum_i \pi^i \eta^{T,i} \left[ \frac{A_C}{A_L} (1 + \nu) \psi^i - (1 - \sigma) \psi^i \right] = \kappa
\]

Hence, (B.6) can be rewritten as

\[
\beta t \frac{C_i^{-\sigma}}{q_t} \kappa = \frac{A_C}{A_L} F_L(K_t, L_t) \frac{\Phi^V C_i^{-\sigma}}{\Phi^V L_t^\nu} - 1
\]

Note that \( \lim_{t \to \infty} \beta t / q_t = 0 \) and \( C_i^{-\sigma} \) is bounded by Lemma 5.1, so taking the limit on both sides gives

\[
\lim_{t \to \infty} \frac{A_C}{A_L} F_L(K_t, L_t) \frac{\Phi^V C_i^{-\sigma}}{\Phi^V L_t^\nu} = 1
\]

Hence, given the definition of \( \tau_i^n \) and the fact that \( A_C, A_L \) are bounded,

\[
\lim_{t \to \infty} \tau_i^n = \lim_{t \to \infty} \left[ 1 - \frac{\Phi^V L_t^\nu}{\Phi^V C_i^{-\sigma} F_L(K_t, L_t)} \right] = 1 - \frac{A_C}{A_L}
\]

In addition, the above argument does not rely on the existence of lump-sum transfers. \( \square \)
B.5 Proof of Proposition 6.2

Proof. $\lambda^i = \phi^i, \forall i \in I$ implies that $A_C = 1$ and $A_L = 1$. Therefore, $\{C^*_t, L^*_t\}_{t=0}^{\infty}, \phi^*, T^*$ solves

$$
\max \sum_{t=0}^{\infty} \beta^t V(C_t, L_t; \phi) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} q_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t] - B_0 \geq 0 \\
\sum_{t=0}^{\infty} \beta^t [V_C(t; \phi)h^c(t; \phi) + V_L(t; \phi)h^l(t; \phi)] \geq V_C(0; \phi) (a^i_0 - T) \\
\sum_{s=1}^{\infty} \beta^{s-t} V(C_t, L_t; \phi) \geq U_t(K_t)
$$

To implement $\{C^*_t, L^*_t\}_{t=0}^{\infty}, \phi^*, T^*$ given the specified tax system, by (21), it must be that

$$
\lim_{t \to \infty} \tau^n_t = 0
$$

\[\square\]

B.6 Proof of Lemma 8.1

Proof. For any competitive equilibrium, there exists market weight $\phi = \{\phi^i\}_{i \in I}$ such that $\forall t$, given $C_t, L_t$, individual assignment $\{c^i_t, l^i_t\}_{i \in I}$ solves

$$
V(C_t, L_t; \phi) = \max \sum_{(c^i_l)_{i \in I}} \phi^i \pi^i \left[u(c^i) - v \left(\frac{l^i}{\theta^i}\right)\right] \\
\text{s.t.} \quad \sum_i \pi^i c^i = C_t; \quad \sum_i \pi^i l^i = L_t
$$

Let $\mu^m$ and $\eta^m$ be the Lagrange multipliers on the consumption and labor constraints. The first-order conditions for interior solutions are

$$
c^i_t = u^{-1}(\mu^m / \phi^i) \quad \text{(B.7)} \\
l^i_t = \theta^i v^{-1}(\theta^i \eta^m / \phi^i) \quad \text{(B.8)}
$$
Substituting for $c_t^i$ and $l_t^i$ in the constraints gives
\[
\sum_{i \in I} \pi^i u_{c}^{-1} \left( \mu^m / \varphi^i \right) = C_t \\
\sum_{i \in I} \pi^i \theta^i v_{l}^{-1} \left( \theta^i \eta^m / \varphi^i \right) = L_t
\]

These equations imply functions $\mu^m(C_t)$ and $\eta^m(L_t)$. Substituting in (B.7) and (B.8), for all $i$ implies that
\[
c_t^i = u_{c}^{-1} \left( \mu^m(C_t) / \varphi^i \right) \\
l_t^i = \theta^i v_{l}^{-1} \left( \theta^i \eta^m(L_t) / \varphi^i \right)
\]

Thus, the time-invariant functions $h_{i,c}^i(\cdot; \varphi), h_{i,l}^i(\cdot; \varphi)$ are
\[
h_{i,c}^i(C_t; \varphi) = u_{c}^{-1} \left( \mu^m(C_t) / \varphi^i \right) \\
h_{i,l}^i(L_t; \varphi) = \theta^i v_{l}^{-1} \left( \theta^i \eta^m(L_t) / \varphi^i \right)
\]

Note that $u_c(\cdot)$ is strictly decreasing, so $u_{c}^{-1}(\cdot)$ is strictly decreasing. This implies that $\mu^m(\cdot)$ is strictly decreasing. Then $h_{i,c}^i(\cdot; \varphi)$ must be strictly increasing. Similarly, one can argue that $h_{i,l}^i(\cdot; \varphi)$ is also strictly increasing.

**B.7 Proof of Lemma 8.2**

*Proof.* Given an efficient allocation $\{C_t^*, L_t^*, K_t^*\}_{t=0}^\infty$, suppose that $\lim \inf_{t \to \infty} C_t^* \leq 0$. Find $\epsilon > 0$ such that $\forall t$,
\[
\sum_{s=t}^\infty \beta^{s-t} \left\{ \sum_{i \in I} \lambda^i \pi^i \left[ u \left( h_{i,c}^i(C_s; \varphi) \right) - v \left( h_{i,l}^i(L_s^*; \varphi) \right) \right] \right\} \leq M_U
\]
with $C_t = \epsilon$ and $C_s = C_s^*$, $\forall s \geq t$. Such $\epsilon$ exists since the utility function is unbounded. Furthermore, there exists $t_0$ such that $C_{t_0}^* < \epsilon$. Then since $u(\cdot)$ and $h_{i,c}^i(\cdot; \varphi)$ are strictly increasing,
\[
\sum_{s=t_0}^{\infty} \beta^{s-t_0} \left\{ \sum_{i \in I} \lambda^i \pi^i \left[ u \left( h^{i,c}(C_s^*, \varphi) \right) - v \left( h^{i,l}(L_s^*, \varphi) \right) \right] \right\} < \sum_{s=t_0}^{\infty} \beta^{s-t_0} \left\{ \sum_{i \in I} \lambda^i \pi^i \left[ u \left( h^{i,c}(C_s^*, \varphi) \right) - v \left( h^{i,l}(L_s^*, \varphi) \right) \right] \right\} \\
\leq M_U \\
\leq U_*(K_t^*)
\]

which is a contradiction. \(\square\)

### B.8 Proof of Proposition 8.1

**Proof.** The proof follows a similar structure of the proof of Proposition 5.1. Let \(\{C_t^*, L_t^*, K_t^*\}_{t=0}^{\infty}, \varphi^*, T^*\) be an interior efficient allocation. Then there exist \(\lambda\) such that \(\{C_t^*, L_t^*, K_t^*\}_{t=0}^{\infty}, \varphi^*, T^*\) solves the planning problem (P). For any interior allocation \(\{C_t, L_t, K_t\}_{t=0}^{\infty}, \varphi, T\) from problem (P), define the followings

\[
A_C(t) = \sum_{i \in I} \pi^i \frac{\lambda^i}{\varphi^i} \frac{\partial h^{i,c}(t; \varphi)}{\partial C_t} \tag{B.9}
\]

\[
A_L(t) = \sum_{i \in I} \pi^i \frac{\lambda^i}{\varphi^i} \frac{\partial h^{i,l}(t; \varphi)}{\partial L_t} \tag{B.10}
\]

Then the following lemma holds.

**Lemma B.3.** Given an interior allocation, for all \(t, 0 < \frac{\partial h^{i,c}(t; \varphi)}{\partial C_t}, \frac{\partial h^{i,l}(t; \varphi)}{\partial L_t} < \infty\), and so \(0 < A_C(t), A_L(t) < \infty\)

**Proof.** First, it must be that \(\varphi^i > 0, \forall i\). Suppose there exists an \(i\) such that \(\varphi^i = 0\). Then from the static sub-problem, it is optimal to set \(c_t^i = 0\) for all \(t\), which contradicts the assumption of interior allocation.

Note that from the proof of Lemma 8.1, using implicit function derivatives, one has

\[
\frac{\partial h^{i,c}(t; \varphi)}{\partial C_t} = \frac{1}{\varphi^i u_{ic}(h^{i,c}(t; \varphi))} \sum_{i} \pi^i \frac{1}{\varphi^i u_{ic}(h^{i,c}(t; \varphi))} \theta^i \tag{B.9}
\]

\[
\frac{\partial h^{i,l}(t; \varphi)}{\partial L_t} = \frac{1}{\varphi^i v_{il}(h^{i,l}(t; \varphi))} \sum_{i} \pi^i \frac{1}{\varphi^i v_{il}(h^{i,l}(t; \varphi))} \theta^i \tag{B.10}
\]
Given \( u_{cc} < 0, v_i > 0 \) by assumption 4, and \( \phi^i > 0, \forall i \), it must be that \( \frac{\partial \bar{h}^{i,c}(t;\phi)}{\partial \phi^i}, \frac{\partial \bar{h}^{i,l}(t;\phi)}{\partial \phi^i} > 0 \). Moreover, \( \sum_{i=1}^{n} \frac{\partial \bar{h}^{i,c}(t;\phi)}{\partial \phi^i} = \sum_{i=1}^{n} \frac{\partial \bar{h}^{i,l}(t;\phi)}{\partial \phi^i} = 1 \). This implies that \( \frac{\partial \bar{h}^{i,c}(t;\phi)}{\partial \phi^i}, \frac{\partial \bar{h}^{i,l}(t;\phi)}{\partial \phi^i} < 1 \).

Since all the terms are positive and bounded, by definition, \( A_C(t) \) and \( A_L(t) \) are positive and bounded.

Define \( (P^T) \) the same problem as \( (P) \) with the restriction that \( (C_t, L_t) = (C^*_t, L^*_t) \), \( \forall t > T, \phi = \phi^* \), \( T = T^* \), and \( K_t = K^*_t, \forall t \). Note that \( \{C^*_t, L^*_t, K^*_t\}_{t=0}^{\infty} \) is a solution to \( (P^T) \), and \( (P^T) \) has a finite number of constraints. By a Lagrangian theorem in Luenberger (1969), there exists non-negative, not identically zero vector \( \{r^T, \mu^T, \eta^{T,1}, \ldots, \eta^{T,l}, \gamma_0^T, \ldots, \gamma_T^T\} \) such that the first-order and complementarity conditions hold, i.e. \( \forall t \geq 1 \)

\[
\begin{align*}
\frac{\beta_t}{q_t} \left( r^T A_C(t) + \sum_i \pi^i \eta^{T,i} \left[ \frac{V_C(t;\phi) h^{i,c}(t;\phi)}{V_C(t;\phi)} \frac{\partial h^{i,c}(t;\phi)}{\partial C_t} \right] + \sum_s \gamma^T_s A_C(t) \right) & \quad \text{for } V_C(t;\phi) = \mu^T \tag{B.11}
\end{align*}
\]

\[
\begin{align*}
\frac{\beta_t}{q_t} \left( r^T A_L(t) + \sum_i \pi^i \eta^{T,i} \left[ \frac{V_L(t;\phi) h^{i,l}(t;\phi)}{V_L(t;\phi)} \frac{\partial h^{i,l}(t;\phi)}{\partial L_t} \right] + \sum_s \gamma^T_s A_L(t) \right) & \quad \text{for } V_L(t;\phi) = -\mu^T F_L(K_t, L_t) \tag{B.12}
\end{align*}
\]

Using the Envelope conditions of the static sub-problem, one can show that

\[
\frac{V_C(t;\phi) h^{i,c}(t;\phi)}{V_C(t;\phi)} = \frac{u_{cc}}{u_c} \left[ \frac{h^{i,c}(t;\phi)}{h^{i,c}(t;\phi)} \right] \frac{\partial h^{i,c}(t;\phi)}{\partial C_t}
\]

\[
\frac{V_L(t;\phi) h^{i,l}(t;\phi)}{V_L(t;\phi)} = \frac{v_i}{v_l} \left[ \frac{h^{i,l}(t;\phi)}{h^{i,l}(t;\phi)} \right] \frac{\partial h^{i,l}(t;\phi)}{\partial L_t}
\]

Define \( \sigma^i_t = -\frac{u_{cc}}{u_c} \left[ h^{i,c}(t;\phi) \right] h^{i,c}(t;\phi) \) and \( \nu^i_t = \frac{v_i}{v_l} \left[ h^{i,l}(t;\phi) \right] h^{i,l}(t;\phi) \), then equations (B.11) and (B.12) become

\[
\begin{align*}
\frac{\beta_t}{q_t} \left( r^T A_C(t) + \sum_i \pi^i \eta^{T,i} \left( 1 - \sigma^i_t \right) \frac{\partial h^{i,c}(t;\phi)}{\partial C_t} \right) + \sum_s \gamma^T_s A_C(t) & \quad \text{for } V_C(t;\phi) = \mu^T \tag{B.13}
\end{align*}
\]

\[
\begin{align*}
\frac{\beta_t}{q_t} \left( r^T A_L(t) + \sum_i \pi^i \eta^{T,i} \left( 1 + \nu^i_t \right) \frac{\partial h^{i,l}(t;\phi)}{\partial L_t} \right) + \sum_s \gamma^T_s A_L(t) & \quad \text{for } V_L(t;\phi) = -\mu^T F_L(K_t, L_t) \tag{B.14}
\end{align*}
\]

Since the allocation is interior and \( A_C(t), A_L(t) > 0 \) by Lemma B.3, one can combine
Lemma B.5. an important property of

Proof. binds.

Lemma B.4. In any subproblem \( (P^T) \) with \( T \geq 1 \), \( \mu_T > 0 \), i.e. the resource constraint binds.

Proof. Follows directly from the proof of Lemma B.2. \( \square \)

Given Lemma B.4 and interior allocation, (B.15) becomes

\[
\frac{1}{\mu_T} \left\{ \sum_i \pi^i \eta^{T,i} \left[ (1 - \sigma^i_t) \frac{\partial h^{i,c}(t; \varphi)}{\partial C_t} - \frac{A_C(t)}{A_L(t)} (1 + \nu_t^i) \frac{\partial h^{i,l}(t; \varphi)}{\partial L_t} \right] \right\} V_C(t; \varphi)
\]

\[
= \mu_T \left[ 1 + F_L(K_t, L_t) \frac{V_C(t; \varphi) A_C(t)}{V_L(t; \varphi) A_L(t)} \right]
\]

Define the left-hand side of the above equation as \( \kappa(t) \), then the following lemma gives an important property of \( \kappa(t) \).

Lemma B.5. For any sub-problem \( (P^T) \) with \( T \geq 1 \), \( \kappa(t) \) is bounded \( \forall t \geq 1 \).

Proof. Note that \( \forall t, \forall i \), by assumption 5, \( \sigma^i_t \) and \( \nu^i_t \) are bounded.

Any sub-problem \( (P^T) \) with \( T \geq 1 \) has

\[
\sum_i \frac{\eta^{T,i}}{\mu_T} \pi^i \left[ (1 - \sigma^i_t) \frac{\partial h^{i,c}(s; \varphi)}{\partial C_t^*} - \frac{A_C^*(s)}{A_L^*(s)} (1 + \nu^i_t) \frac{\partial h^{i,l}(s; \varphi)}{\partial L_t^*} \right]
\]

\[
= \frac{q_t}{\beta^s} \frac{1}{V_C^*(s; \varphi)} \left[ 1 + F_L^*(s) \frac{V_C^*(s; \varphi) A_C^*(s)}{V_L^*(s; \varphi) A_L^*(s)} \right]
\]

for \( s = 1, \ldots, ||I|| \).

The above equations formulate a linear system with respect to \( ||I|| \) variables \( \left\{ \frac{\eta^{T,i}}{\mu_T} \right\}_{i \in I} \). By Lemma B.3 and interior allocation, the right-hand sides and the coefficients are bounded. Therefore, for any \( T \), \( \left\{ \frac{\eta^{T,i}}{\mu_T} \right\}_{i \in I} \) are functions of \( \left\{ C^*_s, L^*_s, K^*_s \right\}_{s=0}^{||I||}, \varphi^* \) and bounded.

So \( \forall t \geq 1 \),

\[
\kappa(t) = \sum_i \frac{\eta^{T,i}}{\mu_T} \pi^i \left[ (1 - \sigma^i_t) \frac{\partial h^{i,c}(t; \varphi)}{\partial C_t} - \frac{A_C(t)}{A_L(t)} (1 + \nu^i_t) \frac{\partial h^{i,l}(t; \varphi)}{\partial L_t} \right]
\]

is bounded. \( \square \)
Substituting for $\kappa(t)$ into equation (B.15) provides

$$\frac{\beta}{q_t} \kappa(t)V_C(t; \varphi) = \left[ 1 + F_L(K_t, L_t) \frac{V_C(t; \varphi) A_C(t)}{V_L(t; \varphi) A_L(t)} \right]$$

Assumption 1 implies that $\lim_{t \to \infty} \frac{\beta}{q_t} = 0$. By Lemma B.5, $\kappa(t)$ is bounded. Since $\lim_{t \to \infty} C_t > 0$ from Lemma 8.2, $V_C(t; \varphi) = \varphi' u_c(h^c(t; \varphi))$ is bounded. Then taking the limit as $t \to \infty$ on both sides of the above equation gives

$$\lim_{t \to \infty} \left[ 1 + F_L(K_t, L_t) \frac{V_C(t; \varphi) A_C(t)}{V_L(t; \varphi) A_L(t)} \right] = 0$$

From Lemma B.3, it must be true that as $t$ approaches infinity, we have that $0 < A_C(t), A_L(t) < \infty$, so $-\infty < \lim_{t \to \infty} A_C(t) / A_L(t) < \infty$. Define $\tau = 1 - \limsup_{t \to \infty} A_C(t) / A_L(t)$ and $\bar{\tau} = 1 - \liminf_{t \to \infty} A_C(t) / A_L(t)$. Then using the definition of $\tau^n_t$ gives

$$\lim_{t \to \infty} \tau^n_t = \liminf_{t \to \infty} \left[ 1 + \frac{1}{F_L(K_t, L_t)} \frac{V_L(t; \varphi)}{V_C(t; \varphi)} \right] = 1 - \limsup_{t \to \infty} A_C(t) / A_L(t) = \tau$$

$$\limsup_{t \to \infty} \tau^n_t = \limsup_{t \to \infty} \left[ 1 + \frac{1}{F_L(K_t, L_t)} \frac{V_L(t; \varphi)}{V_C(t; \varphi)} \right] = 1 - \liminf_{t \to \infty} A_C(t) / A_L(t) = \bar{\tau}$$

In the case of steady states, it must be true that $A_C(\infty), A_L(\infty)$ exist and that $0 < A_C(\infty), A_L(\infty) < \infty$. Hence, $\lim_{t \to \infty} \tau^n_t = 1 - A_C(\infty) / A_L(\infty)$. Similarly, the argument of the proof does not rely on lump-sum transfers.

B.9 Proof of Lemma 7.1

Proof. Note that the sustainability constraint is rewritten as $\forall t$,

$$\sum_{s=t}^{\infty} \beta^{s-t} \left[ \Phi_C^P \frac{C_s^{1-\sigma}}{1-\sigma} - \Phi_L^P \frac{L_s^{1+\nu}}{1+\nu} \right] \geq U$$

Define $u_t = \Phi_C^P \frac{C_t^{1-\sigma}}{1-\sigma} - \Phi_L^P \frac{L_t^{1+\nu}}{1+\nu}$. Then the proof is similar to Lemma 2 in Aguiar and Amador (2016).

C Numerical Appendix

This section explains the numerical algorithm that is implemented in Section 7 for a simple environment with no capital, and additional plots.
C.1 Deviation Utility

The deviation utility \( U \) is calculated as the maximum weighted utility attained from a competitive equilibrium with taxes where the government does not issue both external and domestic debts. Given that output is equal to the total effective labor supply, one has

\[
U \equiv \max_{c_t, l_t, \tau^n_t, T_t} \sum_{t=0}^{\infty} \beta^t \sum_{i \in I} \lambda^i \pi^i U^i(c^i_t, l^i_t)
\]

subject to

\[
c^i_t = (1 - \tau^n_t) l^i_t - T_t
\]

\[
G_t \leq \tau^n_t L_t + T_t
\]

There exist a vector of market weights \( \varphi \) such that

\[
U \equiv \max_{C_t, L_t, \varphi, C_t, L_t} \sum_{t=0}^{\infty} \beta^t \left[ \hat{\Phi}^W C_t^{1-\sigma} - \hat{\Phi}^W L_t^{1+\nu} \right]
\]

subject to

\[
C_t + G_t \leq L_t
\]

where \( \hat{\psi}_C, \hat{\psi}_L, \hat{\Phi}_C, \hat{\Phi}_L, \hat{\Phi}^W_C, \hat{\Phi}^W_L \) are calculated using \( \varphi \).

C.2 Algorithm

Define \( U^p(C_t, L_t; \varphi) = \sum_{i \in I} \lambda^i \pi^i [U^i(C_t, L_t; \varphi)] \).

Fix \( B_0^* \), the initial external debt level.

Step 1: Guess \( S \), the number of unconstrained-debt periods

Guess \( \varphi \) such that \( \sum_{i \in I} \pi^i \varphi^i = 1 \)

Step 2: Calculate \( \eta \) as functions of \( \varphi \) by the optimal condition

\[
\eta^i = \sum_j \pi^j \lambda^j / \varphi^j - \lambda^i / \varphi^i
\]

Solve for \( C_t, L_t \) and \( \gamma_s \), \( \forall s \geq S \) as functions of \( \varphi, \eta \) by using the first-order conditions and that \( U^p(C_s, L_s; \varphi) = (1 - \beta)U \), \( \forall s \geq S \)

Step 3: Iterate on \( \varphi \) and repeat step 2 until the left-hand sides of the implementability constraints are equal across individuals

Step 4: Solve for steady states \( C_\infty, L_\infty \) that satisfy \( U^p_C(C, L; \varphi) = -U^p_L(C, L; \varphi) \) and \( U^p(C, L; \varphi) = (1 - \beta)U \)

Calculate steady-state external debt level

\[
B_\infty = r^* (L_\infty - C_\infty - G) / (1 + r^*)
\]

Calculate \( B_0 \) by iterating backwards

\[
B_t = L_t - C_t - G + B_{t+1} / (1 + r^*)
\]
Step 5: Iterate on $S$ and repeat step 2-4 until $B_0$ is equal to $B_0^*$

C.3 Additional figures

Figure 5: Time paths of aggregate allocation by relative inequality

(a) Aggregate effective labor

(b) Aggregate consumption

- $\theta^H/\theta^L = 1$
- $\theta^H/\theta^L = 2$
- $\theta^H/\theta^L = 3$
- $\theta^H/\theta^L = 4$
- $\theta^H/\theta^L = 5$