Income Taxation and Stochastic Interest Rates

Preliminary and Incomplete: Please Do Not Quote or Circulate

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Abstract

Note to NTA conference organizers: This is a very preliminary set of notes that form the basis for a paper I am in the process of writing. I anticipate that it will be substantially more developed by the time of the conference in November but that it would still benefit greatly from the feedback that I would be able to receive at the conference.

The basic idea is relatively straightforward. I model a two-period world with a stochastic risk-free rate. This rate is fixed over a single period, but its value in the second period is uncertain from the perspective of the start of the first period. If mark-to-market taxation is applied in this world, then the present value of the tax burden on a fixed-rate risk-free bond is different than a floating rate risk-free bond, and in particular there can be a positive expected payoff to engaging in a zero-cost receive-fixed / pay-floating swap.

The positive expected payoff is not a true arbitrage opportunity, because it has risk associated with it. However, I will show in the next draft of the paper that there is an opportunity for taxpayers subject to different tax rates (such as tax exempts and taxable entities) to coordinate to achieve an aggregate positive cash flow at the government's expense on an after-tax basis by engaging in interest rate swaps. The extent to which parties will do this depends upon their risk aversion, but for relatively risk averse parties, the engagement in such behavior could be quite large.

This is important because mark-to-market taxation is often proposed as an alternative to realization-based taxes and may be applied to derivatives (under recent and current Congressional proposals). Such a change would indeed alleviate tax game-playing associated with realization-based taxes, but it would then put the focus on a new sort of game based on interest rate swaps. I believe that I am the first to describe the existence of such a game in the academic literature, although it may well play out in the real world already for taxpayers already subject to mark-to-market taxation.

I also will provide a discussion of the severity of this newly identified problem faced by mark-to-market taxation, as well as describing techniques for mitigating the problem.

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1 Introduction

In a two period model, with an intervening time interval for investment, a flat income tax with full loss offsets is equivalent to an up-front wealth tax. The wealth tax does not depend upon the investment choices made by the taxpayer, and so there is no distortion of investment choices, beyond of course any distortion resulting from levying the wealth tax itself. This was proven in a powerful sense in a general equilibrium setting by Louis Kaplow.[1] These notes investigate the extent to which the equivalence holds when more than two periods are allowed, so that investment choices can be revised over time, with the additional assumption that the income tax is levied at the end of each time interval.

2 Three-period / Two-interval Model

To start with, consider a model involving three times, 0, 1, and 2, with a first interval for investment between times 0 and 1 and a second interval for investment between times 1 and 2. Let s_1 index the possible states of the world at time 1, and let q_{s_1} be the price of a payoff of \$1 in state s_1, and \$0 in any other state. Any investment made at time 0 for the first interval can be expressed as a set of choices α_{s_1}, with the constraint that $\sum_{s_1} \alpha_{s_1} q_{s_1} = 1$. Each individual α_{s_1} represents the fraction of initial wealth invested in the outcome for state s_1. Write $Z(1)$ for the zero-coupon bond that pays \$1 for sure at the end of the first interval. The price of this bond at time 0 is $Z_0(1) = \sum_{s_1} q_{s_1}$, and the risk-free rate of return for the first interval satisfies $\frac{1}{1+r_1} = \sum_{s_1} q_{s_1}$. We can rearrange this to see that $r_1 = \left(\sum_{s_1} q_{s_1}\right)^{-1} - 1$.

For the second interval, the notation is similar, except that "1" is replaced with "2" throughout. The states available to invest in for the second interval may depend upon the outcomes from the first interval, i.e., the list of s_2 possibilities may depend upon which state s_1 came to pass. It is of particular note that the risk-free rate in the second interval, $r_2 = \left(\sum_{s_2} q_{s_2}\right)^{-1} - 1$, generally depends upon the state s_1 that was realized, and it is not known ex ante. Write $Z(2)$ for the zero-coupon bond that pays \$1 for sure at the end of the second interval, and let $Z_i(2)$ be the price of this asset at time i. Evidently $Z_2(2) = 1$, and $Z_1(2) = 1/(1 + r_2)$, which is an uncertain quantity. The initial price $Z_0(2)$ is not uncertain. It is

$$
\begin{aligned}
Z_0(2) &= \sum_{s_1} \left(\sum_{s_2} q_{s_2} \right) q_{s_1} \\
&= \sum_{s_1} \left(\frac{1}{1+r_2} \right) q_{s_1}.
\end{aligned}
$$

Thus we can speak of a market-implied risk-free rate for the second interval, r_2^*, with the property that $Z_0(2) = \left(\frac{1}{1+r_1}\right)\left(\frac{1}{1+r_2^*}\right)$. The value of r_2^* can be expressed as

$$
r_2^* = \left(\sum_{s_1} \left(\sum_{s_2} q_{s_2} \right) \frac{q_{s_1}}{\sum_{s_1} q_{s_1}} \right)^{-1} - 1.
$$

[1] Louis Kaplow, "Taxation and Risk Taking: A General Equilibrium Perspective", 46(4) National Tax Journal 789 (1994).

This ex ante value will of course not necessarily be the same as the actual future stochastic value r_2.

Consider an investor with an up-front amount of wealth, W, to invest, and suppose that he chooses asset allocations for the two time intervals given by α_{s_1} and α_{s_2}. The pre-tax return is

$$W \left(\sum_{s_1} \alpha_{s_1} \delta_{s_1} \right) \left(\sum_{s_2} \alpha_{s_2} \delta_{s_2} \right),$$

where δ_{s_i} is a delta function that pays \$1 in state s_i at the end of the i-th interval, and \$0 in all other states. The values of the α_{s_i} are constrained by the conditions that $\sum_{s_i} \alpha_{s_i} q_{s_i} = 1$ for each $i = 1, 2$, so that exactly 100% of the available wealth is allocated during each interval.

3 Adding Tax to the Model

Under an income tax levied at the end of each period, the investor ends with

$$W \left((1-t) \sum_{s_1} \alpha_{s_1} \delta_{s_1} + t \right) \left((1-t) \sum_{s_2} \alpha_{s_2} \delta_{s_2} + t \right), \tag{1}$$

and this can be written as

$$W \left((1-t)^2 \sum_{s_1} \alpha_{s_1} \delta_{s_1} \sum_{s_2} \alpha_{s_1} \delta_{s_2} + t(1-t) \sum_{s_1} \alpha_{s_1} \delta_{s_1} + t(1-t) \sum_{s_2} \alpha_{s_2} \delta_{s_2} + t^2 \right). \tag{2}$$

How can this income tax be converted into an up-front wealth tax? In other words, what amount of initial wealth would allow an investor to reproduce exactly this after-tax result in a no-income-tax world? It helps to think separately about each of the four terms in the parenthetical. For the first term, if the investor had a fraction $(1-t)^2$ of his initial wealth, he could reproduce the result by simply using the same α_{s_1} and α_{s_2} as before. Let us skip the second term for a moment. For the third term, if the investor had a fraction $\frac{t(1-t)}{1+r_1}$ of his initial wealth, he could reproduce the result by investing this amount in the risk-free asset during the first interval and then investing using α_{s_2} for the second interval. For the fourth term, if the investor had a fraction $\frac{t^2}{(1+r_1)/(1+r_2^*)}$, he could reproduce the result by investing this amount in $Z(2)$, the zero-coupon bond that pays \$1 for sure at time 2. So far, for all terms but the second, the investor would need a fraction

$$(1-t)^2 + \frac{t(1-t)}{1+r_1} + \frac{t^2}{(1+r_1)(1+r_2^*)} = \left(1 - \frac{tr_1}{1+r_1}\right) \left(1 - \frac{tr_2^*}{1+r_2^*}\right) - \frac{t(1-t)}{1+r_2^*}. \tag{3}$$

This final expression is close to what we might expect from the familiar one-interval results, which have the form $1 - \frac{tr}{1+r}$. The difference is that there is an extra term involving $t(1-t)/(1+r_2^*)$, and this corresponds to the second term in the parenthetical in (2).

How much wealth does the taxpayer need to replicate the second term from (2), which is a fraction of initial wealth equal to

$$t(1-t) \sum_{s_1} \alpha_{s_1} \delta_{s_1}.$$

In a sense, this is easy, because you could get by with simply a fraction $t(1-t)$ of your original wealth. If this is invested according to α_{s_1} and then just left "under the mattress" through the second interval, the investor gets what he wants. This is more wealth than the investor actually needs, however. What he can do instead is to invest in such a way that he has

$$t(1-t)\sum_{s_1}\left(\frac{1}{1+r_2}\right)\alpha_{s_1}\delta_{s_1}.$$

at the end of the first interval, and then he can reinvest at the risk-free rate r_2 for the second interval. This will get him exactly what he wants. The question is how much this costs. The up-front price is a fraction of wealth equal to

$$t(1-t)\sum_{s_1}\left(\frac{1}{1+r_2}\right)\alpha_{s_1}q_{s_1}.$$

If r_2 were a known constant, we could factor the term $1/(1+r_2)$ out of the sum, and use the constraint that $\sum_{s_1}\alpha_{s_1}q_{s_1}=1$ to simplify the term to $t(1-t)/(1+r_2)$. If, however, r_2 is stochastic because it depends upon the realized state s_1, the problem is more complex.

What can we say about the nature of the price of the second term? We can rewrite it as an expectation

$$t(1-t)\left(\frac{1}{1+r_1}\right)\mathbf{E}\left[\left(\frac{1}{1+r_2}\right)\alpha_{s_1}\right],$$

where the probability of state s_1 is given by $\pi_{s_1}=\frac{q_{s_1}}{\sum_{s_1}q_{s_1}}$. Combining this with (3), we find that the up-front wealth necessary to reproduce the desired final outcome is

$$\left(1-\frac{tr_1}{1+r_1}\right)\left(1-\frac{tr_2^*}{1+r_2^*}\right)-\frac{t(1-t)}{1+r_2^*}+\frac{t(1-t)}{1+r_1}\mathbf{E}\left[\left(\frac{1}{1+r_2}\right)\alpha_{s_1}\right]. \tag{4}$$

It is important to emphasize that it is possible to have an ex ante wealth tax that is equivalent to the income tax levied. The quantity in (4) expresses the fraction of the initial wealth that would remain for the investor after the up-front wealth tax is levied, and the balance would be the ex ante tax collected by the government. The investor and the government could exactly replicate the after-tax positions that they would have in an income tax world, and they would choose do so, assuming those positions reflected a state of general equilibrium. What is different from the two-period / one-interval situation, however, is that the final term in (4) depends explicitly on the portfolio choice of the investor. As a result, we cannot say that the income tax is only distortionary in the way a wealth tax would be.

4 Strategic Behavior by the Taxpayer

If r_2 is not stochastic, then because $\mathbf{E}[\alpha_{s_1}]=1+r_1$, we see that (4) simplifies to

$$\left(1-\frac{tr_1}{1+r_1}\right)\left(1-\frac{tr_2^*}{1+r_2^*}\right). \tag{5}$$

In this case, the up-front wealth tax does not depend upon on the portfolio choices α_{s_1} and α_{s_2}, and so we have a result similar to the one that holds in the single investment interval model.

We can write (4) as

$$\left(1 - \frac{tr_1}{1+r_1}\right)\left(1 - \frac{tr_2^*}{1+r_2^*}\right) - \frac{t(1-t)}{(1+r_1)(1+r_2^*)}\,\mathbf{E}\left[\left(\frac{r_2 - r_2^*}{1+r_2}\right)\alpha_{s_1}\right]. \tag{6}$$

This is the fraction of his initial wealth that a taxpayer would be left with if an up-front wealth tax were levied. The up-front wealth tax would be equivalent to an income tax levied on an ex post basis, conditional upon no change in the portfolio choice α_{s1}.

We can also express the quantity in (6) as

$$\left(1 - \frac{tr_1}{1+r_1}\right)\left(1 - \frac{tr_2^*}{1+r_2^*}\right) - \frac{t(1-t)}{1+r_1}\,\mathbf{E}\left[\left(Z_0(2)(1+r_1) - Z_1(2)\right)\alpha_{s_1}\right], \tag{7}$$

where $Z_i(2)$ is the price of the zero-coupon bond that pays \$1 for sure at time 2. The price $Z_1(2)$ is stochastic, but the values of $Z_0(2)$ and r_1 are deterministic. This alternative version will be convenient to have below.

If r_2 is deterministic rather than stochastic, it is immediate that (6) simply becomes

$$\left(1 - \frac{tr_1}{1+r_1}\right)\left(1 - \frac{tr_2^*}{1+r_2^*}\right). \tag{8}$$

This is also the result if r_2 is stochastic but α_{s_1} is orthogonal to $Z_0(2)(1+r_1) - Z_1(2)$. It is thus useful to decompose α_{s_1} into separate components, one of which is in the null space of the vector $Z_0(2)(1+r_1) - Z_1(2)$, and the other of which is in the orthogonal complement of the null space. In this way, we can write each α_{s_1} as

$$\alpha_{s_1} = \alpha_{s_1}^{\text{null}} + \alpha_{s_1}^{\text{comp}},$$

and we see that (7) becomes

$$\left(1 - \frac{tr_1}{1+r_1}\right)\left(1 - \frac{tr_2^*}{1+r_2^*}\right) - \frac{t(1-t)}{1+r_1}\,\mathbf{E}\left[\left(Z_0(2)(1+r_1) - Z_1(2)\right)\alpha_{s_1}^{\text{comp}}\right]. \tag{9}$$

The complement of the null space of $Z_0(2)(1+r_1) - Z_1(2)$ is simply a one-dimensional space spanned by multiples of $Z_0(2)(1+r_1) - Z_1(2)$, and every portfolio choice α_{s_1} will have some component, possibly zero, in this space. The coefficient of the component is given by $\alpha_{s_1}^{\text{comp}} = \mathbf{E}[(Z_0(2)(1+r_1) - Z_1(2))\alpha_{s_1}]$. An investor has considerable flexibility to choose the size of this component, because an investment in the payoff $Z_0(2)(1+r_1) - Z_1(2)$ is costless, thanks to the fact that the prices at time 0 of $Z_0(2)(1+r_1)$ and $Z_1(2)$ are both $Z_0(2)$. As a result, an investor can add as much of this component as he likes to his portfolio without changing its initial cost. Consider a fixed baseline choice of portfolio α_{s_1}, with fixed baseline component $\alpha_{s_1}^{\text{comp}}$, and allow an investor to add a multiple f of $Z_0(2)(1+r_1) - Z_1(2)$ to the baseline portfolio. The fraction in (9) becomes

$$\left(1 - \frac{tr_1}{1+r_1}\right)\left(1 - \frac{tr_2^*}{1+r_2^*}\right) - (f + \alpha_{s_1}^{\text{comp}})\frac{t(1-t)}{1+r_1}\,\mathbf{E}\left[\left(Z_0(2)(1+r_1) - Z_1(2)\right)^2\right]. \tag{10}$$

If the investor selects a negative f, then he increases his ex ante after-tax wealth, under a wealth tax that is equivalent to the income tax, conditional upon the chosen portfolio. The negative sign of f corresponds to engagement by the investor in receive-fixed / pay-floating interest rate swaps, in which the investor receives a payment corresponding to $Z_1(2)$ in exchange for making a payment corresponding to $Z_0(2)(1 + r_1)$. The investor essentially finances the purchase of a long-term bond with short-term borrowing. The more the investor does of this, the larger the magnitude of the negative f value, and the greater the ex ante after-tax wealth.

If there is no restriction, the investor may increase his initial wealth without bound. Of course there will be a restriction, likely because of either lack of counterparty availability or because of restrictions on tax refunds permitted by the tax system. Assuming that there is an ability to engage in the costless transaction up to some limit, however, it appears that a taxpayer can effectively reduce the tax burden on himself by engaging essentially in receive-fixed / pay-floating interest rate swaps.

If this analysis is correct, we would expected mark-to-market taxpayers, such as financial institutions, to engage in receive-fixed / pay-floating interest rate swaps. These swaps might be supplied by tax-insensitive parties, such as tax-exempt institutions or foreign investors. We would also anticipate that this strategic behavior could substantially reduce the effective tax burden on these parties. Note that it would not necessarily reduce the level of observed tax receipts, because these could still be collected to a meaningful degree, even if there was essentially no burden from the tax. (For example, the tax on a pure bet can still collect money, even though the bet may be rescaled by the taxpayer.)

5 Income Tax Perspective

The foregoing analysis has focused on the wealth tax perspective. It is natural to ask how the interest rate swap strategy would play out under an income tax. The results are equivalent, of course, but it is useful to understand both perspectives.

We can rewrite (1) as

$$\begin{aligned}
&W\left((1-t)\sum_{s_1}\alpha_{s_1}\delta_{s_1}+t\right)\left((1-t)\sum_{s_2}\alpha_{s_2}\delta_{s_2}+\left(\frac{1+r_2}{1+r_2^*}\right)t-\left(\frac{r_2-r_2^*}{1+r_2^*}\right)t\right)\\
&\quad=W\left(\sum_{s_1}\alpha_{s_1}\delta_{s_1}\right)\left(\sum_{s_2}\alpha_{s_2}\delta_{s_2}\right)\left(1-\frac{tr_1}{1+r_1}\right)\left(1-\frac{tr_2^*}{1+r_2^*}\right)\\
&\qquad-tW\left(1-\frac{tr_1}{1+r_1}\right)\sum_{s_1}\left(\frac{r_2-r_2^*}{1+r_2^*}\right)\alpha_{s_1}\delta_{s_1}.
\end{aligned}$$

This decomposition is informative because the expression after the equality in the middle line is the result that would be obtained if r_2 were not stochastic, while the expression in the third line reflects the difference resulting from the fact that r_2 is stochastic.

Now suppose that we adjust the baseline portfolio α_{s_1} by adding a multiple f of the costless asset with payoff $Z_0(2)(1+r_1) - Z_1(2)$. The result is

$$\begin{aligned}
W &\left(\sum_{s_1} \alpha_{s_1}\delta_{s_1}\right)\left(\sum_{s_2} \alpha_{s_2}\delta_{s_2}\right)\left(1-\frac{tr_1}{1+r_1}\right)\left(1-\frac{tr_2^*}{1+r_2^*}\right) \\
&\qquad - t\alpha_{s_1}^{\mathrm{comp}} W\left(1-\frac{tr_1}{1+r_1}\right)\sum_{s_1}\left(\frac{r_2-r_2^*}{1+r_2^*}\right)^2 \delta_{s_1} \\
&\qquad + fW(Z_0(2)(1+r_1)-Z_1(2))\left(\sum_{s_2} \alpha_{s_2}\delta_{s_2}\right)\left(1-\frac{tr_1}{1+r_1}\right)\left(1-\frac{tr_2^*}{1+r_2^*}\right) \\
&\qquad - tfW\left(1-\frac{tr_1}{1+r_1}\right)\sum_{s_1}\left(\frac{r_2-r_2^*}{1+r_2^*}\right)^2 \delta_{s_1}.
\end{aligned}$$

The first two lines are the result that would be obtained from the baseline portfolio alone. The last two lines are the additional impact from the costless portfolio adjustment. The third line has zero expected value, precisely because $Z_0(2)(1+r_1) - Z_1(2)$ is costless, but this line may give rise to additional volatility in results. The last line is the source of added value from strategic taxpayer behavior, and it will be positive if $f < 0$, so that receive-fixed / pay-floating interest rate swaps are purchased by the taxpayer.

The additional value to the taxpayer results from the fact that the government implicitly promises at time 0 to pay the future risk-free rate of return between time 1 and time 2. Because the value of r_2 is uncertain, this promise is valuable, and the strategy employed by the taxpayer takes advantage of this value in an specific way.

6 Additional Thoughts

The above results seem important because they call into question the wisdom of imposing mark-to-market taxation as a cure-all to problems with the realization-based income tax. It seems that a purely mark-to-market system lends itself to considerable game-playing, contrary to what is often believed, and it may also cause distortions in the interest rate yield curve. (Note, though, that the yield curve is already distorted for many other reasons, including those that give rise to the term premium.)

The problem seems to come from the fact that the historical information about basis is carried over from the start of the period to the end of the period, without adjustment for time. This could be fixed by "tweaking" basis in a certain way so as to eliminate the opportunity for strategic tax minimization by taxpayers. Doing so would be complex, and correct tweaks would ultimately transform the income tax into a wealth tax in any event. Thus, perhaps a better route would be simply to abandon the attempt to tax income and rather impose a regular wealth tax at a low rate, without any offset for basis. This would be simply and avoid the sorts of interest rate games described herein.