INEFFICIENCIES IN NETWORKED MARKETS

MATTHEW ELLIOTT*

Abstract. In many markets relationship specific investments are necessary to enable trade. Typically there are multiple buyers, multiple sellers and heterogeneous gains from trade. In some markets a buyer and seller must make different and separate investments to trade, in others investments are jointly made and negotiated. These investments are subject to inefficiencies: under-investment, due to potential hold-up, and over-investment to generate “outside options”. When investments are separate over-investment is limited. In contrast, inefficiency from under-investment cannot be bounded. This result reverses when investments are negotiated. In this case there is no under-investment. However, inefficiency from over-investment cannot now be bounded.

Key Words: Trade networks, network formation, bargaining, outside option, inefficiency, relationship specific investment, hold up.

1. Motivation

In many markets a buyer and seller must make a relationship specific investment before they can trade: Countries invest in pipelines to trade natural gas; suppliers are taught the specific requirements of manufacturers; railroads and shippers invest in sidetracks; acquirers pay due diligence costs whilst takeover targets reveal private information; there are interviews prior to recruitment; art dealers authenticate paintings; illegal goods handlers risk transacting with under-cover law enforcers; etc. This paper analyzes inefficiencies in these investment decisions when long term contracts cannot be enforced.

*Stanford University, Department of Economics. I am grateful to Matt Jackson for extensive discussions, advice and guidance and to Rachel Kranton for very helpful comments on an earlier draft. I also thank Manuel Amador, Attila Ambrus, Doug Bernheim, Francis Bloch, Itay Fainmesser, Guillaume Fréchette, Sanjeev Goyal, Bob Hall, Fuhito Kojima, Pete Kyle, Jon Levin, Paul Milgrom, David Myatt, Muriel Niederle, Michael Ostrovsky, Al Roth, Ilya Segal, Andy Skrzypacz and Fernando Vega-Redondo as well as numerous seminar and conference participants for valuable feedback.

1 Examples include the New York garment industry, Uzzi (1996) and the Japanese engineering and electronics industry, Nishiguchi (1994).

2 Sidetracks are railway line extensions required by a shipper. See Pittman (1991).

3 This assumption is critical and how reasonable it is will depend on the market being considered. However, it will be argued that it many markets long term contracts either cannot be written or are not written in practice. See section 6.4 for a discussion of this assumption applied to several different markets.
Inefficiencies in Networked Markets

Relationship specific investments are subject to well understood inefficiencies. A party may fail to make an efficient up-front investment for fear that they will be held-up and unable to recover their costs. This problem can lead to under-investment. A party may also make costly investments into alternative trading relationships just to improve their terms of trade. This can result in over-investment. This paper considers the efficiency of relationship specific investment decisions that enable trade in a market context. Which parties will make the required investments to enable trade? Who will actually trade with whom? How large are inefficiencies? Which type of inefficiency is most severe and when?

This paper departs from much of the contract theory literature by considering investment incentives in markets with many buyer and many sellers. In Section 4 it will be shown that the network structure of the market, who can trade with who, can have substantial non-local affects on payoffs. This theoretical prediction is backed up by experimental evidence in Charness, Corominas-Bosch and Fréchette (2007). Further, in many markets the presence of additional buyers and additional sellers affects investment decisions. For example, there are currently proposals to build natural gas pipelines from the Middle East to Europe (Nabucco pipeline), from Russia to Europe (South Stream) and from the Middle East to India and Pakistan (IIP pipeline). The decisions of whether or not to build these pipelines are not being made in isolation. When industries are consolidated, acquisition decisions are also interdependent.

The literature to date modeling relationship specific investments with many buyers and many sellers has assumed that a buyer and seller must make separate, exogenously determined investments before they can trade. In some cases this is reasonable. When one firm wants to acquire another the acquirer bears the direct due diligence costs - advisory fees and such like - whilst the target bears the cost of making their private information available. Further, these investments are not directly substitutable - the acquirer cannot pay higher due diligence costs to allow the target to reveal less information. In these cases the inefficiency problem of most concern is the hold-up problem. With separate investments over-investment inefficiency is bounded but under-investment inefficiency cannot be bounded and may consume all the gains from trade.

---

5For example, Google recently purchased mobile advertising group Admob and further acquisitions are expected soon. Potential acquirers include Microsoft and AOL. Potential targets include AdMarvel, Millennial Media and Medialets. (Reuters (12/04/09))
7The literature to date has typically assumed investments are separate and non-substitutable and that the ex-ante gains from trade are homogeneous. There is then no over-investment inefficiency whenever buyers and seller must both make positive relationship specific investments to enable trade.
Whilst in some markets relationship specific investments are naturally separate and non-substitutable, in other situations joint and substitutable relationship specific investment are made. When two countries have to build a pipeline to trade natural gas with each other they will negotiate the share of the investment each makes. This is done in anticipation of bargaining over the formed network. Nonetheless, investment shares can always be picked to overcome the hold-up problem. The investment made by the party at risk of being held up can be reduced until they are willing to participate. Are investments then efficient or close to efficient? Although negotiation overcomes the hold-up problem, it exacerbates the over-investment problem. It is easier to form links that will not be used for trade as well as links that will be used for trade. Whilst inefficiency due to over-investment was bounded with separate non-substitutable investments, with negotiated joint investments it is unbounded and can consume all the gains from trade.

So far it has been argued that the negotiated investment model fits some markets better than the separate non-substitutable investment assumption previously made in the literature. However, even when investments appear separate and non-substitutable, if transfers can be made then the problem behaves as if investments are joint and substitutable. The following example is paraphrased from Brandenburger and Nalebuff (1996):

By 1989 Craig McCaw had acquired cellular phone licences covering 50 million potential customers, but he wanted a national network and had no coverage in a number of large cities. LIN Broadcasting Corporation owned licences covering 18 million potential customers in many of these cities. McCaw wanted to acquire LIN. Although there were many potential acquirers of LIN, McCaw valued LIN much more than any of them. As with any take over there were substantial costs associated with the due diligence process - advisory fees would have to be paid and managerial time foregone to permit any of these companies...
to bid for LIN. Aware that they would very likely be outbid by McCaw no other potential
acquirers entered the bidding. LIN responded by agreeing to pay the bidding costs for
Bell South, the company they expected to value them most highly after McCaw. LIN paid
about $75 Million to Bell South which included compensation for indirect expenses like
management time. Bell South are estimated to have bid up the acquisition price McCaw
paid for LIN by about $1 Billion to about $6.5 Billion.

In the case of the acquisition of LIN, the ability of LIN to make a transfer to Bell
South turned the environment into one of negotiated investments from one of separate
investments. With negotiated investments it is predicted that there will be no under-
investment and that over-investment is the main inefficiency concern. The $75 Million
investment that enabled Bell South to bid for LIN is one such over-investment. It resulted
in a transfer of rents from McCaw to LIN without generating any additional surplus.

To the best of my knowledge this is the first paper to consider negotiated relationship
specific investments to enable trade in markets. Allowing for negotiated investments
changes the source of inefficiency. This will matter when considering market interven-
tions. Methodologically the paper also makes the following contributions. The size of
inefficiencies in this literature has not previously been considered. Two concepts imported
from the computer science literature permit such an analysis: The cost of anarchy and
the price of anarchy which measure the efficiency loss on the best and worst stable net-
works respectively (Koutsoupias and Papadimitriou (2009) and Roughgarden and Tardos
(2004)). The paper also models bargaining once investment have been made. The pairs
of parties that have made the required investments to trade with each other can be rep-
resented as links in a network. A bargaining solution for any network with any gains
from trade driven by the requirement that no two parties have a jointly profitable devi-
ation is presented. This fills a gap in the bargaining over networks literature. Further,
by utilizing and extending results from the matching literature an intuitive algorithm is
constructed which decomposes any network to identify (i) the role of each link in the
network - whether it is used for trade or as an outside option to affect the terms of trade
- and (ii) the affect of each link on each party’s bargained outcome - who they trade with
and at what price. This bargaining solution is useful in other applications too.

The paper proceeds as follows. Section 2 provides a simple example illustrating the main
points of the paper. Section 3 then sets up the model before Section 4 analyzes bargaining

11See, for example, Myerson (1977), Kranton and Minehart (2000a), Navarro and Perea (2001),
12Elliott (2009) applies this bargaining solution to networked labor markets.
over a formed network. Section 5 then identifies the efficient network(s) providing a useful benchmark for the set of stable networks which are considered in Section 6. Section 6 examines the inefficiency present in stable networks and contains the main results of the paper. Section 7 then places these results in the context of the related literature before Section 8 concludes.

2. Example

This section presents a very simple example to illustrate the main points of the paper. Suppose that there are two producers of natural gas with surpluses they want to export, suppliers $s_1$ and $s_2$, and a single importer of natural gas buyer $b_1$. To enable trade between either supplier and $b_1$ a pipeline must be built between them at cost $c = \frac{1}{2} - \varepsilon$, where $\varepsilon$ is small and positive. Neither supplier directly values their surplus of gas and $b_1$ values gas from $s_1$ at 1 and from $s_2$ at $1 - \varepsilon$. Figure 1a shows these potential gains from trade.

![Networked Market](image)

**Figure 1.** Networked Market

The efficient network maximizes the net gains from trade (the realized gains from trade less resources spent forming the gas pipelines) and is shown in Figure 1b. The net gains from trade generated by this network are $1 - \left(\frac{1}{2} - \varepsilon\right) = \frac{1}{2} + \varepsilon$. Suppose that sellers have all the bargaining power and can make take-it-or-leave offers.

If the buyer had to pay a separate and non-substitutable share of the cost of building either pipeline $\gamma c$, where $\gamma \in (0, 1)$, and the sellers had to pay $(1 - \gamma)c$ the efficient network would not be stable. Buyer $b_1$ would have no alternative but to trade with $s_1$ and as $s_1$ can make a take-it-or-leave it offer they would sell gas to $b_1$ at a price of 1 leaving $b_1$ with a payoff of $-\gamma c$. Buyer $b_1$ will not invest for fear of hold-up. The complete

---

13This might reflect a political need of $b_1$ to import gas.
network is not stable either. Suppliers $s_1$ and $s_2$ will compete with each other to supply $b_1$ driving the price they receive below the point at which they are able to recover their (sunk) investment. The empty network is the unique stable network. Under-investment inefficiency, due to the hold-up problem, consumes all the gains from trade.

Suppose instead that the buyer and sellers can negotiate their shares of the pipeline costs. These negotiations are bilateral and contingent contracts cannot be written. The empty network is no longer stable: $b_1$ and $s_1$ could agree to split the cost $c$ such that $s_1$ paid the entire cost, leaving $b_1$’s payoff unaffected but strictly increasing $s_1$’s payoff. However, the efficient network is not stable either. Buyer $b_1$ will want to also build a pipeline to $s_2$ and would be willing to pay the entire cost of doing so. Once this pipeline is built $s_1$ and $s_2$ will compete to supply $b_1$ increasing $b_1$’s payoff from 0 to $1 - \varepsilon - c = \frac{1}{2} - \varepsilon > 0$.

With negotiation the unique stable network is the complete network.

When buyers and sellers had to make separate non-negotiable contributions towards the pipeline investment the unique stable network was the empty network and all gains from trade were lost to under-investment inefficiency. Allowing buyers and sellers to negotiate their cost shares overcame this hold-up problem. However, the unique stable network was then the complete network. A pipeline was constructed by $b_1$ to $s_2$ just to affect their term of trade with $s_1$. From an efficiency perspective this is a waste of resources. The investment generated no additional surplus, it only results in a transfer of rents. Over-investment inefficiency now consumes all the gains from trade as $\varepsilon \to 0$.

Eliminating the under-investment inefficiency by permitting parties to negotiate their investment contributions exacerbated over-investment inefficiency.

In Appendix A the size of under-investment inefficiency with separate investment and the size of over-investment inefficiency with negotiated investment is shown for one buyer and two seller networks for a range of parameter values.

3. Model Set Up

There is a set of $m$ buyers denoted $P$ and a set of $n$ sellers denoted $Q$. The value of trade between a buyer $i$ and a seller $j$ is given by $a_{ij} \geq 0$. Each buyer demands one unit of the good and each seller supplies one unit of the good. The $m \times n$ dimensional matrix $a$ describes the value of all potential bi-lateral trades.

In stage one buyers and sellers make relationship specific investments to enable trade, where the gains from trade $a$ are common knowledge. A link is formed enabling trade between $i$ and $j$ ($l_{ij} = 1$) if and only if their joint investment is greater than $c$, otherwise
no link is formed \((l_{ij} = 0)\). Two investment protocols are considered: (i) following the
literature buyers and sellers choose whether or not pay an exogenously fixed proportion
of the cost \(c\), \(\gamma\) and \(1 - \gamma\) respectively; or (ii), as has not been previously been considered,
buyers and sellers negotiate over how the cost \(c\) is split between them such that the link
is formed whenever they jointly benefit from it. It is assumed that these investments are
non-contractible.\(^{14}\)

In stage two buyers and sellers bargain over the network. The trades implemented are
described by a \(n \times m\) matrix \(x\), where an element \(x_{ij}\) indicates that buyer \(i\) purchases
\(x_{ij} \in [0, 1]\) units from seller \(j\). All rows and all columns of \(x\) must sum to no more than
1 as each buyer demands only a single unit and each seller supplies only a single unit.\(^{15}\)
For \(x_{ij} > 0\), the price buyer \(i\) pays seller \(j\) per unit is \(p_{ij}\). Bargaining solutions will be
considered that map the potential gains form trade \(a\) and network \(L\) into trades \(x\) and
prices \(p\). Buyer \(i\)'s payoff is denoted by \(u_i(a, L) : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^+\) and seller \(j\)'s payoff by
\(v_j(a, L) : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^+\). The utilities of buyer \(i\) and seller \(j\) are the surpluses they extract
from each of their transactions.\(^{16}\)

\[
\begin{align*}
    u_i &= \sum_{k \in Q} x_{ik}(a_{ik} - p_{ik}) \\
    v_j &= \sum_{k \in P} x_{kj}p_{kj}
\end{align*}
\]

The symmetry of buyers and sellers means that for all the results found for buyers there
are equivalent results for sellers. Some results will be stated for just buyers or just sellers.

4. BARGAINING OVER A FORMED NETWORK

To analyze network formation a backward induction approach will be applied and trade
over a given network is considered first. This section proposes a mapping from the formed
network \((L)\) and potential gains from trade \(a\) into outcomes (quantities traded by different
pairs and payoffs) and identifies how the network structure affects these outcomes.

A bargaining outcome is a triple \((x, u, v)\). Bargaining outcomes are required to sat-
ify pairwise stability such that no buyer-seller pair can form a profitable coalition and

\(^{14}\)Modeling network formation prior to bargaining precludes firms from improving their terms of trade by
threatening to invest in alternative relationships without actually having to. Whether this is reasonable
or not is an empirical question. However, in the example in the introduction LIN was not able to increase
their acquisition price by simply threatening to pay Bell South $75 Million to enter the bidding, they
actually had to pay Bell South. In the case of the Nabucco pipeline, even if it is not ultimately built,
much money has already been spent on it.

\(^{15}\)It may be possible to relax this assumption utilizing results from Sotomayor (1999).

\(^{16}\)Sellers' values from keeping their goods can be normalized to zero and reflected in their value of trading
with each buyer.
participation constraints are satisfied: An outcome \((x, u, v)\) is pairwise stable\(^{17}\) if \(\forall i, j, u_i + v_j \geq a_{ij}l_{ij}, u_i \geq 0\) and \(v_j \geq 0\). Denote the set of pairwise stable outcomes by \(\Omega\).

Generically, the quantity traded \(x_{ij}\) is an element of \(\{0, 1\}\) in all pairwise stable outcomes. This follows from the linearity in the utility functions\(^{18}\). Let the matrix \(\alpha\), with elements \(\alpha_{ij} = a_{ij}l_{ij}\), summarize the values of feasible trades after investments have been made. The bargaining over a network problem with potential gains from trade \(a\) and possible trades given by the network \(L\) is then homomorphic to a matching with transferable utility (assignment) problem with gains from trade \(\alpha\). In the bargaining problem buyers and sellers choose to match one-to-one with each other. A match \(\mu(\alpha)\) is a function from the set of all buyers and sellers into itself, \(\mu(\alpha) : P \cup Q \rightarrow P \cup Q\), such that: (i) \(\mu(i, \alpha) \in Q \cup i\) (a buyer is matched to themselves if they are not matched to a seller); (ii) \(\mu(j, \alpha) \in P \cup j\); (iii) if \(\mu(i, \alpha) = j\), then \(\mu(j, \alpha) = i\) and \(l_{ij} \in L\); and (iv) if \(\mu(j, \alpha) = i\), then \(\mu(i, \alpha) = j\) and \(l_{ij} \in L\). It is then without loss of generality that bargaining outcomes consisting of the triple \((\mu, u, v)\) are considered. Denote the set of all possible matches \(M(L)\). Where there can be no confusion notation will be abused and the arguments dropped from the above notations.

There is generically a unique match \(\mu^*(\alpha)\) that maximizes the gains from trade.

\[
\mu^*(\alpha) = \arg\max_{\mu \in M(L)} \left\{ \sum_{i \in P} \alpha_{i\mu(i)} \right\}
\]

It will be shown that there are buyer and seller payoffs for which this match is pairwise stable. Furthermore, no other match is ever pairwise stable. Requiring pairwise stability therefore precludes any inefficiency occurring at the bargaining stage. This permits analysis of inefficiencies at the investment stage without confounding effects.

Having pinned down which buyers buy from which sellers on a formed network the terms of trade can be considered. Let the worst case scenario for a buyer \(i\) - the lowest payoff they could receive whilst the outcome is still pairwise stable - be termed \(i\)'s ‘outside option’:

\[
u_i \equiv \min_{(\mu, u, v) \in \Omega} u_i
\]

Suppose that \(i\) is matched to \(j\) on the network \(\alpha\): \(\mu^*(i) = j\). Consider terms of trade in which \(i\) and \(j\) receive their outside option value for sure and then the remaining gains

\(^{17}\) In the matching literature such outcomes are sometimes simply referred to as stable outcomes.

\(^{18}\) See Roth and Sotomayor, (1990).
from trade are split according to a parameter $\beta \in [0, 1]$, which can be interpreted as buyers’ relative bargaining power:

$$u_i = u_i + \beta(\alpha_{ij} - u_i - v_j)$$  \hspace{1cm} (1)

$$v_j = v_j + (1 - \beta)(\alpha_{ij} - u_i - v_j)$$  \hspace{1cm} (2)

These terms of trade will be motivated later. These bargaining outcomes are shown for a simple example below:

Figure 2 shows a simple formed network. On this network $b_1$ will be matched to seller $s_1$, otherwise $b_1$ and $s_1$ would have a profitable deviation. The worst case scenario for $b_1$ is to receive a payoff of 5. If they received any worse terms of trade they would have a profitable deviation by instead trading with $s_2$. Thus $u_{b_1} = 5 + 5\beta$ whilst $s_1$ receives a payoff $v_{s_1} = 5(1 - \beta)$.

Results from the matching literature are now utilized to show that the match $\mu^*$ with terms of trade proposed in Equations 1 and 2 (i) is pairwise stable; (ii) always exist and (iii) is in the core such that no coalition of players has a profitable deviation. Shapley and Shubik (1972) showed for this environment that an outcome is pairwise stable if and only if it is in the core. Requiring that there are no profitable pairwise deviations to the bargaining over a network problem implies that there are no profitable coalitional deviations. Any coalitional would arrange itself into pairs and if none of these pairs are individually profitable then the coalition as a whole cannot be profitable either. Shapley and Shubik (1972) also characterize the structure of the core in these environments: First, the core is non-empty. Second, as already utilized, there is generically a unique match

---

Outside options in these payoffs act similarly to outside options in an alternating offer bargaining game where there is some probability that the trading opportunity will be lost each period and parties then receive their outside option payoffs. Outside options play a different role in an alternating offer bargaining game where the cost of delay is captured by time preference. In that case outside option only affect the terms of trade when they bind. See Binmore, Rubinstein and Wolinsky (1986).
and this match maximizes the gains from trade. If the match did not maximize the gains from trade the grand coalition of all buyers and sellers would have a profitable deviation. Third, the payoffs in the core, partially ordered by buyers preferences, have a lattice structure with a buyer optimal point and a seller optimal point. At the buyer optimal point all buyers receive the highest payoff they can at any point in the core whilst all sellers receive the lowest payoff they possible can at any point in the core. Thus at the buyer optimal point of the core all sellers simultaneously receive their outside option payoff. Fourth, if buyer $i$ and seller $j$ are matched then $u_i + v_j = \alpha_{ij}$. If $u_i + v_j < \alpha_{ij}$ then $i$ and $j$ would have a profitable deviation and if $u_i + v_j > \alpha_{ij}$ then the coalition of all other buyers and sellers would have a profitable deviation. 

Using these results the payoffs received by buyer $i$ and seller $j$ can then be rewritten:

$$u_i = \beta u_{BO}^i + (1 - \beta)u_{SO}^i$$
$$v_j = \beta v_{BO}^j + (1 - \beta)v_{SO}^j$$

where $u_{BO}^i$ is $i$’s payoff at the buyer optimal point of the core, that is $u_{BO}^i = \pi_i = \alpha_{ij} - v_j$. The core is convex and so the payoffs proposed are in the core (and pairwise stable). 

Demange, Gale and Sotomayor (1986) identify an algorithm to find the buyer optimal and seller optimal points of the core. However, not much is known about how the structure of the network $L$ affects payoffs. It will now be shown precisely how the network structure affects each party’s terms of trade.

It is helpful to introduce some notation associated with the removal of a party from a network. Once a party is removed they cannot be matched to their trade partner and these gains from trade are lost. There may then be some optimal rematching to maximize the gains from trade on the reduced network. Denote the links that are no longer matched over following the removal of $k$ by $L_k^-$ and links that are newly matched over by $L_k^+$. 

The network decomposition algorithm identifies a simple way of decomposing any bilateral trade network into a directed network from which the affect of every link on each party’s payoff can be easily found. 

---

20The results from Shapley and Shubik (1972) cited in the paragraph above are afforded a concise and careful exposition in Chapter 8 of Roth and Sotomayor (1990).
21The convexity of the core also follows from Shapley and Shubik (1972) who showed that the core is characterized by the solution to a linear programming problem.
22As the proposed solutions are a subset of the core they may be considered a refinement of the core. For $\beta = \frac{1}{2}$ the proposed solution sometimes corresponds to the Nucleolus, but not always.
23Generically there is a unique network decomposition. When there are multiple directed network representations any one can be selected and all results will carry through.
(1) Identify trade partners. Find the match that maximizes the possible gains from trade given a network $\alpha$. These matches defines each party $k$’s trade partner $\mu^*(k)$. Represent these relationships by directed solid links.

(2) Identify outside trade partners. To find a party $k$’s outside trade partner $\nu(k) \equiv \mu(k, L/\mu^*(k, L))$, remove their trade partner $\mu^*(k)$ from the network and identify the party they rematch to for the gains from trade to maximized over the new network. Repeat for all buyers and sellers. Represent these relationships (outside option links) by directed dashed links from $k$ to $\nu(k)$.

To see how this decompositions is implemented consider the network in figure 3a. The links traded over are found first by considering the match the maximizes the gains from trade. These are relationships are represented by directed solid links in Figure 3b. For example $b_1$’s trade partner is $s_1$. To find each party’s outside trade partner their trade partner is removed from the network and the match that maximizes the gains from the trade on this reduced network is found. If $s_1$ was removed from the network $b_1$ would have to rematch to $s_2$ for the gains from trade to be maximized and so $b_1$’s outside trade partner is $s_2$ as shown by the dashed link. It is interesting to note that, in this example, the highest value link does not feature in the decomposed network and so is redundant.

Party $k$’s outside option chain is a sequence of links that can be easily identified from the network decomposition. Start at the node $k$. Then alternately follow the directed dashed links and then directed solid links, until this is no longer possible. Let the set of dashed links in $k$’s outside option chain be given by $L^d_k$ and the set of solid links in

If $k$ is matched to themself it will be said that they have no outside trade partner.
k’s outside option chain be given by $L^+_{i,k}$\footnote{Outside option chains, although defined and motivated in a very different way, are similar to the opportunity paths identified by Kranton and Minehart (2000a) for networks with homogeneous gains form trade. Outside option chains can be viewed as a generalization of opportunity paths.} The outside option chains for the network in Figure 3a are shown in Figure 4b.

Outside option chains have a number of useful properties:

**Lemma 1.**

(i) $L^\mu_+(k) = L^d_k$ and $L^\mu_-(k) = L^d_k \cup l_k^\mu(k)$.

(ii) For a given network decomposition, each party has a unique finite outside option chain.

The proof of lemma 1 is in Appendix B. Part (i) of Lemma 1 states that the dashed links in k’s outside option correspond to the links that would be rematched over were their trade partner $\mu^*(k)$ removed from the network and that the solid links in k’s outside option chain, along with the link from k to $\mu^*(k)$, correspond to the links that would no longer be traded over were $\mu^*(k)$ removed from the network. Lemma 1 can be used to show how each party’s outside option payoff can be found from the directed network decomposition.

**Proposition 1.** Each party’s outside option payoff can be found by alternatively adding and then subtracting the values of the links in their outside option chain:

$$u_i = \sum_{l \in L^d_i} \alpha_l - \sum_{l \in L^s_i} \alpha_l$$
$$v_j = \sum_{l \in L^d_j} \alpha_l - \sum_{l \in L^s_j} \alpha_l$$

**Proof.** Demange (1982) and Leonard (1983) showed that a parties’ maximum core payoff is equal to their Vickrey payoff - their marginal contribution to the grand coalition. Utilizing Lemma 1: $v^{SO}_j = \sum_{l \in L^d_j} \alpha_l - \sum_{l \in L^s_j} \alpha_l = \sum_{l \in L^d_{\mu^*(i)} \cup L^s_{\mu^*(i)}} \alpha_l - \sum_{l \in L^d_{\mu^*(j)}} \alpha_l$.

A buyer receives their outside option payoff (their minimum core payoff) when their trade partner receives their maximum core payoff: $u_i = \alpha_{i\mu^*(i)} - v^{SO}_{\mu^*(i)} = \sum_{l \in L^d_i} \alpha_l - \sum_{l \in L^s_i} \alpha_l$.

Equivalent calculations establish the result for sellers’ outside options.  

$$\square$$

Proposition 1 identifies how each party’s outside option payoff can be found from the network decomposition. Outside option chains for the network given in Figure 3 are
Each party’s outside option payoff is then found by alternately adding and subtracting the values in these links. For example, \( b_1 \)’s outside option chain consists of the sequence of links \( \{ l_{b_1s_2}, l_{b_2s_2}, l_{b_2s_3}, l_{b_3s_3} \} \) and so their outside option is given by \( \alpha_{b_1s_2} - \alpha_{b_2s_2} + \alpha_{b_2s_3} - \alpha_{b_3s_3} \) as shown in Figure 4b.

In the network decomposition algorithm a party \( k \)’s outside trade partner \( \nu(k) \) was defined as the party \( k \) would rematch to were their trade partner removed from the network. This terminology can now be further motivated.

**Corollary 1.** Consider any pairwise stable outcome. Suppose party \( k \)'s terms of trade are now weakened to the extent that they receive a payoff below their outside option payoff. Party \( k \) would then have a profitable deviation by trading with their outside trade partner \( \nu(k) \).

**Proof.** From the lattice structure of the core \( v_{\nu(i)} = \alpha_{\mu^*(\nu(i))\nu(i)} - u_{\mu^*(\nu(i))} \). From Lemma \( \mu^*(\nu(i)) \)'s outside option option chain will be a strict subset of \( i \)'s outside option chain. Proposition \( \square \) can then be used to relate \( i \) and \( \mu^*(\nu(i)) \)'s outside option payoffs to each other: \( u_i = u_{\mu^*(\nu(i))} - \alpha_{\mu^*(\nu(i))\nu(i)} + u_{\mu^*(\nu(i))} \). Combining the above equalities \( v_{\nu(i)} + u_i = \alpha_{\nu(i)} \). As \( v_{\nu(i)} \geq v_{\nu(i)} \) whenever \( \nu(i) \) receives a core payoff, \( v_{\nu(i)} + u_i \leq \alpha_{\nu(i)} \). Thus if \( i \) ever received a payoff less than \( u_i \), \( i \) and \( \nu(i) \) would have a profitable deviation by trading with each other. An equivalent calculation can be shown for a seller \( j \).

**Corollary \( \square \)** shows that it is the potential for a party to trade with their outside trade partner that generates a binding constraint on their minimum possible pairwise stable payoff and establishes their outside option payoff.
With Proposition 1 in hand it is straightforward to apply equations 1 and 2 to find how each link in the network affects each party’s payoffs. This is summarized in corollary 2 and will be useful when considering party’s incentives to invest in forming links.

Corollary 2. Suppose \( i \) is matched to \( j \) (\( \mu^*(i) = j \)). Then the marginal effect on \( i \)’s payoff of increasing the value of a link in the network is as follows:

<table>
<thead>
<tr>
<th>link</th>
<th>( \Delta u_i )</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{i,j} )</td>
<td>( (1 - \beta) )</td>
<td>+</td>
</tr>
<tr>
<td>dashed link in ( i ) and not ( j )’s outside option chain</td>
<td>( \beta )</td>
<td>+</td>
</tr>
<tr>
<td>dashed link in ( j ) and not ( i )’s outside option chain</td>
<td>( -(1 - \beta) )</td>
<td>-</td>
</tr>
<tr>
<td>dashed link in both ( i ) and ( j )’s outside option chains</td>
<td>( 2\beta - 1 )</td>
<td>+ iff ( \beta \geq \frac{1}{2} )</td>
</tr>
<tr>
<td>solid link in ( i ) and not ( j )’s outside option chain</td>
<td>( -\beta )</td>
<td>-</td>
</tr>
<tr>
<td>solid link in ( j ) and not ( i )’s outside option chain</td>
<td>( (1 - \beta) )</td>
<td>+</td>
</tr>
<tr>
<td>solid in both ( i ) and ( j )’s outside option chains</td>
<td>( 1 - 2\beta )</td>
<td>+ iff ( \beta \leq \frac{1}{2} )</td>
</tr>
<tr>
<td>any other link</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

5. Efficient Network Formation

When considering network formation it is useful to have the benchmark of efficient networks. The efficient network maximizes the net gains from trade (\( NGT(L) \)) - the gains from trade generated less the costs of forming the links in \( L \):

\[
L^e \equiv \max_{L \in \mathcal{L}} NGT(L) = \max_{L \in \mathcal{L}} \left\{ \sum_{i \in P} \alpha_{i\mu^*(i,L)} - c|L| \right\}
\]

where \( \alpha_{ii} = 0 \) for each buyer \( i \in P \), \( |L| \) is the number of links in a network \( L \) and \( \mathcal{L} \) is the set of all possible bi-partite networks.\(^\text{26}\)

The efficient network only identifies a subset of the Pareto frontier. However, consider a network \( L \) that is not efficient but is Pareto efficient. There will always exist transfers that could be made on the efficient network that would constitute a Pareto improvement to \( L \).\(^\text{27}\) In efficient networks there are no links formed and not traded over.

\textsuperscript{26} This notion of efficiency was introduced in Jackson and Wolinsky (1996) and referred to as ‘strong efficiency’.

\textsuperscript{27} Further, it is not necessary for transfers to be possible between all buyers and sellers to achieve this Pareto improvement. For the Pareto improvement to occur transfers only have to be made over the links traded over on network \( L \). See Jackson (2003).
6. Network formation

This section compares networks that are likely to be formed (stable networks) to the efficient network. Broadly, the following questions will be considered: When are efficient networks stable? Can the inefficiency present in stable networks be bounded? What types of inefficiencies can be present? When are different types of inefficiency important? In order to address these questions the highest and the lowest percentage of the possible net gains from trade that can be lost on any stable network will be used to bound inefficiency. These inefficiency measures will then be adapted to identify (i) under-investment, due to hold up problems and (ii) over-investment in relationships to generate ‘outside options’. Networks will be defined as stable when they are pairwise Nash stable. The main results of the paper are then presented.

6.1. Measuring the size of inefficiencies. The efficiency loss $EL(L)$ of a network $L$ will be measured as the percentage of the possible net gains from trade that are lost:

$$EL = \frac{NGT(L^e) - NGT(L)}{NGT(L^e)} \in [0, 1].$$

There are then two subtleties to deal with. First, when the supremum of possible efficiency losses, for all potential gains from trade, is 1 it will be said that the efficiency losses are unbounded (100%). Second, when there are multiple stable networks a conservative approach will be taken and results only stated that apply to all stable networks.28

6.2. Different types of inefficiency. Over-investment inefficiency $OII(L)$ on a network $L$ is the efficiency loss due to the formation of links that are not used for trade. Denoting the links in a network $L$ used for trade by $K \subseteq L$, the efficiency loss due to over-investment in non-trade links is:29

$$OII(L) = \frac{c(|L| - |K|)}{NGT(L) + c(|L| - |K|)} \in [0, 1]$$

Over-investment inefficiency can occur when a party forms a link they don’t trade over but instead use to establish an outside option and affect their terms of trade with their trade partner.

28This approach is imported from the computer science literature. The “price of anarchy” measures the efficiency loss on the best stable network and the “cost of anarchy” measures the efficiency loss on the worst stable network (Koutsoupias and Papadimitriou (2009) and Roughgarden and Tardos (2004)). Stating results for all stable networks is equivalent to considering the price of anarchy when bounding the efficiency losses from above and the cost of anarchy when bounding efficiency losses from below.

29If there is only over-investment then $NGT(L^e) = NGT(L) + c(|L| - |K|)$ and $OII(L) = EL(L)$. 
Under-investment inefficiency \((\text{UII}(L))\) on a network \(L\) is the efficiency loss due to foregone net gains from trade that could be realized by forming links between unmatched buyers and unmatched sellers. Consider a network \(L\) such that buyers \(\hat{P} \subseteq P\) and sellers \(\hat{Q} \subseteq Q\) will end up unmatched. For this subset of buyers and sellers denote the efficient network by \(\hat{L}^e \equiv \max_{L' \in \hat{L}} \text{NGT}(L')\), where \(\hat{L}\) is the set of all possible bi-partite networks between the buyers \(\hat{P}\) and sellers \(\hat{Q}\). Under-investment inefficiency on \(L\) is then:

\[
\text{UII}(L) = \frac{\text{NGT}(\hat{L}^e)}{\text{NGT}(L) + \text{NGT}(\hat{L}^e)} \in [0, 1]
\]

Under-investment inefficiency can occur due to a hold-up problem. Unmatched Parties may have net gains from trade but remained unlinked on a stable network for fear of hold up once investment costs have been sunk.\(^{31}\)

Having identified an efficiency measure and adapted it to the under-investment and over-investment problems, stable networks need to be identified. The exact definition of which networks are stable will depend on the investment protocol. First stable networks will be defined for costs shares that are separate and exogenous, following the literature, then stable networks will be defined for negotiated cost shares.

6.3. **Stable networks - Exogenous cost shares.** Assume, for now, that buyers and sellers have to make separate non-substitutable investments to enable trade: For every link formed the buyer pays \(\gamma c\) and the seller pays \((1 - \gamma)c\), \(\gamma \in [0, 1]\).\(^{32}\)

The simultaneous link formation game is a simultaneous move game of complete and perfect information. Buyers and sellers are the players. Their strategy sets consist of the different sets of links they could pay to form. Payoffs are given by the vectors \(u\) and \(v\) following trade on the formed network less parties’ investments. A link is formed if and only if an investment of \(c\) is made. Links therefore require investment from both the buyer and seller except when \(\gamma \in \{0, 1\}\).

If stable networks were defined to be a Nash equilibrium of the simultaneous link formation game there would be stable networks where a buyer and seller fail to coordinate on forming a link that would benefit them both. Further the empty network would be stable.

---

\(^{30}\)If there is only under-investment then \(\text{NGT}(L^e) = \text{NGT}(L) + \text{NGT}(\hat{L}^e)\) and \(\text{UII}(L) = \text{EL}(L)\).

\(^{31}\)For any network \(L\) there are weakly positive levels of inefficiency remaining once over-investment inefficiency and under-investment inefficiency have been taken into account. This inefficiency can be attributed to coordination problems.

\(^{32}\)It is (implicitly) assumed that buyers and sellers cannot make transfers based on these investments or more generally contract over these investments.
∀γ ∈ (0, 1). To eliminate these types of networks from the set of stable networks buyers and sellers will be allowed to coordinate on forming a link that benefits them both.

A network will be considered stable when it is pairwise Nash stable. A network \( L \) is pairwise Nash stable\(^{33}\) if and only if it is a Nash equilibrium of the simultaneous link formation game and unformed links would not benefit both the buyer and seller were they formed.\(^{34}\) Unformed links would not benefit the buyer and seller if and only if:

(i) If \( u_i(L \cup l_{ij}) - \gamma c > u_i(L) \), then \( v_j(L \cup l_{ij}) - (1 - \gamma)c < v_j(L) \), \( \forall l_{ij} \notin L \); and

(ii) If \( v_j(L \cup l_{ij}) - (1 - \gamma)c > v_j(L) \), then \( u_i(L \cup l_{ij}) - \gamma c < u_i(L) \), \( \forall l_{ij} \notin L \).

Necessary and sufficient conditions for the efficient network to be stable are identified and discussed in the additional material.\(^{36}\) Of particular interest is that the efficient network is stable for all potential gains from trade \( a \) when buyers unilaterally form links (\( \gamma = 1 \)) and have all the bargaining power \( \beta = 1 \),\(^{37}\) whilst no non-empty efficient network is ever stable for \( \beta = 1 \) and \( \gamma < 1 \). In general, the gains from trade will matter.

Having defined a stable network under-investment and over-investment inefficiencies can be considered more precisely. Consider again network formation for the potential gains from trade shown in Figure 1a in Section 2 and the case where both buyers and sellers had to make a positive investment for links to form, \( \gamma = \in (0, 1) \), and sellers have all the bargaining power, \( \beta = 0 \). It was shown that the unique stable network was the empty network such that all the gains from trade that would have been generated by the efficient network were lost on the stable network. The efficiency loss on all stable networks is therefore 100% and it is due to under-investment inefficiency. In this example there is no inefficiency due to over-investment in non-trade links. However, there can also be over-investment inefficiency.

\(^{33}\)Jackson and Wolinsky (1996) introduced the concept of Pairwise stability.

\(^{34}\)Pairwise Nash stability is a relatively minimal requirement for a network to be stable. In particular a network can be pairwise Nash stable even if a buyer and seller could profit from forming a link between themselves and simultaneously deleting some other link. However permitting a larger set of possible deviations only reduces the set of stable networks and so results bounding inefficiency from above are strengthened. Further, it can be verified that the constructive equilibria found in proofs (with 100% efficiency loss) are still equilibria under this stronger definition of stability.

\(^{35}\)For all potential gains from trade (\( \forall a \)), for all levels of bargaining power (\( \forall \beta \)) and for cost shares \( \gamma \in (0, 1) \) there exists a pairwise Nash stable network where all inefficiency can be attributed to either under-investment or over-investment. This helps motivate the focus on under-investment and over-investment inefficiency and not coordination problems. This is proved in Section AM-3 of the additional material.

\(^{36}\)Section AM-4.

\(^{37}\)This corresponds to the main result of Kranton and Minehart (2001).
Assume that the buyers bargaining power is $\beta = \frac{1}{2}$, costs are shared evenly ($\gamma = \frac{1}{2}$) and the cost of link formation is $c = 1 - \varepsilon$.

The efficient network is again unstable, but now it is because of over-investment in non-trade links. Buyer $b_1$ and seller $s_2$ can increase their payoffs by forming a non-trade link between themselves. Furthermore this is the unique stable network. Here not all the net gains from trade generated by the efficient network are lost. However, half the gains from trade are lost as $\varepsilon \to 0$.

Under-investment inefficiency in the first example is a direct result of the assumption that cost shares are exogenous. If buyers and sellers could negotiate over the costs of link formation they would always be able to negotiate a cost share that eliminated any under-investment inefficiency. This possibility is considered in the next section.

6.4. **Negotiated cost shares.** Assume now that investments are negotiated and endogenous - a link is now formed whenever a buyer-seller pair jointly benefit from it. This assumption can be motivated in a couple of ways. Most simply buyers and sellers just have to make a joint investment where investment by one of them is perfectly substitutable with investment by the other. Alternatively, buyers and sellers may be able to make transfers to one another based on the non-substitutable investments they make.

Whilst the investment shares of buyer-seller pairs can be negotiated, it is still assumed that contracts cannot be written that specify future terms of trade. This assumption is critical. If such contracts could be written the first best could always be attained. Whether this assumption is reasonable or not will depend on the market being considered. However, it is thought that for many markets the assumption will be appropriate. In the
example in the introduction, when LIN wanted McCaw to bid up their acquisition price they paid Bell South to enter the bidding. Presumably it was not sufficient for LIN to threaten McCaw with the prospect of paying Bell South to enter without actually doing so. When trade is across or between countries, as is the case with oil and natural gas, there is often no effective court to enforce the contract. In labor markets workers can typically leave firms at will. In the New York apparel industry Uzzi (1996) provides evidence for the terms of trade between manufacturers and suppliers being determined after investment decisions have been made. Further, when complete state contingent contracts cannot be written contracting is not necessarily an improvement over ex-post negotiation.

With negotiated cost shares it is necessary to adjust the definition of a stable network. First the strategy space of the simultaneous link formation game needs to be expanded. Each party $k$ can now invest $I_{kk'} \in [0, c]$ in a link to each potential trade partner $k'$. Again a link is only formed if and only if an investment in it of at least $c$ is made. A network is a Nash equilibrium of the expanded simultaneous link formation game if all parties investments are mutual best responses. As in all Nash equilibria joint investments of either 0 or $c$ will be made in each link, parties’ strategy space can be restricted, without loss of generality, to choosing investment shares $\gamma_{ij}, \gamma_{ji} : \gamma_{ij} + \gamma_{ji} \in \{0, 1\}$ rather than absolute investment levels.

A network $L$ is side payment pairwise Nash stable if and only if it is a Nash equilibrium of the expanded simultaneous link formation game and unformed links would not jointly benefit the unconnected parties: $u_i(L) + v_j(L) \geq u_i(L \cup l_{ij}) + v_j(L \cup l_{ij}) - c, \forall l_{ij} \not\in L$.

It is assumed that if an additional link $l' \not\in L$ were added to $L$, the cost shares over each link $l \in L$ would remain the same on the network $L \cup l'$ as they were on $L$. Necessary and sufficient conditions for the efficient network to be side-payment pairwise stable are identified and discussed in the additional material. Of particular interest

---

38 There has been renegotiation of gas and oil contracts in a number of countries post investment. See Walde (2008).
39 See Stole and Zwiebel (1996) for a careful analysis of the assumption that employment is ‘at will’.
40 See for example Che and Hausch (1999), Segal (1999) and Bernheim and Whinston (1998).
41 Whilst this condition restricts the scope of possible profitable deviations (i.e. a link cannot be added and the cost shares of different links simultaneously changed) it is in the same spirit as pairwise Nash stability which considered a network to be stable even if it is profitable for a link to be jointly formed whilst others links are simultaneously deleted. In this respect side payment pairwise stability is a relatively minimal requirement.
42 Section AM-4.
is that a sufficient condition for the efficient network to be stable is that parties have anti-assortative preferences. This condition also becomes necessary as $c \to 0$. Having defined what is meant by a stable network when cost shares are endogenous the size of inefficiencies can be considered. Consider again Figure 1a in Section 2. With negotiated cost shares the empty network will not be stable because $s_1$ will be willing to pay the entire costs of forming the link $l_{11}$. Indeed the unique stable network is now the complete network and as $\varepsilon \to 0$ all the net gains from trade are lost due to over-investment in an outside option link by $b_1$. In this stable network $b_1$ must pay the entire cost of forming the link $l_{12}$ as $s_2$ does not trade and therefore receives no benefit from the link. However, the exact share of costs for link $l_{11}$ is not pinned down. $s_1$ would be willing to pay up to $\varepsilon$ for it and $b_1$ must then pay the rest. Thus $\pi_{b_1} \in [\varepsilon, 2\varepsilon]$ and $\pi_{s_1} \in [0, \varepsilon]$ depending on the cost share agreed. Note that both these payoffs go to zero as $\varepsilon \to 0$. Indeed $OII = \frac{1 + \varepsilon - 2\varepsilon}{\varepsilon} = \frac{1 - 2\varepsilon}{1 + 2\varepsilon}$ which goes to 100% as $\varepsilon \to 0$.

The examples considered so far suggest that under-investment in trade-links may be a more serious problem than inefficiency due to over-investment in non-trade links when cost sharing is exogenous whilst inefficiency due to over-investment in non-trade links will become more problematic when cost shares are endogenous. Proposition 2 in the next section makes this intuition precise.

6.5. The Impact of Negotiation on Inefficiencies. When the cost of link formation is zero the complete network can be costlessly formed and as matching on formed networks is efficient there is then no inefficiency. This corresponds to the traditional analysis of markets in which any buyer can costlessly transact with any seller. When the cost of link formation is larger than the potential gains from trade between any buyer and any seller the empty network will be efficient and uniquely stable. This corresponds to transaction costs being prohibitively high for any trade. There is no inefficiency for both very large costs of link formation and no cost of link formation. This paper is concerned with the case of intermediate costs of link formation relative to the gains from trade.

Proposition 2. When costs shares are exogenous:

(i) for any gains from trade $a$ and for all cost shares $\gamma \in (0, 1)$, over-investment inefficiency on all stable networks is bounded at 50%;

This requires that no two buyers most preferred seller is the same when prices are zero and no two sellers most preferred buyer is the same when prices are the entire gains form trade.
(ii) for any non-empty sets of buyers and sellers $P$, $Q$ and $\beta \neq \gamma$, there exist gains from trade $a$ with unbounded under-investment inefficiency of 100% on all stable networks.

Endogenizing cost shares:

(i) there is no under-investment inefficiency;

(ii) for any non-empty sets of buyers and sellers $P$, $Q$, there exist gains from trade $a$ with unbounded over-investment inefficiency of 100% on all stable networks.

The proof of Proposition 2 is deferred until section 6.6. Proposition 2 emphasizes that inefficiency is not an artifact of the investment protocol assumption, although this does affect which types of inefficiency are likely to be most important and when. Removing under-investment inefficiency problems by permitting buyers and sellers to endogenously negotiate their investment shares exacerbates over-investment inefficiency problems insofar as they are no longer bounded.

**Corollary 3.** When costs shares are exogenous there is no over-investment inefficiency on all stable networks for any cost shares $\gamma \in (0, 1)$ if either:

(i) there are homogeneous gains from trade $a_{ij} \in \{0, 1\}$, $\forall i, j$; or
(ii) there is a single buyer, $|P| = 1$; or
(iii) there is a single seller, $|Q| = 1$.

Corollary 3 emphasizes the sensitivity of results to considering environments with multiple buyers and multiple sellers and heterogeneous gains from trade when analyzing inefficiencies. This helps motivates these generalizations in comparison to previous literature.

**Proposition 3.** When buyers have all the bargaining power $\beta = 1$ and have to pay all the costs of link formation $\gamma = 1$ the efficient network is stable for any potential gains from trade $a$. Furthermore there does not exist a stable network with any inefficiency due to over-investment in non-trade links or under-investment in trade links. However, if either:

(i) buyers’ bargaining power declined ($\beta < 1$); or
(ii) sellers had to pay some costs of link formation ($\gamma < 1$); or
(iii) buyers and sellers could negotiate over the costs of link formation
there would exist potential gains from trade where, on even the most efficient stable network, all the gains from trade are consumed by inefficiencies.

The proof of Proposition 3 is deferred until section 6.6. It is well known that inefficiencies can be present when parties cost shares and bargaining power do not coincide. The contribution of Proposition 3 is identifying the discontinuity in the level of inefficiency present when moving away from the special case where the efficient network is always stable: Moving even slightly away from this special case all the gains from trade generated by the efficient network can be lost on all stable networks.

6.6. Proofs. Propositions 3 and 2 and Corollary 3 follow immediately from three lemmas that are presented in this section. Over-investment inefficiency with separate investments is considered first.

Lemma 2. For exogenous cost shares $\gamma \in (0, 1)$ and any homogeneous potential gains from trade $a : a_{ij} \in \{0, 1\},$ there is no over-investment inefficiency. For any (heterogenous) potential gains from trade $a,$ over-investment inefficiency is bounded by the amounts shown in the table below and this bound is tight:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Bound on over-investment inefficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in (0, 1)$</td>
<td>$\in (0, 1)$</td>
<td>$h(K, \gamma, \beta)^* \leq \frac{1}{2}$</td>
</tr>
<tr>
<td>$\in (0, 1)$</td>
<td>$0$</td>
<td>$0^{**}$</td>
</tr>
<tr>
<td>$\in (0, 0.5)$</td>
<td>$0$</td>
<td>$\frac{\beta}{1-\beta}$</td>
</tr>
<tr>
<td>$[0.5, 1]$</td>
<td>$1$</td>
<td>$0^{**}$</td>
</tr>
<tr>
<td>$[0, 0.5]$</td>
<td>$0$</td>
<td>$\frac{1-\beta}{\beta}$</td>
</tr>
<tr>
<td>$[0.5, 1]$</td>
<td>$1$</td>
<td>$0^{**}$</td>
</tr>
</tbody>
</table>

Table 1. Bounding over-investment inefficiency.

* $h(\tilde{K}, \gamma, \beta) \equiv \frac{\tilde{K}-1}{\max\{\frac{(K-1)(1-\gamma)}{\beta^2} + \frac{(K-1)(1-\gamma)}{\beta^2}, \frac{(K-1)\gamma}{\beta^2} + \frac{(K-1)\gamma}{\beta^2}\} - \tilde{K}}$

** The unique stable network is the empty network.

where $\tilde{K}$ is the number of matches on the component in the network with the most matches when $\beta \leq (1 - \gamma);$ the number of matches on the component in the network with the least number of matches greater than one when $\beta > (1 - \gamma)$ and there is a component with more than one match; and $\tilde{k} = 1$ when all components have only one match.

The proof of Lemma 2 is in Appendix B. These bounds are shown in Figure 6 below.

44 The bounds are independent of the cost of link formation $c$ because the values of $a$ can always be scaled to compensate for a higher or lower $c.$
Figure 6a shows the bound on over investment inefficiency for $\gamma \in [0, 1]$, $\beta \in [0, 1]$, $\tilde{K} = 10$ and $c$ normalized to 1. There are similar bounds for other values of $\tilde{K}$. For all $\tilde{K} > 1$ the bound on the efficiency loss for $\gamma \in (0, 1)$ is largest when $\gamma = \beta = \frac{1}{2}$, and at this point is 50%.

Recall that it is precisely when $\gamma = \beta$ that there is no under-investment inefficiency (Lemma 3). Thus, it is when there is no under-investment inefficiency that the problem of over-investment is most acute.

Figure 6a also shows a discontinuity in the bound at the extreme values of $\gamma$. This is because when $\gamma \in \{0, 1\}$ links can be formed unilaterally. Consider now under-investment inefficiency.

**Lemma 3.** For exogenous cost shares, any non-empty sets of buyers $P$ and sellers $Q$ and for $\beta \neq \gamma$ there exist potential gains from trade $a$ for which all the net gains from trade generated by the efficient network are lost to under-investment inefficiency on all stable networks. When $\beta = \gamma$ there is no under-investment inefficiency in any stable network for any potential gains from trade $a$.

The proof of Lemma 3 is in Appendix B. It is intuitive that there will be no under-investment in efficiency when $\beta = \gamma$. In this case each party’s revenue and cost shares from joint production are aligned. When investment shares are negotiated buyers and sellers can always agree to set $\beta = \gamma$. This is why there is never any under-investment with negotiated investment shares.

The final Lemma required shows that the inefficiency loss due to over-investment in non-trade links is unbounded when cost shares are endogenous.

---

45 This is proved as part of the proof of Lemma 2 in Appendix B.
46 A similar condition is found in Caballero and Hammour (1998).
Lemma 4. Suppose cost sharing is endogenous. Then $\forall \beta$ and for any set of $m \geq 2$ buyers $P$ and $n \geq 2$ sellers $Q$, there exist potential gains from trade $a$ with 100% over-investment inefficiency.

The proof of Lemma 4 is in Appendix B. Propositions 2 and 3 follow immediately from Lemmas 2, 3 and 4.

7. Related Literature

This section places the models of Sections 4 and 6 in the context of three related literatures: the literatures on bargaining over formed networks, forming trade networks, and the considerable literature on making relationship specific investments when contracts are incomplete.\footnote{Some related literatures are not discussed. For network formation outside the context of trade networks see Jackson and Wolinsky (1996), Bala and Goyal (2000), Galeotti et al (2006) and Goyal and Vega-Redondo (2007) amongst others. Jackson (2008) provides an overview. Perhaps most relevant paper in this literature is Currarini and Morelli (2000) who consider a non-cooperative game of network formation in which players form links and propose surplus splits thereby endogenizing these splits. This is somewhat similar to permitting buyers and sellers to negotiate their investment shares. In contrast to the model of Section 6 they find that the efficient network is stable under their condition of ‘size monotonicity’. The search literature is also related but not discussed. Elliott (2009) applies the bargaining model of section 4 to the labor market and discusses the relationship with the search literature in detail. Rogerson, Shimer and Wright (2005) have a broader and excellent survey.}

7.1. Bargaining over formed networks. Although the role of networks in affecting bargaining outcomes has long been recognized (e.g. Emerson (1967)) there is a rapidly expanding literature on bargaining over formed networks.\footnote{In experiments Cook and Emerson (1983) investigated the role of networks in determining the terms of trade and found that the network mattered.} Models of bargaining over a network tend to take one of two approaches: A cooperative or non-cooperative game theoretic approach. There are a number of papers that model non-cooperative alternating offer bargaining games over the network. Navarro and Perea (2001), Corominas-Bosch (2004), Polanski (2007), Manea (2008) and Abreu and Manea (2008) all pursue this methodology. Corominas-Bosch (2004) is closest to this paper. Networks in the Corominas-Bosch model can also be viewed as affecting bargaining outcomes by providing outside options.\footnote{This contrasts with Manea (2008) and Abreu and Manea (2008) where the network can be interpreted as affecting parties bargaining power. The approximate intuition is that when a party has more links (although it also matters who these links are to) the bargaining protocol selects this party more often to receive and make offers reducing the time they wait between making or receiving offers. This has the affect of making parties with more links more patient, in effect increasing their bargaining power.} The alternative methodology employed in this paper replicates the Corominas-Bosch equilibrium payoffs. This provides some non-cooperative justification
for the approach taken, although Corominas-Bosch is only able to deal with the homogeneous gains from trade.\footnote{It is not straight forward to directly extend Corominas-Bosch’s model to include heterogeneous gains from trade. Indeed she concludes that although it is “natural to ask for the introduction of a little bit of heterogeneity in the model […] [w]e believe that this line of research is unlikely to lead to fruitful results.” See Section AM-2 of the additional material.}

Charness, Corominas-Bosch and Fréchette (2007) provide experimental support for the qualitative predictions of this model by virtue of supporting Corominas-Bosch (2004).\footnote{Cahuc Postel-Vinay and Robin (2006) model bargaining between one seller and many buyers as an auction followed by an alternating offer bi-lateral bargaining game. Their payoffs are also replicated by the model of Section 4.}

The second approach to modeling bargaining over a network utilizes cooperative game theory. Myerson (1977) initiated this approach by applying the Shapley value over a network.\footnote{Navarro and Perea (2001) replicate the payoffs identified by Myerson with a non-cooperative bargaining protocol. Kranton and Minehart (2000b), Segal and Whinston (2000) and Rajan and Zingales (1998) all consider investments prior to surpluses being split in accordance with the Shapley value.}

Whilst the Shapley value has many nice properties and provides an equitable way of allocating surpluses within a fixed coalition it is less well suited for considering which coalitions are formed. As determining which buyers buy from which sellers is central to bargaining over buyer-seller networks the core may be a more appealing solution concept. This also explains, to some extent, why the core is also widely utilized in the matching literature.\footnote{See Roth and Sotomayor (1990).}

The analysis of bargaining over a formed network in this paper is closest to Kranton and Minehart (2000a) who also generate outcomes in the core. Kranton and Minehart consider competitive bargaining outcomes (which correspond to core bargaining outcomes) and identify chains of links between buyers and sellers which in effect determine outside options. However, they are only able to deal with the case of identical sellers such that it does not matter which seller a buyer trades with, only whether they trade with a seller or not. Their analysis exploits the homogeneity of sellers, and in particular the fact that they provide a common reference price to all buyers connected to them. To consider heterogenous gains from trade it is necessary to take a different, albeit related approach building on their core insights. This new approach generates new results. Proposition 1 characterizes payoffs as a weighted sum of the potential gains from trade over links identified by the directed network decomposition. This result has no counterpart in Kranton and Minehart (2000a).\footnote{Considering heterogenous gains from trade also permits new situations to be analyzed. For example, the special case of vertically differentiated sellers is considered in Section AM-4 of the additional material, whilst in Kranton and Minehart (2000a) sellers are assumed identical.}

\footnote{It is not straight forward to directly extend Corominas-Bosch’s model to include heterogeneous gains from trade. Indeed she concludes that although it is “natural to ask for the introduction of a little bit of heterogeneity in the model […] [w]e believe that this line of research is unlikely to lead to fruitful results.” See Section AM-2 of the additional material.} 

\footnote{Cahuc Postel-Vinay and Robin (2006) model bargaining between one seller and many buyers as an auction followed by an alternating offer bi-lateral bargaining game. Their payoffs are also replicated by the model of Section 4.} 

\footnote{Navarro and Perea (2001) replicate the payoffs identified by Myerson with a non-cooperative bargaining protocol. Kranton and Minehart (2000b), Segal and Whinston (2000) and Rajan and Zingales (1998) all consider investments prior to surpluses being split in accordance with the Shapley value.} 

\footnote{See Roth and Sotomayor (1990).} 

\footnote{Considering heterogenous gains from trade also permits new situations to be analyzed. For example, the special case of vertically differentiated sellers is considered in Section AM-4 of the additional material, whilst in Kranton and Minehart (2000a) sellers are assumed identical.}
7.2. **Trade network formation.** The main focus of this paper is modeling the formation of trade networks and analyzing inefficiencies in these networks. Most closely related is Kranton and Minehart’s (2001) pioneering paper, that addresses when efficient investments will be made. They show that when buyers have all the bargaining power, $\beta = 1$, buyers unilaterally form links, $\gamma = 1$, and the gains from trade are ex-ante homogeneous, then the stable network is efficient. They also allow for uncertainty in the value of relationships. This positive result contrasts with Proposition 3 which identifies a discontinuity in the inefficiency bound at this point. Specifically, for any other level of bargaining power or any other buyer cost share, there are potential gains from trade for which all the surplus generated is lost on all stable networks.

Kranton and Minehart (2000b) consider the formation of supply networks as an alternative to vertical integration. The efficiency of relationship specific investments that enable trade under (buyer optimal) core bargaining and Shapley value bargaining are compared. As in Kranton and Minehart (2001) relationship specific investments are one sided, that side extracts their maximum core payoff and gains from trade are ex-ante homogeneous. When Shapley value bargaining is utilized too many or too few links may be formed. In contrast when the core bargaining solution is applied there is efficient network formation. However, there are non relationship specific investments that may be made on the other side of the market and these can be inefficient.

Manea (2008) considers both pairwise stable and unilaterally stable networks where gains from trade are split according to his non-cooperative bargaining game and there are no costs to forming links. With no costs of link formation links can no longer be interpreted as relationship specific investments, indeed there are no investments at all. Thus the network formation problem considered by Manea is fundamentally different from the problem considered in this paper: Were $c = 0$ in the model of Section 6 there would be no inefficiencies in network formation.

7.3. **Other models of relationship specific investments.** The hold-up problem and over-investment in outside options in the presence of relationship specific investments has been well understood, in the case of a single buyer and single seller, for a long time. The subsequent literature is voluminous and only a couple of the most relevant papers are discussed here.

---

Caballero and Hammour (1998) consider investments with discontinuous and highly inter-dependent returns, similar to those in this paper. Capital and labour must be combined in fixed proportions to be productive and parties split the surplus generated from investments equally. There is no over-investment inefficiency in Caballero and Hammour as only two parties are modeled. There can be under-investment inefficiency but when capital and labour must be combined in equal proportions to generate a productive unit, efficient investment decisions are made. This is a special case of the condition that $\beta = \gamma$ for no under-investment inefficiency (Lemma 3).

Several papers consider markets with a single party on one side but multiple parties on the other. Moving from bilateral settings to these (limited) multilateral settings introduces important complexities. In McAfee and Schwartz (1994) there are many producers who must make specific investments to enable trade with a monopolist supplier. The terms of trade each producer receives from the supplier affects their marginal cost and ability to compete in the product market. Thus the presence of many buyers creates additional recontracting problems - renegotiation of the terms of trade with one buyer can hurt other buyers. To overcome this problem suppliers would like to commit to publicly observable prices, but such contracts are difficult to enforce in practice. Non-discrimination clauses can also be ineffective.

Cole, Mailath and Postlewaite (2001a) and (2001b) consider markets with many buyers and many sellers. In the first paper investment efficiency is restored for a finite set of buyers and sellers by conditioning which equilibrium is selected from the core on parties’ investments. Inefficient investment can then be punished and the incentives for efficient investment restored. This approach might also restore efficiency in the set up of Section 6. However, the bargained outcomes require careful manipulation to generate the correct incentives and seem unlikely to be provided by a decentralized market. In the second paper efficiency is restored by modeling a continuum of potential trade partners.

---

56See Segal (1999) for a general treatment of inefficiencies that arise when one principal contacts with many agents in the presence of externalities.
57Stole and Zwiebel (1996) model a single firm’s employment decisions where the marginal value of workers depends on the number employed. Wages are determined via an intuitive bargaining protocol with a unique perfect equilibrium corresponding to the Shapley value, which is in the core. The firm may then over-employ to reduce the wages it has to pay each worker. In contrast, with many buyers and many sellers and one-to-one matching the Shapley value need not be in the core.
58Lee and Schwarz (2009) model interviewing decisions, which are required to learn preferences, prior to a centralized matching process. Any worker can be employed by any firm following an interview, firms bear all the costs of interviewing and ex-ante both workers and firms are identical.
In many markets two parties must make a relationship specific investment before they can trade. To date analysis of investment in these networked markets has focused on separate investments and the question of when investments are efficient. Whilst separate investments may be a good approximation for some market they are not for others. Investments in oil or gas pipelines are typically negotiated and even when investments appear separate they will behave as if they are negotiated if transfers are possible. How does allowing investments to be negotiated change the overall size of inefficiencies and the types of inefficiency that are important?

With both separate and negotiated investments inefficiencies can be substantial. Except for a knife edge case inefficiency cannot be bounded and can consume all the gains from trade (Proposition 3), but for very different reasons. With separate investments over-investment inefficiency to establish outside options is limited whilst under-investment inefficiency due to potential hold-up is not. In contrast, when investments are negotiated under-investment inefficiency is eliminated but over-investment inefficiency is exacerbated (Proposition 2). It is then important to understand whether investments are negotiated or not when considering market interventions. For example there may be a case for permitting exclusive dealing contracts to prevent over-investment when investments are negotiated but not when they are separate.

To permit a careful comparison of inefficiencies across investment protocols it is necessary to deal with multiple buyers, multiple sellers and heterogenous gains from trade. For example, with either homogeneous gains from trade or a single buyer or a single seller there can be no over-investment when a buyer and seller must both make separate investments before they can trade. New tools are also required. First, measures of inefficiencies attributable to over and under investment, allowing for multiple stable networks, are introduced. Second, to analyze investment inefficiencies it was necessary to understand precisely how the structure of a networked market - which buyers can trade with which sellers - influences payoffs. The network decomposition algorithm decomposes any bi-partite network into a directed network identifying outside option chains. These outside option chains determine how the network structure affects each party’s payoff (Proposition 1). This decomposition can also be utilized in other settings.

---

59 Segal and Whinston (2000) and McAfee and Schwartz (1994) both recognize potential efficiency benefits from exclusive dealing contracts.

60 Elliott (2009) analyze vacancy creation decisions in networked labor markets. There may also be applications in finance, in particular market microstructure, and advertising.
This section considers the size of inefficiencies, rather than just bounds on the size of inefficiencies, for a simple example: Buyer bargaining power \( \beta = \frac{1}{2} \) and there is one buyer and two sellers with potential gains from trade \( a_{11} \) and \( a_{12} \) respectively. All values of \( a_{11} \) and \( a_{12} \) are considered. In the case of separate investments the size of under-investment inefficiency is shown in Figure 7a for different values of \( \gamma \) and \( \max(a_{11}, a_{12}) \). There is no over-investment inefficiency for \( \gamma \in (0, 1) \). For negotiated investments the size of over-investment inefficiency is shown in Figure 7b for different values of \( a_{11} \) and \( a_{12} \).

![Figure 7](image)

**Figure 7.** The size of inefficiencies with exogenous and endogenous cost shares

Full (100%) under-investment inefficiency is more pervasive than 100% over-investment inefficiency, which requires very specific values of \( a_{11} \) and \( a_{12} \). However, when under-investment inefficiency is not 100% it is 0% whilst for a wide range of parameters there is some over-investment inefficiency.

### Appendix B. Proofs

**B.1. Proof of Lemma 1**

(i) \( L_{\mu^*(k)}^+ = L_{k}^{dl} \) and \( L_{\mu^*(k)}^- = L_{k}^{sl} \cup l_{k\mu^*(k)}. \)

(ii) For a given network decomposition, each party has a unique finite outside option chain.

**Proof.** It will be helpful to first prove the following Lemma.

**Lemma 5.** All the links \( l \in L_{j'}^+ \cup L_{j'}^- \) can be arranged in the sequence:

\( S_{j'} \equiv l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, l_{\mu^*((j'))^*}, \ldots \)

**Proof.** Define \( \hat{L}_{j'}^+ \equiv \{ l \in L_{j'}^+ \cap S_{j'} \}, \hat{L}_{j'}^- \equiv \{ l \in L_{j'}^- \cap S_{j'} \} \) and \( \hat{L}_{j'}^+ \equiv \{ l \in L_{j'}^+ \cap S_{j'} \} \) and \( \hat{L}_{j'}^- \equiv \{ l \in L_{j'}^- \cap S_{j'} \}. \) By the construction of \( S_{j'} \) and the fact that each party can be matched only once if \( l_{\mu^*(i)} \in \hat{L}^{-}_{j'} \) then \( \exists l_{ij} \in \hat{L}^+_{j'} \). Then, as each rematched party can no longer be matched to
their trade partner on $L$, $\forall ij \in \hat{L}^+_j$, $l_{\mu^*}(i) \in \hat{L}^-_j$ and $l_{\mu^*}(j) \in \hat{L}^-_i$, where $l_{kk} = \emptyset \in \hat{L}^-_j$. As $\mu^*(L)$ uniquely maximized the gains from trade on $L$, $\sum_{l \in L^+} a_l - \sum_{l \in L^-} a_l < 0, \forall L^- \cup L^+ \neq \emptyset$. Consider now the gains from trade generated by the rematching: $\sum_{l \in L^+} a_l - \sum_{l \in L^-} a_l = \sum_{l \in L^+} a_l - \sum_{l \in L^-} a_l + \sum_{l \in L^+} a_l - \sum_{l \in L^-} a_l < \sum_{l \in L^+} a_l - \sum_{l \in L^-} a_l, \forall L^- \cup L^+ \neq \emptyset$. Thus for the rematching to maximize the gains from trade on $L/j$, $L^- \cup L^+ = \emptyset$. □

Denote by $\hat{S}_j$ the sequence of links after $l_{\mu^*}(j) \in S_j$ and by $\hat{j} S_j$, the sequence of links before and including $l_{\mu^*}(j)$ in $S_j$. Suppose, in contradiction, that $l_{\mu^*}(i) \in \hat{S}_j$ and $l_{\mu^*}(i) \in S_j$. Define net gains from rematching over the sequence $S_j$ if $j$ is removed from $L$ by $NG(S_j) \equiv \sum_{l \in L^+_j} - \sum_{l \in L^-_j} l_{\mu^*}(j)$. As the rematch following the removal of $\hat{j}$ is optimal $NG(S_j) = NG(\hat{j} S_j) + NG(\hat{j} S_j) \geq NG(\hat{j} S_j)$. As the match $\mu^*(L)$ was optimal $NG(\hat{j} S_j) + NG(\hat{j} S_j) > 0$.

Combining these expressions: $NG(\hat{j} S_j) > NG(S_j)$. This expression states that, following the removal of $j$, higher gains from trade are possible than are obtained in the optimal rematching. This is a contradiction and $l_{\mu^*}(i) \in \hat{j} S_j$ implies $l_{\mu^*}(i) \notin S_j$.

Thus if a link $l_{\mu^*}(j) \in L/j$, then $\hat{S}_j = S_j$, otherwise the rematching on $L/j$ or the rematching on $L/j'$ could not maximize the gains from trade. It follows immediately that $L^+_k = L^d_{\mu^*(k)}$, $L_k = L^d_{\mu^*(k)} \cup l_{kd}(k)$, that outside option chains are finite and that outside option chains do not cycle. □

B.2. Proof of Lemma 3. For exogenous cost shares, any non-empty sets of buyers $P$ and sellers $Q$ and for $\beta \neq \gamma$ there exist potential gains from trade a for which all the net gains from trade generated by the efficient network are lost to under-investment inefficiency on all stable networks. When $\beta = \gamma$ there is no under-investment inefficiency in any stable network for any potential gains from trade $a$.

Proof. Suppose $\beta \neq \gamma$. Consider any gains from trade $a$ with a non-empty associated efficient network $L^e(a)$. Adjust the value of the links in $a$ as follows. Set $a_{i_{\mu^*}(i,L^e)} < c, \max \left\{ \frac{2c}{1-\gamma}, \frac{(1-\gamma)c}{1-\beta} \right\} > c, \forall i \in P$ and set all other links $a_{ij} = 0, \forall j \neq i, L^e$. As $\beta \neq \gamma$, $\max \left\{ \frac{2c}{1-\gamma}, \frac{(1-\gamma)c}{1-\beta} \right\} > c$. Denote these adjusted gains from trade $\hat{a}$.

For potential gains from trade $\hat{a}$ the efficient network is non-empty $(L^e(\hat{a}) = L^e(a))$. However, for all links with positive potential gains from trade either the buyer will not invest, $\beta a_{i_{\mu^*}(i,L^e)} < \gamma c$ or the seller will not invest $(1 - \beta) a_{i_{\mu^*}(i,L^e)} < (1 - \gamma)c$. Thus the unique stable network is the empty network.

Suppose $\beta = \gamma$. For any potential gains from trade $a$, consider, in contradiction, a relationship $a_{ij}$ inefficiently under-invested in on a stable network. From the definition of under-investment inefficiency $a_{ij} > c$. However, this implies that $i$ will wish to invest in $a_{ij} (\beta a_{ij} > \gamma c)$ and $j$ will wish to invest in $a_{ij} ((1 - \beta)a_{ij} > (1 - \gamma)c)$ and so this stable network is not stable generating a contradiction. □
B.3. **Proof of Lemma 2**. For exogenous cost shares $\gamma \in (0, 1)$ and any homogeneous potential gains from trade $a: a_{ij} \in \{0, 1\}$, there is no over-investment inefficiency. For any (heterogeneous) potential gains from trade $a$, over-investment inefficiency is bounded by the amounts shown in the table below and this bound is tight:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Bound on over-investment inefficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\in (0, 1)$</td>
<td>$\in (0, 1)$</td>
<td>$h(\tilde{K}, \gamma, \beta)^* \leq \frac{1}{2}$</td>
</tr>
<tr>
<td>$\in {0, 1}$</td>
<td>$\in (0, 1)$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\in (0, 0.5)$</td>
<td>0</td>
<td>$\frac{1}{\beta}$</td>
</tr>
<tr>
<td>$\in [0.5, 1)$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\in [0, 0.5]$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\in (0.5, 1]$</td>
<td>1</td>
<td>$\frac{1-\beta}{\beta}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

* $h(\tilde{K}, \gamma, \beta)^* = \max\left\{\frac{K-1}{1-\beta}, \frac{K-1}{1-\beta}, \frac{K-1}{1-\beta}, \frac{K-1}{1-\beta}\right\}$

** The unique stable network is the empty network.

where $\tilde{K}$ is the number of matches on the component in the network with the most matches when $\beta \leq (1-\gamma)$; the number of matches on the component in the network with the least number of matches greater than one when $\beta > (1-\gamma)$ and there is a component with more than one match; and $\tilde{k} = 1$ when all components have only one match.

*Proof.* This proof is undertaken in five parts. In Part i) the homogeneous gains from trade case is considered. Part ii) corresponds to the first row in the above table, Part iii) to the second row, Part iv) to rows five and six and Part v) to rows three, four, seven and eight. In each part the existence of the bound and it tightness is proved.

**Part i):** When $\beta \in (0, 1)$ and $\gamma \in (0, 1)$ all non-trade links must benefit both parties and as outside option chains cannot cycle (Lemma 1) each component with any non-trade links must take the form of a chain, as shown below for the case of four trade links.

![Figure 8. Component structure: 4 trade links](image)

With homogenous gains from trade Proposition 1 can be directly applied to show that $u_i = 0$, $\forall i \in P$ and $v_j = 0$, $\forall j \in Q$. Thus there are insufficient incentives for any outside option links to be formed and there is no outside over-investment inefficiency in any stable network.

**Part ii):** Label buyers and sellers as in Figure 8 but with $\tilde{K}$ buyers and $\tilde{K}$ sellers. Buyer $b_{\tilde{K}}$ and seller $s_1$ must have sufficient incentives to form their trade links:
\[
\pi_1 \geq \frac{(1 - \gamma)c}{1 - \beta} \quad \quad \pi_{\tilde{K}} \geq \frac{\gamma c}{\beta}
\]

Applying Proposition 1:
\[
\pi_1 = \sum_{k=j}^{k=j} (\alpha_{k-1,k} - \alpha_{k-1,k}) + \pi_j \geq \frac{(1 - \gamma)c}{1 - \beta} \quad (4)
\]
\[
\pi_{\tilde{K}} = \sum_{k=1}^{k=1} (\alpha_{k,k} - \alpha_{k-1,k}) + \pi_i \geq \frac{\gamma c}{\beta} \quad \forall j \in \{2, \ldots, \tilde{K}\}; \forall i \in \{1, \ldots, \tilde{K} - 1\} \quad (5)
\]

As neither \(b_{\tilde{K}}\) or \(s_1\) have an outside trade partner \(u_{\tilde{K}} = 0\) and \(v_1 = 0\). All other buyers and sellers must have sufficient incentives to form their outside option links. For the other sellers \(\beta v_j \geq (1 - \gamma)c\) and for the other buyers \((1 - \beta)v_i \geq \gamma c\). Applying Proposition 1 again:
\[
u_i = \sum_{k=i+1}^{k=i+1} (\alpha_{k-1,k} - \alpha_{k,k}) \geq \frac{\gamma c}{1 - \beta} \quad (6)
\]
\[
u_j = \sum_{k=2}^{k=2} (\alpha_{k,k} - \alpha_{k-1,k}) \geq \frac{(1 - \gamma)c}{\beta} \quad \forall j \in \{2, \ldots, \tilde{K}\}; \forall i \in \{1, \ldots, \tilde{K} - 1\} \quad (7)
\]

Combining equations 5, 4, 6 and 7:
\[
\pi_j \geq \frac{(1 - \gamma)c}{1 - \beta} + \sum_{k=2}^{k=2} (\alpha_{k-1,k} - \alpha_{k-1,k}) \geq \frac{(1 - \gamma)c}{1 - \beta} \quad (8)
\]
\[
\pi_i \geq \frac{\gamma c}{\beta} + \sum_{k=i}^{k=1} (\alpha_{k,k+1} - \alpha_{k+1,k+1}) \geq \frac{\gamma c}{\beta} \quad \forall j \in \{2, \ldots, \tilde{K}\}; \forall i \in \{1, \ldots, \tilde{K} - 1\} \quad (9)
\]

The gains from trade on a network are equal to the total payoffs received by all parties. Combining equations 3, 6, and 8 and then equations 3, 7 and 9:
\[
\sum_{k=1}^{k=1} \alpha_{k,k} = \sum_{k=1}^{k=1} (u_k + v_k) \geq (\tilde{K} - 1) \left( \frac{1}{1 - \beta} + \frac{1 - \gamma}{\beta} \right) c + \frac{1 - \gamma}{1 - \beta} \equiv f_1(\tilde{K}, \gamma, \beta)
\]
\[
\sum_{k=1}^{k=1} \alpha_{k,k} = \sum_{k=1}^{k=1} (u_k + v_k) \geq (\tilde{K} - 1) \left( \frac{1}{\beta} + \frac{\gamma}{1 - \beta} \right) c + \frac{\gamma}{\beta} \equiv f_2(\tilde{K}, \gamma, \beta)
\]

This provides a lower bound on the gains from trade reached with \(\tilde{K}\) trade links and an upper bound on the over-investment inefficiency
\[
h(\tilde{K}, \gamma, \beta) \equiv \frac{\tilde{K} - 1}{\max\{f_1(\tilde{K}, \gamma, \beta), f_2(\tilde{K}, \gamma, \beta)\} - \tilde{K}}
\]

such that, on any stable component with \(\tilde{K}\) matches, buyer bargaining power \(\beta \in (0, 1)\) and a buyer cost share of \(\gamma \in (0, 1)\): \(OII \leq h(\tilde{K}, \gamma, \beta)\).

\(OII\) on the network must be bounded from above by \(OII\) on the component with the highest \(OII\). When \(\beta \leq 1 - \gamma\) the component upper bound on \(OII\) is weakly increasing in the size of
INEFFICIENCIES IN NETWORKED MARKETS

the component and \( OII \) on the largest component bounds \( OII \) on the network. When \( \beta > 1 - \gamma \) the \( OII \) component bound is weakly decreasing in the size of the component for components with at least two trade links, smaller components are necessarily efficient. \( OII \) on the network is thus bounded from above by inefficiency on the smallest component with at least two trade links.

To show that \( OII \) is always less that 50% the bound derived above can be maximized over \( \beta \) and \( \gamma \):

\[
\hat{OII} \equiv \max_{\beta, \gamma} \left\{ h(\hat{K}, \gamma, \beta) \right\} = h(\hat{K}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}.
\]

This is independent of \( \hat{K} \).

It will now be shown that the derived bound is tight. Consider a component such that \( \min(m, n) = \hat{K} \) and \( \beta \geq \gamma \), such that \( f_1 > f_2 \). Set \( \alpha_{11} = \frac{c}{1 - \beta} \), \( \alpha_{k,k} = \frac{c}{1 - \beta} + \frac{(1 - \gamma)c}{\beta}, \forall k \in \{2, \hat{K}\} \), \( \alpha_{k-1,k} = \frac{-c}{1 - \beta} + \frac{(1 - \gamma)c}{\beta}, \forall k \in \{2, \ldots, \hat{K}\} \) and the value of all other links to zero. It can verified that for these potential gains form trade that the chain network of Figure 8 is stable and that over-investment inefficiency achieves the upper bound. Similar examples can be constructed for \( \beta \leq \gamma \), such that \( f_2 > f_1 \).

Part iii): As \( \gamma \in (0, 1) \) the component structure of any non-empty network with over-investment inefficiency must be of the form shown in Figure 8. On any such component there exists a seller with no outside trade partner who will have to pay to form a link their trade partner but receive none of the gains from trade as \( \beta = 1 \). Thus all such networks are unstable, the unique stable network is the empty network and there is never any over-investment.

Part iv): When \( \beta \geq \frac{1}{2} \) and \( \gamma = 0 \) or \( \beta \leq \frac{1}{2} \) and \( \gamma = 1 \) there exist potential gains form trade where the efficiency loss on all stable networks is 100%. This is shown for \( \beta \leq \frac{1}{2} \) and \( \gamma = 1 \) in Figure 9.

**Figure 9.** Unbounded over-investment inefficiency

The efficiency loss is: \( \frac{c}{2c+2\varepsilon-c} \to 1 \), as \( \varepsilon \to 0 \).

Part v): Let \( \beta < \frac{1}{2} \) and \( \gamma = 0 \). As \( \gamma = 0 \) only sellers can form outside option links. By Proposition 1 seller \( s_j \)'s incentives to form an outside option link to buyer \( b_i \) are given by \( \beta(\alpha_{\nu(j)}, \beta_{\nu(j)}) \leq \beta_{\nu(j)} \). Thus the maximum incentives to form outside option links can always occur on networks where sellers’ outside trade partners do not trade with anyone and \( \beta(\alpha_{\nu(j)}, \beta_{\nu(j)}) = \beta_{\nu(j)} \). Without loss of generality therefore, and as sellers will form at
most one outside option link, inefficiency on network components consisting of two buyers and one seller can be considered: If an over-investment inefficiency bound for all stable networks is found on these networks it will apply to all networks.

Consider then, without loss of generality, the following network component:

![Diagram of network components](image)

(\(A\) Potential links, \(B\) Efficient network, \(C\) Stable network)

**Figure 10.** Bounded over-investment inefficiency

On this network component for any efficiency loss due to over-investment in non-trade links \(s_1\) must form links to both \(b_1\) and \(b_2\). To be incentivised to do this \(s_1\)'s payoff on the network shown in Figure 10c must be greater than their payoff on the efficient network: \(\zeta + (1 - \beta)\varepsilon - 2c \geq (1 - \beta)(\zeta + \varepsilon) - c\). This holds if and only if \(\zeta\beta \geq \varepsilon\). Let \(c = \zeta\beta - \xi\), \(\xi \geq 0\), such that this constraint is satisfied. For this cost, and for the gains from trade above, \(OII = \frac{\zeta\beta - \xi}{\zeta(1 - \beta) + \varepsilon + \xi}\).

The efficiency loss on the worst stable network is maximized by setting \(\xi = \varepsilon = 0\). Thus over-investment inefficiency must be less than or equal to \(\frac{\beta}{1 - \beta}\) for any network with \(\beta < \frac{1}{2}\) and \(\gamma = 0\). This bound is achieved in the above example with \(\xi = \varepsilon = 0\). \(\square\)

B.4. **Proof of Lemma** Suppose cost sharing is endogenous. Then \(\forall \beta\) and for any set of \(m \geq 2\) buyers \(P\) and \(n \geq 2\) sellers \(Q\), there exist potential gains from trade \(a\) with 100% over-investment inefficiency.

**Proof.** This proof is by counter example. Consider gains from trade \(a\) with \(a_{11} = 1\), \(a_{12} = 1 - \varepsilon\), \(a_{ij} = 0\) for all other \(ij\). Let \(c = \frac{1}{2} - \varepsilon\) and consider first \(\beta \in [0, \frac{1}{2}]\). The efficient network, consisting of the single link \(l_{11}\), will not be stable for any such \(\beta\): \(b_1\) will increase their share of the surplus from trade with \(s_1\) by \((1 - \beta)a_{b_1s_2} = (1 - \beta)(1 - \varepsilon)\), at a cost of \(c = \frac{1}{2} - \varepsilon\), if they form the link to \(s_2\). This increases \(b_1\)'s net payoff if and only if \((1 - \beta)(1 - \varepsilon) > \frac{1}{2} - \varepsilon\), or equivalently \(\beta < \frac{1 - \varepsilon}{\frac{1}{2} - \varepsilon} \leq \frac{1}{2}\). It is then straightforward to show that the unique stable network is the complete network with \(b_1\) paying for the link \(b_1s_2\) and paying between \(\frac{1}{2} - (2 - \beta)\varepsilon\) and \(\frac{1}{2} - \varepsilon\) towards the formation of link \(l_{b_1s_1}\). The efficiency loss on all stable networks is therefore \(\frac{1 - \varepsilon}{\frac{1}{2} + \varepsilon}\) and as \(\varepsilon \to 0\) the efficiency loss goes to 100%. To show that the efficiency loss can go to 100% on all stable networks for \(\beta \in [\frac{1}{2}, 1]\) relabel \(b_1\) as \(s_1\), \(s_1\) as \(b_1\) and \(s_2\) as \(b_2\). \(\square\)

\(^{61}\) Varying \(\varepsilon \geq 0\) and \(\zeta > 0\) this example accounts for all possible two buyers one seller networks, for any \(c\).
REFERENCES


Reuters (05/29/08): “Nabucco pipeline cost rises to 7.9 bln euros,” http://uk.reuters.com/article/idUKL2911967120080529/20080529

——— (12/04/09): “Google’s AdMob takeover to spark mobile ad M&A wave,” http://www.reuters.com/article/idUSGEE5AM1NF20091204


Stählin, I. (1972): Bargaining theory. (Ekonomiska forskningsinstitutet vid Handelshögskolan i Stockholm (EFI)).


