A Kernel Weighted Smoothed Maximum Score Estimator for the Endogenous Binary Choice Model

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Abstract

This paper considers a local control function approach for the binary response model under endogeneity. The objective of the Smoothed Maximum Score estimator (SMSE)(Horowitz 1992) is modified by weighting the observations with a kernel. Under some mild regularity conditions similar in nature to those of the SMSE, the consistency of this "Kernel Weighted Smoothed Maximum Score" estimator is established. Furthermore, under some reasonable smoothness conditions the estimator's asymptotic normality is derived with a convergence rate at least $n^{-3/8}$ which can be rendered arbitrary close to $n^{-1/2}$ as the regularity conditions improve. Additionally, the covariance of the limiting distribution can be estimated consistently from the sample at hand permitting convenient inferences. Under stronger regularity

conditions, an alternative C.A.N. estimator using a two stage procedure via Sieves is shown to achieve a faster rate of convergence. Some Monte Carlo experiments are conducted highlighting the robust advantage of these estimators.

Key words: Smoothed maximum score, Endogenous binary choice model, Control function.

JEL codes: C14,C31,C35.

1 Introduction

This paper considers the endogenous linear binary choice model:

$$\text{(i) } U = \dot{X}'\beta + \varepsilon,$$

$$\text{(ii) } A = \Pi'W + V,$$

$$Y = d(U) \text{ with } d(.) = 1[. \geq 0],$$

where Y is the observable response variable, $\dot{X}' = (Z', A)$ is a $1 \times K$ observable vector, W a $q \times 1$ observable vector, (ε, V) a couple of unobservable error terms, Π is a $q \times 1$ unknown parameter and β a $K \times 1$ parameter of interest. Write $\tilde{W}$ the components of W which are excluded from $\dot{X}$. Here the vector $S = (Z', \tilde{W}')$ contains exogenous "instruments" while A is the endogenous variable due to the stochastic relationship between ε and V. For simplicity assume that $\dot{X}$ contains no intercept since a later is not identifiable under the estimation technique which is to be exposed soon (unless a very specific condition holds i.e. $Med(\varepsilon|\bar{v}) = 0$ for some $\bar{v}$ in the support of V). It is important to point that A needs not be one dimensional. Under appropriate identification restrictions the results put

forth in this article may be easily generalizable when A contains distinct variables. Quite importantly, the proposed estimator allows for powers of the endogenous variable which is an advantage of the approach employed in this paper.

In the Economics literature the latent variable U usually represents the agent's willingness to pay or the difference in utility between two mutually exclusive alternatives. This model may have an omitted variable interpretation where A is correlated with ε through some unobservable factors. The model has also an "errors in variables" interpretation when A represents a misreported variable.

In principle when either $(\varepsilon, V)|S$ or $\varepsilon|S, V$ has a distribution function known up to some finite dimensional parameter one may estimate β consistently via maximum likelihood(ML). A vast literature assumes this is the case with a normal homoscedastic distribution posited for $(\varepsilon, V)|S$ such as in Heckman (1978), Amemiya (1978), Lee (1981), Newey (1987) or for $\varepsilon|S, V$ as in Smith and Blundell (1986) and Rivers and Vuong (1988). If the parametrization of the distribution in question is incorrect those estimators will be inconsistent. Since an assumption involving the parametrization of the distribution of ε is not testable per see (Newey 1985, Pagan and Vella 1989), new "semi parametric" estimators have been proposed relaxing this parametric requirement. For instance, the quasi ML estimator proposed in Rothe (2009) is consistent for β whenever the distribution function of $\varepsilon|\dot{X}, V$ depends only on $\dot{X}'\beta$ and V. Also, the two stage least square estimator developed in Lewbel (2000) is consistent for β provided there exists a "special regressor" in $\dot{X}$ meeting a certain conditional independence restriction. Even though these semi parametric estimators offer a robust advantage, they present some limitations in terms of either the permitted form of heteroscedastic-

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ity (Rothe 2009) or which variables affect the scedasticity of both ε and V (Lewbel 2000). This is due to the very nature of their distributional oriented assumptions.

Estimators that are robust to unknown heteroscedasticity are based instead on some conditional median restrictions which loosely speaking only require the center of the distribution of ε to remain unaffected by the covariates. For instance, Newey (1985) provided a consistent asymptotically normally distributed two stage maximum score estimator for β under the requirement that (V, ε) be symmetrically distributed around the origin conditional on S. Also, Hong and Tamer (2003) proposed a consistent minimum distance estimator for β under the less restrictive condition that $Med(\varepsilon|S) = 0$ *almost surely* (a.s.). However, in Newey (1985) a consistent estimator for the asymptotic covariance is not provided (see Newey 1985, page 228) while Hong and Tamer's estimator has an unknown limiting distribution.

The main motivation behind this article is to remedy this inferential problem offering a consistent estimator of β which only imposes a weak median restriction but does allow for testing. The main estimator presented in this article, named the "Kernel Weighted Smoothed Maximum Score" (KWSMS) estimator, meets these objectives. The KWSMS estimator is constructed by imposing a restriction on $Med(\varepsilon|S, V)$ which must not vary with the instruments. This ensures the existence of some random variable ϕ and unobservable term e such that $Y = d(\dot{X}'\beta + \phi + e)$ where now e satisfies the classic median restriction introduced for Maximum Score estimation (Manski 1985). Then, a smoothed Maximum Score estimation (Horowitz 1992) is performed as if ϕ were a constant correcting this approximation by means of a kernel. Doing so facilitates the asymptotic analysis using the framework laid out in Horowitz 1992. An interesting additional contri-

bution of this article is in fact to offer a robust estimation procedure for a semi linear random utility model.

Not surprisingly, this estimation approach imposes stronger assumptions than those required from the SMSE albeit similar in essence. The KWSMS estimator's consistency for β (up to a positive scale) requires that one element of $\dot{X}$ be fully supported and that the endogenous variable be continuous. Additionally, if certain cumulative distribution functions involving the random variables V and $\dot{X}'\beta$ are sufficiently differentiable then the KWSMS estimator is asymptotically normally distributed provided the fourth moments of $\dot{X}$ exist. Finally, the KWSMS estimator is $O_p(n^{-\frac{1}{2}+\epsilon})$ for some chosen $\epsilon \in (c, 1/8)$ where c is a positive constant becoming arbitrary small under adequate regularity conditions. Hence, the parametric rate is potentially achievable.

This paper relates to the previous literatures using the control function approach which has already been employed to handle endogeneity in the context of binary choice models (Blundell and Powell 2004), triangular equation models (Newey, Powell and Vella 1999) and quantile regression models (Lee 2007). Also, the technique used to derive the asymptotic results is similar to that of the SMSE using non parametric convolution based arguments. Finally, its local nature can be thought as a smoothed analogue of the local quantile regression estimator (Chaudhuri 1991, Lee 2003) in the context of the random utility model.

As explained in section 2.2, a KWSMS estimator in effect uses only observations of V close to a given value. This local nature suggests that the rate of convergence can be accelerated by using more observations of V instead. Thus, in this paper a second stage estimation is offered with a "Score Approximation Smoothed Maximum Score" (SASMS) estimator which uses

the information content from various KWSMS estimators retrieved in a first stage estimation. Under stronger regularity conditions the SASMS estimator is still consistent and asymptotically normally distributed while achieving a faster rate of convergence. Additionally, the SASMS estimator is a plug-in estimator which does not require solving a non linear optimization problem once the first stage estimation is completed.

The rest of the paper is organized as follows. Section 2 provides a rapid review of the control function approach in the context of this binary choice model and defines the KWSMS estimator. Section 3 presents some sufficient conditions for identification. Section 4 covers the KWSMS estimator's asymptotic properties. Section 5 gives some generic assumptions for the SASMS estimator's asymptotic properties. Finally, section 6 exhibits some Monte Carlo simulations to illustrate the finite sample qualities of the suggested estimators. All the proofs are to be found in the appendix.

At this point, it is convenient to introduce some notations used throughout the reminding sections:

(1)For f:$\mathbb{R} \longrightarrow \mathbb{R}$ define $f^{(j)}(t)$ its jth derivative at t whenever this later exists. Also, when the function is defined everywhere use $||f||_{sup} \equiv sup_{t \in \mathbb{R}}|f(t)|$.

(2)For a joint couple of real valued random variables (A, B) define $f_b(a)$ as the Lebesgue density of A conditional on $B = b$ whenever this later exists and $F_{A|B}(.) = P[A \leq .|B]$. When T is a measurable random variable, a property will be said to be met "a.e in t" if that the property is true except maybe when $t \in \mathfrak{N}$ with $P[T \in \mathfrak{N}] = 0$.

(3)For $s \in \mathbb{N}*$, define $\mathcal{K}_s = \{f : \mathbb{R} \longrightarrow \mathbb{R}$, f Borel, $||f||_{sup} < \infty$, $\int f(t)dt = 1$, $\int t^u f(t)dt = 0$ for $u = 1, ..., s - 1$ and $\int |t^u f(t)|dt < \infty$ for $u = 0, s\}$. Also, given a positive number M, define $C_\infty^s(M) = \{f : \mathbb{R} \longrightarrow \mathbb{R}, f^{(j)}$ exists

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and is continuous for $j = 0, 1, ..., s$ everywhere with $|f^{(j)}| < M$ for $j = 0, 1, ..., s\}$, and given a real number t define $C_\infty^s(t, M) = \{f : \mathbb{R} \longrightarrow \mathbb{R}$, f exists everywhere with $|f| < M$ and there exists an open neighborhood of t where $f^{(j)}$ exists, is continuous with $|f^{(j)}| < M$ for $j = 1, ..., s\}$.

(4) The triplet $(\Omega, \Im, P)$ refers to a probability space where Ω is the space of states of nature, $\Im$ is the sigma field of measurable events and P the probability measure.

(5) Given a strictly positive deterministic sequence $\{a_n\}_{n \geq 1}$ and a deterministic sequence $\{c_n\}_{n \geq 1}$ the notation $c_n = o(a_n)$ is used if $lim\ c_n/a_n = 0$ as $n \to \infty$. The notation $c_n = O(a_n)$ is used if c_n/a_n is a bounded sequence. When $\{c_n\}_{n \geq 1}$ is a sequence of measurable random variables defined on some probability space $(\Omega, \Im, P)$ the convention $c_n = o_p(a_n)$ is employed whenever $plim\ c_n/a_n = 0$ as $n \to \infty$. Finally, $c_n = O_p(a_n)$ is used if for any $\epsilon > 0$ there exists a positive number M and natural number N such that $n \geq N$ implies $P[|c_n/a_n| > M] < \epsilon$.

2   Informal Summary of the KWSMS Estimator

Section 2.1 provides an intuitive account of the control function approach in the context of the binary model presented above which is instructive to understand the essence of the estimation strategy pursued in this article. Section 2.2 offers an informal description of the KWSMS estimator.

2.1. Estimation Strategy

The key condition introduced in this paper is that there exists some $\bar{v}$ in the support of V satisfying:

$$[1] \ Med(\varepsilon|Z, W, V = \overline{v}) = Med(\varepsilon|V = \overline{v}) \ a.s.,$$

Loosely speaking, [1] imposes that once V has been fixed at $\overline{v}$, the exogenous variables becomes uninformative to alter the center of the distribution of ε. This holds for instance when $(Z, W) \perp (\varepsilon, V)$ or under a "global" restriction $F_{\varepsilon|Z,W,V} \equiv F_{\varepsilon|V}$ $a.s.$ but those are not necessary. This key median assumption, which can be tested from data as given in section C of the Appendix, is neither stronger nor weaker than that assumed in Hong and Tamer (2003) because each restriction can be implied by the other under certain conditions. This median restriction can accommodate heteroscedasticity in V of unknown form in the error term. However, in some cases heteroscedasticity in some variables of $\dot{X}$ let's say X_s only permits a more general restriction $Med(\varepsilon|Z, W, V) = Med(\varepsilon|X_s, V)$ a.s. violating [1]. Fortunately, the methodology developed in this paper can be extended to identify and estimate the coefficients of the variables "purged" by the control as explained in section A of the appendix.

Let suppose now that [1] holds for almost every arbitrary $\overline{v}$. As will be explained shortly, this is stronger than required for the KWSMS estimator but is needed for the SASMS estimator (at least over a range of values for v). Invoking this last condition and the fact $(\dot{X}, V)$ is one to one with $(Z, \Pi'W, V)$ yields:

$$Med(\varepsilon|\dot{X}, V) = Med(\varepsilon|V) \ a.s.$$

Noting $\phi(V) = Med(\varepsilon|V)$ thus provides:

$$[2] \ Med(U|\dot{X}, V) = \dot{X}'\beta + \phi(V) \ a.s.$$

The conditional median in [2] becomes the starting point for consistent

estimation since by the quantile invariance property to monotonic transfor-
mations (Powell 1986) one derives :

$$Med(Y|\dot{X}, V) = d(\dot{X}'\beta + \phi(V)) \ a.s.$$

This conditional median restriction on the response variable Y is, up to
$\phi(.)$, identical to Manski's 1985 restriction for Maximum Score estimation.
A priori "the control function" $\phi(.)$ has an unknown form. However, when
V is fixed at some given v the nuisance term $\phi(.)$ becomes a constant
and the lack of knowledge on $\phi(.)$ is no longer a problem. This fixing
is the foundation of the estimation procedure elaborated in this article.
This is the analogue principle used in the literature for unspecified quantile
regression (Chaudhuri 1991) or semi linear quantile regression (Lee 2003).
Given some consistent residuals $\{\hat{V}_i\}_{i=1}^n$ and a sampling $\{Y_i, \dot{X}_i\}_{i=1}^n$, this
suggests estimating the parameter β by running a local version of Manski's
Maximum Score maximizing in (b, c) the following:

$$[3] \ \frac{1}{nh_q} \sum_{i=1}^n (2Y_i - 1) d(\dot{X}_i'b + c) k(\frac{\hat{V}_i - \overline{v}}{h_q}),$$

where $k(.)$ is a kernel, h_q a bandwidths sequence and $\overline{v}$ some given value.
Yet, even when V is truly fixed at $\overline{v}$ one can not expect (given the cur-
rent level of mathematical knowledge) a viable limiting distribution for
inferential purposes (Pollard 1990) nor does bootstrapping offer hope of a
converging distribution (Abrevaya and Huang 2005). These asymptotic ab-
normalities have to do with the discontinuity of the indicator function $d(.)$
in the objective function which forbids classic asymptotic analysis using
a Taylor's representation for the score. Yet, in a seminal paper Horowitz
(1992) showed that the maximization for the Maximum Score estimator can
be modified smoothing $d(.)$ with the antiderivative of a kernel to obtain a

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Smoothed Maximum Score estimator which permits to get back into this classic asymptotic framework whenever the kernel in question is smooth. This suggests constructing a consistent and asymptotically normally distributed estimator by using a smoothed version of [3]. This is the estimation's path adopted in this paper.

This differs from the previous Control Function literature (Newey, Powell and Vella 1999) and (Lee 2007) where the control function is not fixed but rather expanded via Sieves. However, the specific problem here is different because of the indicator variable in the objective and using the conventional approach would involve stronger assumptions for both identification purposes and asymptotic purposes.

2.2. Description of the KWSMS Estimator

Define the parameters $\Pi_{\tilde{w}}$ and Π_z from $\Pi'W = \Pi'_{\tilde{w}}\tilde{W} + \Pi'_z Z$ where $\tilde{W}$ contains exogenous variables excluded from Z.

The parameter of interest β is only identifiable up to a positive scale since $d(cU) = d(U)$ for any $c > 0$. Identification up to a positive scale requires three main conditions. The distribution function of $V|\dot{X}$ needs to admit a density (with respect to the Lebesgue measure) which exists at the chosen $\bar{v}$ ($a.s.$). Consequently, another prerequisite for identification up to scale is a "rank condition" demanding $\tilde{W}$ to contain one component which is not a function of Z and whose associated slope coefficient is non null. To understand this recall that $\dot{X}' = (Z', A)$ so that $V = A - \Pi'W$ becomes a deterministic function of $\dot{X}$ should the last mentioned "rank condition" fail making $V|\dot{X}$ a single atom with a degenerated Dirac distribution. Finally, one element of $\dot{X}$ conditional on its reminding elements must possess a distribution function absolutely continuous with respect to the Lebesgue

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measure ($a.s.$). Let $(X_1, \widetilde{X}')$ be a partition of $\dot{X}'$ such that the scalar variable X_1 satisfies this property and write β_1 its associated slope coefficient. These conditions combined with [1] and some conventional regularity conditions suffice for identification up to the scaling factor $1/|\beta_1|$ whenever $\beta_1 \neq 0$. Thus, assume without loss of generality that β_1 is known to be strictly positive.[1]

Let $\{Y_i, \dot{X}_i\}_{i=1}^n$ be a random sample from $(Y, \dot{X})$ and let $\{\hat{V}_i\}_{i=1}^n$ be residuals induced by some root n consistent estimator of Π. In this paper, such an estimator is given leaving the choice of this first stage estimation open since under the mild assumptions for M estimators root n consistency will be attained. Write $\alpha_i \equiv 2Y_i - 1$ and $X' \equiv (1, \tilde{X}')$. The KWSMS estimator, noted $\widetilde{\theta_n}$, is defined as the maximizer in θ of the following objective:

$$\widetilde{S_n}(\theta) = \frac{1}{nh_q} \sum_{i=1}^n \alpha_i D\left(\frac{X_{1,i}+X_i'\theta}{h}\right)k\left(\frac{\hat{V}_i-\overline{v}}{h_q}\right),$$

where $(\{h_q\}_n, \{h\}_n)$ is a given pair of strictly positive bandwidths sequences vanishing to 0 as n approaches infinity and $D(.)$ is some chosen bounded function from the real line into itself meeting:

$$lim_{t\to-\infty}D(t) = 0, \; lim_{t\to\infty}D(t) = 1,$$

and

$$D' = K \text{ everywhere with } ||K||_{sup} < \infty.$$

This function $D(.)$ introduces the building block for deriving an asymptotic theory requiring to pick a differentiable function covering the real line

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[1]All the asymptotic results can be conducted using the fact that the estimator of $\frac{\beta_1}{|\beta_1|} \in \{-1, 1\}$ is equal to $\frac{\beta_1}{|\beta_1|}$ with probability approaching one, see Horowitz 1992, Theorem 2.

whose tail behavior mimics that of a cumulative distribution function. This permits to approximate (after tuning with the bandwidth h) the indicator variable. Because of the subsequent asymptotic conditions, a natural choice for $D(.)$ is to use the antiderivative of a kernel that is compactly supported (see Müller 1984). Apart from the lack of differentiability for $|t| = 1$, a good example for such function is given by:

$$D(t) = [0.5 + \tfrac{105}{64}(t - \tfrac{5}{3}t^3 + \tfrac{7}{5}t^5 - \tfrac{3}{7}t^7)]1[|t| \leq 1] + 1[t > 1].$$

The function $k(.)$ is a given kernel satisfying notably:

$$k \text{ belongs to } \mathcal{K}_m \text{ for some } m \geq 2, \int |k(t)|^2 dt < \infty,$$

and

$$k \text{ is differentiable everywhere with } ||k^{(1)}||_{sup} < \infty.$$

That is $\widetilde{S_n}$ is similar to the objective of the SMSE (had V been fixed at $\overline{v}$) apart from our weighting the ith observation with $\frac{1}{h_q}k(\frac{\hat{V}_i - \overline{v}}{h_q})$. The choice for m is dictated by the regularity conditions (see assumptions 9). Suppose that $\phi(\overline{v}) \equiv Med(\varepsilon|V = \overline{v})$ exists. Define $\tilde{\beta}$ the slope coefficient associated to $\tilde{X}$ and write $\ell \equiv X_1 + X'\theta_0$ where $\theta_0' \equiv \frac{1}{\beta_1}(\phi(\overline{v}), \tilde{\beta}')$. Noting $\Delta \equiv \theta - \theta_0$, the KWSMS estimator can be regarded as the maximizer of:

$$\frac{1}{nh_q}\sum_{i=1}^n \alpha_i D(\frac{\ell_i + X'\Delta}{h})k(\frac{\hat{V}_i - \overline{v}}{h_q}).$$

Viewing the maximization problem in this manner is enlightening to understand the nature of the regularity conditions involved. Introduce $F_{X,\ell,V}[.]$ the cumulative distribution function of $\varepsilon|X, \ell, V$ and $f_{X,\ell}(.)$ the density of $V|X, \ell$. In this paper identification requires the distribution function

of $V|\dot{X}$ to be (almost surely) absolutely continuous with respect to the Lebesgue measure implying the (almost sure) existence of the density of $V|X, \ell$ because $\dot{X}$ is one to one with (X, ℓ).

Suppose that $f_{X,\ell}(v)$ exists everywhere with $|f_{X,\ell}(v)| < M$ for some finite M (almost surely). Also assume that in some open neighborhood of $\bar{v}$, $F_{X,\ell,v}[-\beta_1 \ell + \phi(\bar{v})]$ and $f_{X,\ell}(v)$ are continuous as functions of v (almost surely).[2] Finally suppose that the bandwidth sequence h_q is chosen to satisfy $\lim nh_q^4 = \infty$ and $\lim \frac{nh^2 h_q^2}{log(n)} = \infty$ as $n \to \infty$. Under these and some mild regularity conditions the KWSMS estimator will be consistent for θ_0 (established in proposition 2). Unlike the consistency for the SMSE which does not demand a particular smoothness to be met, some local continuity for the above functions of v is needed.

The asymptotic normality of the KWSMS estimator is established by the same argument as that used for the SMSE. That is, suppose that $K(.)$ is selected to be a Kernel satisfying notably:

$$K \text{ belongs to } \mathcal{K}_r \text{ for some } r \geq 2,$$

$$K \text{ is symmetrical and twice differentiable everywhere,}$$

and

$$||K^{(j)}||_{sup} < \infty \text{ for } j = 1, 2 \text{ and } \int |K^{(1)}(t)|^2 dt < \infty.$$

K being everywhere differentiable ensures that $\nabla \widetilde{S_n}$ the gradient of $\widetilde{S_n}$ and $H\widetilde{S_n}$ its Hessian (with respect to θ) exist everywhere. Assume that θ_0 is an interior point of some compact set so that the same applies to $\widetilde{\theta_n}$

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[2] In section 3 stronger conditions are imposed for simplifying the proofs but those are not needed for consistency purposes, see assumption 9.

with probability approaching one as $n \to \infty$. Hence, by the Mean Value Theorem one finds:

$$0 = \nabla \widetilde{S_n}(\theta_0) + H\widetilde{S_n}(\bar{\theta})(\widetilde{\theta_n} - \theta_0) \text{ wpa.1,}$$

for some $\bar{\theta}$ in the line segment joining $\widetilde{\theta_n}$ and θ_0.

Now introduce $F_{X,\ell,\bar{v}}[.]$ the distribution function of $\varepsilon|X, \ell, V = \bar{v}$ and $f_X(.)$ the density of $\ell|X$. This last density is well defined under the identification assumptions which demand that the distribution function of $X_1|\widetilde{X}$ be (almost surely) absolutely continuous with respect to the Lebesgue measure.[3] Also, define $\mu_X(\ell) \equiv f_{X,\ell}(\bar{v})f_X(\ell)$ and $F^{(1)}_{X,\ell,\bar{v}}[-\beta_1\ell + \phi(\bar{v})] \equiv \partial F_{X,\ell,\bar{v}}[-\beta_1\ell + \phi(\bar{v})]/\partial\ell$ whenever these quantities exist.

When as function of v, $F_{X,\ell,v}[-\beta_1\ell + \phi(\bar{v})]$ and $f_{X,\ell}(v)$ are m times differentiable in some open neighborhood of $\bar{v}$ for some $m \geq 7$ in the sense of assumptions 9(a) and as functions of ℓ, $F_{X,\ell,\bar{v}}[-\beta_1\ell + \phi(\bar{v})]$, $f_{X,\ell}(\bar{v})$ and $f_X(\ell)$ are r times differentiable everywhere for some $r \geq 2$ in the sense of assumptions 14, conditions are provided (see lemma 3 and lemma 7 combined to assumptions 14 and 16) under which:

$$plim \ H\widetilde{S_n}(\bar{\theta}) \equiv H_0,$$

where $H_0 \equiv 2E[XX'F^{(1)}_{X,0,\bar{v}}[\phi(\bar{v})]\mu_X(0)]$. Let further suppose that H_0 is negative definite. The plausibility of this definiteness is discussed on page 23. Furthermore, if the bandwidths are selected appropriately according to assumption 17 and if the kernels k and K satisfy some mild integrability conditions one can further establish (see lemma 5 and lemma 8):

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[3]This is because (for almost every x) the cumulative distribution function (cdf) of ℓ conditional on $X = x$ is just the translated of the cdf of X_1 conditional on $\tilde{X} = \tilde{x}$.

$$\sqrt{nhh_q}\nabla\widetilde{S_n}(\theta_0) \to_d \mathcal{N}(0,\Sigma_0),$$

where $\Sigma_0 \equiv \int |k|^2 \int |K|^2 E[XX'\mu_X(0)]$. It will hence follow that:

$$[4] \quad \sqrt{nhh_q}(\widetilde{\theta_n} - \theta_0) \to_d \mathcal{N}(0,\Omega),$$

where $\Omega \equiv H_0^{-1}\Sigma_0 H_0^{-1}$ can be estimated consistently from data as given in proposition 4.

As explained on page 24 the bandwidths can be selected according to the followings:

pick $h \propto n^{-a}$ and $h_q \propto n^{-a_q}$ where a and a_q are some chosen constants satisfying:

$$a \in (Max\{\tfrac{1}{1+\eta+2\eta m}; \tfrac{1}{1+\eta+2r}\}, \tfrac{1}{4+4\eta}), \text{ and}$$

$$a_q = \eta a \text{ for some } \eta \in (\tfrac{3}{2m-3}, \tfrac{1}{3}).$$

The asymptotic result from [4] and the above bandwidths criteria imply that the KWSMS estimator, under relatively weak smoothness conditions, satisfies at least $\widetilde{\theta_n} - \theta_0 = O_p(n^{-3/8})$. However, this rate may improve when $\lambda = Min\{m,r\}$ augments eventually reaching the parametric rate i.e. $O_p(n^{-1/2})$ whenever λ approaches infinity.

The KWSMS estimator has an asymptotically centered normal distribution because the bandwidths pair has been selected purposefully such that the asymptotic bias vanishes which translates here into $\lim nh_q^{2m+1}h = 0$ and $\lim nh^{2r+1}h_q = 0$ as $n \to \infty$. In the non parametric jargon "under smoothing" is employed. As established in Horowitz (1992) this is not optimal from an asymptotic mean squared error perspective which requires some strictly positive finite bias. This choice is driven by two considerations. First, the

construction of an asymptotically biased KWSMS estimator would impose additional regularity conditions. Secondly, the unbiased SMSE has superior bootstrapping properties than the biased SMSE (see Horowitz 2002) in terms of the accuracy of its bootstrapped critical values which suggests the analogue for the KWSMS estimator since the objective of the KWSMS estimator is just a weighted version of SMSE's objective.

It is worth pointing a practical concern. The maximization of the objective function will be carried out by an iterative procedure such as the quadratic hill climbing (Goldfeld, Quandt and Trotter 1966). Additionally, the starting value for the iterative search may be better chosen as a result of some annealing procedure (see Horowitz 1992 and Szu et al. 1987).

3   Identification

The identification of β (up to a positive scale) is ensured under the followings:

Assumption 1:

$\tilde{W}$ *has one component which is not measurable[4] in Z and whose associated slope coefficient is non null.*

Assumption 2:

There exists a partition of $\dot{X}' = (X_1, \tilde{X}')$ where dim X_1 =1 and such that its corresponding coefficient, noted β_1, is strictly positive.

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[4]A random variable is said to be measurable in Z if it has the form $f(Z)$ for some Borel function f. The function is Borel if for any real number a the preset $f^{-1}(a, \infty)$ is a Borel set. Most functions of Z encountered in applied work are measurable in Z such as powers of Z, intercept, the indicator involving the level of Z and the conditional mean $E[T|Z]$ provided $E|T| < \infty$.

Assumption 3:

(a) There exists some given $\bar{v} \in \mathbb{R}$ and some $\phi(\bar{v}) \in \mathbb{R}$ such that:

$$P[\varepsilon \leq \phi(\bar{v}) | Z = z, W = w, V = \bar{v}] = \tfrac{1}{2} \ a.e.in \ z,w.$$

(b) The distribution function of $\varepsilon | \dot{X} = \dot{x}, V = \bar{v}$ has everywhere positive density with respect to the Lebesgue measure a.e.in $\dot{x}$.

Assumption 4:

(a) The distribution function of $X_1 | \tilde{X} = \tilde{x}$ has everywhere positive density with respect to the Lebesgue measure a.e.in $\tilde{x}$.

(b) The distribution function of $V | \dot{X} = \dot{x}$ is absolutely continuous with respect to the Lebesgue measure a.e.in $\dot{x}$ and its density evaluated at $\bar{v}$ exists a.e.in $\dot{x}$. Furthermore, there exists some finite constant C such that $0 < f(\bar{v} | \dot{x}) < C$ a.e.in $\dot{x}$.

Assumption 5:

$E[XX']$ is positive definite where $X' = (1, \tilde{X}')$.

Comments: Assumption 1 is a rank condition requiring at least one excluded instrument which is not a function of Z having an impact on the endogenous variable (see Lee 2007 and Newey, Powell and Vella 1999). Consider for instance the simple case where Z is a scalar variable and $W = (Z, Z^2)$. Even though Z^2 is not part of $\dot{X}$ assumption 1 fails. More generally, adding functions of the exogenous variables including in (i) to the reduced form equation (ii) is not a viable strategy in the context of our estimation problem. Assumptions 2 demands one variable whose marginal impact on the the latent index is positive. As pointing out earlier merely β_1 non null suffices because our parameter of interest is estimated

up to the constant $\frac{1}{|\beta_1|}$ and all of our results can be generalized by adding $\frac{\beta_1}{|\beta_1|} \in \{-1, 1\}$ as an additional unknown parameter. Assumption 3(a) is a classic control function condition except that only a local restriction at some $\overline{v}$ is imposed. Assumption 3(b), introduced similarly to Manski's 1985 assumption 2b, prevents the binary outcome Y from being perfectly predictable by $(\dot{X}, \overline{v})$ with some strictly positive probability.[5] Assumption 4 contains classic slack conditions permitting LMDR-identification (see Manski 1985, lemma 2) in the context of our control function approach. This is a prerequisite to identification which requires the existence of a significant (in the sense of having a coefficient non null) variable in $\dot{X}$ that must be fully supported. The additional presence of V in the controlled model imposes that $V|\dot{X}$ be supported is some neighborhood (albeit small) of $\overline{v}$. Finally, assumption 5 prevents identification of an intercept in $\dot{X}$. Now write $\phi(\overline{v}) \equiv Med(\varepsilon|V = \overline{v})$ and $\theta_0' \equiv \frac{1}{\beta_1}(\phi(\overline{v}), \tilde{\beta}')$ where $\tilde{\beta}$ denotes the slope coefficient associated to $\tilde{X}$.

Proposition 1 (Identification)

Under assumptions 1 through 5,

$$\theta_0 \equiv Argmax_{\theta \in \mathbb{R}^K} E[d(\ell + X'(\theta - \theta_0))g_{X,\ell}(\overline{v})],$$

where $\ell \equiv X_1 + X'\theta_0$, $g_{X,\ell}(\overline{v}) \equiv (1 - 2F_{X,\ell,\overline{v}}[-\beta_1\ell + \phi(\overline{v})])f_{X,\ell}(\overline{v})$, $F_{X,\ell,\overline{v}}[.]$ indicates the cumulative distribution function of $\varepsilon|X, \ell, V = \overline{v}$ and $f_{X,\ell}(\overline{v})$ indicates the density of $V|X, \ell$ evaluated at $\overline{v}$.

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[5]Assumption 3(b) is equivalent to $P[Y = 1|\dot{X} = \dot{x}, V = \overline{v}] \in (0, 1)$ a.e. in $\dot{x}$.

4 Asymptotic Properties of the KWSMS Estimator

Let $\{Y_i, \dot{X}_i\}_{i=1}^n$ be a sequence of observations and let $\hat{\Pi}$ be some given estimator from a first stage estimation inducing $\hat{V}_i \equiv A_i - \hat{\Pi}'W_i$ for $i = 1...n$. Also, let h_q and h be two strictly positive bandwidths sequences, $D(.)$ some given function from the real line into itself and $k(.)$ a kernel. For any $\theta \in \mathbb{R}^K$ define the following objective:

$$\widetilde{S_n}(\theta) = \frac{1}{nh_q} \sum_{i=1}^n \alpha_i D(\frac{X_{1,i}+X_i'\theta}{h}) k(\frac{\hat{V}_i-\bar{v}}{h_q}).$$

Sufficient conditions for weak consistency are given next.

Assumption 6:

$\{Y_i, \dot{X}_i, W_i\}_{i=1}^n$ *is an iid sequence from* $(Y, \dot{X}, W)$ *satisfying* $Y = d(\dot{X}'\beta + \varepsilon).$

Assumption 7:

The support of W *is a bounded subset of* $\mathbb{R}^q$ *with* $q \geq 1.$

Assumption 8:

θ_0 *is an interior point of* $\Theta \subset \mathbb{R}^K$ *compact.*

Assumption 9:(Define $F_{x,l,v}[.]$ *the cumulative distribution function of* $\varepsilon|X = x, \ell = l, V = v$ *and* $f_{x,l}(.)$ *the density of* $V|X = x, \ell = l$ *whenever this later exists. Also define* $\Psi(\tilde{x})$ *the essential supremum of the density of* $X_1|\tilde{X} = \tilde{x}$ *whenever this later exists i.e.* $\Psi(\tilde{x}) \equiv \{inf M \in \mathbb{R} : f_{\tilde{x}}(x_1) \leq M \, a.e. x_1\}.)$

(a) The function $v \mapsto F_{x,l,v}[-\beta_1 l + \phi(\bar{v})]$ *and* $v \mapsto f_{x,l}(v)$ *belong to* $C_\infty^m(\bar{v}, M_1)$ *for some* $M_1 < \infty$ *and some* $m \geq 2$ *a.e.in* $x, l.$

(b) The density of $X_1|\tilde{X} = \tilde{x}$ is essentially bounded a.e.in $\tilde{x}$ and the function

$\tilde{x} \mapsto \Psi(\tilde{x})$ is bounded on its domain.

Assumption 10:

There exists some $\hat{\Pi}$ such that $n^{1/2}(\hat{\Pi} - \Pi) = O_p(1)$.

Assumption 11:

(a)$D : \mathbb{R} \longrightarrow \mathbb{R}$. (b)$D$ is bounded. (c) $lim_{t \to -\infty} D(t) = 0$ and $lim_{t \to \infty} D(t) = 1$. (d) D is differentiable everywhere and its derivative noted K satisfies $||K||_{sup} < \infty$.

Assumption 12:

(a) k belongs to $\mathcal{K}_m$. (b) $\int |k(t)|^2 dt < \infty$. (c) k is differentiable everywhere with $||k^{(1)}||_{sup} < \infty$. (d) $\int |t^j k(t)| dt < \infty$ for $j = 1, 2, ..., m - 1$ and for any $\sigma > 0$ and any deterministic sequence $c_n = o(1)$,

$$lim \ c_n^{j-m} \int_{|t|>\sigma/c_n} |t^j k(t)| dt < \infty \ as \ n \to \infty \ for \ j = 0, 1, ..., m - 1.$$

Assumption 13:

The deterministic sequences of strictly positive real numbers $\{h_q\}_n$ and $\{h\}_n$ satisfy $lim \ h = lim \ h_q = 0$ and $lim \ nh_q^4 = lim \ \frac{nh^2 h_q^2}{log(n)} = \infty$ as $n \to \infty$.

Comments: Assumption 7 is imposed for simplicity. Merely, the first moments of W must exist. The bounded support, introduced for deriving the subsequent asymptotic results, may also be dropped if one is willing to assume extra regularity conditions for the distribution of X_1 conditional on $\tilde{X}$ and W. Assumption 8 is technical identically to assumption 4 in Horowitz (1992) because proposition 1 covers $\mathbb{R}^K$ while consistency is easier to establish for a compact set. Assumption 9(a) will be met for instance when both $F_{\varepsilon|\dot{x},v}$ and $f_{\dot{x}}(v)$ are twice differentiable in v with some bound

on the first and second derivatives (a.e.in $\dot{x}$). Assumption 9(b) is technical but is needed to get a uniform convergence for the empirical moment $\widetilde{S_n}$. Assumption 10 is verified under the mild assumptions for M estimators. Assumption 11 introduces the building block for smoothing the indicator function. As explained in the introduction, an easy manner to construct such a function is by integrating a kernel but for consistency purposes this is not needed. Assumption 12 is for the most part a typical condition which demands to select the order of the kernel $k(.)$ to match the smoothness of the function it will convolute with.

Proposition 2 (KWSMS Consistency)

Under the assumptions of proposition 1 and assumptions 6 through 13,

$$\widetilde{\theta_n} \equiv Argmax_{\Theta} \widetilde{S_n}(\theta) \ is \ (weakly) \ consistent \ for \ \theta_0.$$

To derive a normal limiting distribution for the estimator introduce the following conditions:

Assumption 14:(Define $g_{x,l}(\overline{v}) \equiv (1 - 2F_{x,l,\overline{v}}[-\beta_1 l + \phi(\overline{v})])f_{x,l}(\overline{v})$ where $F_{x,l,\overline{v}}[.]$ indicates the cumulative distribution function of $\varepsilon|X = x, \ell = l, V = \overline{v}$ and $f_x(.)$ the density of $\ell|X = x$ whenever this later exists.). The function $l \mapsto g_{x,l}(\overline{v})$ and $l \mapsto f_x(l)$ belong to $C_{\infty}^r(M_2)$ for some $M_2 < \infty$ and some $r \geq 2$ a.e.in x.

Assumption 15:

(a)$E||X||^4 < \infty$.

(b)$E[XX'T_X^{(1)}(0)]$ is positive definite where $T_X(l) \equiv g_{X,l}(\overline{v})f_X(l)$ and $T_X^{(1)}(u) \equiv \frac{\partial}{\partial l}T_X \ |_{l=u}$.

Assumption 16:

(a)K belongs to $\mathcal{K}_r$ and is symmetrical.

(b)$\int |K(t)|^{2+\delta}dt < \infty$ and $\int |k(t)|^{2+\delta}dt < \infty$ for some $\delta > 0$.

(c)$\int |t||K(t)|^2dt < \infty$,$\int |t||k(t)|^2dt < \infty$ and $\int |tK(t)|dt < \infty$.

(d)For any $\sigma > 0$ and any deterministic sequence $c_n = o(1)$,

$$lim \; c_n^{-1} \int_{|t|>\sigma/c_n} |K^{(1)}(t)|dt = 0 \; as \; n \to \infty,$$

and

$$lim \; c_n^{-1} \int_{|t|>\sigma/c_n} |k(t)|^2dt < \infty \; as \; n \to \infty,$$

(e)K is twice differentiable everywhere, $||K^{(j)}||_{sup} < \infty$ for $j = 1,2$, and $\int |K^{(1)}(t)|^2dt < \infty$.

Assumption 17:

$lim \; nh_q^{2m+1}h = lim \; nh^{2r+1}h_q = lim \; \frac{h}{h_q^3} = lim \; \frac{h_q^m}{h} = 0 \; as \; n \to \infty$, and

$lim \; \frac{nh_q^4 h^4}{log(n)} = \infty \; as \; n \to \infty$.

Comments: Assumption 14 is the key condition needed to derive the asymptotic result for the KWSMS estimator using the classic Taylor's expansion. The stringency in terms of the domain of smoothness may be construed as demanding but this is imposed for simplifying the proofs, a smoothness in a neighborhood of $l = 0$ would suffice (see Horowitz 1992, assumption 8 and assumption 9) using a lengthier argument. Assumption 15(b), is needed for deriving an asymptotic theory for the KWSMS estimator similarly to the SMSE (see Horowitz 1992, assumption 11). In fact, under assumption 14 the positive definiteness of such matrix would be implied automatically under the identification conditions if assumption 3(a)

is strengthened to $F_{\varepsilon|Z,W,\bar{v}} \equiv F_{\varepsilon|\bar{v}}$ a.s..[6] However, assumption 3(a) does not forbid some degree of heteroscedasticity for ε in which case assumption 15(b) is not ensured by the identification assumptions. Assumption 15(a) is needed for A is necessarily continuously distributed (by assumption 4) and the support of X is not assumed bounded. The existence of the fourth moment permits some control to show the convergence of certain expected values notably the collapse of the limiting bias. Assumption 16(a) is a reflection of assumption 14 since various convolutions involving $K(.)$ need to converge in some senses. Assumptions 16(b) and 16(c) are stability conditions for obtaining asymptotic Normality and are satisfied by many kernels, a clear example of which being polynomials compactly supported kernels which are smooth at boundary points. Finally, assumptions 16(d) and 16(e) are needed for the Hessian to converge in probability to some finite quantity and is related to assumptions 7 of Horowitz (1992), which demands the first two derivatives of $K(.)$ to be well behaved. Finally, assumption 17 dictates the bandwidths' rate which must be selected for the asymptotic to be met with $\lim nh_q^{2m+1}h = \lim nh^{2r+1}h_q = 0$ collapsing the asymptotic bias while $\lim \frac{h}{h_q^3} = 0$ allows the usage of the estimated nuisance $V(A,W)$ to be asymptotically irrelevant.

Proposition 3 (KWSMS Asymptotic Normality)

Under the assumptions of proposition 2 and assumptions 14 through 17,

$$\sqrt{nhh_q}(\widetilde{\theta_n} - \theta_0) \rightarrow_d \mathcal{N}(0, H^{-1}\Sigma H^{-1}),$$

where

$$H \equiv E[XX'T_X^{(1)}(0)], \Sigma \equiv \int |k|^2 \int |K|^2 E[XX'\mu_X(0)] \text{ and}$$

$$\mu_X(\ell) \equiv f_{X,\ell}(\bar{v})f_X(\ell).$$

---

[6]In that case $E[XX'T_X^{(1)}(0)] = 2\beta_1 E[XX'f_{\bar{v}}[\phi(\bar{v})]\mu_X(0)]$ where $f_{\bar{v}}(.)$ is the density of $\varepsilon|V = \bar{v}$. This matrix is positive definite by assumptions 2,3(b),4(b)and 5.

Comments: So far it is implicitly assumed that both assumptions 13 and 17 are met. However, this imposes some smoothness conditions beyond those assumed in assumptions 9. When $h \propto n^{-a}$ and $h_q \propto n^{-a_q}$ for some strictly positive constants a and a_q, the bandwidths requirement put forth in proposition 3 will hold as long as $a \in (max\{\frac{1}{1+\eta+2\eta m}; \frac{1}{1+\eta+2r}\}, \frac{1}{4+4\eta})$ and $a_q = \eta a$ for some $\eta \in (\frac{3}{2m-3}, \frac{1}{3})$.[7] Thus, the asymptotic conclusion needs a strengthening to $m \geq 7$ in assumption 9. Under this last condition and $r \geq 2$, one can therefore obtain a rate on convergence for the KWSMS estimator at least $n^{-3/8}$. However, this rate may improve when $\lambda = Min\{m,r\}$ augments eventually reaching the parametric rate if λ approaches infinite.

As stressed in the introduction, one of the important practical advantage of the KWSMS estimator for the endogenous binary choice model is its ability to conduct inferences from a large sample of observations. The next proposition offers the consistent estimators for the covariance of the above limiting distribution. The notation $\alpha'_i = -\alpha_i$ is adopted.

Proposition 4 (KWSMS Inferential Feasibility)

Let

$$\widetilde{H_n} = \frac{1}{nh^2 h_q} \sum_{i=1}^{n} \alpha'_i X_i X'_i K^{(1)}\left(\frac{X_{1,i}+X'_i \widetilde{\theta_n}}{h}\right) k\left(\frac{\hat{V}_i-\overline{v}}{h_q}\right),$$

and

$$\widetilde{\Sigma_n} = \frac{1}{nh^{\gamma_1} h_q^{\gamma_2}} \sum_{i=1}^{n} X_i X'_i |K\left(\frac{X_{1,i}+X'_i \widetilde{\theta_n}}{h^{\gamma_1}}\right)|^2 |k\left(\frac{\hat{V}_i-\overline{v}}{h_q^{\gamma_2}}\right)|^2,$$

for some constant $\gamma_1 \in (0, 3/4]$ and $\gamma_2 \in (0, 1]$. Then under the assumptions of proposition 3,

$$\widetilde{H_n} \longrightarrow_p H.$$

---

[7]It is clear that Assumptions 13 and 17 both hold as long as $\lim nh_q^{2m+1}h = \lim nh^{2r+1}h_q = 0$, $\lim \frac{h}{h_q^3} = \lim \frac{h_q^m}{h} = 0$ and $\lim \frac{nh_q^4 h^4}{log(n)} = \infty$. Solving these implied inequalities directly yields the bandwidths spectrum given above.

Furthermore, if $\int |K(t)|^4 dt < \infty$ and $\int |k(t)|^4 dt < \infty$,

$$\widetilde{\Sigma_n} \longrightarrow_p \Sigma.$$

Comments: The rational behind $\widetilde{\Sigma_n}$ not using the bandwidths on which the KWSMS estimator is based upon is to avoid having to add additional bandwidths constraints on the already substantial list.

5 Accelerating Convergence with a Score Approximation Smoothed Maximum Score Estimator

As explained in the previous section, a KWSMS estimator's rate of convergence will be as rapid as the order of differentiability of certain cumulative distribution functions and densities. Under low differentiability one may thus desire to rely on an alternative estimator with a faster rate of convergence which uses more observations of V. In this section, the SASMS estimator is shown to attain that target provided some stronger assumptions hold. In section 5.1 an informal description of the SASMS estimator is presented. In section 5.2 the asymptotic's properties of a SASMS estimator are covered.

5.1. **Description of the SASMS Estimator**

In this section we present an informal description for constructing the SASMS estimator. Suppose now that [1] holds for an arbitrary $\bar{v} \in [0,1]$ which will be simply noted from now on v. Define $e'_K \equiv [O, I_{K-1}]$ the $K-1 \times K$ matrix where the first column is the zero vector while I_{K-1}

represents the $K - 1 \times K - 1$ identity matrix and e_1' the $1 \times K$ vector whose first entry is 1 and zero elsewhere. Let Θ be some compact set and for a given v introduce the followings:

$$\tilde{\theta}(v) \equiv Argmax_\Theta \frac{1}{nh_q} \sum_{i=1}^n \alpha_i D(\frac{X_{1,i} + X_i'\theta}{h}) k(\frac{\hat{V}_i - v}{h_q}),$$

and

$$\tilde{\beta}(v) \equiv e_K' \tilde{\theta}(v) \text{ while } \tilde{\phi}(v) \equiv e_1' \tilde{\theta}(v),$$

where $D(.)$, $k(.)$ and the bandwidths pair (h, h_q) are as described in section 4. Let $\{f_j\}_{j \geq 1}$ be a known basis of functions such that $\sum_{j=1}^\rho b_j f_j$ can approximate a smooth function of $[0, 1]$ arbitrary well using some real sequence $\{b_j\}_{j \geq 1}$ and natural number ρ large enough. There are many candidates for such basis depending on the assumed topology of the smooth function involved. Classic examples include power series, splines, trigonometric series and wavelets. Here are some easy to implement basis taken from Chen (2007):

Power series: Let $Pol(\rho) = \{f : [0, 1] \to \mathbb{R}, f(v) = \sum_{j=0}^\rho b_j v^j, b_j \in \mathbb{R}\}$ the space of polynomials on $[0, 1]$ of degree less or equal to ρ. A differentiable function on $[0, 1]$ can be approximated arbitrary well by some element of $Pol(\rho)$ with ρ large enough. Thus, here $f_j(v) = v^{j-1}$ for $j \geq 1$.

Trigonometric cosinus: Let $cosPol(\rho) = \{f : [0, 1] \to \mathbb{R}, f(v) = b_1 + \sum_{j=2}^\rho b_j \sqrt{2} cos(2\pi(j - 1)v), b_1, b_j \in \mathbb{R}\}$ the space of cosinus polynomials on $[0, 1]$ of degree less or equal to ρ. A differentiable function on $[0, 1]$ (or merely a square integrable function on $[0, 1]$) can be approximated arbitrary well by some element of $cosPol(\rho)$ with ρ large enough. Thus, here $f_j(v) = \sqrt{2} cos(2\pi(j - 1)v)$ for $j \geq 2$ and $f_1(v) = 1$. This choice is particular suited

for the SASMS estimator because $\{f_j\}_{j \geq 1}$ forms an orthonormal basis of $L_2[0,1]$ the space of square integrable functions on $[0,1]$.

Splines: For a given natural number d define $Spl(d+1, \rho) = \{f : [0,1] \rightarrow \mathbb{R}, f(v) = \sum_{j=0}^{d} a_j v^j + \sum_{j=1}^{\rho} b_j[(v - t_j)_+]^d, a_j, b_j \in \mathbb{R}\}$ the space of Splines on $[0,1]$ of order $d+1$ where $(.)_+ = Max(.,0)$ and $(t_1, t_2, ...t_\rho)$ is a given increasing sequence of "knots" partitioning $[0,1]$ such that $t_1 = 0$ and $t_\rho = 1$. Here $\sum_{j=1}^{\rho} b_j[(v - t_j)_+]^d$ is a piecewise polynomial shifter which permits to adjust the approximation of a baseline polynomial on each interval $I_j = [t_j, t_{j+1}]$. Define $|I_j| = t_{j+1} - t_j$ for $j = 1, ..., \rho - 1$. A differentiable function on $[0,1]$ can be approximated arbitrary well by some element of $Spl(d+1, \rho)$ with ρ large enough provided the mesh ratio $Max|I_j|/Min|I_j|$ stays bounded. Thus, here $f_j(v) = v^{j-1}$ if $1 \leq j \leq d+1$ and $f_j(v) = [(v - t_{j-d-1})_+]^d$ if $d+2 \leq j \leq d+1+\rho$.

Define $p_n(.)' = (f_1(.), ..., f_{\rho(n)}(.))$ where $\rho(n)$ is some chosen deterministic sequence of natural numbers satisfying $\rho(n) \rightarrow \infty$ as $n \rightarrow \infty$ but $\rho(n) < n$. Now write Λ_n the $n \times \rho(n)$ matrix whose i^{th} row is $p_n(i/n)'$ and $\tilde{\phi}_n$ the $n \times 1$ vector whose i^{th} entry is $\tilde{\phi}(i/n)$. That is, running a first stage estimation with n "local" KWSMS estimators at $v = 1/n, 2/n, ..., 1$ (where n still indicates the sample size) permits to collect the vector $\tilde{\phi}_n$ and retrieve the following:

$$b_n \equiv Argmin_{b \in \mathbb{R}^{\rho(n)}} ||\tilde{\phi}_n - \Lambda_n b|| \equiv (\Lambda_n' \Lambda_n)^{-1} \Lambda_n' \tilde{\phi}_n.$$

This constitutes the essence of the SASMS estimator since for $\rho(n)$ well chosen and under some regularity conditions involving notably the sufficient differentiability of $\phi(.)$ then $b_n' p_n(.)$ is consistent for $\tilde{\phi}_0(.) = \frac{1}{|\beta_1|}\phi(.)$ in the sense that,

$$plim \ sup_{v \in [0,1]} |b'_n p_n(v) - \tilde{\phi}_0(v)| = 0.$$

However, $\{V_i\}_{i=1}^n$ is not observed but only $\{\hat{V}_i\}_{i=1..n}$. Hence, a natural way to proceed is to estimate $\tilde{\phi}_0(V_i)$ with $b'_n p_n(\hat{V}_i)$ for $i = 1...n$. Let $K(.)$ be some kernel (possibly different from the function $D'(.)$ used in the first stage) from the real line into itself whose derivative exists everywhere. Now define for an arbitrary β the followings:

$$G_n[\beta] \equiv \frac{1}{nh_*} \sum_{i=1}^n \tau(\hat{V}_i) \alpha_i \tilde{X}_i K\left(\frac{X_{1,i} + \tilde{X}_i'\beta + b'_n p_n(\hat{V}_i)}{h_*}\right),$$

and

$$H_n[\beta] \equiv \frac{1}{nh_*^2} \sum_{i=1}^n \tau(\hat{V}_i) \alpha_i \tilde{X}_i \tilde{X}_i' K^{(1)}\left(\frac{X_{1,i} + \tilde{X}_i'\beta + b'_n p_n(\hat{V}_i)}{h_*}\right),$$

where $\tau(.) = 1[0 \leq . \leq 1]$ and h_* is a deterministic strictly positive sequence of real numbers meeting $\lim h_* = 0$ as $n \to \infty$. The SASMS estimator, noted $\bar{\beta}$, is given by:

$$\bar{\beta} \equiv \tilde{\beta}(v) - H_n[\tilde{\beta}(v)]^{-1} G_n[\tilde{\beta}(v)],$$

where $\tilde{\beta}(v)$ is the slope coefficient estimator of a KWSMS estimator using some fixed $v \in [0,1]$. The reader familiar with Horowitz's 1992 paper would have noticed that $\bar{\beta}$ is an approximation for a feasible SMSE based upon [2] which would use $b'_n p_n(\hat{V})$ in lieu of $\phi(V)$ (up to a scale). This estimator belongs to the class of score approximation based estimators (Stone 1975, Bickel 1982, Lee 2003).

The SASMS estimator exists only with probability approaching 1 as $n \to \infty$ because the "pseudo hessian" $H_n[\tilde{\beta}(v)]$ has an inverse with probability approaching 1. Section 5.2 explains in further details some regularization scheme to mitigate this problem.

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5.2. **Asymptotic Results**

Assumption S1:

Assumptions 3, 4(b), 9 and 14 hold for all $\bar{v} \in [0,1]$ as well as other assumptions of proposition 3.

Comments: This ensures that the conclusion of proposition 2 and 3 holds using any fixed value of v chosen in $[0,1]$. The choice of $[0,1]$ is purely symbolic and can be replaced by any compact set of $\mathbb{R}$ for which the above assumptions hold.

Assumption S2:

There exists a sample size N such that for each v in $[0,1]$ the sequence $\{E|\widetilde{\theta}(v) - \theta(v)|^2\}_{n \geq N}$ is monotone.

Comments: This is a dominance condition which ensures a uniform rate of convergence for the KWSMS estimator $\widetilde{\theta}(v)$ over $[0,1]$. Under assumption S1 we already know that for each v, the sequence of mean squared errors converges to 0. This however requires no oscillations if the sample size is large enough.

Assumption S3:

(a) $\phi(.)$ is p times continuously differentiable on $[0,1]$ for some $p \geq 1$. (b) There exists some finite constant C and some $\gamma \in (0,1]$ such that $|\phi^{(p)}(v_1) - \phi^{(p)}(v_2)| \leq C|v_1 - v_2|^\gamma$ for all $(v_1, v_2) \in [0,1] \times [0,1]$.

Comments: Condition (a) is explicit with the additional slightly stronger requirement in (b) that the pth derivative of $Med(\varepsilon|V = v)$ be Hölder continuous. Then the nuisance function $\phi(.)$ can be approximated (up to scale) arbitrary well by many linear Sieves methods.

Assumption S4:

$\rho(n)$ *is a given sequence of natural numbers such that $\rho(n)/n < 1$ for all n and $\rho(n) \to \infty$ as $n \to \infty$.*

Comments: Let $||f||_{sup}$ for a real valued function $f : [0,1] \to \mathbb{R}$ denotes the sup norm on $[0,1]$. Under assumption S3 and assumption S4 there exists a known basis of functions $\{f_j\}_{j\geq 1}$ such that its linear span $E_\rho = \{f : [0,1] \to \mathbb{R}, f = \sum_{j=1}^{\rho} a_j f_j, a_j \in \mathbb{R}\}$ can approximate the control function $\phi(.)$ arbitrary well in the sense that $inf_{E_{\rho(n)}}||f - \phi||_{sup} \to 0$ as $n \to \infty$ (see Chen 2007). That is, defining $p_n(.)' = (f_1(.), ..., f_{\rho(n)}(.))$ there exists $B_n' = (b_{0,1}, ..., b_{0,\rho(n)})$ such that $B_n' p_n$ provides a good approximation of the unknown control function on $[0,1]$ for n large enough.

Let Λ_n be the $n \times \rho(n)$ matrix whose i^{th} row is $p_n(i/n)'$. Also, under assumption S3 one can introduce $||p_n||_{sup} \equiv sup_{v\in[0,1]}||p_n(v)||$ and given a naturel number ρ use $L[\rho] \equiv \sum_{j=1}^{\rho} ||f_j^{(1)}||_{sup}$.

Assumption S5:

For n large enough the largest eigenvalue of $\Lambda_n' \Lambda_n / n$ is bounded from above and its smallest eigenvalue is bounded away from 0.

Comments: This can be viewed as a dominance condition which permits the discrepancy between b_n and B_n to be imposed only by the "mistakes" committed by the various KWSMS estimators on the first stage and on the approximation error from truncating the basis up to the first $\rho(n)^{th}$ terms.

Assumption S6:

The distribution function of $X_1|\tilde{X} = \tilde{x}, V = v$ has everywhere positive density with respect to the Lebesgue measure a.e in $\tilde{x}, v$.

Comments: Let $L \equiv X_1 + \tilde{X}' \frac{\tilde{\beta}}{\beta_1} + \frac{\phi(V)}{\beta_1}$. This assumption permits the existence of the density of $L|\tilde{X} = \tilde{x}, V = v$ (a.e.$\tilde{x}$,v) which is needed to derive an asymptotic. Define $F_{\tilde{x},l,v}[.]$ the cumulative distribution function of $\varepsilon|X = \tilde{x}, L = l, V = v$ and $f_{\tilde{x},v}(.)$ the density of $L|\tilde{X} = \tilde{x}, V = v$. Also, use the convention $F_{\tilde{x},l,v}^{(1)}[-\beta_1 l + \phi(v)] \equiv \partial F_{\tilde{x},l,v}[-\beta_1 l + \phi(v)]/\partial l$ whenever this derivative exists.

Assumption S7:

The function $l \mapsto F_{\tilde{x},l,v}[-\beta_1 l + \phi(v)]$ and $l \mapsto f_{\tilde{x},v}(l)$ belong $C_\infty^s(0, M)$ for some $M < \infty$ and some $s \geq 4$ a.e.in $\tilde{x}, v$.

Comments: Under this the classic asymptotic is permitted via non parametric convolution arguments to show consistency and normality. Also, assumption S7 along with assumption S1 ensures the existence of $Q \equiv 2E[\tau(V)\tilde{X}\tilde{X}'F_{\tilde{X},0,V}^{(1)}[\phi(V)]f_{\tilde{X},V}(0)]$ where $\tau(V) = 1[0 \leq V \leq 1]$. Now impose the followings:

Assumption S8:

Q is negative definite.

Assumption S9:

(a) $K(.)$ belongs to $\mathcal{K}_s$.

(b) $K(.)$ is twice differentiable everywhere and $||K^{(j)}||_{sup} < \infty$, for $j = 1, 2$.

(c) $\int |K(t)|^4 dt < \infty$ and $\int |K^{(1)}(t)|^2 dt < \infty$.

(d) $\int |t^j K(t)| dt < \infty$ for $j = 1, 2, ..., s - 1$.

(e) For any $\sigma > 0$ and any deterministic sequence $c_n = o(1)$,

$$\lim c_n^{-1} \int_{|t|>\sigma/c_n} |K^{(1)}(t)| dt = 0 \text{ as } n \to \infty,$$

and

$$\lim c_n^{j-s} \int_{|t|>\sigma/c_n} |t^j K(t)| dt < \infty \ \text{as } n \to \infty \ \text{for } j = 0, 1, ..., s-1.$$

Assumption S10:

$h_* \to 0 \ \text{and} \ \frac{nh_*^8}{log(n)} \to \infty \ \text{as } n \to \infty.$

Assumption S11: (Using $L_n = L[\rho(n)]$)

(a)$nhh_q h_*^6 \to \infty \ \text{as } n \to \infty.$

(b) $L_n = o(\sqrt{n}h_*^3).$

(c)$||p_n||_{sup} = O(n^{(1-\gamma)/2}h_*^3 hh_q)$ for some strictly positive γ.

(d) $inf_{E_{\rho(n)}}||f - \phi||_{sup}||p_n||_{sup} = o(h_*^3).$

Proposition 5 (SASMS consistency)

Under assumptions S1 though S11,

$$\bar{\beta} \ \text{is (weakly) consistent for } \tilde{\beta}_0 \equiv \frac{\tilde{\beta}}{\beta_1}.$$

Comments: To make the SASMS estimator more appealing than the KWSMS estimator one needs to show its asymptotic normality and construct consistent estimators for its asymptotic covariance. In order to derive the asymptotic normality a few more assumptions are needed. Introduce the followings:

$$\Xi \equiv \int |K|^2 E[\tau(V)\tilde{X}\tilde{X}'f_{\tilde{X},V}(0)],$$

and

$$\bar{G} \equiv \frac{1}{nh_*} \sum_{i=1}^{n} \tau(V_i)\alpha_i\tilde{X}_i K(\frac{L_i}{h_*}) \ \text{where} \ L_i \equiv \frac{1}{\beta_1}Med(U|\dot{X}_i, V_i).$$

Assumption S12:

$$h_*/hh_q \to \infty \ as \ n \to \infty.$$

Assumption S13:

$$\sqrt{nh_*}(G_n[\tilde{\beta}(v)] - \bar{G}) = o_p(1).$$

Assumption S14:

$$nh_*^{2s+1} \to 0 \ as \ n \to \infty.$$

Comments: Assumption S12 permits an estimator asymptotically centered. Assumption S13 can be ensured by a stochastic equicontinuity assumption whose sufficient conditions are provided in Andrews (1994). Finally, assumption S14 enables the researcher to collapse the asymptotic bias. Define the following:

$$\hat{\Xi} \equiv \frac{1}{nh_*} \sum_{i=1}^{n} \tau(\hat{V}_i)\tilde{X}_i\tilde{X}_i' |K\big(\frac{X_{1,i}+\tilde{X}_i'\tilde{\beta}(v)+b_n'p_n(\hat{V}_i)}{h_*}\big)|^2.$$

The key result of section 6 is now provided next.

Proposition 6

Under assumptions S1 though S14,

$$\sqrt{nh_*}(\bar{\beta} - \tilde{\beta}_0) \to_d \mathcal{N}(0, Q^{-1}\Xi Q^{-1}).$$

Furthermore,

$$H_n[\tilde{\beta}(v)] \longrightarrow_p Q \ and \ \hat{\Xi} \longrightarrow_p \Xi.$$

Comments: Proposition 6 implies that the SASMS estimator achieves a faster rate of convergence than the KWSMS estimator while still allowing

for hypothesis testing. To be more specific, the SASMS estimator's rate of convergence is $(\frac{hh_q}{h_*})^{1/2}$ times that achieved on the KWSMS estimator which is faster since $\lim\frac{hh_q}{h_*} = 0$ as $n \to \infty$ by assumption S12. It turns out that this is not the most efficient estimator (in the asymptotic sense) under the assumptions of proposition 6. It is not very difficult to show that a more efficient CAN estimator is given by:

$$\bar{\beta}_E \equiv \tilde{\beta}(v) + \hat{\Xi}^{-1} G_n[\tilde{\beta}(v)],$$

which yields,

$$\sqrt{nh_*}(\bar{\beta}_E - \tilde{\beta}_0) \to_d \mathcal{N}(0, \Xi^{-1}).$$

This will be subsequently referred to as the "efficient" SASMS estimator.[8]

It is important to bear in mind that the SASMS estimator exists only with probability approaching one as $n \to \infty$ since the matrix $H_n[\tilde{\beta}(v)]$ defined in section 5.1 has an inverse only with probability approaching one. In finite sample the SASMS estimator may exhibit a large variance because of the instability of the inverse in question which may be singular with strictly positive probability. In practice, this poses the same problem as that induced by collinearity where a small change in data produces a substantial variation in estimates. When the kernel of assumption S9 has the form $K(t) = P(t)1[|t| \leq 1]$ for some finite degree polynomial P (see Muller 1984), one way to mitigate this problem is to compute $H_n[\tilde{\beta}(v)]$ replacing

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[8]Indeed, this efficient SASMS estimator requires milder assumptions than those imposed in propositions 5-6. Clearly, assumption S8 is not needed but also assumptions S9(b), S9(c),S9(e) can be shown to be stronger than required for deriving consistency and asymptotic normality.

$K^{(1)}(t)$ with $K_c^{(1)}(t) = P^{(1)}(t)1[|t| \leq 1 + c_n]$ where c_n is a deterministic sequence of positive real numbers satisfying $\frac{c_n}{h_*} \to 0$ as $n \to \infty$.[9]

The selection of the bandwidths is not covered in proposition 6 owing to the fact that only a generic case for any basis $\{f_j\}_{j \geq 1}$ is treated. However, in application one needs to select an appropriate basis for smooth functions and pick three bandwidths sequences h, h_q and h_* meeting the assumptions of proposition 6. The next proposition establishes for the power series basis and trigonometric cosinus basis how the bandwidths and sieves's sequence $\rho(n)$ may be selected up to a scale. The symbol $[\kappa]$ for a real number κ will refer to the least lower integer of κ.

Corollary (Bandwidths Admissibility For Power series and Trigonometric cosinus)

Suppose that assumption S1 holds with $r > m/3$, assumption S7 holds for some $s \geq 5$ and assumption S3 holds for some $p > 4$. Also, suppose that others assumptions of proposition 6 hold but assumptions S4,S10,S11,S12,S14. When $p_n(v)' = (f_1(v), ..., f_{\rho(n)}(v))$ is chosen from Power series or Trigonometric cosinus then the assumptions of proposition 6 are satisfied under the followings:

(a)$h \propto n^{-a}$ and $h_q \propto n^{-\lambda a}$,

for some $a \in (\frac{1}{1+\lambda+2\lambda m}, \frac{1}{10(1+\lambda)})$ and some $\lambda \in (\frac{3}{2m-3}, min\{\frac{9}{2m-9}, 1/3\})$.

(b) $h_* \propto n^{-a*}$,

for some $a* \in (max\{a', \frac{1}{2s+1}\}, min\{\frac{1-4a'}{6}, \frac{p'}{6p'+12}\})$ where $a' = a(1+\lambda)$ and $p' = p - 1$.

[9]This "regularized" version for the SASMS estimator has the same limiting distribution because $K^{(1)}$ and $K_c^{(1)}$ differ only when $1 \leq |t| \leq 1 + c_n$ under assumption 9, see Lemma 13-14 and proof of proposition 5.

(c) $\rho(n) = C_0[n^\nu]$,

for some $\nu \in (\frac{3a*}{p-1}, \frac{1-6a*}{4})$ and some $C_0 \in (0, \frac{n}{[n^\nu]})$.

Comments: This corollary is based upon the fact that with power series or trigonometric series on has $||p_n||_{sup} = O(\rho(n))$ and $L_n = O(\rho(n)^2)$ while $inf_{E_{\rho(n)}}||f - \phi||_{sup} = O(1/\rho(n)^p)$ (see Chen 2007). Some lengthy algebra can show that (a),(b) and (c) are sufficient for the conditions of proposition 6 to hold. However, those are not necessary and assumptions S4,S10,S11,S12,S14 may hold under different set of conditions which can be found by the researcher on a case to case basis.

6 Monte Carlo Simulations

This section examines the finite sample properties of the estimators put forth in this paper using Monte Carlo experiments. These estimators are used to estimate the parameter $\beta = 1$ when the data generating process obeys:

$$Y = 1 \; if \; Z + \beta A + \varepsilon \geq 0 \text{ and } Y = 0 \text{ otherwise,}$$

$$A = \Pi W + V,$$

$$\varepsilon = \phi(V) + e,$$

where (Z, W) is a standard bivariate Normal couple of correlation coefficient ϱ , $V \sim \mathcal{N}(0, 1)$, and Π is set equal to 1. In this experiment three designs are used satisfying the followings:

Design ST:

$\varrho = 0.5$; $\phi(V) = exp(-V^2)$; $e = (1 + Z^2 + Z^4)T$ where T is Student with 3 degrees of freedom.

Design PR:

$\varrho = 0.5$; $\phi(V) = 0.5V$; $e \sim \mathcal{N}(0,1)$.

Design LG:

$\varrho = 0$; $\phi(V) = cos(\pi V)$; $e \sim$ *Logistic*.

In addition, two other estimators addressing endogeneity for the binary choice model are used. The first one is the limited information ML estimator[10] (LIML) proposed in Rivers and Vuong (1988) and the second is the artificial two stage least square estimator[11] (2SLS) suggested in Lewbel (2000). Design ST has a non linear control function with an heteroscedastic error term. Design PR has a linear control function with a normally distributed (conditional on V) error term, which satisfies the parametric theory laid out in Rivers and Vuong (1988). Design LG has Z independent of W which makes Z a "special regressor" as defined in Lewbel (2000).

In all designs the variable e is normalized to have a 0.5 standard deviation. A simulation for a sample size $n = 250, 500$ and 1000 consists of 1000 replications for all estimators but the SASMS estimator. For that later, experiments with $n = 1000$ are not performed and 500 replications are completed due to the long computational time required. The simulations

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[10]Under the assumptions of Rivers and Vuong (1988) the coefficients are identified up to a different scaling factor. In our context, the LIML refers thus to the ratio between the LIML estimator of A slope coefficient and Z slope coefficient since this is how a researcher would estimate our coefficient of interest.

[11]One choice left to the researcher for computing this estimator is the Kernel which is needed for estimating the density of Z given W, see Lewbel (2000). The Monte Carlo experiments are performed with a normal kernel along with the bandwidths $n^{-1/6}$.

are conducted in Gauss.

For the KWSMS estimator the smoothing of the indicator function is carried out using:

$$D(t) = [0.5 + \tfrac{105}{64}(t - \tfrac{5}{3}t^3 + \tfrac{7}{5}t^5 - \tfrac{3}{7}t^7)]1[|t| \leq 1] + 1[t > 1].$$

This choice violates assumption 11(d) due to lack of differentiability for $|t| = 1$ but the results are not sensitive to smoothing $D(.)$ around those two points. The derivative of $D(.)$ (almost everywhere) is a kernel meeting assumption 16 for $r = 4$ (Müller 1984). Also, the weighting of the objective is performed using:

$$k(t) = \tfrac{1}{48}(105 - 105t^2 + 21t^4 - t^6)\tfrac{1}{\sqrt{2\pi}}exp(-\tfrac{1}{2}t^2),$$

providing a kernel of order $m = 7$ (Pagan and Ullah 1999) which satisfies assumption 12. The first stage estimation of the nuisance parameter Π is conducted via least square which meets assumption 10. The local choice $\bar{v} = 0$ is selected satisfying assumption 3. The bandwidths conditions explained at the end of proposition 3 are only qualitative. Since the optimal bandwidths selection is not covered in this article, a simple Silverman's like rule of thumb (see Silverman 1986) is adopted. This consists of using $h = \hat{\sigma}_l n^{-3/16}$ and $h_q = \hat{\sigma}_v n^{-3\eta/16}$ where $\eta = 1/3$, $\hat{\sigma}_v$ is to the sample standard deviation of $\{\hat{V}_i\}_{i=1..n}$ and $\hat{\sigma}_l$ is the sample standard deviation of $\{X_{1,i}+X_i'\tilde{\theta}\}_{i=1..n}$ with $\tilde{\theta}$ a KWSMS estimator retrieved in a first stage using $(h, h_q) = (n^{-3/16}, n^{-3\eta/16})$. This "plug in" method is of course arbitrary in that it depends on the bandwidths selected originally. Even though this choice for the bandwidths does not a priori satisfy any optimal criteria in the context of our specific problem, it has the benefit of being easy to implement while performing reasonably well compared to other choices used in

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preliminary experiments. The covariance matrix estimator of proposition 4 relies on $\gamma_1 = 3/8$ and $\gamma_2 = 1$. Other choices for (γ_1, γ_2) meeting proposition 4 were employed in a preliminary study but this did not materially alter the quality of the sizes.

Finally, the KWSMS estimator is computed maximizing the objective by quadratic hill climbing (Goldfeld, Quandt and Trotter 1966). A search for the global maximum consists of selecting out of 10 iterative searches, the local maximum maximizing the objective[12] as there is no guaranty in finite sample that the local maximum is unique.

For the SASMS estimator, the first stage uses n local KWSMS estimators which are retried as above but for the value $\bar{v}$. The "plug in" SASMS estimator is then computed as described in section 5.1 using a KWSMS estimator with $v = 1/n$ and the following:

$$K_6(t) = \tfrac{315}{2048}(15 - 140t^2 + 378t^4 - 396t^6 + 143t^8)1[|t| \leq 1],$$

which is a kernel of order 6 (Müller 1984) meeting assumption S9. The sieves' dimensionality sequence $\rho(n) \propto n^{1/11}$ meets the assumptions of proposition 6. The optimal choice for $\rho(n)$ is beyond the scope of this paper. One natural way to proceed in practice is to use $\rho(n) = \Delta[n^{1/11}]$ where Δ is some measure of variation for the control function such as the empirical standard deviation of $\{\tilde{\theta}(i/n)\}_{i=1..n}$ obtained in the first stage. Here we have the advantage of knowing that the smoothness of the functions involved in all designs is very large so we simply use $\rho(n) = 2[n^{1/11}]$, which amounts to using a polynomial of degree 3 for our displayed simulations. The kernel bandwidths $h_* = \hat{\sigma}_L n^{-1/10}$ is chosen where $\hat{\sigma}_L$ refers

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[12]The different starting values are drawn from a Uniform distribution of mean $\theta_0' = (1, 1)$ and variance 5.

now to the sample standard deviation of $\{X_{1,i} + \tilde{X}_i'\tilde{\beta}(v) + b_n'p_n(\hat{V}_i)\}_{i=1..n}$ with b_n computed as explained on page 27. Finally, the regularization sequence $c_n = 20n^{-1/8}$ is selected to enlarge the trimming inside $H_n[\tilde{\beta}(v)]$ as explained on page 35.

Table 1 contains loss measures enabling to assess the quality of the estimators $\hat{\beta}$ of β. The Bias refers to absolute value of the bias i.e. $|E(\hat{\beta}) - \beta|$. The RM refers to the root mean squared error i.e. $(E|\hat{\beta} - \beta|^2)^{1/2}$. **Table 2** provides the size of the t-test for β relying on the estimators put forth in this article using the asymptotic critical values for a 1 percent, 5 percent and 10 percent type I error level.

As displayed on **Table 2**, the qualitative behaviors of the proposed estimators agree with the asymptotic theory developed in this paper. For all designs the bias and RM of the KWSMS estimator (hereafter noted KWSMSE) consistently shrink as n increases. The same applies to the SASMS estimator (hereafter noted SASMSE). For the KWSMSE, on average across bandwidths, a doubling of the sample size from 500 observations leads close to a 30 percent decrease in the loss measures (i.e. bias and RM) which is slightly faster than a 24 percent decrease hinted by asymptotic theory.[13] For all designs, the SASMSE performs poorly when $n = 250$ relative to the KWSMSE. As suggested by asymptotic theory the performance gap between the SASMSE and KWSMSE narrows for all designs if $n = 500$ where the SASMSE outperforms the KWSMSE (in terms of the RM) except for the LG design. That is, the SASMSE needs a large

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[13]proposition 3 suggests that the rate of convergence on the loss is $1/\sqrt{n^{1-a-a\eta}}$ which here implies a 24 percent decrease in losses for a doubling of the sample size. This discrepancy does not undermine our theory because the moments of $\sqrt{nhh_q}(\tilde{\theta} - \theta_0)$ need not to converge unless strong uniform integrability conditions hold, see Chung page 100-101.

enough sample to reach its asymptotic regime. As explained on page 34 the SASMSE defined in section 5.1 may not even exist in finite sample. The regularization scheme employed for the SASMSE is one out of many possible means to solve this existence problem at the origin of the larger RM experienced for $n = 250$. Motivated by these simulations and those of **Table 2** (discussed soon) there seems to be a need to develop in future research optimal regularization criteria for the SASMSE.

With respect to the overall competitiveness of the proposed estimators, the ST design clearly favors the KWSMSE (or SASMSE provided n large enough) for every sample size. In that case, the LIML is inconsistent with a RM twice larger when $n = 1000$. As expected the PR design unambiguously supports the LIML, which shows all its efficiency power. In that instance, the KWSMSE (respectively SASMSE) exhibits a RM approximately 3 times larger for $n = 1000$ (respectively for $n = 500$). Finally, the LG design still favors the LIML (which in not too surprising owing to the fact that the logistic distribution and normal distribution have relatively close shapes). In that logistic design, the second most performing estimator when $n = 250$ is the 2SLS, which is eventually slightly outperformed by the KWSMS for $n \geq 500$.

As exhibited on **Table 2**, the sizes of the test with the KWSMSE using the asymptotic critical values are systematically above the asymptotic sizes (even for a sample of 1000 observations). For instance, the size using the 5 percent critical value ranges from 10 to 29 percent across designs. Hence, one requires a much larger sample for the asymptotic critical values to provide an accurate probability coverage for the t-statistic. The same inferential problem affects the smoothed maximum score estimator (see Horowitz 1992). Even though one can not yet affirm whether the theory of boot-

strapping applies to the KWSMS, the result established in Horowitz (2002) concerning the SMSE does suggest that the critical value of a bootstrapped t-statistics will provide a more reliable coverage in finite sample for the KWSMSE. Alternatively, the SASMSE seems to offer somewhat superior testing capability in terms of sizes, which for $n = 500$ are closer to the ones promised by asymptotic theory. This is notably true for the ST design where the type I error of the null hypothesis is fairly accurately provided by the asymptotic critical value.

Overall, our Monte Carlo simulations hint that in small sample a KWSMS estimator can be severally bias when endogeneity is present in the binary choice model. However, if the sample size is sufficiently large this estimator offers a simple robust estimation technique to obtain accurate estimates as illustrated by the loss measures. However, conducting inferences using the estimated standard errors relying on proposition 4 may be misleading if the data set is not very large. Also, a second stage estimation with a SASMS estimator presents clear advantages when the sample size is sufficiently large especially in terms of testing.

Conclusion

This paper has presented a local version of the control function approach for the binary choice model to reach consistency when one of the explanatory variables is endogenous. This article has explained how the objective function of the SMSE can be weighted by means of a kernel taking the control variables' estimates as arguments in order to derive an asymptotically centered normal estimator. Finally, a consistent estimator for the asymptotic covariance matrix has been offered enabling expedient inferences for

Table 1: Losses

n=250	LIML	2SLS	KWSMS	SASMS
	Bias—RM	Bias—RM	Bias—RM	Bias—RM
ST	0.135—0.300	0.625—0.638	0.081—0.240	0.151—0.456
PR	0.005—0.178	0.666—0.676	0.256—0.939	0.377—0.985
LG	0.007—0.141	0.298—0.318	0.127—0.434	0.220—1.484
n=500	LIML	2SLS	KWSMS	SASMS
	Bias—RM	Bias—RM	Bias—RM	Bias—RM
ST	0.132—0.236	0.588—0.596	0.044—0.146	0.033—0.134
PR	0.006—0.118	0.623—0.630	0.115—0.355	0.120—0.348
LG	0.000—0.104	0.256—0.270	0.040—0.244	0.038—0.277
n=1000	LIML	2SLS	KWSMS	
	Bias—RM	Bias—RM	Bias—RM	
ST	0.133—0.184	0.554—0.560	0.034—0.098	
PR	0.000—0.082	0.580—0.584	0.075—0.255	
LG	0.001—0.070	0.227—0.236	0.028—0.168	

Table 2: Sizes

n=250	KWSMS	SASMS
Nominal level	0.01—0.05—0.10	0.01—0.05—0.10
ST	0.11—0.20—0.27	0.06—0.12—0.17
PR	0.23—0.34—0.42	0.14—0.21—0.26
LG	0.26—0.38—0.45	0.12—0.17—0.25
n=500	KWSMS	SASMS
Nominal level	0.01—0.05—0.10	0.01—0.05—0.10
ST	0.07—0.12—0.19	0.03—0.05—0.10
PR	0.17—0.26—0.33	0.02—0.09—0.17
LG	0.24—0.36—0.42	0.10—0.18—0.22
n=1000	KWSMS	
Nominal level	0.01—0.05—0.10	
ST	0.04—0.10—0.16	
PR	0.13—0.23—0.30	
LG	0.19—0.29—0.35	

applied work whenever a large data set is available. An alternative score approximation based smoothed maximum score estimator has also been described combining many first stage estimators to obtain a faster rate of convergence. Our Monte Carlo simulations hint that both of these estimators can provide new tools to estimate the coefficients of interest and conduct hypothesis testing in the binary choice model when endogeneity is present without having to impose strong distributional assumptions. We foresee three topics for future research. First, the optimal choice for the bandwidths. Secondly, the extension of the KWSMS to the case when more than one variable is endogenous and V is estimated at a non parametric rate. Finally, the testing of endogeneity in a model satisfying (i) and (ii).

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Appendix

Section A-Relaxing the Median Restriction and Heteroscedastic Endogenous Models

A1. **Semi Linear Heteroscedasticity**

To clarify the conflict between the key median restriction in [1] and heteroscedasticity considers a fairly general example taken from Lee 2007:

$$\varepsilon = H(V) + \vartheta(X_s, V)R,$$

where $R|Z, W, V \sim R$ and (X_s, X_g) forms a partition of X_* with Dim $X_g \geq 2$. That is $U = X_g'\beta_g + X_s'\beta_s + \varepsilon$ where $\beta = (\beta_s, \beta_g)$. Here $H(.)$ is some unknown function and $\vartheta(.,.)$ may not be known a priori. For simplicity suppose that $\vartheta(X_s, V) > 0$ $a.s.$ and that the cumulative distribution function of R noted $F_R(.)$ is strictly increasing. Assuming that R has a second moment the error term ε is thus heteroscedastic in (X_*, V) and it is rapid to show:

$$Med(\varepsilon|Z, W, V) = H(V) + \vartheta(X_s, V)F_R^{-1}(1/2) \ a.s.$$

When $Med(R) = 0$ holds then the restriction in [1] is met. Conversely, when $Med(R) \neq 0$ it is possible to employ our estimation procedure whenever $\vartheta(X_s, V)$ is semi linear of the form $T(V) + X_s'\varphi$ for some unknown $T(.)$ and parameter φ. To see this transformed the model into $U = X_g'\beta_g + X_s'(\beta_s + \varphi F_R^{-1}(1/2)) + e$ where now $Med(e|Z, W, V) = Med(e|V) = H(V) + T(V)F_R^{-1}(1/2)$ $a.s.$. In this case, β_g and the composite slope $\beta_s + \varphi F_R^{-1}(1/2)$ can be estimated consistently via our estimation method. Thus, unless further restrictions on the parameter $\beta_s + \varphi F_R^{-1}(1/2)$ are imposed, the estimator presented in this paper offers a viable approach to estimate and answer inferential questions pertaining only to β_g. Yet, our asymptotic results permit some testing on β_s. For instance, letting $\beta_{s,1}$ and $\beta_{s,2}$ being two slope coefficients of X_s one can test the null $\beta_{s,1} > \beta_{s,2}$ or $\beta_{s,1} = \beta_{s,2}$ which may be the only question of interest in some applied work when $\beta_{g,1}$ and $\beta_{g,2}$ represent some elasticities.

Finally, when $Med(R) \neq 0$ it is possible to extend our estimation procedure when $\vartheta(X_s, V)$ is unknown by using the partial median restriction $Med(\varepsilon|Z, W, V) = Med(\varepsilon|X_s, V)$ $a.s.$ and viewing (X_s, V) as the new artificial control variable (see section A2 below).

As when $\vartheta(X_s, V)$ is known to be semi linear, only the slope coefficient of X_g can be estimated consistently and tested for significance.

A2. **Unknown Heteroscedasticity**

Let suppose that there exists a partition of $\dot{X} = (X^s, X^g)$ for which:

$$Med(\varepsilon|\dot{X}, V) = Med(\varepsilon|X^s, V).$$

That is, the control is only "partial" because only X^g is purged after controlling for V. For instance, an heteroscedastic error term $\varepsilon = \sigma(X^s)e$ where $e|Z, W, V \sim e|V$ may generate such a partial restriction. As will be explained shortly, one can without much difficultly extend the framework laid out in section 3-4-5 to identify and consistently estimate the slope coefficients of the variables purged from the control i.e. X^g. In this section we only treat the extension for the KWSMS estimator since the adaptation for the SASMS estimator follows straightforwardly from that of the former.

Similarly to section 2, only a local control is needed for the median restriction. Let suppose that $\varepsilon|\overline{x^s}, \overline{v}$ has an absolutely continuous distribution function and that $\phi(\overline{x^s}, \overline{v}) = Med(\varepsilon|\overline{x^s}, \overline{v})$ is well defined. The extension in this context requires the following restriction:

$$Med(\varepsilon|X^g, \overline{x^s}, \overline{v}) = Med(\varepsilon|\overline{x^s}, \overline{v}) \text{ a.s.,}$$

Suppose that $p = dim X^g$ is greater than 2. Furthermore, assume that there exists a partition of $X^g = (X_1^g, \widetilde{X^g})$ with $X_1^g|\widetilde{X^g}$ having an absolutely continuous distribution (a.s.) and such that β_1^g, the slope coefficient of X_1^g is non null (which we take to be positive without loss of generality). Finally, suppose the distribution of $V, X^s|X^g$ is absolutely continuous (a.s.) with respect to the relevant product measure. Under some regularity conditions similar to those of proposition 1 one can show:

$$\theta_0 = (\beta_0, c_0) = Argmax_{\beta, c} E[d(\ell + \widetilde{X^g}(\beta - \beta_0) + c - c_0) f_{\widetilde{X^g}, \ell}(\overline{x^s}, \overline{v}) M_{\overline{x^s}, \overline{v}}(\ell)],$$

where

$\ell = X_1^g + \widetilde{X^g}\beta_0 + c_0,$

$\beta_0 = \dfrac{\widetilde{\beta^g}}{\beta_1^g},\ c_0 = \dfrac{\overline{x^s}'\beta_s + \phi(\overline{x^s}, \overline{v})}{\beta_1^g},$

$f_{\widetilde{X^g},\ell}(\overline{x^s},\overline{v})$ is the density of $X^s, V | \widetilde{X^g}, \ell$ evaluated at $(\overline{x^s}, \overline{v})$,

and

$M_{x^s,v}(\ell) = 1 - 2F_{x^s,v}[-\beta_1^g \ell + \phi(\overline{x^s},\overline{v})]$ with $F_{x^s,v}[.]$ the cumulative distribution function of $\varepsilon | X^s = x^s, V = v$.

Hence, under a random sampling from (Y, X^s, X^g), assuming the analogue of assumption 9 for $v, x^s \mapsto 1 - 2F_{\ell,\widetilde{x^g},x^s,v,}[-\beta_1^g \ell + \phi(\overline{x^s},\overline{v})]f_{\widetilde{x^g},\ell}(\overline{x^s},\overline{v})$ a.e.in $\widetilde{x^g}, \ell$ ($F_{\ell,\widetilde{x^g},x^s,v,}[.]$ indicating the cumulative distribution function of $\varepsilon | \ell, \widetilde{X^g}, X^s, V$) and given some root n consistent residuals for the control variable one can establish the weak consistency of:

$$Argmax_{\beta,c} \frac{1}{nh_q^{K-p+1}} \sum_{i=1}^n \alpha_i D(\frac{X_{1,i}^g + \widetilde{X_i^g}'\beta + c}{h})k(\frac{\hat{V}_i - \overline{v}}{h_q}, \frac{X_i^s - \overline{x^s}}{h_q}),$$

where $k(t_1, t_2)$ is now a multivariate kernel. Additionally, one can without major obstacles show the asymptotic normality of such an estimator adapting the assumptions of proposition 3 to the multivariate case for $k(.,.)$. Define $f_{\widetilde{x^g}}(\ell)$ the density of $\ell | \widetilde{x^g}$, the main changes would involve the analogue of assumptions 14 for $\ell \mapsto T_{\widetilde{x^g}}(\ell) = M_{\overline{x^s},\overline{v}}(\ell)f_{\widetilde{x^g},\ell}(\overline{x^s},\overline{v})f_{\widetilde{x^g}}(\ell)$ a.e.in $\widetilde{x^g}$ and assuming that $E[XX'T_{\widetilde{x^g}}^{(1)}(0)]$ is positive definite. We leave to the reader the details of this generalization.

Section B-Proof

This section provides the proofs of the propositions. Some notations will be used. $||X||$ denotes the Euclidean norm of a vector $X \in \mathbb{R}^p$ where $p \in \mathbb{N}$ and $|||M||| = \sqrt{traceMM'}$ for a real valued Matrix M. For $r > 0$ and $z \in \mathbb{R}^p$ where $p \in \mathbb{N}$ define $B(z,r) = \{x \in \mathbb{R}^p : ||x - z|| < r\}$. The least upper integer of a real number t is noted $int[t]$.

For a given multivariate real value function twice differentiable say $F(\theta)$ the symbol $\nabla F(\theta)$ denotes its gradient and $HF(\theta)$ its hessian evaluated at θ. Also the sequences of real value functions $D_n(t) = D(t/h)$, $K_n(t) = \frac{1}{h}K(t/h)$ and $k_n(t) = \frac{1}{h_q}k(t/h_q)$ are used. However, the notations $k_n(V) = \frac{1}{h_q}K(\frac{V-\overline{v}}{h_q})$ and $k_n(\hat{V}) = \frac{1}{h_q}K(\frac{\hat{V}-\overline{v}}{h_q})$ are employed which should be kept in mind.

Moreover, the objectives,

$$\widetilde{S_n}(\theta) = \frac{1}{nh_q} \sum_{i=1}^n \alpha_i D(\frac{X_{1,i} + X_i'\theta}{h})k(\frac{\hat{V}_i - \overline{v}}{h_q}),$$

and

$$S_n(\theta) = \frac{1}{nh_q} \sum_{i=1}^n \alpha_i D(\frac{X_{1,i} + X_i'\theta}{h})k(\frac{V_i - \overline{v}}{h_q}) \text{ are used.}$$

For an arbitrary real number v use:

$$\widetilde{S_n}(\theta, v) = \frac{1}{nh_q} \sum_{i=1}^{n} \alpha_i D(\frac{X_{1,i}+X_i'\theta}{h}) k(\frac{\hat{V}_i - v}{h_q})$$

and

$$S_n(\theta, v) = \frac{1}{nh_q} \sum_{i=1}^{n} \alpha_i D(\frac{X_{1,i}+X_i'\theta}{h}) k(\frac{V_i - v}{h_q}).$$

The gradient of $\widetilde{S_n}(\theta, v)$ with respect to θ is noted $\nabla \widetilde{S_n}(\theta, v)$ and its Hessian $H\widetilde{S_n}(\theta, v)$. Similarly, $\nabla S_n(\theta, v)$ and $HS_n(\theta, v)$ are used. Write $\theta_0(v)' = \frac{1}{\beta_1}(\phi(v), \tilde{\beta})$ whenever $\phi(v)$ exists. The notation $\lambda_{Min}[A]$ and $\lambda_{Max}[A]$ for a symmetric matrix A will refer to the smallest (respectively largest) eigenvalue of A. Define $P^\star$ the outer probability measure i.e. $P^\star(E) = inf\{\sum P(E_i)|E \subseteq \cup E_i, \{E_i\} \subseteq \Im\}$. Given a sequence of random variables X_n (not necessarily measurable) define $plim^\star X_n = 0$ if for any $\delta > 0$ there exists a natural number N such $n \geq N$ implies $P^\star[|X_n| > \delta] < \delta$. When unspecified the term lim is to be understood with respect to $n \to \infty$. Finally, the complement of a set E will be noted E'.

Lemma 1: Under assumptions 2-4,6,9,11-13 and 15

(i) plim $||S_n - ES_n||_{sup\Theta} = 0$. (ii) lim $||ES_n - S||_{sup\Theta} = 0$.

proof (i): Let $g_n(\theta) = S_n(\theta) - ES_n(\theta)$. We have:

$$g_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i D_n(\ell_i + X'\Delta)k_n(V_i) - E[\alpha_i D_n(\ell_i + X'\Delta)k_n(V_i)]$$

Notice that $|\alpha_i D_n(\ell_i + X'\Delta)k_n(V_i) - E[\alpha_i D_n(\ell_i + X'\Delta)k_n(V_i)]| \leq ||D_n||_{sup}||k||_{sup}\frac{1}{h_q}$ where $||D_n||_{sup}||k||_{sup}$ is a constant by assumption 11 and 12. Also using a change of variable provides :

$$E_{X,\ell}|k_n(V_i)|^2 = \frac{1}{h_q} \int |k(t)|^2 f_{X,\ell}(\overline{v} + th_q)dt = O(\frac{1}{h_q}) a.s.$$

due to assumption 12 and 9. Using this last finding and the fact the $D(.)$ is a bounded function provides:

$$\text{Var } [\alpha_i D_n(\ell_i + X'\Delta)k_n(V_i)] \leq E|D_n(\ell_i + X'\Delta)|^2 |k_n(V_i)|^2 = O(\frac{1}{h_q}).$$

It follows by the Bennett's inequality (1962) that given $\delta > 0$ arbitrary there exists a strictly positive constant $C(\delta)$ such that:

$$P[|g_n(\theta)| > \delta] \le 2e^{-nh_q C(\delta)}$$

(1)

and $\lim |g_n(\theta)| = 0$ a.s. follows by the Borel-Cantelli lemma because of assumption 13.
Finally, to show that the convergence is uniform consider the standard argument using
non overlapping coverings (Horowitz 1992 lemma 7 or Spady and Klein 1993 lemma 1) of
our compact set (assumption 8) with subsets of $\mathbb{R}^K$ such that the distance between two
points in each subset is strictly less than a positive sequence r_n. Let $C_{k,n}$ for $k = 1, ..., \Gamma_n$
denotes such collection of subsets where the number of coverings Γ_n will depend on the
length of the radius r_n. Let $\{\theta_{k,n}\}_{k=1...\Gamma_n}$ be some selected finite grid of points with
$\theta_{k,n} \in C_{k,n}$. Noticing first that:

$$||\nabla g_n(\theta)||_{sup\Theta} \le c_1 \frac{1}{hh_q}(\frac{1}{n}\sum ||X_i|| + c_2)$$

(where c_1 and c_2 are constants by assumption 12 and 13 by $E[||X||]$ existence i.e. assumption
15) and that any θ in some $C_{k,n}$ implies $||\theta - \theta_{k,n}|| < r_n$ yields:

$$||g_n(\theta)||_{sup\Theta} \le r_n c_1 \frac{1}{hh_q}(\frac{1}{n}\sum ||X_i|| + c_2) + sup_{k=1..\Gamma_n}|g_n(\theta_{k,n})|$$

It then suffices to set the decreasing radius such that $r_n \propto \frac{log(n)}{nhh_q}$ yielding:

$$\lim r_n c_1 \frac{1}{hh_q}(\frac{1}{n}\sum ||X_i|| + c_2) = 0 \text{ a.s.,}$$

because $\lim \frac{1}{n}\sum ||X_i|| = E[||X||]$a.s. by Kolmogorov's strong law of large numbers due to
our iid assumption and $r_n \frac{1}{hh_q} = o(1)$ by assumption 13. Finally, plim $sup_{k=1..\Gamma_n}|g_n(\theta_{k,n})|=0$
follows since for $\gamma > 0$ arbitrary and using (1) one can bound $P[sup_{k=1..\Gamma_n}|g_n(\theta_{k,n})| > \gamma]$
owing to:

$$P[\bigcup_{k=1..\Gamma_n} |g_n(\theta_{k,n})| > \gamma] \le \sum_{k=1}^{\Gamma_n} P[|g_n(\theta_{k,n})| > \gamma] \le 2\Gamma_n e^{-nh_q C(\gamma)}$$

where $\lim \Gamma_n e^{-nh_q C(\gamma)} = 0$ because $\Gamma_n \propto int[(1/r_n)^K]$. Hence, plim$||g_n(\theta)||_{sup\Theta} = 0$
is established.

proof(ii): step1: Our iid assumption and iterated expectation provide:

$$ES_n(\theta) = E[D_n(\ell + X'\Delta)k_n(V)E_{X,\ell,V}(\alpha)]$$

where

$$E_{X,\ell,V}(\alpha) = 1 - 2F_{X,\ell,V}(-\beta_1\ell + \phi(\overline{v}))$$

$F_{X,\ell,V}(.)$ indicating the distribution function of $\varepsilon|X,\ell,V$. Iterating again gives:

$$ES_n(\theta) = E[D_n(\ell + X'\Delta)E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\}]$$

where,

$$E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\}] = \int g_{X,\ell}(v)k_n(v)dv, \text{ and}$$

$$g_{X,\ell}(v) = [1 - 2F_{X,\ell,v}(-\beta_1\ell + \phi(\overline{v}))][f_{X,\ell}(v)].$$

Using a change of variable with $t = \frac{v-\overline{v}}{h_q}$ and assumptions 2 and 4 further provides:

$$E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\} = \int g_{X,\ell}(\overline{v} + th_q)k(t)dt \text{ a.s}$$

Also by assumption 9(a), there exists $\sigma > 0$ and a natural number $m \geq 2$ such that on $I_n = \{|t| < \sigma/h_q\}$:

$$g_{x,\ell}(\overline{v} + th_q) = g_{x,\ell}(\overline{v}) + \sum_{j=1}^{m-1} \frac{1}{j!}g_{x,\ell}^{(j)}(\overline{v})(th_q)^j + \frac{1}{m!}g_{x,\ell}^{(m)}(\xi(x,\ell))(th_q)^m \text{ a.e.in } x,\ell,$$

for some $\xi(x,\ell)$ meeting $|\xi(x,\ell) - \overline{v}| < \sigma$ where for $|v - \overline{v}| < \sigma$,

$$g_{x,\ell}^{(j)}(v) = \sum_{k=0}^{j} \frac{1}{k!(j-k)!}j![1 - 2F_{x,\ell,v}(-\beta_1\ell + \phi(\overline{v}))]^{(k)}[f_{x,\ell}(v)]^{(j-k)}$$

with

$$[1 - 2F_{x,\ell,v}(-\beta_1\ell + \phi(\overline{v})]^{(j)} = \frac{\partial^j}{\partial^j v}1 - 2F_{x,\ell,v}(-\beta_1\ell + \phi(\overline{v})) \text{ and}$$
$$[f_{x,\ell}(v)]^{(j)} = \frac{\partial^j}{\partial^j v}f_{x,\ell}(v) \text{ for } j = 1...m.$$

Simplifying and using assumption 12 offers:

$$\int g_{x,\ell}(\overline{v} + th_q)k(t)dt = g_{x,\ell}(\overline{v}) - g_{x,\ell}(\overline{v})\int_{I'_n} k(t)dt - \sum_{j=1}^{m-1} \frac{g_{x,\ell}^{(j)}(\overline{v})}{j!}h_q^j \int_{I'_n} t^j k(t)dt +$$
$$\frac{h_q^m}{m!}\int_{I_n} g_{x,\ell}^{(m)}(\xi(x,\ell))t^m k(t)dt + \int_{I'_n} g_{x,\ell}(\overline{v} + th_q)k(t)dt \text{ a.e.in } x,\ell.$$

Furthermore, $|g_{x,\ell}(v)| < M_1^*$ for all v, $|g_{x,\ell}^{(j)}(\overline{v})| < M_1^*$ for $j = 1,...,m-1$ and $|g_{x,\ell}^{(m)}(\xi(x,\ell))| < M_1^*$ a.e in x,ℓ for some finite constant M_1^* by assumption 9(a). It follows that:

$$h_q^{-m}|\int g_{x,\ell}(\overline{v} + th_q)k(t)dt - g_{x,\ell}(\overline{v})| \leq M_1^*\beth_n \text{ a.e in } x,\ell,$$

where $\beth_n = 2h^{-m}\int_{I'_n}|k(t)|dt + \sum_{j=1}^{m-1}\frac{h_q^{j-m}}{j!}\int_{I'_n}|t^jk(t)|dt + \frac{1}{m!}\int|t^mk(t)|dt$ is a bounded sequence by assumption 12(d).

Consequently,

$$E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\}] = g_{X,\ell}(\overline{v}) + R_n \text{ a.s}$$

where $g_{X,\ell}(\overline{v})$ is given as in proposition 1 due to $E_{X,\ell,V=\overline{v}}(\alpha) = 1 - 2F_{X,\ell,\overline{v}}[-\beta_1\ell + \phi(\overline{v})]$ and $|R_n| = O(h_q^m)$ a.s. by our finding in (1')and assumptions 12(d). Furthermore, $|g_{X,\ell}(\overline{v})|$ is bounded almost surely by some constant C (under assumption 4) yielding:

$$|ES_n(\theta) - S(\theta)| \le CE[|D_n(\ell + X'\Delta) - d(\ell + X'\Delta)|] + O(h_q^m).$$

Step 2: Subsequently, it is straightforward to establish $\lim D_n = d$ a.e. by assumption 11 (where the convergence may not hold at the origin). It follows (by Horowitz 1992, lemma 4) that given $\varepsilon > 0$ arbitrary there exists some Borel set $\mathfrak{B}$ of Lebesgue measure strictly less than ε where $\lim ||D_n - d||_{sup\mathfrak{B}'} = 0$ holds. Consequently:

$$|ES_n(\theta) - S(\theta)| \le C(||D||_{sup} + 1)P[\ell + X'\Delta \in \mathfrak{B}] + C||D_n - d||_{sup\mathfrak{B}'} + O(h_q^m).$$

step 3: Finally, the cumulative distribution function of $\ell|X = x$ is absolutely continuous with respect to the Lebesgue measure a.e.in x by assumption 4(a) with furthermore a density whose essential supremum is bounded by some constant M a.e.in x by assumption 9(b)implying:

$$P[\ell + X'\Delta \in \mathfrak{B}] < M\varepsilon \text{ uniformly over } \Theta,$$

where we used $P[\ell + X'\Delta \in \mathfrak{B}] = EP_X[\ell \in \mathfrak{B} - X'\Delta]$ and the invariance of the Lebesgue measure to translation. Hence:

$$|ES_n(\theta) - S(\theta)| \le C(||D||_{sup} + 1)M\varepsilon + C||D_n - d||_{sup\mathfrak{B}'} + O(h_q^m),$$

with $O(h_q^m) = o(1)$ by assumption 13. It follows that for any $\delta > 0$ one can pick $\mathfrak{B}$ to have measure $\varepsilon < \frac{\delta}{3M(||D||_{sup}+1)C}$ so there exists a sample size $N(\delta)$ such that $n \ge N(\delta)$ implies $|ES_n(\theta) - S(\theta)| < \delta$ uniformly over Θ concluding (ii). QED

Lemma 2: Let G be some function in $C^2_\infty(M)$ for some finite real number M, $K(.)$ satisfying assumption 16 and h_n some strictly positive sequence converging to 0 as n approaches infinity. Then we have:

$$\lim ||\mu_n(x) - G^{(1)}(x)||_{sup} = 0 \text{ where } \mu_n(x) = \frac{1}{h} \int -K^{(1)}(t)G(x + th)dt$$

proof: Define $E_n = \{t \in \mathbb{R} : |t| \leq \frac{1}{h}\}$ and use the indicator function $1_E(t) = 1$ if t belongs to a real Borel set E. Given an arbitrary real number x we have:

$$\mu_n(x) = I_{n1}(x) + I_{n2}(x)$$

where

$$I_{n1}(x) = \frac{1}{h} \int -K^{(1)}(t)G(x + th)1_{E'_n}(t)dt$$

and

$$I_{n2}(x) = \frac{1}{h} \int -K^{(1)}(t)G(x + th)1_{E_n}(t)dt$$

The first part is easy as:

$$|I_{n1}(x)| \leq ||G||_{sup} \frac{1}{h} \int |K^{(1)}(t)|1_{E'_n}(t)dt$$

This results in $\lim |I_{n1}(x)| = 0$ uniformly in x by the tail property of the Kernel $K(.)$ (assumption 16(d)). Furthermore, integrating by part over E_n yields:

$$I_{n2}(x) = I_{n3}(x) + I_{n4}(x)$$

where

$$I_{n3}(x) = -\frac{1}{h}\{K(t)G(x + th)|_{t \in E_n}\} \text{ and } I_{n4}(x) = \int K(t)G^{(1)}(x + th)1_{E_n}(t)dt$$

Moreover, $|I_{n3}(x)| \leq 2||G||_{sup}\frac{1}{h}|K(\frac{1}{h})|$ because the Kernel is symmetric by assumption. As $|t||K(t)|$ tends to 0 as t tends to infinity by assumption 16 we obtain $\lim |I_{n3}(x)| = 0$ uniformly in x. Consequently we have :

$$\mu_n(x) - G^{(1)}(x) = \int [G^{(1)}(x + th)1_{E_n}(t) - G^{(1)}(x)]K(t)dt + e_n(x)$$

where the function $e_n(x) = I_{n1}(x) + I_{n3}(x)$ meets $||e_n||_{sup} = o(1)$ by our previous findings. Simplifying gives:

$$|\mu_n(x) - G^{(1)}(x)| \leq \int |G^{(1)}(x)||K(t)|1_{E_n'}\,dt + \int |G^{(1)}(x+th) - G^{(1)}(x)||K(t)|dt + |e_n(x)|$$

$$\leq ||G^{(1)}||_{sup} \int |K(t)|1_{E_n'}\,dt + hL \int |t||K(t)|dt + ||e_n||_{sup}$$

where L is a constant as the derivative of $G(.)$ is Liptchitz due to G belonging to $C_\infty^2(M)$. Finally, using $\lim \int |K(t)|1_{E_n'}\,dt = 0$ (by the Lebesgue's Dominated Convergence Theorem) and $\int |t||K(t)|dt < \infty$ by assumption 16 finishes the proof. QED

Lemma 3: Under assumptions 1-4,6,8,9,12-17

$$plim\ HS_n(\theta) = -E[XX'T_X^{(1)}(-X'\Delta)]\ where\ \Delta = \theta - \theta_0\ uniformly\ over\ \Theta.$$

proof: We will first use $HS_n^*(\theta) = \frac{1}{n}\sum_{i=1}^n \alpha_i X_i X_i' 1_{\{|X_iX_i'|\leq a_n\}} K_n^{(1)}(\ell_i + X_i'\Delta)k_n(V_i)$ (where $|.|$ here is to be understood component wise) where $a_n \propto h^{-2}log(n)$. We will start showing the uniform consistency of HS_n^* since it is easier to establish. Then, we will have left to show $plim\ HS_n(\theta) - HS_n^*(\theta) = 0$ uniformly over Θ. The notation $H(\theta) = -E[XX'T_X^{(1)}(-X'\Delta)]$ is adopted.

step1: Let 's show $plim\ HS_n^*(\theta) - EHS_n^*(\theta) = 0$ uniformly over Θ. We have:
$HS_n^*(\theta) - EHS_n^*(\theta) =$

$$\frac{1}{n}\sum_{i=1}^n \alpha_i X_i X_i' 1_{\{|X_iX_i'|\leq a_n\}} K_n^{(1)}(\ell_i + X_i'\Delta)k_n(V_i) -$$
$$E[\alpha_i X_i X_i' 1_{\{|X_iX_i'|\leq a_n\}} K_n^{(1)}(\ell_i + X_i'\Delta)k_n(V_i)].$$

By assumption 12 and 16 we get :

$$|\alpha_i X_i X_i' 1_{\{|X_iX_i'|\leq a_n\}} K_n^{(1)}(\ell_i+X_i'\Delta)k_n(V_i) - E[\alpha_i X_i X_i' 1_{\{|X_iX_i'|\leq a_n\}} K_n^{(1)}(\ell_i+X_i'\Delta)k_n(V_i)]| =$$
$$O(\frac{a_n}{h^2 h_q})$$

$$(2)$$

Also, by assumption 12 and 9 and a change of variable it is rapid to find $E_{X,\ell}[|k_n(V)|^2] \leq \frac{C_1}{h_q}$ a.s. for some finite constant C_1 and similarly by Assumption 14 and 16 that $E_X[|K_n^{(1)}(\ell_i + X_i'\Delta)|^2] \leq \frac{C_2}{h^3}$ a.s. for some finite constant C_2. Hence, by iterated expectations first with respect to X, ℓ and then with respect to X we obtain the inequality:

$$E[|X_iX_i'|^2 1_{\{|X_iX_i'|\le a_n\}}|K_n^{(1)}(\ell_i + X_i'\Delta)|^2|k_n(V_i)|^2] \le \frac{C_1 C_2}{h^3 h_q} E[|X_iX_i'|^2]$$

and because $E[|X_iX_i'|^2]$ exists by assumption 15:

$$Var[\alpha_i X_iX_i'1_{\{|X_iX_i'|\le a_n\}}K_n^{(1)}(\ell_i + X_i'\Delta)k_n(V_i)] = O(\frac{1}{h^3 h_q})$$

$$(3)$$

Combining (2) and (3) suffices for applying again the Bennett's inequality implying that for any $\delta > 0$ arbitrary real number there exist two strictly positive constants ν_1 and ν_2 such that:

$$P[|HS_n^*(\theta) - EHS_n^*(\theta)| > \delta] \le 2e^{\frac{-nh^3 h_q \delta^2}{\nu_1 + \nu_2 a_n h}}$$

and plim $|HS_n^*(\theta) - EHS_n^*(\theta)| = 0$ follows since $\lim \frac{nh^4 h_q}{log(n)} = \infty$ by assumption 17. Subsequently, we have the bounding:

$$\frac{\partial}{\partial\theta}|\alpha_i X_iX_i'1_{\{|X_iX_i'|\le a_n\}}K_n^{(1)}(\ell_i + X_i'\Delta)k_n(V_i)| \le O(\frac{a_n}{h^3 h_q})$$

due to $||K^{(2)}||_{sup}||k||_{sup}$ being a finite constant by assumptions 12 and 16. Hence, choosing a non overlapping covering with balls whose side length r_n satisfies $r_n \frac{a_n}{h^3 h_q} = o(1)$ will provide plim $|HS_n^*(\theta) - EHS_n^*(\theta)| = 0$ uniformly over Θ by a similar argument as that used for lemma 1.

step2: Let 's now show $\lim EHS_n^*(\theta) - H(\theta) = 0$ uniformly over Θ.

By assumption 6 we obtain:

$$E[HS_n^*(\theta)] = E[XX'1_{\{|XX'|\le a_n\}}K_n^{(1)}(\ell + X'\Delta)E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\}]$$

Invoking assumptions 9 and 12 and employing the same approach as in lemma 1(ii) provides:

$$E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\} = g_{X,\ell}(\overline{v}) + R_n a.s. \text{ with } R_n = O(h_q^m)\text{a.s.}$$

it follows that:

$$E[HS_n^*(\theta)] = A_{1,n}(\theta) + A_{2,n}(\theta)$$

where

$$A_{1,n}(\theta) = E[XX'1_{\{|XX'|\leq a_n\}}K_n^{(1)}(\ell + X'\Delta)g_{X,\ell}(\overline{v})]$$

and

$$A_{2,n}(\theta) = E[XX'1_{\{|XX'|\leq a_n\}}K_n^{(1)}(\ell + X'\Delta)R_n]$$

First, by assumption 15 and 16 we can use the fact that (where $f_X(.)$ indicates the density of $\ell|X$):

$$E_X\{|K_n^{(1)}(\ell + X'\Delta)| = \int \frac{1}{h}|K^{(1)}(t)|f_X(th - X'\Delta)dt \leq M_2 \int \frac{1}{h}|K^{(1)}(t)|dt \text{ a.s. for some}$$
$$\text{finite constant } M_2,$$

and the existence of $E|XX'|$ (i.e.assumption 15) to derive:

$$|E[XX'1_{\{|XX'|\leq a_n\}}K_n^{(1)}(\ell + X'\Delta)R_n]| = O(\frac{h_q^m}{h}).$$

This proves $\lim A_{2,n}(\theta) = 0$ uniformly over Θ since $O(\frac{h_q^m}{h}) = o(1)$ by assumption 17.

Secondly, using $\mu_n(X, \Delta) = E_X\{K_n^{(1)}(\ell + X'\Delta)g_{X,\ell}(\overline{v})\}$ and some simplifications furnishes:

$$A_{1,n}(\theta) - H(\theta) =$$
$$E[XX'1_{\{|XX'|\leq a_n\}}\{\mu_n(X, \Delta) + T_X^{(1)}(-X'\Delta)\}] + E[XX'1_{\{|XX'|>a_n\}}T_X^{(1)}(-X'\Delta)]$$

But notice that $E_X\{K_n^{(1)}(\ell + X'\Delta)g_{X,\ell}(\overline{v})\} = \frac{1}{h}\int T_X(th - X'\Delta)K^{(1)}(t)dt$ where $T_x(\ell) = g_{x,\ell}(\overline{v})f_x(\ell)$. Under assumption 14 and assumption 16 the conditions of lemma 2 holds (a.e. in x) yielding:

$$|\frac{1}{h}\int T_X(th - X'\Delta)K^{(1)}dt + T_X^{(1)}(-X'\Delta)| \leq M_2^2 b_n + 2M_2^2 c_n + 4hM_2^2 \text{ a.s for some finite}$$
$$\text{constant } M_2,$$

where b_n and c_n are deterministic sequences vanishing to 0 as n approaches infinity. This last finding along with $E|XX'|$ existence establishes that:

$$\lim E[XX'1_{\{|XX'|\leq a_n\}}\{\mu_n(X, \Delta) + T_X^{(1)}(-X'\Delta)\}] = 0 \text{ uniformly over } \Theta.$$

Finally, $|T_X^{(1)}(-X'\Delta)|$ is almost surely bounded by a finite constant (independently of θ) by assumption 14, $E|XX'|$ exists and the sequence a_n meets $\lim a_n = \infty$. Thus, the Dominated Convergence Theorem directly yields:

$$\lim E[XX'1_{\{|XX'|>a_n\}}T_X^{(1)}(-X'\Delta)] = 0$$

Hence, $\lim A_{1,n}(\theta) = H(\theta)$ uniformly over Θ is established and thus $\lim EHS_n^*(\theta) - H(\theta) = 0$ uniformly over Θ.

step 3: Using basic inequalities we find:

$$||HS_n(\theta) - HS_n^*(\theta)||_{sup\Theta} \leq \frac{1}{h^2}||K^{(1)}||_{sup}\frac{1}{n}\sum_{i=1}^{n}|X_iX_i'|1_{\{|X_iX_i'|>a_n\}}|k_n(V_i)|$$

so that:

$$E[||HS_n(\theta) - HS_n^*(\theta)||_{sup\Theta}] \leq \frac{1}{h^2}||K^{(1)}||_{sup}E[|X_iX_i'|1_{\{|X_iX_i'|>a_n\}}|k_n(V_i)|]$$

where $E_{X,\ell}[|k_n(V_i)|] \leq M_1\int|k(t)|dt$ a.s. which is rapid to show by change of variable in the integral along with assumption 9. Consequently:

$$E[||HS_n(\theta) - HS_n^*(\theta)||_{sup\Theta}] \leq M_1(\int|k(t)|dt)\frac{1}{h^2}E[|X_iX_i'|1_{\{|X_iX_i'|>a_n\}}]$$

But $E|X_iX_i'|^2 < \infty$ from assumption 7 so by the Cauchy-Schwartz's inequality we can assert:

$$E[|X_iX_i'|1_{\{|X_iX_i'|>a_n\}}] \leq \{E|X_iX_i'|^2\}^{1/2}\{P[|X_iX_i'| > a_n]\}^{1/2}$$

and by the Tchebychev's inequality:

$$P[|X_iX_i'| > a_n] \leq \frac{E|X_iX_i'|^2}{a_n^2}.$$

Since $\int|k(t)|dt < \infty$ (i.e.assumption 12) and $\lim a_nh^2 = \infty$ we have established:

$$\lim E[||HS_n(\theta) - HS_n^*(\theta)||_{sup\Theta}] = 0$$

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and lemma 3 follows by a triangular inequality using step 1 and step 2. QED

Lemma 4:Under assumptions 9,12,14-16

$$E[\nabla S_n(\theta_0)] = O(h_q^m) + O(h^r).$$

proof: Under the iid sampling (assumptions 6)we obtain:

$$E[\nabla S_n(\theta_0)] = E[\alpha X K_n(\ell) k_n(V)] = E[X K_n(\ell) k_n(V) E_{X,\ell,V}(\alpha)]$$

where

$$E_{X,\ell,V}(\alpha) = 1 - 2F_{X,\ell,V}[-\beta_1\ell + \phi(\overline{v})]$$

and similarly to lemma 1 using assumption 9 and 12 permits to show :

$$E_{X,\ell}\{k_n(V)E_{X,\ell,V}(\alpha)\}] = g_{X,\ell}(\overline{v}) + R_n \text{ a.s}$$

where $|R_n| = O(h_q^m)$ a.s. which henceforth returns:

$$E[\nabla S_n(\theta_0)] = B_{1,n} + B_{2,n}$$

where

$$B_{1,n} = E[X K_n(\ell) g_{X,\ell}(\overline{v})]$$

and

$$B_{2,n} = E[X K_n(\ell) R_n]$$

First notice that $|E[X K_n(\ell) R_n]| \leq O(h_q^m) E[|X||K_n(\ell)|]$ and that $E[|X||K_n(\ell)|] = E[|X|E_X\{|K_n(\ell)|\}]$ is bounded due to:

$$E_X\{|K_n(\ell)|\} = \int f_X(th)|K(t)|dt \leq M_2 \int |K(t)|dt \text{ a.s.,}$$

for some finite constant M_2 ($f_X(.)$ indicating the density of $\ell|X$) by assumptions 14 and 16. Hence, $B_{2,n} = O(h_q^m)$ is established. Secondly, we can rewrite $B_{1,n}$ by iterating with respect X yielding:

$$B_{1,n} = E[X\rho_n(X)]$$

where $\rho_n(X) = E_X\{K_n(\ell)g_{X,\ell}(\overline{v})\} = \int T_X(\ell)K_n(\ell)d\ell$ with $T_X(\ell) = g_{X,\ell}(\overline{v})f_X(\ell)$. Since by assumptions 14, $T_x(\ell)$, as a function of ℓ, is $r \geq 2$ times continuously differentiable everywhere with bounded j^{th} derivatives for $j = 1...r$ (a.e.in x) we can use the same approach as in lemma 1 but this time with a change of variable $t = \frac{\ell}{h}$ "Taylorizing" $T_x(th)$ around 0 at order $r - 1$ and invoking assumption 15 to find:

$$\rho_n(X) = T_X(0) + R'_n \text{ a.s.,}$$

where $T_X(0) = 0$ a.s. since $F_{X,0,\overline{v}}[\phi(\overline{v})] = 1/2$ a.s. by assumption 3. Also $R'_n = O(h^r)$a.s. is straightforward to establish using the existence of some constant C such that $|T_x^{(r)}(\ell)| < C$ a.e.in x (from assumptions 14) and the same bounding principle as given in equation(1) of lemma 1. Because $E|X|$ exists by assumption 15 we have also $B_{1,n} = O(h^r)$ which concludes lemma 4. QED

Lemma 5: Under assumptions 9,11,12,14-17

$$\sqrt{nhh_q}\nabla S_n(\theta_0) \to_d \mathcal{N}(0,\Sigma)$$

proof: It will be convenient to note $s_{i,n} = \sqrt{hh_q}\alpha_i X_i K_n(\ell_i)k_n(V_i)$ and $u_{i,n} = E[s_{i,n}]$ for i=1...n. The structure of the proof is as follows. First, we will show that $\sqrt{nhh_q}(\nabla S_n(\theta_0) - E[\nabla S_n(\theta_0)]) \to_d \mathcal{N}(0,\Sigma)$. Then, we will prove $\lim \sqrt{nhh_q}E[\nabla S_n(\theta_0] = 0$. We have thus:

$$\sqrt{nhh_q}(\nabla S_n(\theta_0) - E[\nabla S_n(\theta_0)]) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} s_{i,n} - u_{i,n}$$

step1: We will first show some preliminary results. Let $\delta > 0$ be some arbitrary constant. Notice that under assumptions 9,11,12,16 and using a change of variable in the integral as in lemma 1 we have:

$$E_{X,\ell}|\frac{1}{\sqrt{h_q}}k(\frac{V-\overline{v}}{h_q})|^{2+\delta} = \frac{1}{h_q^{\delta/2}}\int |k(t)|^{2+\delta}f_{X,\ell}(\overline{v} + th_q)dt \leq \frac{M_1}{h_q^{\delta/2}}\int |k(t)|^{2+\delta}dt \text{ a.s. for}$$
$$\text{some constant } M_1$$

(4)

Similarly but using assumptions 14 15 and 16 returns:

$$E_X|\tfrac{1}{\sqrt{h}}K(\tfrac{\ell}{h})|^{2+\delta}| \leq \tfrac{M_2}{h_q^{\delta/2}}\int|K(t)|^{2+\delta}dt \text{ a.s. for some constant } M_2$$

$$(5)$$

Letting $L_n = \sum_{i=1}^n |\tfrac{s_{i,n}-u_{i,n}}{\sqrt{n}}|^{2+\delta}$ we obtain under assumption 6:

$$E[L_n] \leq n^{-\delta/2}E|s_{i,n}-u_{i,n}|^{2+\delta} \leq 2^{1+\delta}n^{-\delta/2}E|s_{i,n}|^{2+\delta}$$

where $E|s_{i,n}|^{2+\delta} = E[|X|^{2+\delta}|\tfrac{1}{\sqrt{h}}K(\tfrac{\ell}{h})|^{2+\delta}|\tfrac{1}{\sqrt{h_q}}k(\tfrac{V-\overline{v}}{h_q})|^{2+\delta}]$. Using (4) and (5) along with assumptions 15(a) and 16(b) ensures that there exists some $\delta > 0$ meeting:

$$E|s_{i,n}|^{2+\delta} = O(\tfrac{1}{h_q^{\delta/2}h^{\delta/2}})$$

and consequently $\lim E[L_n] = 0$ for some some $\delta > 0$ holds because of our choice for the bandwidths meeting $\lim nh_qh = \infty$ by assumption 17.

step 2: Additionally,

$$E[s_{i,n}s'_{i,n}] = E[XX'\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2P_n(X,\ell)]$$

where

$$P_n(X,\ell) = E_{X,\ell}\{\tfrac{1}{h_q}|k(\tfrac{V-\overline{v}}{h})|^2\} = \int \tfrac{1}{h_q}|k(\tfrac{v-\overline{v}}{h_q})|^2f_{X,\ell}(v)dv.$$

Moreover, a change of variable $t = \tfrac{v-\overline{v}}{h_q}$ and a similar reasoning used to derive $(1')$ of lemma 1 invoking assumption $9(a)$, $12(b)$ and $16(c)-(d)$ provides:

$$P_n(X,\ell) = \int f_{X,\ell}(\overline{v}+th_q)|k(t)|^2dt = f_{X,\ell}(\overline{v})\int|k(t)|^2dt + R_n, \text{ where } R_n = O(h_q) \text{ a.s.}$$

Thus, we obtain:

$$E[s_{i,n}s'_{i,n}] = \int|k|^2E[XX'\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2f_{X,\ell}(\overline{v})] + E[XX'\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2R_n]$$

Moreover, it is easy to show from assumption 14 and a change of variable that $|E[XX'\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2]|$ is bounded by $M_2\int|K|^2E|XX'| < \infty$ from assumptions 15(a)and 16(b). As a result we find :

$$E[s_{i,n}s'_{i,n}] = \int |k|^2 E[XX'\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2 f_{X,\ell}(\overline{v})] + o(1)$$

Lastly, assumption 9 and assumption 16 yield:

$$E_X[\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2 f_{X,\ell}(\overline{v})] = \int \mu_X(\ell)\tfrac{1}{h}|K(\tfrac{\ell}{h})|d\ell$$

where $\mu_x(\ell) = f_{x,\ell}(\overline{v})f_x(\ell)$ is continuous and meets $|\mu_x(.)| < C$ for some finite constant

C (a.e. in x) by assumptions 14. Thus, changing the variable into $t = \frac{\ell}{h}$ provides:

$$E_X[\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2 f_{X,\ell}(\overline{v})] = \int \mu_X(th)|K(t)|^2 dt$$

and two consecutive applications of the Dominated Convergence Theorem furnishes:

$$\lim E_X[\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2 f_{X,\ell}(\overline{v})] = \mu_X(0)\int |K(t)|^2 dt \text{ a.s.,}$$

and

$$\lim E[XX'E_X\{\tfrac{1}{h}|K(\tfrac{\ell}{h})|^2 f_{X,\ell}(\overline{v})\}] = \int |K|^2 E[XX'\mu_X(0)].$$

This subsequently offers:

$$\lim E[s_{i,n}s'_{i,n}] = \int |k|^2 \int |K|^2 E[XX'\mu_X(0)]$$

Notice also that $\lim E[(s_{i,n}-u_{i,n})(s_{i,n}-u_{i,n})'] = \lim E[s_{i,n}s'_{i,n}]$ due to $u_{i,n} = \sqrt{hh_q}E[\nabla S_n(\theta_0)] = o(1)$ by lemma 4 and assumptions 14. Hence, using the conclusion of step 1 and step 2 permits to apply the Lyapunov's Central Limit Theorem (Chung p 208) to affirm:

$$\sqrt{nhh_q}(\nabla S_n(\theta_0) - E[\nabla S_n(\theta_0)]) \rightarrow_d \mathcal{N}(0,\Sigma)$$

Finally, $\sqrt{nhh_q}E[\nabla S_n(\theta_0)] = O(\sqrt{nhh_q}h_q^m) + O(\sqrt{nhh_q}h^r)$ by lemma 4 and $\sqrt{nhh_q}E[\nabla S_n(\theta_0)] = o(1)$ follows by assumptions 17. QED

Lemma 6: Under assumptions 6,7 and 10-14

$$plim||\widetilde{S_n} - S_n||_{sup\Theta} = 0$$

proof: Using the fact that $D(.)$ is bounded by assumption 11 first let us find :

$$||\widetilde{S_n} - S_n||_{sup\Theta} \leq ||D||_{sup} \frac{1}{n} \sum_{i=1}^{n} |k_n(\hat{V_i}) - k_n(V_i)|$$

and $||k^{(1)}||_{sup}$ is finite by assumption 12(iii) so the mean value theorem further provides:

$$||\widetilde{S_n} - S_n||_{sup\Theta} \leq ||D||_{sup}||k^{(1)}||_{sup} \frac{1}{nh_q^2} \sum_{i=1}^{n} |\hat{V_i} - V_i|$$

finally, using $|\hat{V_i} - V_i| = |W_i'(\hat{\Pi} - \Pi)|$ and noting $C = ||D||_{sup}||k^{(1)}||_{sup}$ yields:

$$||\widetilde{S_n} - S_n||_{sup\Theta} \leq C||\hat{\Pi} - \Pi||h_q^{-2} \frac{1}{n} \sum_{i=1}^{n} ||W_i||$$

where $\frac{1}{n} \sum_{i=1}^{n} ||W_i|| = O_p(1)$ by assumption 7 and $||\hat{\Pi} - \Pi||h_q^{-2} = O_p(h_q^{-2}n^{-1/2}) = o_p(1)$ by assumption 10 and 13 which shows the claim.

Lemma 7: under assumptions 6,7,10,12,16 and 17

$$plim \ ||H\widetilde{S_n} - HS_n||_{sup\Theta} = 0$$

proof: Since $||K^{(1)}||_{sup}$ is finite by assumption 16 we have:

$$||H\widetilde{S_n} - HS_n||_{sup\Theta} \leq ||K^{(1)}||_{sup} \frac{1}{nh^2} \sum_{i=1}^{n} |X_i X_i'||k_n(\hat{V_i}) - k_n(V_i)|$$

where $|k_n(\hat{V_i}) - k_n(V_i)| \leq \frac{1}{h_q^2}||k^{(1)}||_{sup}|W_i'(\Pi - \hat{\Pi})|$ by assumption 12. Noting $C = ||K^{(1)}||_{sup}||k^{(1)}||_{sup}$ and simplifying further yields:

$$||H\widetilde{S_n} - HS_n||_{sup\Theta} \leq C \frac{1}{nh^2 h_q^2}||\Pi - \hat{\Pi}|| \sum_{i=1}^{n} |X_i X_i'|||W_i||$$

where $\frac{1}{n} \sum_{i=1}^{n} |X_i X_i'|||W_i|| = O_p(1)$ by assumption 6-7 and 15 and $\frac{1}{h^2 h_q^2}||\Pi - \hat{\Pi}|| = O_p(\frac{1}{h^2 h_q^2 n^{1/2}})$ by assumption 10. Consequently $||H\widetilde{S_n} - HS_n||_{sup\Theta} = o_p(1)$ by assumption 17. QED

Lemma 8: Under assumptions 6,7,10,11,13,14,16 and 17

$$plim \ \sqrt{nhh_q}||\nabla\widetilde{S_n}(\theta_0) - \nabla S_n(\theta_0)|| = 0$$

proof: Using assumption 11 and $|k_n(\hat{V}_i) - k_n(V_i)| \leq \frac{1}{h_q^2}||k^{(1)}||_{sup}|W_i'(\hat{\Pi} - \Pi)|$ easily shows that for some constant C:

$$\sqrt{nhh_q}||\nabla\widetilde{S_n}(\theta_0) - \nabla S_n(\theta_0)|| \leq C\sqrt{nhh_q}\frac{||\Pi - \hat{\Pi}||}{h_q^2}T_n$$

where $T_n = \frac{1}{nh}\sum_{i=1}^n ||X_i|||K(\frac{\ell_i}{h})|$. Now assumptions 13-14-16-17 and a double application of the Dominated Convergence Theorem easily yields $\lim E[T_n] = \int |K|E[||X||f_X(0)]$ (where $f_X(0)$ is the density of $\ell|X$ evaluated at 0). Also, under the iid sampling (assumptions 6), $Var(T_n) \leq \frac{1}{nh^2}E[||X||^2|K(\frac{\ell}{h})|^2]$ and the classic change of variable subsequently offers:

$$Var(T_n) \leq \frac{1}{nh}E[||X||^2 \int |K(t)|^2 f_X(th)dt]$$

where again the by Dominated Convergence Theorem applied twice establishes that $E[||X||^2 \int |K(t)|^2 f_X(th)dt]$ is bounded for:

$$\lim E[||X||^2 \int |K(t)|^2 f_X(th)dt] = E[||X||^2\lim$$

$$\{\int |K(t)|^2 f_X(th)dt\}] = \int |K(t)|^2 E[||X||^2 f_X(0)]$$

Since $\lim nh = \infty$ by assumption 17 we conclude that T_n is bounded in probability. Therefore we have:

$$\sqrt{nhh_q}||\nabla\widetilde{S_n}(\theta_0) - \nabla S_n(\theta_0)|| = O_p(\frac{\sqrt{hh_q}}{h_q^2})$$

and the choice of bandwidths from assumption 17 finalizes the proof. QED

Lemma 9: Under assumptions S1 and S2

Let $\theta_n(v)$ in the line segment between $\tilde{\theta}(v)$ and $\theta_0(v)$ for any v $\in [0,1]$. Then there exists $H_0(v)$ negative definite such that:

$$plim^\star H\widetilde{S_n}(\theta_n(v), v) \equiv H_0(v) \text{ uniformly over } [0,1]$$

proof: Under assumption S1 we know (from lemma 3) that for all v $\in [0,1]$ and almost every x there exists a bounded function $\Psi_x(.,v)$ such that plim $H\widetilde{S_n}(\theta_n(v),v) \equiv E[XX'\Psi_X(\theta_0(v),v)]$ is negative definite. Let introduce $H(\theta,v) \equiv E[XX'\Psi_X(\theta,v)]$ for any θ and let $\theta_n(v)$ in the line segment between $\tilde{\theta}(v)$ and $\theta_0(v)$. Using 2 consecutive triangular inequalities yields:

$$|H\widetilde{S_n}(\theta_n(v),v) - H(\theta_0(v),v)| \leq |H\widetilde{S_n}(\theta_n(v),v) - HS_n(\theta_n(v),v)| + |HS_n(\theta_n(v),v) - E[HS_n(\theta_n(v),v)]| + |E[HS_n(\theta_n(v),v)] - H(\theta_n(v),v)| + |H(\theta_n(v),v) - H(\theta_0(v),v)|$$

By lemma 7 we obtain $plim^\star |H\widetilde{S_n}(\theta_n(v),v) - HS_n(\theta_n(v),v)| = 0$ uniformly over [0,1]. Also, invoking assumption S1 and a similar approach as in lemma 2 (or lemma 3) results in:

$$sup_{(\theta,v)\in\Theta\times[0,1]}|HS_n(\theta,v) - E[HS_n(\theta,v)]| = o_p(1)$$

and

$$\lim sup_{(\theta,v)\in\Theta\times[0,1]}|E[HS_n(\theta,v)] - H(\theta,v)| = 0$$

It therefore follows that,

$plim^\star |HS_n(\theta_n(v),v) - E[HS_n(\theta_n(v),v)]| + |E[HS_n(\theta_n(v),v)] - H(\theta_n(v),v)| = 0$ uniformly over $[0,1]$.

Finally, under S1, $sup_{l,v\in[0,1]}|\partial\Psi_x(l,v)\partial l|$ exists and is bounded by some constant constant M (a.e in x). It follows by the mean value theorem along with the Cauchy-Schwartz inequality that:

$$|H(\theta_n(v),v) - H(\theta_0(v),v)| \leq ME[|XX'|^2||X||^2]^{1/2}E[||\theta_n(v) - \theta_0(v)||^2]^{1/2}$$

Since $\theta_n(v)$ in the line segment between $\tilde{\theta}(v)$ and $\theta_0(v)$ we have plim $||\theta_n(v)-\theta_0(v)|| = 0$ under assumption S1 implying lim $E[||\theta_n(v)-\theta_0(v)||^2]^{1/2}$=0 by dominated convergence since both $\theta_n(v)$ and $\theta_0(v)$ lie in a compact set by assumption S1. It follows under assumption S2 that lim $E[||\theta_n(v)-\theta_0(v)||^2]^{1/2}$=0 uniformly over [0,1] by Dini's Theorem establishing $plim^\star |H(\theta_n(v),v) - H(\theta_0(v),v)| = 0$ uniformly over [0,1]. QED

Lemma 10: Under assumptions S1, S2 and S3

$$plim^\star n^{\frac{1-\gamma}{2}} h h_q sup_{v\in[0,1]}||\Delta_n(v)|| = 0 \text{ for all } \gamma > 0 \text{ where } \Delta_n(v) \equiv \tilde{\theta}(v) - \theta_0(v).$$

proof: We use $\widetilde{g}(v) \equiv \nabla\widetilde{S_n}(\theta_0(v), v)$ as well as $\bar{g}(v) \equiv \nabla S_n[\theta_0(v), v]$. Since $[0,1]$ is compact we can invoke assumption S1 and assumption S3 to show in a similar fashion as in lemma 1-3 that:

$$n^{\frac{1-\gamma}{2}} h h_q sup_{v\in[0,1]}||\bar{g}(v) - E\bar{g}(v)|| = o_p(1) \text{ for all } \gamma > 0.$$

Also, by assumption S1 we have Assumption 9 and 14 holding uniformly for an arbitrary $\bar{v} \in [0,1]$. Thus, by lemma 4 we obtain:

$$sup_{v\in[0,1]}||E\bar{g}(v)|| = O(h_q^m + h^r)$$

Hence, the bandwidths conditions of proposition 3 (i.e. assumption 17) shows that:

$$n^{\frac{1-\gamma}{2}} h h_q sup_{v\in[0,1]}||\bar{g}(v)|| = o_p(1) \text{ for all } \gamma > 0$$

$$(6)$$

Additionally, lemma 8 provides:

$$n^{\frac{1-\gamma}{2}} h h_q sup_{v\in[0,1]}||\bar{g}(v) - \widetilde{g}(v)|| = o_p(1)$$

$$(7)$$

Now with wpa.1 as $n \to \infty$, the mean value theorem gives:

$$-H\widetilde{S_n}(\overline{\theta(v)}, v).\Delta_n(v) \equiv \bar{g}(v) + E_n(v)$$

where plim $\overline{\theta(v)} = \theta_0(v)$ for all v in $[0,1]$ due to assumption S1. Using the triangular inequality furnishes:

$$|| - H\widetilde{S_n}(\overline{\theta(v)}, v).\Delta_n(v)|| \leq ||\bar{g}(v)|| + ||E_n(v)||$$

since $||\Delta_n(v)||. \ |\lambda_{Min}[-H\widetilde{S_n}(\overline{\theta(v)}, v)]| \leq || - H\widetilde{S_n}(\overline{\theta(v)}, v).\Delta_n(v)||$ by the spectral decomposition of $-H\widetilde{S_n}(\overline{\theta(v)}, v)$ we further obtain:

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$$Min_{v \in [0,1]} |\lambda_{Min}[-H\widetilde{S_n}(\overline{\theta(v)}, v)]|.|sup_{v \in [0,1]}||\Delta_n(v)||$$

$$\leq sup_{v \in [0,1]}||\bar{g}(v)|| + sup_{v \in [0,1]}||E_n(v)||$$

where $plim^\star Min_{v \in [0,1]}|\lambda_{Min}[-H\widetilde{S_n}(\overline{\theta(v)}, v)]|$ is some finite strictly positive constant by lemma 9. This last fact along with (6) and (7) combined yield the result. QED

Lemma 11: Under assumptions S1 through S5

(a) For n large enough there exists $B_n \in \mathbb{R}^{\rho(n)}$ and a given $p_n(.)' = (f_1(.), ..., f_{\rho(n)}(.))$ such that:

$$||b_n - B_n|| = O(||\Delta_n||_{sup}) + O(||R_n||_{sup})$$

where $\Delta_n(v) \equiv \tilde{\theta}(v) - \theta_0(v)$ and $||R_n||_{sup} = inf_{E_{\rho(n)}}||f - \phi||_{sup} \to 0$ as n $\to \infty$.

(b) $||b_n' p_n - \tilde{\phi_0}||_{sup} = O(||p_n||_{sup}||b_n - B_n||) + O(||R_n||_{sup})$

proof(a): From assumption S3 and assumption S4 there exists(see Chen 2007,Timan 1963)$B_n \in \mathbb{R}^{\rho(n)}$ and a basis of function $p_n(.)'$ such that:

$$inf_{E_{\rho(n)}}||f - \phi||_{sup} = ||B_n' p_n - \tilde{\phi_0}||_{sup}$$

where $||B_n' p_n - \tilde{\phi_0}||_{sup} = o(1)$. Let define $\tilde{\phi_{0,n}} \in \mathbb{R}^{\rho(n)}$ the vector whose i^{th} element is $\tilde{\phi_0}(i/n)$ and $\delta_n = \tilde{\phi_n} - \tilde{\phi_{0,n}}$. From (1) we have $\tilde{\phi_{0,n}} = \Lambda_n B_n + r_n$ where $||r_n||/n = O(||R_n||_{sup})$. It is also easy to show that:

$$b_n - B_n \equiv (\Lambda_n' \Lambda_n/n)^{-1}(\Lambda_n' \delta_n/n) + (\Lambda_n' \Lambda_n/n)^{-1}(\Lambda_n' r_n/n)$$

and consequently,

$$||b_n - B_n|| \leq ||(\Lambda_n' \Lambda_n/n)^{-1}(\Lambda_n' \delta_n/n)|| + ||(\Lambda_n' \Lambda_n/n)^{-1}(\Lambda_n' r_n/n)||$$

Now use assumption S5 which permits to use the spectral decomposition of $\Lambda_n' \Lambda_n/n$ yielding:

$$||(\Lambda_n' \Lambda_n/n)^{-1}(\Lambda_n' \delta_n/n)||^2 \leq \frac{\lambda_{max}[\Lambda_n' \Lambda_n/n]}{\lambda_{min}[\Lambda_n' \Lambda_n/n]}||\delta_n||^2/n \text{ for n large}$$

and likewise

$$||(\Lambda_n'\Lambda_n/n)^{-1}(\Lambda_n'r_n/n)||^2 \leq \frac{\lambda_{max}[\Lambda_n'\Lambda_n/n]}{\lambda_{min}[\Lambda_n'\Lambda_n/n]}||r_n||^2/n \text{ for n large}$$

But $||\delta_n||^2/n = O(||\Delta_n||_{sup}^2)$, $||r_n||^2/n = O(||R_n||_{sup}^2)$ and $\frac{\lambda_{max}[\Lambda_n'\Lambda_n/n]}{\lambda_{min}[\Lambda_n'\Lambda_n/n]}$ is bounded by assumption S5 for n large. QED

proof(b): use the decomposition $\tilde{\phi}_0 = B_n'p_n + e_n$ where $e_n(.)$ meets $||e_n||_{sup} = inf_{E_{\rho(n)}}||f - \phi||_{sup}$. Then from $|b_n'p_n(v) - \tilde{\phi}_0(v)| \leq |(b_n - B_n)'p_n(v)| + |e_n(v)|$ use the Cauchy-Schwartz inequality $|(b_n - B_n)'p_n| \leq ||b_n - B_n||.||p_n(v)||$ and take the supremum over $[0, 1]$ on both sides. QED

Lemma 12: Under assumptions S1 through S5

$$sup_{i=1...n}\tau(V_i)|b_n'p_n(\hat{V}_i) - \tilde{\phi}_0(V_i)| = O_p(1)O(||\hat{\Pi} - \Pi||.L_n) + O(||p_n||_{sup}||R_n||_{sup})$$

proof: For all $i = 1...n$ we have:

$$\tau(V_i)|b_n'p_n(\hat{V}_i) - \tilde{\phi}_0(V_i)| \leq \tau(V_i)|b_n'p_n(\hat{V}_i) - b_n'p_n(V_i)| + \tau(V_i)|b_n'p_n(V_i) - \tilde{\phi}_0(V_i)|$$

where

$$\tau(V_i)|b_n'p_n(V_i) - \tilde{\phi}_0(V_i)| \leq ||b_n'p_n - \tilde{\phi}_0||_{sup} \text{ (1)}$$

and

$$\tau(V_i)|b_n'p_n(\hat{V}_i) - b_n'p_n(V_i)| \leq ||b_n||.||p_n(\hat{V}_i) - p_n(V_i)|| \text{ (2)}$$

Notice that $||b_n||$ is bounded in probability by lemma 11. Also the mean value theorem for each function f_j comprising p_n relying on $\hat{V}_i - V_i = W_i'(\hat{\Pi} - \Pi)$ and $||W_i|| < C$ a.s. for some constant C by assumption S1 establishes $||p_n(\hat{V}_i) - p_n(V_i)|| \leq CL_n||\hat{\Pi} - \Pi||$. Using this last finding into (2) and the results of lemma 11 into (1) shows the claim. QED

Lemma 13: Under assumptions S6,S7 and S9

(a) $E[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)] = O(h_*^s)$ for some natural number $s \geq 2$

(b) Var $[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)] = O(\frac{1}{nh_*^2})$

proof(a):Iterating first with respect to $\tilde{X}, L, V$ and then with respect to $\tilde{X}, V$ yields:

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$$E[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)] = E[\tau(V)\tilde{X}\mu_n(\tilde{X}, V)]$$

where $\mu_n(\tilde{X}, V) = E_{\tilde{X}, V}\{(1 - 2F_{\tilde{X}, L, V}[-\beta_1 L + \phi(V)])K_n(L)\}$ a.s. and using S6 along with a change of variable yields:

$$\mu_n(\tilde{x}, v) = \int p_{\tilde{x}, v}(th_*)K(t)dt \text{ a.e. } \tilde{x}, v$$

where $p_{\tilde{x}, v}(l) = 1 - 2F_{\tilde{x}l, v}[-\beta_1 l + \phi(v)]f_{\tilde{x}, v}(l)$. Now considers the expression for $\mu_n(x, v)$. Under assumption S7 there exists $\sigma > 0$ such that on $E_n = \{|t| < \sigma/h_*\}$:

$$p_{\tilde{x}, v}(th_*) = p_{\tilde{x}, v}(0) + \sum_{j=1}^{s-1} \frac{p_{\tilde{x}, v}^{(j)}(0)}{j!}(th_*)^j + \frac{p_{\tilde{x}, v}^{(s)}(\xi(\tilde{x}, v))}{s!}(th_*)^s$$

for some $s \geq 2$ and $|\xi(x, v)| < \sigma$ a.e. $\tilde{x}, v$. Since $p_{\tilde{x}, v}(0) = 0$ a.e $\tilde{x}, v$ by assumption S1 we can further obtain:

$$\mu_n(\tilde{x}, v) = \mu_{n,1}(\tilde{x}, v) + \mu_{n,2}(\tilde{x}, v) \text{ a.e.} \tilde{x}, v.$$

where

$$\mu_{n,1}(\tilde{x}, v) = \sum_{j=1}^{s-1} \frac{p_{x,v}^{(j)}(0)}{j!}h_*^j \int_{E_n} t^j K(t)dt + \frac{h_*^s}{s!}\int_{E_n} p_{x,v}^{(s)}(\xi(x, v))t^s K(t)dt \text{ a.e. } \tilde{x}, v.$$

and

$$\mu_{n,2}(\tilde{x}, v) = \int_{E_n'} K(t)p_{\tilde{x}, v}(th_*)dt \text{ a.e.} \tilde{x}, v.$$

notice that for $j = 1, 2, ..., s - 1$:

$$\int_{E_n} t^j K(t)dt = -\int_{E_n'} t^j K(t)dt \text{ by assumption 9(a)}$$

and that there exists a finite M for which:

$$|p_{x,v}^{(j)}(0)| < M \text{ for } j = 1, 2, ..., s - 1, |p_{\tilde{x}, v}(.)| < M \text{ and } |p_{x,v}^{(s)}(\xi(x, v))| < M \text{ a.e. } \tilde{x}, v \text{ by}$$

$$\text{assumption S7.}$$

Thus we obtain the following bounding:

$$|\mu_{n,1}(\tilde{X},V)|+|\mu_{n,2}((\tilde{X},V)| \leq M(\sum_{j=1}^{s-1}\frac{h_*^j}{j!}\int_{E_n'}|t^jK(t)|dt+\frac{h_*^s}{s!}\int|t^sK(t)|dt+\int_{E_n'}|K(t)|dt)$$

a.s.

and subsequently:

$$h_*^{-s}|E[\nabla S_*(\tilde{\beta}_0,\tilde{\phi}_0)]|\leq h_*^{-s}E[|\tilde{X}\mu_n(\tilde{X},V)|]\leq M\beth_nE|\tilde{X}|,$$

where $\beth_n = \sum_{j=1}^{s-1}\frac{h_*^{j-s}}{j!}\int_{E_n'}|t^jK(t)|dt + \frac{1}{s!}\int|t^sK(t)|dt + h_*^{-s}\int_{E_n'}|K(t)|dt$. But $\beth_n$ is a bounded sequence by assumption S1,S9(a)and S9(e) while $E|\tilde{X}|$ exists by assumption S1. QED

proof(b): This is immediate under the iid sampling assumption since $\frac{1}{n}E[|\tilde{X}|^2|K_n(L)|^2] \leq \frac{||K||_{sup}^2}{nh_*^2}E|\tilde{X}|^2$ where $E|\tilde{X}|^2$ exists by assumption S1 and $||K||_{sup}$ exists by S9(b). QED

Lemma 14: Under assumptions S6,S7 and S9

(a) $\lim E[HS_*(\tilde{\beta}_0,\tilde{\phi}_0)] = Q$ as n $\rightarrow \infty$

(b) Var $[HS_*(\tilde{\beta}_0,\tilde{\phi}_0)] = O(\frac{1}{nh_*^4})$

proof(a): By the same approach as in lemma 12 we get:

$$E[HS_*(\tilde{\beta}_0,\tilde{\phi}_0)] = E[\tau(V)\tilde{X}\tilde{X}'A_n(\tilde{X},V)]$$

where

$$A_n(\tilde{X},V) = E_{\tilde{X},V}\{(1 - 2F_{\tilde{X},L,V}[-\beta_1 L + \phi(V)])K_n^{(1)}(L)\} \text{ a.s.},$$

and invoking assumptions S7,S9(e) and a similar argument as in lemma 3 one can easily derive:

$$\lim A_n(\tilde{X},V) = -p_{\tilde{X},V}^{(1)}(0) \text{ a.s.},$$

where $p_{\tilde{X},V}(.)$ is as defined in lemma 12. The claim follows by Dominated convergence since $E|\tilde{X}\tilde{X}'|$ exists by assumption S1. QED

proof(b): This is immediate using the same bounding principle as in proof(b) of lemma 12 invoking instead the existence of both $||K^{(1)}||_{sup}$ (by assumption S9(b)) and $E[\tilde{X}\tilde{X}'\tilde{X}\tilde{X}']$ (by assumption S1). QED

Proposition 1

proof: under assumption 4(b), $g_{X,\ell}(\overline{v})$ is well defined and $S(\theta)$ exists uniformly over $\mathbb{R}^K$. For any $\theta \in \mathbb{R}^K$ such that $||\Delta|| > 0$ where $\Delta = \theta - \theta_0$ we have:

$$S(\theta_0) - S(\theta) = E[1[|X'\Delta| > 0](d(\ell) - d(\ell + X'\Delta))g_{X,\ell}(\overline{v})]$$

Using iterated expectation yields:

$$S(\theta_0) - S(\theta) = E[1[|X'\Delta| > 0]E_X\{(d(\ell) - d(\ell + X'\Delta))g_{X,\ell}(\overline{v})\}].$$

Using $Med(\varepsilon|X,\ell,\bar{v}) = \phi(\bar{v})$ a.s. by assumption 3 subsequently offers :

$(d(\ell) - d(\ell + x'\Delta))g_{x,\ell}(\overline{v}) = |(1 - 2F_{x,\ell,\overline{v}}[-\beta_1\ell + \phi(\overline{v})]|f_{x,\ell}(\overline{v}) > 0$ a.e.in x whenever $|d(\ell) - d(\ell + x'\Delta)| > 0$,

because of assumption 2, assumption 3(b) and assumption 4(b). Lastly, $f_x(\ell) > 0$ a.e.in x (by assumption 4(a)) implies by Manski's 1985 lemma 2 that:

$$P[|d(\ell) - d(\ell + x'\Delta)| > 0] > 0 \text{ provided } |x'\Delta| > 0.$$

Thus, the random variable $E_X\{(d(\ell) - d(\ell + X'\Delta))g_{X,\ell}(\overline{v})\} > 0$ a.s. on the event $|X'\Delta| > 0$ which has a strictly positive probability by assumption 5 and $S(\theta_0) - S(\theta) > 0$ follows. QED

Proposition 2

proof: By a triangular inequality $||\widetilde{S_n} - S||_{sup\Theta} \leq ||\widetilde{S_n} - S_n||_{sup\Theta} + ||S_n - S||_{sup\Theta}$ where $||\widetilde{S_n} - S_n||_{sup\Theta} = o_p(1)$ by lemma 6 and $||S_n - S||_{sup\Theta} = o_p(1)$ by lemma 1. Hence, $plim||\widetilde{S_n} - S||_{sup\Theta} = 0$ with in addition $S(.)$ continuous everywhere under the assumptions of proposition 2 (see Manski's 1985 lemma 5) and admitting a unique global maximizer at θ_0 by proposition 1. Invoking assumption 8 concludes the proof of Proposition 2 by Theorem 4.1.1 of Amemiya (1985). QED

Proposition 3

proof: By assumption 8 and proposition 2, the estimator $\widetilde{\theta_n}$ is an interior point of Θ with probability approaching 1 as $n \to \infty$. Since $\widetilde{S_n}$ is twice differentiability everywhere (by assumption 16(e))and attains a maximum over Θ at $\widetilde{\theta_n}$ one can use a mean value expansion yielding:

$$0 = \nabla\widetilde{S_n}(\theta_0) + H\widetilde{S_n}(\bar{\theta})(\widetilde{\theta_n} - \theta_0) \text{ wpa.1},$$

for some $\bar{\theta}$ in the line segment joining $\widetilde{\theta_n}$ and θ_0 which may vary from row to row. Also, combining lemma 3 and lemma 7 furnishes:

$$H\widetilde{S_n}(\bar{\theta}) = H(\bar{\theta}) + o_p(1),$$

where $H(\theta) = -E[XX'T_X^{(1)}(-X'(\theta - \theta_0))]$ is continuous at θ_0 by assumption 14 and 16(a). Hence, proposition 2 implies $plim - H\widetilde{S_n}(\bar{\theta}) = H$. Moreover, $-H\widetilde{S_n}(\bar{\theta})^{-1}$ exists wpa.1 by assumption 16(b) and $\sqrt{nhh_q}\nabla\widetilde{S_n}(\theta_0) = O_p(1)$ by lemma 8 and lemma 5 yielding:

$$\sqrt{nhh_q}(\widetilde{\theta_n} - \theta_0) = H^{-1}\sqrt{nhh_q}\nabla\widetilde{S_n}(\theta_0) + o_p(1),$$

where lemma 8 further yields:

$$\sqrt{nhh_q}(\widetilde{\theta_n} - \theta_0) = H^{-1}\sqrt{nhh_q}\nabla S_n(\theta_0) + o_p(1),$$

and proposition 2 follows from lemma 5. QED

Proposition 4

proof: The first part of the proposition is straightforward by simply combining lemma 3 and lemma 7. For the second part,we will introduce some notation. Given a bandwidth pair $\sigma* = (h* = h^{\gamma_1}, h_{q*} = h_q^{\gamma_2})$ we note:

$$\Sigma_n(\sigma*) = \frac{1}{nh_*h_{q*}} \sum_{i=1}^n X_iX_i'|K(\frac{X_{1,i}+X_i'\theta_0}{h_*})|^2|k(\frac{V_i-\overline{v}}{h_{q*}})|^2$$

Hence we have:

$$\widetilde{\Sigma_n} = \widetilde{\Sigma_n} - \Sigma_n(\sigma*) + \Sigma_n(\sigma*)$$

Using the same approach as in lemma 5, it is rapid to show $\lim E[\Sigma_n(\sigma*)] = \Sigma$ as long as both $h*$ and h_{q*} converge to 0 as n approaches infinity. Furthermore, using a similar bounding method as in lemma 7 one has $\text{Var}\,[\Sigma_n(\sigma*)] \le \frac{M_1 M_2}{nh_* h_{q*}} \int |K|^4 \int |k|^4$ where M_1 and M_2 are finite constants. Hence, if both $\int |K|^4$ and $\int |k|^4$ exist, one needs the additional condition that $\lim nh_* h_{q*} = \infty$ to ensure $\text{plim}\,\Sigma_n(\sigma*) = \Sigma$. Under the assumption of proposition 4 this condition holds for h and h_q by assumption 13 and $\lim nh^4 h_q^4 = \infty$ by assumption 17 so a fortiori for h^{γ_1} and $h_q^{\gamma_2}$. Secondly, we have:

$$\widetilde{\Sigma_n} - \Sigma_n(\sigma*) = \frac{1}{nh_* h_{q*}} \sum_{i=1}^n X_i X_i'[|\hat{K}_i|^2|\hat{k}_i|^2 - |K_i|^2|k_i|^2]$$

where

$$\hat{K}_i = K(\tfrac{X_{1,i}+X_i'\widetilde{\theta}_0}{h_*})$$

and

$$\hat{k}_i = k(\tfrac{\hat{V}_i - \overline{v}}{h_{q*}})$$

while K_i, k_i are their counterparts when both θ_0 and Π are used instead. Doing some simplifications with a triangular inequality and using the fact that $k(.)$ and $K(.)$ are bounded functions yields:

$$|\widetilde{\Sigma_n} - \Sigma_n(\sigma*)| \le R_{1,n} + R_{2,n},$$

where

$$R_{1,n} = 2(||k||_{sup})(||K||_{sup})^2 \frac{1}{nh_* h_{q*}} \sum_{i=1}^n |X_i X_i'||\hat{k}_i - k_i|,$$

and

$$R_{2,n} = 2(||K||_{sup})(||k||_{sup})^2 \frac{1}{nh_* h_{q*}} \sum_{i=1}^n |X_i X_i'||\hat{K}_i - K_i|.$$

Finally, by assumption 17 the mean value theorem gives:

$$|\hat{k}_i - k_i| \le \frac{1}{h_{q*}}||k^{(1)}||_{sup}|\hat{V}_i - V_i|,$$

and

$$|\hat{K}_i - K_i| \le \frac{1}{h_*}||K^{(1)}||_{sup}|X_i'\hat{\Delta}|,$$

where $\hat{\Delta} = O_p(\frac{1}{\sqrt{nhh_q}})$ by proposition 3. Hence, there exists two finite constants ζ_1 and ζ_2 such that:

$$R_{1,n} \leq \zeta_1 \frac{1}{nh_* h_{q*}^2} ||\hat{\Pi} - \Pi|| \sum_{i=1}^n |X_i X_i'|||W_i||$$

and

$$R_{2,n} \leq \zeta_2 \frac{1}{nh_*^2 h_{q*}} ||\hat{\Delta}|| \sum_{i=1}^n |X_i X_i'|||X_i||$$

But under the assumption of proposition 3 we have $\frac{1}{n} \sum_{i=1}^n |X_i X_i'|||W_i|| = O_p(1)$ and $||\hat{\Pi} - \Pi|| = O_p(n^{-1/2})$ leading to:

$$\frac{1}{nh_* h_{q*}^2} ||\hat{\Pi} - \Pi|| \sum_{i=1}^n |X_i X_i'|||W_i|| = O_p(\frac{1}{h_* h_{q*}^2 n^{1/2}}) = o_p(1)$$

because $\lim nh^4 h_q^4 = \infty$ by assumption 17, a fortiori $\lim nh^{2\gamma_1} h_q^{4\gamma_2} = \infty$ when $\gamma_1 \in (0, 3/4]$ and $\gamma_2 \in (0, 1]$. Additionally, $\frac{1}{n} \sum_{i=1}^n |X_i X_i'|||X_i|| = O_p(1)$ and $||\hat{\Delta}|| = O_p(\frac{1}{\sqrt{nhh_q}})$ yielding:

$$\frac{1}{nh_*^2 h_{q*}} ||\hat{\Delta}|| \sum_{i=1}^n |X_i X_i'|||X_i|| = O_p(\frac{1}{h_*^2 h_{q*}}) O_p(\frac{1}{\sqrt{nhh_q}}) = o_p(1)$$

because assumption 17 implies $\lim nh^{4\gamma_1+1} h_q^{2\gamma_2+1} = \infty$ whenever $\gamma_1 \in (0, 3/4]$ and $\gamma_2 \in (0, 1]$. We conclude that $\widetilde{\Sigma_n} - \Sigma_n(\sigma*) = o_p(1)$. QED

Proposition 5

proof: For any function $f(.)$ defined on $[0, 1]$ and parameter β let us introduce the followings:

$$\nabla S_*(\beta, f) \equiv \frac{1}{nh_*} \sum_{i=1}^n \tau(V_i) \alpha_i \tilde{X}_i K(\frac{X_{1,i} + \tilde{X}_i'\beta + f(V_i)}{h_*}),$$

and

$$HS_*(\beta, f) \equiv \frac{1}{nh_*^2} \sum_{i=1}^n \tau(V_i) \alpha_i \tilde{X}_i \tilde{X}_i' K^{(1)}(\frac{X_{1,i} + \tilde{X}_i'\beta + f(V_i)}{h_*}).$$

It is not too difficult using assumption S9(b)to establish (componentwise):

$$|HS_*(\tilde{\beta}_0, \tilde{\phi}_0) - H_n[\tilde{\beta}(v)]| \leq R_{1,n} + R_{2,n} + R_{3,n}$$

where,

$$R_{1,n} \equiv ||K^{(2)}||_{sup} h_*^{-3} ||\tilde{\beta}(v) - \tilde{\beta}_0|| \frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i \tilde{X}_i'| . ||\tilde{X}_i||,$$

$$R_{2,n} \equiv ||K^{(2)}||_{sup} h_*^{-3} \sup_{i=1..n} \tau(V_i) |b_n' p_n(\hat{V}_i) - \tilde{\phi}_0(V_i)| \frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i \tilde{X}_i'|,$$

and

$$R_{3,n} \equiv ||K^{(1)}||_{sup} \frac{1}{nh_*^2} \sum_{i=1}^{n} |\tau(\hat{V}_i) - \tau(V_i)| |\tilde{X}_i \tilde{X}_i'|.$$

First, $\frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i \tilde{X}_i'| . ||\tilde{X}_i|| = O_p(1)$ and $\sum_{i=1}^{n} |\tilde{X}_i \tilde{X}_i'| = O_p(1)$ by assumption S1. Also $h_*^{-3} ||\tilde{\beta}(v) - \tilde{\beta}_0|| = O_p(\frac{1}{\sqrt{nhh_q h_*^3}})$ by assumption S1. Consequently $R_{1,n} = o_p(1)$ by assumption S11(a).

Secondly, $\frac{1}{n} \sum_{i=1}^{n} |\tilde{X}_i \tilde{X}_i'| = O_p(1)$ by assumption S1 and $\sup_{i=1..n} \tau(V_i) |b_n' p_n(\hat{V}_i) - \tilde{\phi}_0(V_i)| = O_p(1) O(||\hat{\Pi} - \Pi||_{L_n}) + O(||p_n||_{sup} ||R_n||_{sup})$ by lemma 12. It follows that $R_{2,n} = o_p(1)$ by assumption S11(b), S11(c) and S11(d).

Lastly, writing $C = ||K^{(1)}||_{sup}$ yields:

$$R_{3,n} \equiv M_{1,n} + M_{2,n},$$

where,

$$M_{1,n} = C \frac{1}{nh_*^2} \sum_{i=1}^{n} |\tau(\hat{V}_i) - \tau(V_i)| |\tilde{X}_i \tilde{X}_i'| 1\{|\tilde{X}_i \tilde{X}_i'| < a_n\},$$

and

$$M_{2,n} \equiv C \frac{1}{nh_*^2} \sum_{i=1}^{n} |\tau(\hat{V}_i) - \tau(V_i)| |\tilde{X}_i \tilde{X}_i'| 1\{|\tilde{X}_i \tilde{X}_i'| \geq a_n\},$$

for any positive deterministic sequence a_n. It is rapid to establish $M_{1,n} = O_p(\frac{a_n}{\sqrt{n}h_*^2})$ by Newey et al.(1999) lemma A3. Also, a Cauchy Schwartz's inequality followed by a Tchebychev's inequality as in step 3 of lemma 3 gives $EM_{2,n} = O(\frac{1}{a_n h_*^2})$. So pick $a_n \propto \frac{log(n)^{1/2}}{h_*^2}$ and $R_{3,n} = o_p(1)$ follows by assumption S10.

Hence, $HS_*(\tilde{\beta}_0, \tilde{\phi}_0) \equiv H_n[\tilde{\beta}(v)] + o_p(1)$ is established and a fortiori $\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0) \equiv G_n[\tilde{\beta}(v)] + o_p(1)$. Lastly, invoking lemma 13 and 14 along with assumption 10 yields:

$$\text{plim } \nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0) = 0 \text{ and plim } HS_*(\beta_0, \phi_0) = Q$$

The conclusion of proposition 5 arises since plim $\tilde{\beta}(v) = \tilde{\beta}_0$ by assumption S1 and Q^{-1} exists by assumption S8. QED

Proposition 6

proof: Since $\sqrt{nh_*}(\tilde{\beta}(v) - \tilde{\beta}_0) \equiv o_p(1)$ by assumption S1 and assumption S12, we obtain:

$$\sqrt{nh_*}(\bar{\beta} - \tilde{\beta}_0) \equiv -H_n[\tilde{\beta}(v)]^{-1}\sqrt{nh_*}G_n[\tilde{\beta}(v)] + o_p(1)$$

(8)

Also, by assumption S13 we get:

$$\sqrt{nh_*}G_n[\tilde{\beta}(v)] - \sqrt{nh_*}\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0) = o_p(1)$$

(9)

Furthermore, one can use the analogue of lemma 5 invoking this time assumption S6,S7,S9 S10 to allow the usage of the Lyapunov's Central Limit Theorem yielding:

$$\sqrt{nh_*}\{\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0) - E[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)]\} \to_d N(0, \Xi)$$

(10)

Since $\sqrt{nh_*}E[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)] = O(\sqrt{nh_*}h_*^s)$ by lemma 13 we conclude using assumption S14 that:

$$\sqrt{nh_*}E[\nabla S_*(\tilde{\beta}_0, \tilde{\phi}_0)] = o(1)$$

(11)

Now use plim $H_n[\tilde{\beta}(v)]^{-1} = Q^{-1}$ under the assumptions of proposition 5 and the claim directly follows combining (8),(9),(10)and(11). QED

Section C

Assume that the assumptions of proposition 3 hold. Write $\ell_i = X_{1,i} + X_i'\theta_0(\bar{v})$ where $\theta_0(v)' = \frac{1}{\beta_1}(\phi(v), \tilde{\beta}')$ and $\hat{\ell}_i = X_{1,i} + X_i'\tilde{\theta}_n$. Here $\bar{v}$ is the value chosen to compute the KWSMS estimator. Suppose that there exists a partition of $\tilde{W} = (W_1, W_2')$ where W_1 is a scalar variable and W_2 is non empty. Let $\mu \bigotimes \mu$ indicates the product measure on $\mathbb{R}^2$ where μ is the Lebesgue measure. Define the following statistic:

$$T_n = \frac{(n\xi^2)^{-1}\sum \varphi(\frac{\hat{\ell}_i}{\xi})\varphi(\frac{\hat{V}_i - \bar{v}}{\xi})\alpha(Y_i)}{(n\xi^2)^{-1}\sum \varphi(\frac{\hat{\ell}_i}{\xi})\varphi(\frac{\hat{V}_i - \bar{v}}{\xi})},$$

where φ is a kernel and ξ a deterministic sequence. Also, define $M(l, v) = E[\alpha(Y)|\ell = l, V = v]$ and $f(.,.)$ the joint density of (ℓ, V) with respect to $\mu \bigotimes \mu$ whenever the density exists. Suppose that the following assumptions hold:

C1. $\partial M(l,v)\backslash\partial l$ and $\partial M(l,v)\backslash\partial v$ exist and are continuous in some open neighborhood of $(0,\bar{v})$. Also, $\partial^2 M(l,v)\backslash\partial^2 l$, $\partial^2 M(l,v)\backslash\partial^2 v$ and $\partial^2 M(l,v)\backslash\partial l\partial v$ exist in some open neighborhood of $(0,\bar{v})$.

C2. φ is a strictly positive kernel belonging to $\mathcal{K}_2$ and meets the same conditions as K in assumption 16.

C3. ξ_n is a strictly positive sequence of real numbers satisfying $\xi \propto n^{-\omega}$ for some $\omega \in (sup\{1/10; a(1+\eta)\}, 1/5)$ where a and η are the bandwidths parameters selected to compute the KWSMS estimator as defined on page 28.

C4. The cdf of (ℓ, V) is absolutely continuous with respect to $\mu \bigotimes \mu$, its density at $(l,v){=}(0,\bar{v})$ exists and is strictly positive. Also, there exists some open neighborhood of $(0,\bar{v})$ where $\partial f(l,v)\backslash\partial l$, $\partial f(l,v)\backslash\partial v$, $\partial^2 f(l,v)\backslash\partial^2 l$, $\partial^2 f(l,v)\backslash\partial^2 v$ and $\partial^2 f(l,v)\backslash\partial l\partial v$ exist and are continuous with $|\partial^2 f(l,v)\backslash\partial^2 l| < M$, $|\partial^2 f(l,v)\backslash\partial^2 v| < M$ and $|\partial^2 f(l,v)\backslash\partial l\partial v| < M$ for some $M < \infty$.

C4. The (cdf of) $X_1|\tilde{x}, v, w$ is absolutely continuous with respect to the Lebesgue measure a.e in $\tilde{x}, v, w$ and $W_1|\dot{x}, w_2$ is absolutely continuous with respect to the Lebesgue measure a.e in $\dot{x}, w_2$.

C5. (Define $F[.|x, l, v, w]$ the cdf of $\varepsilon|x, l, v, w$. Also, write $f(.|x, v, w)$ the density of $\ell|x, v, w$ and $f(.|x, l, w_2)$ the density of $V|x, l, w_2$ whenever those densities exist.)

(i) As functions of l:

(ia)$f(l|x, v, w)$ and $F[-l|x, l, v, w]$ belong to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,v,w.

(ib)$f(l|x, w)$ and $F[-l|x, l, -\lambda'w+\bar{v}, w]$ belong to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,w for all λ parameter having the dimension of W. Furthermore, $f(-\lambda'w + \bar{v}|x, l, w_2)$ belongs to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,w_2 for all λ parameter having the dimension of W.

(ii) As functions of v:

(iia) $F[-l|x, l, v, w]$ belongs to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,l,w. Also, $f(v|x, l, w_2)$ belongs to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,l,w_2

(iib) $F[\lambda'x|x, -\lambda'x, v, w]$ and $f(-\lambda'x|x, v, w)$ belong to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,w for all λ parameter having the dimension of X. Also, $f(v|x, w_2)$ belongs to $C^2_\infty(M)$ for some $M < \infty$ a.e in x,w_2 for all λ parameter having the dimension of X.

then under H_o: $Med(\varepsilon|\dot{X}, \bar{v}) = Med(\varepsilon|\bar{v})$ a.s.,

$$\sqrt{n\xi^2}T_n \to_d \mathcal{N}(0, f(0,\bar{v})^{-1}(\int |\varphi|^2)^2),$$

and,

$$(n\xi^2)^{-1}\sum \varphi(\tfrac{\hat{\ell}_i}{\xi})\varphi(\tfrac{\hat{V}_i-\bar{v}}{\xi}) \to_p f(0,\bar{v}).$$

proof: The structure of this proof is analogous to that provided in Horowtiz (1993), proposition 2. The only difference deals with the number of variables conditioning Y and the presence of an additional nuisance term Π from the reduced form. The test is based upon the fact that under H_o: $Med(\varepsilon|\dot{X}, \bar{v}) = Med(\varepsilon|\bar{v})$ $a.s$ one must have $M(0, \bar{v}) = 0$. The proof for the consistent estimator of $f(0, \bar{v})$ is omitted since it stems directly from what is to follow.

For any $\Delta' = (\Delta_1', \Delta_2')$ where Δ_1 is $K \times 1$ and Δ_2 is $d \times 1$ introduce the following:

$$\bar{M}(\triangle) = \frac{(n\xi^2)^{-1}\sum \varphi(\frac{\ell_i+X_i'\Delta_1}{\xi})\varphi(\frac{v_i+W_i'\Delta_2}{\xi})\alpha(Y_i)}{(n\xi^2)^{-1}\sum \varphi(\frac{\ell_i+X_i'\Delta_1}{\xi})\varphi(\frac{v_i+W_i'\Delta_2}{\xi})},$$

where $v_i = V_i - \bar{v}$. The key is to notice that $T_n = \bar{M}(\hat{\triangle})$ where $\hat{\Delta}' = ((\tilde{\theta} - \theta_0(\bar{v}))', (\Pi - \hat{\Pi})')$. Applying Theorem 3.5-3.6 of Pagan and Ullah (1999) using assumptions C1 through C4 yields:

$$\sqrt{n\xi^2}\bar{M}(0) - M(0,\bar{v}) \to_d \mathcal{N}(0, f(0,\bar{v})^{-1}(\int |\varphi|^2)^2).$$

Also using a Taylor's expansion furnishes:

$$\bar{M}(\hat{\triangle}) = \bar{M}(0) + \tfrac{\partial \bar{M}}{\partial \Delta}|_{\ddot{\Delta}}'\hat{\Delta}$$

where plim $\ddot{\triangle} = 0$ by the assumptions of proposition 3. Writing $\bar{a}$ the numerator of $\bar{M}$ and $\bar{b}$ its denominator gives:

$$\tfrac{\partial \bar{M}}{\partial \Delta} = \bar{b}^{-2}(\tfrac{\partial \bar{a}}{\partial \Delta}\bar{b} - \bar{a}\tfrac{\partial \bar{b}}{\partial \Delta}).$$

Under C2 and C5 one can apply lemma 2 as in lemma 3 to derive plim $\tfrac{\partial \bar{a}}{\partial \Delta} = \lim \mathrm{E}\tfrac{\partial \bar{a}}{\partial \Delta} < \infty$ and plim $\tfrac{\partial \bar{b}}{\partial \Delta} = \lim \mathrm{E}\tfrac{\partial \bar{b}}{\partial \Delta} < \infty$ uniformly over a compact set of $\mathbb{R}^{K+q}$ which contains 0. Likewise, by the same token as in lemma 4 using a classic convolution argument invoking C1-C4 returns plim $\bar{a} = \lim \mathrm{E}\bar{a} < \infty$ and plim $\bar{b} = \lim \mathrm{E}\bar{b} < \infty$ uniformly over a compact set of $\mathbb{R}^{K+q}$ which contains 0. This establishes $\tfrac{\partial \bar{M}}{\partial \Delta}|_{\ddot{\Delta}} = O_p(1)$ and $\sqrt{n\xi^2}\bar{M}(\hat{\triangle}) - \bar{M}(0) = o_p(1)$ follows because $\sqrt{n\xi^2}\hat{\triangle} = o_p(1)$ by proposition 3 and C3 (i.e. $\xi^2/hh_q = o(1)$).
QED