Assortative Learning

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Abstract

Because of sorting, more skilled workers are more productive in higher type firms. They also learn at different rates about their productivity. Unlike most standard models of learning and continuous time games of incomplete monitoring, we propose a model without a deterministic outside option (absorbing state): agents always learn in all firms, albeit at different rates. As a result, we derive a new condition to ensure there is no deviation, in addition to smooth-pasting and value-matching. This no-deviation condition imposes restrictions on the second derivative of the value function. We offer a complete characterization of the equilibrium, including the distribution of beliefs and the fact that under supermodularity there is always Positive Assortative Matching, even if the speed of learning is higher in the low type firm. The model provides realistic predictions that existing learning models of the labor market cannot generate. For example, wage dispersion increases with tenure and an indifferent type will accept lower wages at a high type firm because learning is faster.


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1 Introduction

High ability workers sort into more productive jobs. Due to complementarities in production, their higher marginal product allows them to command higher wages. The Beckerian model of assortative matching is very well suited to explain those patterns of sorting. Unfortunately, it is mute on the issue of turnover of workers between different jobs. Instead, the Jovanovic (1979) learning model has long been the canonical framework for turnover in the labor market over the life cycle.\footnote{Of course, also the search model inherently exhibits turnover, but with observable types turnover is constant over the life cycle. Moscarini (2005) brings together search and learning in the Jovanovic framework, though without sorting nor ex ante heterogeneity.} Workers and firms learn about match-specific human capital and will tend to stay in a match if learning reveals the match is good. Experimentation tends to occur early on which leads to decreasing turnover over the life cycle. Because learning is about the match and not about the worker, there is neither worker heterogeneity nor sorting. In this paper, we offer a unified approach in which learning about the worker characteristics depends on which firm the worker is matched with, i.e., the speed of learning differs across firms. Our objective is double: 1. to provide a solution method for a market equilibrium in a continuous time economy with multiple learning opportunities (multi-armed bandit); and 2. to offer predictions about wages, sorting and turnover that can be reconciled with the stylized facts.

In the labor market, the learning experiences of workers are most likely to differ across different firms. Starting in a top law firm or a multinational will induce different paths of information revelation than working in a local family business. The worker now faces a trade-off between different experimentation experiences: take a lower wage at a high productivity firm where information may be revealed at a different rate or accept higher wage and learn more slowly.

Modeling the labor market as a multi-armed bandit problem and solving it is challenging. Most existing learning models and continuous time games are tractable because they are essentially one-armed bandit problems with a fixed outside option that acts as an absorbing state. One-armed bandit problems typically have attractive properties, including reservation strategies. Instead, multi-armed bandits in general do not have reservation strategies even if the learning rate is the same across firms.\footnote{See for example Chernoff (1968) and Rosenberg et al. (2007).} But our labor market is not exactly identical to the canonical bandit problem. There are a continuum of experimenters, and as a result of two-sided heterogeneity, deviations and off-equilibrium path beliefs non-trivially affect equilibrium. And because of competitive wage determination, the payoffs are endogenous. In addition, because workers learn about general human capital instead of match-specific human capital, the arms are positively correlated.

We find that it is the combination of competitive wage determination (endogenous payoffs) and the incentives needed to avoid a deviation that give rise to a new condition which we call the no-deviation condition. This condition must be satisfied in addition to the common equilibrium...
conditions of value-matching and smooth-pasting. The no-deviation condition can be interpreted as the continuous time version of the one-shot deviation principle. In other words, there is no assumption of commitment on the part of the firm and the equilibrium wage must be self-enforcing.\(^3\) We prove that the no-deviation condition implies that the second derivative at the cut-off belief is the same for the value functions of the high type as well as the low type firms. Recall that value matching requires that at the cut-off the value functions take the same value, the smooth-pasting condition requires that the first derivative is the same, and now the no-deviation requires equal second derivatives as well.

We show that supermodularity of the production technology is a necessary and sufficient condition for positive assortative matching, and that the equilibrium allocation is unique. Those workers with the highest beliefs about their ability will in equilibrium sort into those firms that are most productive. Moreover, we can analytically solve for the equilibrium allocation in terms of the cut-off belief, and we derive in closed form the stationary distribution of beliefs.

While in most of the analysis we consider common noise across firms, it turns out that the sorting result holds for different learning rates (noise) across firms, even if the rate of learning is slower in the high type firm. It is easy to see that with supermodularity and a learning rate no smaller in high types firms there will be positive sorting. The high type firm is both superior in the signal and in the noise. But if high type firms learn at a sufficiently slower rate (the noise is sufficiently high), then the signal-to-noise ratio in the high type firm may well be lower. The reason why this nonetheless does not affect the learning is because the value of learning also depends on the degree of convexity of the value function (from Ito’s Lemma), in addition to the signal-to-noise ratio. But by the no-deviation condition, at the cut-off belief, the degree of convexity is the same in both firms and therefore the equilibrium value of learning is the same, no matter the difference in signal-to-noise ratios. Key here is that wages are endogenous and determined competitively. That is why this property does not hold in the canonical multi-armed bandit problem.

The technical contribution of the no-deviation condition allows us to actually solve the model. We can now employ the framework and generate realistic predictions about labor market variables over the life cycle. Like in the existing learning models, turnover decreases with tenure. Unlike most existing models, we can also explain that the variance of wages of a cohort is maximized asymptotically. This can provide a rationale for the fact that the variance of wages increases over the life cycle. Like existing models, the signal precision for a given type increases – eventually the type becomes exactly known – and as a result, the variance conditional on a type goes to zero. However, because of sorting the variance between types is maximized. With a fixed outside option, the second source of variance is absent and existing models find decreasing variance in wages over the life cycle. There is conclusive evidence that the life cycle variance of wages is maximized asymptotically.

\(^3\)This approach differs from Harris and Holmström (1982) whose wage setting in a learning model with risk aversion assumes commitment by the firms.
increasing and concave (see amongst others Heathcoate, Violante and Perri (2009)).

We can fully characterize the stationary distribution of posterior beliefs and therefore also of the distribution of wages. At the cut-off belief, there is a discontinuity in the wage schedule due to the fact that the worker is indifferent between a low wage and a high option value of learning in the high type firm and a high wage with a low option value.

The motivation of our analysis and the results are obviously closest related to the labor market learning literature (Jovanovic (1979, 1984), Harris and Holmström (1982) and Moscarini (2005)). Yet, there is a close relation to both the experimentation literature (Bolton and Harris (1999), Rady and Keller (1999), Rosenberg et al. (2007)) and the literature on continuous time games (Sannikov (2007, 2008), Faingold and Sannikov (2009)). Most models of learning have a finite set of players and have an absorbing state. Ours has a continuum of agents and there is learning in all states. Moreover, it is essentially a competitive model with equilibrium prices and therefore payoffs from learning are endogenous.

The idea of analyzing a matching model where the current allocation determines the future type is first explored in Anderson and Smith (2009). They predict that even under supermodularity, there will not be Positive Assortative Matching. However, our models are difficult to compare. They analyze a two-sided matching model of reputations with imperfect information about both matched types. Moreover, because the belief space is continuous, theirs is a model of two-sided matching from a continuum of types. At the same time, their time dimension is discrete. Here instead, we build on the labor learning literature: time is continuous and there is imperfect information on only one side of the market (the workers) and the type space of the firms is discrete. Finally, while the payoffs in their model are determined competitively, the wage depends on the realization of future values. As a result, there is an implicit assumption of commitment as in Harris and Holmström (1982) which is absent our model since we impose sequential rationality. Because we cannot analyze a continuous time model of learning with a continuum of types (agents would instantaneously switch partners), it is impossible to frame our results in their context.

2 The Model Economy

*Population of Firms and Workers.* The economy is populated by a unit measure of workers and a unit measure of firms. Both firms and workers are *ex ante* heterogeneous. The firm’s type $y \in H, L$ represents its productivity. The type $y$ is observable to all agents in the economy. The fraction of $H$ type firms is $\pi$ and all firms are infinitely lived. The worker ability $x \in \{H, L\}$ is not observable, both to firms and workers, i.e., information is symmetric.\(^4\) Nonetheless, both hold a common belief about the worker type, denoted by $p \in [0, 1]$. Upon entry, a newly born worker is of type $H$ with

\(^4\)This substantially simplifies the problem at hand. With private signals Cripps, Ely, Mailath and Samuelson (2008) show that with a finite signal space there will be common learning, but not necessarily with an infinite signal space as is the case in our model here.
probability $p_0$ and of type $L$ with probability $1 - p_0$. Workers die with exogenous probability $\delta$. New workers are born at the same rate.\(^5\)

**Preferences and Production.** Workers and firms are risk-neutral and discount future payoffs at rate $r > 0$. Utility is perfectly transferable. Output is produced in pairs of one worker and one firm $(x, y)$. Time is continuous. Positive output produced consists of a divisible consumption good and is denoted by $\mu_{xy}$. We assume that more able workers are more productive in any firm, $\mu_{Hy} \geq \mu_{Ly}, \forall y$ and refer to it as worker monotonicity. While it is often useful, we do not in general assume firm monotonicity, which would be $\mu_{xH} \geq \mu_{xL}, \forall x$. Strict supermodularity is defined in the usual way:

$$\mu_{HH} - \mu_{LH} > \mu_{HL} - \mu_{LL}, \quad (1)$$

and with the opposite sign for strict submodularity. In the entire page, we will refer to strict supermodularity when we just mention supermodularity, likewise for submodularity.

**Information.** Because worker ability is not observable to both the worker and the firm, parties face an information extraction problem. They observe noisy measure of productivity, denoted by $X_t$. Cumulative output is assumed to be a Brownian motion with drift $\mu_{xy}$ and common variance $\sigma^2$

$$X_t = \mu_{xy} t + \sigma Z_t \quad (2)$$

where $Z_t$ is a standard Wiener process and as a result, $X_t$ is normally distributed with mean $\mu_{xy} t$ and variance $\sigma^2 t$. By Girsanov’s Theorem the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are equivalent, that is, they have the same zero-probability events. Since the volatility of a continuous-time diffusion process is effectively observable, the worker’s type could be learned directly from the observed volatility if $\sigma$ depends on workers’ types.\(^6\)

**Equilibrium.** We consider a (stationary) competitive equilibrium in this economy. With two types of firms and a continuum of $p$’s in this market, take a competitive wage schedule $w_y(p)$ as given which specifies wage for every possible type $p$ worker working in firm $y$. Denote by $V_y$ the stationary discounted present value of the competitive profits for firm $y$. The flow profit can be written as $rV_y$.\(^7\) Now we are ready to define the notion of competitive equilibrium:

\(^5\)Without death, we know the posterior belief will converge with probability one to $p = 1$ or $p = 0$. Death here actually acts as a shuffling device to guarantee a non-trivial stationary distribution of posterior beliefs. Another modeling approach is to assume that such $p_0$ is drawn from some exogenous given distribution $f_0$. But as we will show later, this makes it quite difficult to solve the ergodic distribution.

\(^6\)However, we can allow $\sigma$ to be firm-specific. In section 5 we analyze the general case of firm-dependent $\sigma_y$.

\(^7\)Notice since there is no free entry, $V_y$ need not to be zero. We could model free entry as long as in equilibrium there is a non-degenerate distribution of firm types in the economy. We consider this does not add to the insights of our model.
Definition 1: In a (stationary) competitive equilibrium, there is a competitive wage schedule \( w_y(p) = \mu_y(p) - rV_y \), where \( \mu_y(p) = p\mu_H + (1-p)\mu_L \) denotes worker \( p \)'s expected productivity for firm \( y = H, L \) and worker \( p \) chooses the firm \( y \) with the highest discounted present value. The market clears such that the measure of workers working in the \( L \) firm is \( 1 - \pi \) and the measure of workers working in the \( H \) firm is \( \pi \).

We would like to point out several things about this definition. First, the definition of competitive equilibrium implies identical types will obtain the same payoff. A firm \( y \) earns the same flow profit for every \( p \). Our notion of competitive equilibrium puts restrictions on the off-equilibrium prices, as does the Beckerian definition of a matching equilibrium. In the current context this is comparable to the notion of subgame perfect equilibrium. Recall that subgame perfect equilibrium requires that agents behave optimally on any possible subgame. Similarly, here we require: Although type \( p \) worker is not employed by firm \( y \) on equilibrium path, the hypothetical wage is still \( w_y(p) = \mu_y(p) - rV_y \) to guarantee the firm cannot make or lose money if the employment suddenly happens. In the Appendix, we consider an alternative set of off-the-equilibrium path wages. Second, unlike Anderson and Smith (2008), our wage definition concerns a spot market wage. They parse the wage into a static wage plus a dynamic human capital effect. Instead, our spot wage approach imposes sequential rationality and therefore requires that the wage contract is self-enforcing. Finally, like all price taking economies, the wage schedule essentially transforms our problem into a decision problem for the workers.

3 Preliminaries

3.1 Benchmark: No Learning

Workers differ in the common beliefs \( p \) of being a high type. We shut down learning so that beliefs are invariant. This can be viewed as a special case of the learning model with the variance \( \sigma^2 \) going to infinity. We assume that there is no birth or death so we essentially have a static problem. Suppose without of generality that \( p \) is uniformly distributed on \([0,1]\). We continue to maintain the assumption that the worker does not know her true type or that she has no private information about it. Denote \( \mu_y(p) = p\mu_H + (1-p)\mu_L \) for \( y = H, L \) and \( r \) as the discount rate.

Under the above notion of competitive equilibrium, it is easy to verify the following claim (All of the results in this paper are in the sense of “almost surely” because we allow a zero measure of players to behave differently):

Claim 1: Under strict supermodularity, PAM is the unique (stationary) competitive equilibrium allocation: \( H \) firms match with workers \( p \in [1 - \pi, 1] \), \( L \) firms match with workers \( p \in [0, 1 - \pi] \). The opposite (NAM) holds under strict submodularity: \( H \) firms match with workers in \([0, \pi]\).
Since there is no learning, essentially this result is the similar to Becker’s (1973) result, but with uncertainty. Noteworthy about this Bayesian version of Becker is that even though for PAM there is supermodularity of the ex-post payoffs \( \mu_{HH} + \mu_{LL} > \mu_{HL} + \mu_{LH} \), there need not be monotonicity in expected payoffs, i.e., \( \mu_H (1 - \pi) \) may be smaller than \( \mu_L (1 - \pi) \). In fact, that will be reflected in the firm’s equilibrium payoffs: \( V_H \geq V_L \) if and only if \( \mu_H (1 - \pi) \geq \mu_L (1 - \pi) \).

As in Becker, the equilibrium allocation is unique, but there may be multiple splits of the surplus. In the case of PAM, we only require that \( w_H (1 - \pi) = w_L (1 - \pi) \). There are multiple equilibrium payoffs if the surplus of a match between \( L \) and \( p = 0 \) is positive. Instead, if \( \mu_L (0) = 0 \), there is a unique equilibrium payoff.

### 3.2 Belief Updating

In the presence of learning we can now derive the beliefs and subsequently the value functions. A sufficient statistic for output history, which determines the future prospects of a match, thus also the natural state variable of the bargaining game, is the posterior belief \( p_t \) that the worker had a high productivity. Now, it is well-known that we can have the following important result: conditional on the output process \( (X_t)_{t \geq 0} \), \( (p_t)_{t \geq 0} \) is a martingale diffusion process. Moreover, this process could be represented as a Brownian motion.

Based on the framework of our model, denote \( s_y = (\mu_{Hy} - \mu_{Ly})/\sigma \), \( y = H, L \), \( \Sigma_y(p) = \frac{1}{2} p^2 (1 - p)^2 s_y^2 \) and then we can get:

**Lemma 1 (Belief Consistency)** Consider any worker who works for firm \( y \) between \( t_0 \) and \( t_1 \). Given a prior \( p_{t_0} \in (0, 1) \), the posterior belief \( (p_t)_{t_0 < t \leq t_1} \) is consistent with the output process \( (X_{y,t})_{t_0 < t \leq t_1} \) if and only if it satisfies

\[
dp_t = p_t (1 - p_t) s_y d\tilde{Z}_{y,t}
\]

where

\[
d\tilde{Z}_{y,t} = \frac{1}{\sigma} [dX_{y,t} - (p_t \mu_{Hy} + (1 - p_t) \mu_{Ly}) dt].
\]

The proof of this Lemma is in Faingold and Sannikov (2007) or Daley and Green (2008). The basic idea behind the proof is a combination of Bayes’ rule and Ito’s lemma. Given the period \( t \) posterior belief \( p_t \) and \( dX_t \), we know the posterior belief at period \( t + dt \) is:

\[
p_{t+dt} = \frac{p_t \exp\left\{-\frac{[dX_t - \mu_{Hy} dt]^2}{2\sigma^2 dt}\right\}}{p_t \exp\left\{-\frac{[dX_{t_0} - \mu_{Hy} dt]^2}{2\sigma^2 dt}\right\} + (1 - p_t) \exp\left\{-\frac{[dX_{t_0} - \mu_{Ly} dt]^2}{2\sigma^2 dt}\right\}}.
\]

\( \ast \)And there is limited liability, i.e., workers and firms cannot receive negative payoffs.
Hence,
\[
dp_t = pt_{t+dt} - pt = pt(1 - pt) \exp\left\{ \frac{-|dX_{t} - \mu_{Hy} dt|^2}{2\sigma^2 dt} \right\} - \exp\left\{ \frac{-|dX_{t} - \mu_{Ly} dt|^2}{2\sigma^2 dt} \right\}.
\]

Apply Ito’s Lemma and we obtain the above result.

Lemma 1 establishes that \(dp\) depends on three elements: \(p(1-p)\), which peaks at 1/2; the signal-to-noise ratio of output, \(s_y = (\mu_{Hy} - \mu_{Ly})/\sigma\) and \(d\bar{Z}_y\), the normalized difference between realized and unconditionally expected flow output, which is a standard Wiener process with respect to the filtration \(\{X_{y,t}\}\). Obviously, beliefs move faster the more uncertain about worker’s quality (\(p\) close to 1/2); the less variation in the output process (smaller \(\sigma\)) and the larger productivity difference (higher \(\mu_{Hy} - \mu_{Ly}\)).

Now the learning consideration will change the results. Moreover, the supermodularity not only affects the value of the output created as in the standard Beckerian model, but it also changes the speed of learning. For example, under supermodularity \((\mu_{HH} - \mu_{HL} > \mu_{LH} - \mu_{LL})\), the learning speed is faster in high type firm, which is especially significant for \(p\) close to 1/2. Intuitively speaking, learning makes it more attractive to match with a high type firm even though it is better for her to match with a low type firm without learning.

### 3.3 Value Functions

Consider any interval for the posterior belief \(p \in [p_1, p_2]\) where the worker accepts the offer from a type \(y\) firm, then the value function is given by\(^9\):

\[
rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W''_y(p) - \delta W_y(p),
\]

from Ito’s Lemma. The term \(\mu_y(p) - V_y\) is equal to the flow wage payoff and corresponds to the deterministic component of the diffusion \(X_{y,t}\), and the term \(\Sigma_y(p)W''_y(p)\) is the second-order term from the transformation \(W\) of the diffusion process \(X_{y,t}\). All higher-order terms vanish as the time interval shrinks to zero. The general solution to this differential equation is:

\[
W_y(p) = \frac{\mu_y(p) - V_y}{r + \delta} + k_{y1}p^{1-\alpha_y}(1 - p)^{\alpha_y} + k_{y2}p^{\alpha_y}(1 - p)^{1-\alpha_y},
\]

where

\[
\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r + \delta)}{s^2_y}} \geq 1.
\]

First notice that the boundedness of the value function implies that if 0 is included in the

\(^9\)Note that we critically need the assumption that the worker doesn’t know any private information about his type. If this assumption is violated, the worker’s value functions could not be written like this.
domain, then \( k_y1 = 0 \) and if 1 is included in the domain, then \( k_y2 = 0 \). Second, \( \Sigma_y(p)W_y''(p) \) is the value of learning and this is an option value in the sense that the worker has the choice to change his job as he learns his type \( p \). It is easy to verify that this value is zero if the worker never changes his job.\(^{10}\) From the Martingale property of the Brownian motion, at any \( p \) the expected value of \( p \) in the next time interval is equal to \( p \). There is as much good news as bad news to be expected in the next period. It is the option value of switching to a more suitable match that generates the value of learning.

### 4 Analysis and Results

#### 4.1 Characterization of the Equilibrium Allocation

Now consider any candidate equilibrium where a type \( p \) worker is indifferent between matching with either firm \( y \). Then equilibrium requires the equal-value condition (the worker gets the same value at the cutoff) and the smooth-pasting condition (the marginal of both value functions is identical). For example, if for \( p \in [p_1, p_2) \), the worker works in the low type firm and for \( p \in [p_2, p_3) \), the worker works in the high type firm, then we must have:\(^{11}\)

\[
W_L(p_2) = W_H(p_2) \quad \text{and} \quad W_L'(p_2) = W_H'(p_2).
\]  

It is important to point out that both the equal value-condition and the smooth-pasting condition are on-equilibrium path conditions. It has nothing to do with the off-equilibrium path (i.e., instead of accepting offers from low type firms, worker with \( p \in [p_1, p_2) \) are tempted to accept offers from high high type firms).

In the following lemmas we characterize the value functions establishing convexity and monotonicity:

**Lemma 2** The equilibrium value functions \( W_y \) are strictly convex for \( p \in (0, 1) \).

**Proof.** In Appendix. \( \blacksquare \)

The intuition for this Lemma is the following. Since the measure of both types of firms are strictly positive, market clearing requires that workers with some \( p \)'s will be employed by high type firms while workers with other \( p \)'s will be employed by low type firms. This implies that some worker has to change jobs at some point and the option value of learning is non-zero. Hence we have \( W_y''(p) > 0 \), for all \( p \in (0, 1) \). On the other hand, when \( p = 0 \) or 1, the posterior belief will

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\(^{10}\)In that case, \( p \) can take both the values 0 and 1. So the boundedness of the value function requires that both \( k_y1 \) and \( k_y2 \) are zero and hence \( W_y''(p) = 0 \) for every \( p \).

\(^{11}\)We slightly abuse notation here since \( W_L \) is not defined on \( p_2 \). A more precise way of writing the equations is \( W_L(p_2+) = W_H(p_2) \) and \( W_L'(p_2+) = W_H'(p_2) \). In what follows, we will continue to use the expression in the text in order to economize on notation.
always stay at 0 or 1 by Bayes’ rule such that learning never happens. It is easy to verify that
\( W''_y(p) = 0 \) for \( p = 0 \) or 1.

Given the strict convexity of equilibrium value functions and the smooth pasting condition, we
can immediately derive the following Lemma:

**Lemma 3** The equilibrium value functions \( W_y \) are strictly increasing.

**Proof.** In Appendix. ■

One important implication is that if we define \( W(p) \) as the envelope of all equilibrium value
functions \( W_y(p) \). Then this envelope function \( W(p) \) is continuous, strictly increasing and strictly
convex for \( p \in (0, 1) \). Suppose workers with \( p \in [0, \bar{p}] \) are employed by type \( y \) firm and workers with
\( p \in (\bar{p}, 1] \) are employed by type \(-y\) firm. Then we should have:
\[
W'_y(0) = \frac{\mu_{H,-y} - \mu_{L,-y}}{r + \delta} < W'_{-y}(1) = \frac{\mu_{H,-y} - \mu_{L,-y}}{r + \delta}.
\]

This gives us another result:

**Lemma 4** Under supermodularity, in any equilibrium \( p = 0 \) worker matches with \( L \) firm; \( p = 1 \)
worker matches with \( H \) firm. The opposite under strict submodularity. Moreover,
\[
min(\Delta_H, \Delta_L) \over r + \delta < W'(p) < max(\Delta_H, \Delta_L) \over r + \delta,
\]

where \( \Delta_H = \mu_{HH} - \mu_{LH} \) and \( \Delta_L = \mu_{HL} - \mu_{LL} \).

The properties derived above are mainly concerned with on-equilibrium path behavior. We
also need to specify what happens in the event of deviations and behavior is off-equilibrium path.
We consider the equivalence of a one-shot deviation in continuous time because we think of the
continuum as an idealization of discrete time. This amounts to a worker playing the deviant
action over an interval \([t, t + dt]\) according to the belief \( p \) at time \( t \), and considering the limit as
\( dt \to 0 \).\(^{12}\) This is very important because it allows us to derive the value function for deviation.
On the contrary, if the deviation only takes place at a single time point \( t \), then the value function
for deviation is essentially the same as the one without deviation because no information will be
extracted from just a single time point.

The next Lemma establishes that if we consider off-the-equilibrium path deviations, we actually
need one additional condition, which we call the no-deviation condition.

**Lemma 5** To deter possible deviations, a necessary condition is:
\[
W''_H(p) = W''_L(p) \quad \text{(No-deviation condition)}
\]

for any possible cutoff \( p \).

\(^{12}\)This notion is also implicitly used in Sannikov (2007, Proposition 2), and in Cohen and Solan (2009) who consider
deviations from Markovian strategies in bandit problems.
Proof. Without loss of generality, we assume that on equilibrium path, worker with $p > p$ accepts offers from high type firms and worker with $p < p$ accepts offers from low type firms. Consider one possible one-shot deviation: a $p > p$ worker matches with a low type firm for $dt$ and then switch back. Then the new value function is defined as:

$$\tilde{W}_L(p) = w_L(p)dt + e^{-(r+\delta)dt}W_H(p + dp)$$

where $dp = p(1-p)s_Ld\bar{Z}$. (7)

Apply Ito’s Lemma and we get:

$$\tilde{W}_L(p) = w_L(p)dt + e^{-(r+\delta)dt}[W_H(p) + \sum L(p)W_H''(p)dt].$$ (8)

This implies:

$$\lim_{dt \to 0} \frac{\tilde{W}_L(p) - W_H(p)}{dt} = w_L(p) - w_H(p) + [\sum L(p) - \sum H(p)]W_H''(p).$$ (9)

The RHS of the above equation must be smaller than zero for any $p > p$. Let $p \to p$ and we have:

$$w_L(p) - w_H(p) + [\sum L(p) - \sum H(p)]W_H''(p) \leq 0$$

$$\Rightarrow w_L(p) + \sum L(p)W_L''(p) - (w_H(p) + \sum H(p)W_H''(p)) + (W_H''(p) - W_L''(p))\sum L(p) \leq 0$$

$$\Rightarrow W_H''(p) \leq W_L''(p).$$ (10)

Similarly, we can consider another possible one-shot deviation: a $p < p$ worker matches with a high type firm for $dt$ and then switch back. Then the new value function is defined as:

$$\tilde{W}_H(p) = w_H(p)dt + e^{-(r+\delta)dt}W_L(p + dp), \text{ where } dp = p(1-p)s_Hd\bar{Z}.$$ (11)

Apply Ito’s Lemma once again and we get:

$$\tilde{W}_H(p) = w_H(p)dt + e^{-(r+\delta)dt}[W_L(p) + \sum H(p)W_L''(p)dt].$$ (12)

taking the limit

$$\lim_{dt \to 0} \frac{\tilde{W}_H(p) - W_L(p)}{dt} = w_H(p) - w_L(p) + [\sum H(p) - \sum L(p)]W_L''(p) < 0$$

for $p < p$. Therefore as $p$ goes to $p$, we should have:

$$w_H(p) - w_L(p) + [\sum H(p) - \sum L(p)]W_L''(p) \leq 0 \Rightarrow W_H''(p) \geq W_L''(p).$$ (13)

(10) and (13) imply that $W_H''(p) = W_L''(p)$. ■
This no-deviation condition is quite unique for the two-armed bandit problem. This condition is absent in an one-armed bandit problem. Most of the models in the literature on continuous time learning models (Jovanovic (1979) and Moscarini (2005)) and continuous time games (see amongst others, Sannikov (2009)) are essentially investigating a one-armed bandit problem. There, we can directly look at equilibria in cutoff strategies, a theoretical foundation for which is given by Rosenberg et al. (2007). In the one-armed bandit problems, the safe arm essentially is an absorbing state so we only need to worry about the potential deviation from the risky arm to the safe arm. Then the no-deviation condition becomes \( W''_H(p) \geq W''_L(p) = 0 \) but this is already implied by the convexity property.

But in our two-armed bandit problem, we first need to answer the question whether there exist non-cutoff stationary equilibria, i.e., a worker with \( p \in [p_1, p_2] \) accepts the offer from a high type firm, with \( p \in [p_2, p_3] \) accepts the offer from a low type firm and with \( p \in [p_3, p_4] \) accepts the offer from a high type firm again. Surprisingly, Lemmas 2–5 imply that all stationary competitive equilibria must be in cutoff strategies.

**Theorem 1** PAM is the unique stationary competitive equilibrium allocation under strict supermodularity. Likewise for NAM under strict submodularity.

To prove this theorem, we only need to prove the following Claim:

**Claim 2** Under strict supermodularity, it is impossible to have \( p_1 < p_2 \) and equilibrium value functions \( W_H \) (for \( p \in [p_1, p_2] \)), \( W_{L1} \) (for \( p < p_1 \)), \( W_{L2} \) (for \( p > p_2 \)) such that:

\[
W_H(p_1) = W_{L1}(p_1) \quad \text{and} \quad W''_H(p_1) = W''_{L1}(p_1)
\]

\[
W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W''_H(p_2) = W''_{L2}(p_2)
\]

are satisfied simultaneously.

Under strict submodularity, it is impossible to have \( p_1 < p_2 \) and equilibrium value functions \( W_L \) (for \( p \in [p_1, p_2] \)), \( W_{H1} \) (for \( p < p_1 \)), \( W_{H2} \) (for \( p > p_2 \)) such that:

\[
W_L(p_1) = W_{H1}(p_1) \quad \text{and} \quad W''_L(p_1) = W''_{H1}(p_1)
\]

\[
W_L(p_1) = W_{H2}(p_2) \quad \text{and} \quad W''_L(p_2) = W''_{H2}(p_2)
\]

are satisfied simultaneously.

**Proof.** In Appendix. ■

---

\(^{13}\)For example, in our model assume \( \mu_{HL} = \mu_{LL} \) and the return in the low type firm is deterministic.

\(^{14}\)In a model of option pricing by Dumas (1991), there does exist a condition on the second derivative called the “super contact” condition. It arises as the optimal solution to the option pricing problem with proportional cost.
Given the above claim, it is easy to prove the theorem:

**Proof.** Under supermodularity, by Lemma 5, worker with sufficiently low \( p \)'s will accept low type firm’s offer and worker with sufficiently high \( p \)'s will accept high type firm’s offer. But 2 implies it is impossible to have worker first accept low type firm’s offer, then accept high type firm’s offer and finally accept low type firm’s offer again. Hence, we must have some cutoff \( p \) such that \( p < p \) will accept low type firm’s offer and \( p > p \) will accept high type firm’s offer. This is exactly a PAM allocation. Use the same logic and you can see NAM is the only possible stationary competitive equilibrium allocation under strict submodularity.

Before we turn to the equilibrium distribution, we show that the no-deviation condition in Lemma 5 is not just necessary but also sufficient under supermodularity:

**Lemma 6** Under strict supermodularity, \( W_H''(p) = W_L''(p) \) implies that no deviation would happen for the PAM equilibrium.

**Proof.** In Appendix.

### 4.2 The Equilibrium Distribution

The previous section shows that under strict supermodularity (submodularity), PAM (NAM) is the unique stationary competitive equilibrium allocation. We still need to construct such an equilibrium. To do that, we assume supermodularity and worker and firm monotonicity: \( \mu_{HH} > \mu_{HL} \) and \( \mu_{LH} > \mu_{LL} \). Now consider a strictly positive assortative matching equilibrium such that workers with beliefs less than \( p \) will choose L firms and workers with beliefs higher than \( p \) will choose H firms. From equation (4) we hence have \( k_{LL} = 0 \) and \( k_{L2} > 0 \) for \( y = L \) and \( k_{H2} = 0 \) and \( k_{H1} > 0 \) for \( y = H \). Let \( k_L = k_{L2} \), \( k_H = k_{H1} \) and worker’s value functions become:

\[
W_L(p) = \frac{w_L(p)}{r + \delta} + k_L p^{\alpha_L} (1 - p)^{1 - \alpha_L} \tag{14}
\]

and

\[
W_H(p) = \frac{w_H(p)}{r + \delta} + k_H p^{1 - \alpha_H} (1 - p)^{\alpha_H}, \tag{15}
\]

where

\[
\alpha_y = \frac{1}{2} + \frac{1}{4} + \frac{2(r + \delta)}{s_y^2} \geq 1.
\]

To discuss market clearing conditions, we need to consider the ergodic distribution of \( p \)'s. From the Fokker-Planck (Kolmogorov forward) equation, the stationary and ergodic density \( f_y \) should satisfy the following differential equation:

\[^{15}\text{Monotonicity is just to help us find one particular way to divide the surplus. The whole construction of equilibrium also goes through if we do not make this assumption.}\]
\[ 0 = \frac{df_y(p)}{dt} = \frac{d^2}{dp^2} \left[ \Sigma_y(p) f_y(p) \right] - \delta f_y(p). \]  
(16)

The general solution to this differential equation is (see also Moscarini (2005)):

\[ f_y(p) = \left[ f_{y0} p^{\gamma_{y1}} (1 - p)^{\gamma_{y2}} + f_{y1} (1 - p)^{\gamma_{y1} p^{\gamma_{y2}}} \right] \]  
(17)

where

\[ \gamma_{y1} = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{2\delta}{s^2_y}} > -1 \]

and

\[ \gamma_{y2} = -\frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2\delta}{s^2_y}} < -2. \]

First, the integrability of \( f_y \) requires that \( f_{y1} = 0 \) if 0 is included in the domain and \( f_{y0} = 0 \) if 1 is included in the domain. Second, the Fokker-Planck (Kolmogorov forward) equation is only valid for \( p \neq p_0 \). Since there is a flow in of new workers, for \( p = p_0 \) we should have a kink in the density function. This also raises the issue of relative position between \( p_0 \) and \( \underline{p} \). We first consider the case where \( \underline{p} < p_0 \). We then derive in abbreviated format the result when \( \underline{p} > p_0 \).

Given any \( p_0 \in (0, 1) \), if \( \underline{p} < p_0 \), then the density functions are:

\[ f_H(p) = \left[ f_{H0} p^{\gamma_{H1}} (1 - p)^{\gamma_{H2}} + f_{H1} (1 - p)^{\gamma_{H1} p^{\gamma_{H2}}} \right] (\underline{p} < p \leq p_0) + f_{H2} (1 - p)^{\gamma_{H1} p^{\gamma_{H2}}}(p > p_0) \]  
(18)

and

\[ f_L(p) = f_{L0} p^{\gamma_{L1}} (1 - p)^{\gamma_{L2}}. \]  
(19)

The density functions are subject to the following boundary conditions. First, once the posterior belief reaches the equilibrium separation point \( \underline{p} \), we should have the cutoff condition:

\[ \Sigma_H(\underline{p}+) f_H(\underline{p}+) = \Sigma_L(\underline{p} -) f_L(\underline{p} -). \]  
(20)

This condition guarantees that the flow of agents who cross \( \underline{p} \) from below is equal to the flow of agents who cross from above. The implication is that since the speed from above \( \Sigma_H \) is larger than \( \Sigma_L \), the densities satisfy \( f_H(\underline{p}+) < f_L(\underline{p} -) \). It is worth comparing this condition to the standard condition when there is an absorbing state (Cox-Miller (1965), Dixit (1993), and Moscarini (2005)). In the case with only one brownian motion and an absorbing state, what is required is \( \Sigma(p+) f(p+) = 0 \) because the probability of obsorption in a time interval \( dt \) must equal speed of the flow in of the brownian motion which is proportional to \( \sqrt{dt} \) (see Cox and Miller (1965, p.220)). Therefore the speed must be zero near the absorbing boundary.
Second, total flows in and out of the low type firms must balance:

$$\Sigma_H(p+)f'_H(p+) = \delta\pi + \Sigma_L(p-)f'_L(p-).$$

The left-hand side of the above equation is the total inflows into low type firms, which are workers who reach $p$ and drop from the high type firms. The right-hand side of the above equation is the total outflows from the low type firms, which include workers who reach $p$ and transfer to high type firms and workers who are hit by the death shock.

Third, the density function has to be continuous at $p_0$ (see also Bertola and Caballero (1990)):

$$f_H(p_0-) = f_H(p_0+).$$

This condition is arbitrary in the sense that if it is not satisfied, the distribution will still be ergodic. It is customary to impose this condition as it approximates entry from a non-degenerate distribution instead of entry of identical types $p_0$.

Finally, we have market clearing condition:

$$\int_{\underline{p}}^{1} f_H(p)dp = \pi \quad \text{and} \quad \int_{0}^{\underline{p}} f_L(p)dp = 1 - \pi.$$  

In summary, when $\underline{p} < p_0$, the equilibrium is characterized by a system of eight equations with nine unknowns $(V_L, V_H, k_L, k_H, p, f_{H0}, f_{H1}, f_{H2}, f_{L0})$:\footnote{\textsuperscript{16}Observe that with more unknowns than variables, the solution to our system is indeterminate. In fact, there are potentially a continuum of wages that can be supported in equilibrium, though the allocation will be unique. This indeterminacy is as in Becker: the allocation is unique, but there may be multiple ways to split the surplus. In all that follows, when we use the term uniqueness of equilibrium, we refer to the allocation, not to the wages.}

\begin{align*}
W_H(p) &= W_L(p) & \text{(Equal value condition)} \\
W'_H(p) &= W'_L(p) & \text{(Smooth pasting condition)} \\
W''_H(p) &= W''_L(p) & \text{(No deviation condition)} \\
\Sigma_H(p+) f_H(p+) &= \Sigma_L(p-) f_L(p-) & \text{(Boundary condition)} \\
\int_{\underline{p}}^{1} f_H(p)dp &= \pi & \text{(Market clearing $H$)} \\
\int_{0}^{\underline{p}} f_L(p)dp &= 1 - \pi & \text{(Market clearing $L$)} \\
\Sigma_H(p+) f'_H(p+) &= \delta(1 - \pi) + \Sigma_L(p-) f'_L(p-) & \text{(Flow equation at $p$)} \\
f_H(p_0-) &= f_H(p_0+) & \text{(Continuous density at $p_0$)}
\end{align*}
words, the procedure of solving this system of equation could be: first we solve \( p \) jointly with \( f_{H0}, f_{H1}, f_{H2}, f_{L0} \) from Equations (24)–(28) and then we plug \( p \) into Equations (21)–(23) to pin down other unknowns.

**Proposition 1** Equations (24)-(28) imply \( p < p_0 \) if and only if:

\[
\left( \frac{p_0}{1-p_0} \right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s^2_H}{\delta/s^2_L} \int_{p_0}^{1} p^\gamma_{H1} (1-p)^{\gamma_{H1}} dp - \int_{0}^{p_0} p^\gamma_{L1} (1-p)^{\gamma_{L2}} dp < \frac{\pi}{1-\pi}.
\]  

Moreover, if such \( p \) exists, it must be unique.

**Proof.** In Appendix. ■

The proof of Proposition 1 is quite straightforward. The idea of the proof is the following: since we have 5 equations with five unknowns, we can first express \( f_{H0}, f_{H1}, f_{H2}, f_{L0} \) as a function of \( p \) and then use the last equation to pin down \( p \).

The existence and uniqueness of the solution to the system require that \( f_{H0}, f_{H1}, f_{H2}, f_{L0} \) change monotonically with \( p \). Fortunately, this is the case as shown in the appendix. First, from Equation (26), we find \( f_{L0} \) is decreasing in \( p \). Second, Equations (24) and (27) imply that \( f_{H0}, f_{H1} \) are also decreasing in \( p \). Finally, Equation (28) tells us that \( f_{H2} \) is also decreasing in \( p \) given \( f_{H0}, f_{H1} \) are decreasing in \( p \). The monotonicity guarantees that if a solution exists, it must be unique. Furthermore, it enables us to only investigate the boundaries when determining a solution exists.

This gives us Equation (29) given in the Proposition.

In the second case, \( p \geq p_0 \). Given any \( p_0 \in (0,1) \), if \( p \geq p_0 \), then the density functions are:

\[
f_L(p) = f_{L0} p^\gamma_{L1} (1-p)^{\gamma_{L2}} (p < p_0) + [f_{L1} p^\gamma_{L1} (1-p)^{\gamma_{L2}} + f_{L2} (1-p)^\gamma_{L1} p^\gamma_{L2}](p_0 \leq p \leq p) \quad (30)
\]

and

\[
f_H(p) = f_{H0} (1-p)^{\gamma_{H1}} p^\gamma_{H2}. \quad (31)
\]
Then the system of equations to determine the equilibrium is:

\[
\begin{align*}
W_H(p) &= W_L(p) \quad \text{(Equal-value)} \quad (32) \\
W'_H(p) &= W'_L(p) \quad \text{(Smooth-pasting)} \quad (33) \\
W''_H(p) &= W''_L(p) \quad \text{(No-deviation)} \quad (34) \\

\Sigma_H(p) f_H(p) &= \Sigma_L(p) f_L(p) \quad \text{(Boundary condition)} \quad (35) \\
\int_{p_0}^{1} f_H(p) \, dp &= \pi \quad \text{(Market clearing H)} \quad (36) \\
\int_{0}^{p} f_L(p) \, dp &= 1 - \pi \quad \text{(Market clearing L)} \quad (37) \\
\Sigma_L(p) f'_L(p) &= \delta \pi + \Sigma_H(p) f'_H(p) \quad \text{(Flow equation at p)} \quad (38) \\
f_L(p_0) &= f_L(p_0^+) \quad \text{(Continuous density at p_0)} \quad (39)
\end{align*}
\]

We can now prove the following Proposition, the counterpart to Proposition 1:

**Proposition 2** Equations (35)-(39) imply \( p \geq p_0 \) if and only if:

\[
\left( \frac{p_0}{1 - p_0} \right)^{\gamma_{H1} - \gamma_{L2}} \frac{\delta}{s_H^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} \, dp}{\int_{0}^{p_0} p^{\gamma_{L2}}(1-p)^{\gamma_{L1}} \, dp} \geq \frac{\pi}{1 - \pi}. \quad (40)
\]

Moreover, if such \( p \) exists, it must be unique.

The idea for the proof of Proposition 2 is exactly the same as that for the proof of Proposition 1 and the proof is also shown in the appendix. Propositions 1 and 2 together provide a very nice existence and uniqueness result:

**Theorem 2** Under strict supermodularity, for any pair \((p_0, \pi) \in (0,1)^2\), there exists a unique PAM cutoff \( p \). Moreover, \( p < p_0 \) if and only if:

\[
\left( \frac{p_0}{1 - p_0} \right)^{\gamma_{H1} - \gamma_{L2}} \frac{\delta}{s_H^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} \, dp}{\int_{0}^{p_0} p^{\gamma_{L2}}(1-p)^{\gamma_{L1}} \, dp} < \frac{\pi}{1 - \pi}. \quad (41)
\]

One of the nice properties about Equation (41) is that the whole equation only depends on \( p_0 \), \( \pi \), \( \delta/s_H^2 \) and \( \delta/s_L^2 \). This also gives us a feasible way to compute \( p_0 \). Given \( p_0 \), \( \pi \), \( \delta/s_H^2 \) and \( \delta/s_L^2 \), we first need to decide the sign of

\[
\left( \frac{p_0}{1 - p_0} \right)^{\gamma_{H1} - \gamma_{L2}} \frac{\delta}{s_H^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}} \, dp}{\int_{0}^{p_0} p^{\gamma_{L2}}(1-p)^{\gamma_{L1}} \, dp} - \frac{\pi}{1 - \pi}.
\]

If this sign is negative, then we know that \( p \) is smaller than \( p_0 \) and we can use the system of equations in Case 1 to find out \( p \). On the contrary, if this sign is not negative, then we know that
$p$ is larger than $p_0$ and we can use the system of equations in Case 2 to find out $p$. Therefore, we can get a very convenient way to determine the equilibrium cutoff numerically.

Before presenting the numerical results, we have a simple theoretical comparative static result:

**Corollary 1** $p$ is strictly increasing in $p_0$ and decreasing in $\pi$.

This corollary is proved in the appendix. But the intuition is quite clear: $\pi$ decreases means there are more low type firms in the economy and hence $p$ has to increase to make sure that more workers are matched with low type firms; $p_0$ increases means the overall quality of the workers is becoming better in the economy and $p$ has to increase to make sure that the low type firms can also be matched with better workers.

Mathematically, the relationships between $p$ and $\delta/s^2_H$, $\delta/s^2_L$ are not clear. But intuitively speaking, as $s_L$ increases, the degree of supermodularity will decrease while the speed of learning in low type firms will increase. Both of them make the low type firms more attractive such that $p$ should increase in $s_L$. On the other hand, as $s_H$ increases, both the degree of supermodularity and the speed of learning in high type firms will increase, which will lead to a decrease of $p$.

Figure 4.2 plots the value of $p$ as a function of $s_L$, $\pi$, $p_0$, for the case of PAM and with parameter values: $s_H = 0.15$, $s_L = 0.05$, $p_0 = 0.5$, $\pi = 0.5$, $\delta = 0.01$. 
4.3 Equilibrium Analysis: Value Functions

Theorem 2 implies that under supermodularity, the PAM cutoff $p$ can be uniquely determined. But given this $p$, we still have the following conditions to satisfy:

\[ W_H(p) = W_L(p) \]  \hspace{1cm} \text{(Equal-value condition)} \quad (42)

\[ W_H'(p) = W_L'(p) \]  \hspace{1cm} \text{(Smooth-pasting condition)} \quad (43)

\[ W_H''(p) = W_L''(p) \]  \hspace{1cm} \text{(No-deviation condition)} \quad (44)

The last problem is that Equations (42)-(44) are three equations for four unknowns. The equilibrium is indeterminate in the sense that although the allocation $p$ is unique, there could be multiple ways to divide the surplus. One way to get rid of this problem is to assume monotonicity and make $\mu_{LL} = 0$. Then non-negative wage requires that $w_L(0)$ has to be zero and hence we have $V_L = 0$. Equations (42)-(44) then become:

\[
\frac{\mu_L(p)}{r + \delta} + k_L p^\alpha (1 - p)^{1 - \alpha} = \frac{\mu_H(p) - r V_H}{r + \delta} + k_H p^{1 - \alpha} (1 - p)^\alpha \\

k_L p^{\alpha - 2} (1 - p)^{-1 - \alpha} (\alpha L - 1) = k_H p^{-1 - \alpha} (1 - p)^{\alpha - 2} (\alpha H - 1)
\]

This system of equations will give us a unique formula about $V_H$:

\[
r V_H = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H (\alpha_L - 1) (\Delta_H - \Delta_L) p}{\alpha_H (\alpha_L - 1) - (1 - p) (\alpha_L - \alpha_H)}. \quad (45)
\]

Here $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$. Furthermore, it is easy to check that both $k_H$ and $k_L$ are strictly larger than zero such that the option value of learning is strictly positive.

Therefore, we finally reach our main result:

**Theorem 3** Under strict supermodularity, the stationary competitive equilibrium is unique in the sense that all equilibria are PAM and the allocation is uniquely determined by Theorem 2. Moreover, assume monotonicity and normalize $V_L = 0$, we can get a unique formula for $V_H$ given by equation 45.

5 Firm-dependent Volatility: $\sigma_y$

A valid criticism of our approach is that we give the $H$ firms too much of an edge under supermodularity (likewise for the $L$ firms under submodularity). Not only are they superior in the production of output, by assuming that the volatility $\sigma$ is common to both types of firms, effectively the
signal-to-noise ratio is higher in $H$ firms:

$$s_H = \frac{\mu_{HH} - \mu_{HL}}{\sigma} > \frac{\mu_{HL} - \mu_{LL}}{\sigma} = s_L,$$

from supermodularity. With firm-dependent volatility, that need not be the case. In particular, for $\sigma_H$ sufficiently high, it may well be the case that $s_H < s_L$.

Mere observation of the value function in Equation (3), $rW_y(p) = \mu_y(p) - V_y + \Sigma_y(p)W''_y(p) - \delta W_y(p)$, reveals that firm-dependent volatility will play a crucial role here. Since $\Sigma_y = \frac{1}{2}p^2(1-p)^2s_y^2$, for sufficiently high $\sigma_H$ and therefore low $s_H$, it appears intuitive that the value $W_H$ can be smaller than the value of $W_L$ for high $p$. It turns out that this intuition is wrong. First, in this competitive equilibrium, wages are endogenous and therefore as the value of learning changes, so does $\mu_y(p) - V_y$.

Second, the no-deviation condition requires that at the marginal type $p$, $W''_H = W''_L$. It turns out that as a result these two features, in equilibrium the learning effect is the same in both firms, no matter what the volatility $\sigma_y$ is.

To make this argument formal, when $\sigma_H \neq \sigma_L$, we generally define $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma_y$, $y = H, L$. It is trivial to show that belief updating also satisfies the formula:

$$dp_t = p_t(1-p_t)s_ydZ_{y,t}.$$ 

Furthermore, Lemmas 2–5 still hold because none of these results depend explicitly on $\sigma_y$. Then we only need to show that Claim 2 extends. Here we adopt a different approach to prove Claim 2.\(^\text{17}\)

**Proof.** Suppose the situation described by the claim is the case. Then the value-matching conditions imply:

$$w_H(p_1) + \Sigma_H(p_1)W''_H(p_1) = w_L(p_1) + \Sigma_L(p_1)W''_L(p_1)$$

and

$$w_H(p_2) + \Sigma_H(p_2)W''_H(p_2) = w_L(p_2) + \Sigma_L(p_2)W''_L(p_2).$$

The no-deviation conditions imply:

$$\frac{s_H^2 - s_L^2}{s_H^2}(r + \delta)W_H(p_1) = w_L(p_1) - \frac{s_L^2}{s_H^2}w_H(p_1)$$

and

$$\frac{s_H^2 - s_L^2}{s_H^2}(r + \delta)W_H(p_2) = w_L(p_2) - \frac{s_L^2}{s_H^2}w_H(p_2).$$

\(^\text{17}\)It is a little bit tricky to prove the sufficiency of the no-deviation condition though because we have to consider both $s_H \geq s_L$ and $s_H < s_L$ cases. In the appendix, we show a proof for a generalized version of Lemma 6.
It follows then that:

\[
\frac{s_H^2 - s_L^2}{s_H^2} (r + \delta)(W_H(p_2) - W_H(p_1)) = [w_L(p_2) - w_L(p_1)] - \frac{s_H^2}{s_H^2} [w_H(p_2) - w_H(p_1)].
\]

\(W_H\) is convex and \(W'_H(p_1) > \frac{\Delta_L}{r+\delta}\) by Lemma 4. Therefore, we have:

\[
\frac{s_H^2 - s_L^2}{s_H^2} \Delta_L(p_2 - p_1) < \Delta_L(p_2 - p_1) - \frac{s_L^2}{s_H^2} \Delta_H(p_2 - p_1),
\]

which implies: \(\Delta_H < \Delta_L\). Contradiction!

With the proof of Claim 2 in hand, the result of Theorem 1 immediately extends: PAM (NAM) is the unique stationary competitive equilibrium allocation under strict supermodularity (submodularity) thus holds for any combination of \((\sigma_H, \sigma_L)\). Surprisingly, this implies that under strict supermodularity, even if we have an extremely high \(\sigma_H\) such that the learning rate in high type firms is smaller than that in low type firms, we still have PAM. It is equivalent to assert that the direct productivity consideration dominates the learning in our model. The reason comes from the fact that the equilibrium wage schedules adjust to offset the impact of change in learning rate. The key insight here is the no-deviation condition. At \(p\), the no-deviation condition requires that the second-order effect on the value function is the same in both firms. This second-order effect \(W''_y\) exactly captures the effect of learning through \(\Sigma_y(p)W''_y(p)\) where \(\Sigma_y = \frac{1}{2}p^2(1-p)^2s_y^2\). Because equilibrium wages adjust to satisfy the no-deviation condition, the impact of differential learning rates is completely offset by the change of wage schedule, and the equilibrium allocation is solely determined by the productivity consideration.

6 Predictions of the Model

We now turn to some of the predictions of the model.

Wage Gap at \(p\)

We start with an interesting observation:

\[
w_H(p) = \mu_H(p) - rV_H = \Delta_H p + \mu_{LL} - \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)p}{\alpha_H(\alpha_L - 1) - (1 - p)(\alpha_L - \alpha_H)} < \Delta_L p + \mu_{LL} = \mu_L(p).
\]

This implies that the worker with posterior belief slightly higher than \(p\) will accept high type firm’s offer even though the wage provided by the high type firm is lower than the productivity at the low type firm. This obviously comes from the fact that the learning speed in the high type firm is higher and this would compensate the loss in the flow wages.
On the other hand, we can see that the difference in expected productivity at $p$ is

$$\mu_H(p) - \mu_L(p) = (\mu_{HL} - \mu_{LL}) + (\Delta_H - \Delta_L)p < r_V H.$$

This implies the high type firm can enjoy a strictly positive rent from a higher learning speed. Because $V_L = 0$, it immediately follows that the wage at $p$ in $H$ is lower than in $L$. This above result does not depend on the assumption $V_L = 0$ and it could be generalized for any possible division of surplus:

**Lemma 7** Under strict supermodularity, for all stationary competitive equilibrium, we have: $w_H(p) < w_L(p)$ and $rV_H - rV_L > \mu_H(p) - \mu_L(p)$.

**Distribution of Posterior Beliefs**

In a steady state, equilibrium sorting transforms the normal output $X_t$ into a piecewise Lévy distribution of posterior beliefs $p$.

Observe the discontinuity in the density (and the subsequent kink in the cumulative) at $p$. This is unlike the learning environments with an absorbing state where the density is continuous. Recall that this is the result of equation (20), which ensures that the flows around $p$ are balanced. Given learning is faster in the $H$ firms, the flow from $H$ to $L$ is larger than from $L$ to $H$, and therefore the measure of types must be smaller to the right of $p$. At $p_0$ the density is continuous but not differentiable, which is due to the fact that there is a measure $\delta$ of new entrants in each period.
Note also that in this example, the density is increasing in the neighborhood of $p = 1$. Whether or not the density is increasing depends on the relation between the parameter values $\delta, s_y$ and the equilibrium value of $p_0$, as is shown in Moscarini (2005, Proposition 4).

**Wages**

The piece-wise Lévy distribution of beliefs maps one-to-one into a wage distribution because of competitive wage determination and the law of one price: for every $p$ there is exactly one wage $w(p)$. As a result, the wage distribution is also piece-wise Lévy. Figure 3 plots $w(p)$ and the density and cumulative distributions.

First, Figure 3A illustrates the wage gap we established earlier, i.e., here is a discontinuity in the wage function $w(p)$ at $p$. This follows immediately from value matching $W_L(P) = W_H(p)$ and the fact that learning is faster in $H$ firms. As a result, it must be the case that $w_H(p) < w_L(p)$. Of course, the wage schedule is in $H$ type firms is steeper, $w_H'(p) > w_L'(p)$ due to faster learning.

The wage schedule transforms the belief distribution into the wage distribution. The discontinuity in the wage schedule at $p$ happens to coincide with the discontinuity in belief distribution. Not surprisingly, as a result there is exactly one discontinuity in the density of $w$ at $p$.

**Wage Variance of $a$ over the life cycle**

One feature of the standard learning model is that over the life cycle, the variance of wages decreases. As time goes by, workers are increasingly likely to have found a high productivity match, and eventually the posterior belief converges to one. As a result, the variance decreases and goes to zero. Yet, there is ample evidence that the variance of wages over the life cycle increases and is concave. For some of the most recent evidence, see Heathcoate, Violante and Perri (2009). This is
not captured by the standard learning model.

Asymptotically, beliefs about a surviving worker converge to either 1 or 0. If we were to follow a cohort of newly entrants all born at \( t = 0 \) with prior \( p_0 \), none of whom ever dies, then as \( t \to \infty \), the distribution of beliefs has mass points at \( p = 1 \) with weight \( p_0 \) and 0 with weight \( 1 - p_0 \). Wages are \( \mu_{HH} - rV_H \) and 0. As a result, the wage variance is maximal: \( \text{Var}_\infty = p_0(1 - p_0)(\mu_{HH} - rV_H)^2 \).

By assumption that at entry all priors are \( p_0 \), the variance at \( t = 0 \) is minimized at zero.

Calculating the variance along the life cycle path is quite a bit more challenging. While we know the equilibrium wages paid and the stationary distribution of wages, in order to track the non-stationary evolution of beliefs (which in turn will determine the distribution of wages), we need to solve a non-stationary version of the Fokker-Planck equation (16) where the left-hand side is not zero. This being a partial differential equation, it can only be solved in some special cases.

Denote by \( \tau_H(p), \tau_L(p) \) the expected tenure of a worker with belief \( p \). Observe that without endogenous separations (as if a worker were forced to stay in one firm), the expected duration is \( 1/\delta \). With endogenous separations, \( \tau_H \) satisfies: \( \Sigma_H(p)\tau''_H(p) - \delta\tau_H(p) = -1 \). Solving for this differential equation, it follows that (see Moscarini (2005, Lemma 1)):

\[
\tau_H(p) = \frac{1}{\delta} \left\{ 1 - \left( \frac{p}{1 - p} \right)^{1/2 - \sqrt{1/4 + 2\delta/s_H^2}} \left( \frac{1 - p}{1 - p} \right)^{1/2 + \sqrt{1/4 + 2\delta/s_H^2}} \right\}.
\]

Likewise, we can derive the expected tenure in the \( L \) type firm.
Appendix

Proof of Lemma 2

**Proof.** The worker \( p \in (0, 1) \) always has the choice that stays in one firm \( y \) forever. Then the value is \( \frac{\mu_y(p) - rV_y}{r + \delta} \). But obviously, this is not an optimal choice (Suppose not, then all of the workers will stay in one type of firms and the market is not cleared). So we have that the equilibrium value function \( W_y(p) \) must satisfy: \( W_y(p) > \frac{\mu_y(p) - rV_y}{r + \delta} \). This immediately implies:

\[
\Sigma_y(p)W_y''(p) = (r + \delta)W_y(p) - (\mu'(p) - r\nu_y) > 0.
\]

So the equilibrium value functions \( W_y \) convex for \( p \in (0, 1) \).

**Proof of Lemma 3**

**Proof.** Suppose workers with \( p \in [0, p) \) are employed by type \( y \) firm. This implies that \( W_y(p) = \frac{\mu_y(p) - rV_y}{r + \delta} + k_{y2}p^{\gamma_y}(1 - p)^{1-\gamma_y} \) since 0 is included in the domain. It is easy to see that \( W_y'(0) = \frac{\mu_y(0) - rV_y}{r + \delta} > 0 \) and since \( W_y \) is strictly convex, \( W_y'(p) > 0 \) for all \( p \in [0, p) \). At \( p \), worker will transfer to type \(-y \) firm but smooth pasting condition implies \( W_y'(p) = W_y'(p) > 0 \). Strict convexity implies \( W_y'(p) > 0 \) so on and so forth. Therefore, we must have the equilibrium value functions \( W_y \) are strictly increasing.

**Proof of Claim 2**

**Proof.** Under supermodularity, suppose the situation described by the claim is the case. Then we have:

\[
w_H(p_1) + \Sigma(H(p_1)W_H''(p_1) = w_L(p_1) + \Sigma_L(p_1)W_L''(p_1)
\]

and

\[
w_H(p_2) + \Sigma_H(p_2)W_H''(p_2) = w_L(p_2) + \Sigma_L(p_2)W_L''(p_2)
\]

since

\[
W_H(p_2) = W_{L2}(p_2) \text{ and } W_H(p_1) = W_{L1}(p_1).
\]

Using the expression for \( w_H, w_L \) and the fact that

\[
W_H'(p_2) = W_{L2}'(p_2) \text{ and } W_H'(p_1) = W_{L1}'(p_1),
\]

we can get:

\[
(\Delta_H - \Delta_L)(p_2 - p_1) = \frac{\Sigma_H(p_1)W_H''(p_1) - \Sigma_H(p_2)W_H''(p_2)}{\Sigma_H(p_1)W_H''(p_1) - \Sigma_H(p_2)W_H''(p_2)}.
\]

Notice that \( \frac{\Sigma_H(p_1)W_H''(p_1) - \Sigma_H(p_2)W_H''(p_2)}{\Sigma_H(p_1)W_H''(p_1) - \Sigma_H(p_2)W_H''(p_2)} \). Then the above equation can be rewritten as:

\[
(\Delta_H - \Delta_L)(\Delta_H + \Delta_L)(r + \delta)[W_H(p_2) - W_H(p_1)] = \frac{\Delta_L}{\Delta_H}(\Delta_H - \Delta_L)(p_2 - p_1).
\]

Meanwhile,

\[
W_H(p_2) - W_H(p_1) > W_H'(p_1)(p_2 - p_1).
\]

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by strict convexity and
\[ W_H'(p_1) > \frac{\Delta_L}{r + \delta} \]
from Lemma 5. These two inequalities imply:
\[ (r + \delta)(W_H(p_2) - W_H(p_1)) > \Delta_L(p_2 - p_1) \]
under strict supermodularity. Hence, finally we get: \( \frac{\Delta_H + \Delta_L}{\Delta_H} < 1 \). Contradiction!

For the strict submodularity case, it suffices to relabel ‘H’ by ‘L’ and ‘L’ by ‘H’. The claim is obviously correct given we have already proved the supermodularity result.

**Proof of Proposition 1**

**Proof.** First, from Equation (26), we have:
\[ f_{L0} = \frac{1 - \pi}{\int_0^p p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} dp} \]

Denote \( A = f_{H0}p^{\gamma_{H1}}(1 - p)^{\gamma_{H2}} \), \( B = f_{H1}(1 - p)^{\gamma_{H1}}p^{\gamma_{H2}} \) and \( C = f_{L0}p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} \). Then Equations 24 and 27 imply:
\[ A = \frac{\eta_H + \eta_L s_L^2}{2\eta_H} + \frac{\delta(1 - \pi)}{p(1 - p)s_H^2 \eta_H} \]

and
\[ B = -\frac{\eta_L - \eta_H s_L^2}{2\eta_H} - \frac{\delta(1 - \pi)}{p(1 - p)s_H^2 \eta_H} \]

Here,
\[ \eta_L = \sqrt{\frac{1}{4} + \frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_H^2}} > 1/2. \]

Now we have:
\[ f_{H0} = \frac{\eta_H + \eta_L s_L^2}{2\eta_H} \left( \frac{p}{1 - p} \right)^n(1 - p) \]

\[ \frac{1 - \pi}{\int_0^p p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} dp} + \frac{\delta(1 - \pi)}{s_H^2 \eta_H} p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} \]

\( p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} \) must be decreasing in \( p \) since \( \frac{1}{2} - \eta_L < 0 \) and \( \frac{1}{2} + \eta_H > 0 \).

Meanwhile, notice that
\[ \left( \frac{p}{1 - p} \right)^{nL - \eta_H} = \int_0^p \left( \frac{p}{1 - p} \right)^{nL - \eta_H} dp = \int_0^p (\eta_L - \eta_H)(\frac{p}{1 - p})^{nL - \eta_H - 1}(\frac{1}{1 - p})^2 dp. \]

Let \( G_1(p) = p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} \) and \( G_2(p) = \left( \frac{p}{1 - p} \right)^{nL - \eta_H - 1}(\frac{1}{1 - p})^2 \). We have:
\[ \frac{G_1(p)}{G_2(p)} = p^{-\frac{1}{2} + \eta_H}(1 - p)^{-\frac{1}{2} - \eta_H} \]

is increasing in \( p \). Therefore, we must have:
\[ \left( \frac{p}{1 - p} \right)^{nL - \eta_H} \frac{1 - \pi}{\int_0^p p^{\gamma_{L1}}(1 - p)^{\gamma_{L2}} dp} \]
is decreasing in \( p^{18} \). So \( f_0^H \) is decreasing in \( p \).

Similarly,

\[
f_1^H = -\frac{\eta_L - \eta_H}{2\eta_H} s^2_L (\frac{p}{1-p})^{\eta_L + \eta_H} \frac{1 - \pi}{s^2_H} (1 - p)^{\frac{1}{2}} dp - \frac{\delta}{s^2_H \eta_H} p^{\frac{1}{2} + \eta_H} (1 - p)^{\frac{1}{2} - \eta_H}.
\]

Obviously, \(-p^{\frac{1}{2} + \eta_H} (1 - p)^{\frac{1}{2} - \eta_H}\) is decreasing in \( p \).

Meanwhile

\[
(\frac{p}{1-p})^{\eta_L + \eta_H} = \int_0^p (\eta_L + \eta_H) p^{\eta_L + \eta_H - 1} \frac{1}{1 - p^2} dp.
\]

Let \( G_3(p) = (\frac{p}{1-p})^{\eta_L + \eta_H - 1}(\frac{1}{1-p})^2 \) and we have:

\[
\frac{G_1(p)}{G_3(p)} = p^{-\frac{1}{2} - \eta_H} (1 - p)^{-\frac{1}{2} + \eta_H}
\]

is decreasing in \( p \). Therefore, we must have:

\[
-(\frac{p}{1-p})^{\eta_L + \eta_H} \frac{1 - \pi}{\int_0^p p^{\gamma_H} (1 - p)^{\gamma_H} dp}
\]

is decreasing in \( p \) and hence \( f_1^H \) is also decreasing in \( p \).

From Equation 28,

\[
f_2^H = f_0^H (\frac{p_0}{1-p_0})^{\gamma_L - \gamma_H} + f_1^H
\]

is also decreasing in \( p \). Therefore, we can expressing \( f_0^H \), \( f_1^H \) and \( f_2^H \) as \( \xi_0(p) \), \( \xi_1(p) \) and \( \xi_2(p) \) respectively such that \( \xi_0' < 0 \), \( \xi_1' < 0 \) and \( \xi_2' < 0 \).

Hence, the market clearing condition 25 implies:

\[
H(p) = \int_p^{p_0} [\xi_0(p) p^{\gamma_H} (1 - p)^{\gamma_H} + \xi_1(p) p^{\gamma_H} (1 - p)^{\gamma_H}] dp + \int_{p_0}^1 \xi_2(p) p^{\gamma_H} (1 - p)^{\gamma_H} dp = \pi.
\]

It is easy to see that \( H' < 0 \) since \( \xi_0' < 0 \), \( \xi_1' < 0 \) and \( \xi_2' < 0 \). Therefore, there exists \( P \in (0, p_0) \) such that \( H(p) = \pi \) if and only if \( \lim_{x \to 0} H(x) > \pi \) and \( \lim_{x \to p_0} H(x) < \pi \).

It is easy to verify that as \( P \to 0 \), \( f_0^H = \xi_0(p) \to \infty \) and \( f_1^H = \xi_1(p) \to 0 \). We thus have:

\[
\lim_{x \to 0} H(x) \to \infty > \pi.
\]

Meanwhile, when \( P \to p_0 \), it is obvious that \( H(p) \to \int_{p_0}^p f_0^H f_2^H (1 - p)^{\gamma_H} dp \). Notice that

\[
f_2^H = f_0^H (\frac{p_0}{1-p_0})^{\gamma_L - \gamma_H} + f_1^H \to \frac{s^2_L}{s^2_H} (\frac{p_0}{1-p_0})^{\eta_L + \eta_H - 1} \frac{1 - \pi}{\int_0^p p^{\gamma_H} (1 - p)^{\gamma_H} dp}
\]

as \( P \to p_0 \).

\[\text{\textsuperscript{18}}\text{Actually, we are using the result that if } G_3(p) \text{ is decreasing in } p, \text{ then } \int_0^p \frac{G_3(p) dp}{G_1(p) dp} \text{ will also be decreasing in } p. \text{ This is true because by the definition of Riemann integral, } \int_0^p G_1(p) dp \text{ and } \int_0^p G_2(p) dp \text{ could be written as the limit of Riemann sum. The ratio of two Riemann sums is always decreasing in } p \text{ since } \frac{G_2(p)}{G_1(p)} \text{ is decreasing in } p.\]

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Therefore, \( \lim_{x \to p_0} H(x) < \pi \) if and only if:

\[
\frac{s_L^2}{s_H^2} \frac{p_0}{1 - p_0} \gamma_{ul} + \eta_H \int_0^{p_0} p \gamma_{ul} (1 - p)^{\gamma_{ul} - 1} dp < \pi,
\]

which gives us the condition in the proposition. Moreover, since \( H \) is strictly decreasing, the solution to \( H(x) = \pi \) must be at most one. This completes our proof of Proposition 1. ■

**Proof of Corollary 1**

**Proof.** It is also straightforward to prove Corollary 1 from the equation \( H(p; \pi, p_0) = \pi \). Obviously, \( H \) is linear in \((1 - \pi)\). So as \( \pi \) increases, \( \pi/(1 - \pi) \) increases and we have to decrease \( p \) to keep the equation. On the other hand,

\[
\frac{\partial H}{\partial p_0} = \xi_0(p) p_0^\gamma H(1 - p_0) + \xi_1(p) p_0^\gamma L(1 - p_0) - \xi_2(p) p_0^\gamma L(1 - p_0) - \int_{p_0}^1 \frac{\partial \xi_2(p)}{\partial p_0} p_0^\gamma H(1 - p) dp.
\]

It is easy to verify that the first term is zero while the second term is negative. Hence \( H(p; \pi, p_0) \) is decreasing in \( p_0 \) and we have to increase \( p \) to keep the equation as \( p_0 \) increases.

The proof for the comparative statics for \( p > p_0 \) case is similar and hence is omitted. ■

**Proof of Proposition 2**

**Proof.** First, from equation (36), we have:

\[
f_{H0} = \frac{\pi}{\int_0^1 p \gamma H(1 - p) dp}.
\]

Denote \( A = f_{L1} p \gamma L(1 - p) \gamma_{L2}, B = f_{L2} (1 - p) \gamma L(1 - p) \gamma_{L2} \) and \( C = f_{H0} (1 - p) \gamma H(1 - p) \gamma_{H2} \). Then Equations (35) and (38) imply:

\[
A = \frac{\eta_L - \eta_H s_L^2 C}{2 \eta_L s_H^2} + \frac{\delta \pi}{p(1 - p)s_L^2 \eta_L},
\]

and

\[
B = \frac{\eta_L + \eta_H s_H^2 C}{2 \eta_L s_L^2} - \frac{\delta \pi}{p(1 - p)s_L^2 \eta_L}.
\]

Here,

\[
\eta_L = \sqrt{\frac{1}{4} + \frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_H^2}} > 1/2.
\]

It is easy to see that \( f_{H0}, f_{L1}, f_{L2} \) are increasing in \( p \) and hence \( f_{L0} \) is also increasing in \( p \) by Equation (39).

Hence, we can express \( f_{H0}, f_{L1}, f_{L2} \) as \( \xi_0(p), \xi_1(p) \) and \( \xi_2(p) \) respectively such that \( \xi_0' > 0, \xi_1' > 0 \) and \( \xi_2' > 0 \).
Finally, the market clearing condition (37) implies:

$$H(p) = \int_0^{p_0} \xi_0(p)p^{\gamma L_1}(1-p)^{\gamma L_2}dp + \int_{p_0}^{p} [\xi_1(p)p^{\gamma L_1}(1-p)^{\gamma L_2} + \xi_2(p)p^{\gamma L_2}(1-p)^{\gamma L_1}]dp = 1 - \pi.$$  

Obviously, $H(\cdot)$ is strictly increasing, which guarantees the solution is unique if it exists and $\lim_{x \to p_0} H(x) \leq 1 - \pi$ will give us Equation (40) in Proposition 2. 

**Proof of Lemma 6**

**Proof.**

Here we will try to prove a generalized version of Lemma 6. More specifically, we want to show that Lemma 6 is true for any combination of $(s_H, s_L)$.

First of all, we want to show all of the one-shot deviations are ruled out by our no-deviation condition as $dt \to 0$.

Under strict supermodularity, the value functions are given by:

$$W_L(p) = \frac{w_L(p)}{r + \delta} + k_L p^{\alpha_L} (1 - p)^{1 - \alpha_L}$$

and

$$W_H(p) = \frac{w_H(p)}{r + \delta} + k_H p^{1 - \alpha_H} (1 - p)^{\alpha_H}.$$  

Let

$$G_L(p) = k_L p^{\alpha_L} (1 - p)^{1 - \alpha_L} \left( \frac{\alpha_L - p}{p(1-p)} \right) > 0$$

and

$$G_H(p) = k_H p^{1 - \alpha_H} (1 - p)^{\alpha_H} \left( \frac{1 - \alpha_H - p}{p(1-p)} \right) < 0$$

be the first derivatives for the non-linear parts of the value functions. Smooth pasting at $p$ implies:

$$\Delta_L \frac{\Delta}{r + \delta} + G_L(p) = \Delta_H \frac{\Delta}{r + \delta} + G_H(p).$$

For $p < p^*_L$, define:

$$Z_L(p) = \lim_{dt \to 0} \frac{W_H(p) - W_L(p)}{dt} = w_H(p) - w_L(p) + [\Sigma_H(p) - \Sigma_L(p)] W''_L(p)$$

$$= w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2} \Sigma_L(p) W''_L(p)$$

$$= w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2} ((r + \delta) W_L(p) - w_L(p)).$$

Obviously, we have $\lim_{p \to p_L} Z_L(p) = 0$ from Lemma 5. If we can show that $Z_L(p)$ is increasing in $p$ as $p$ increases from $0$ to $p_L$, then we are done. Notice that

$$Z_L'(p) = \Delta_H - \frac{s_H^2}{s_L^2} \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2} (r + \delta) W_L'(p)$$
and $W'_L(p)$ lies between $\frac{\Delta L}{r+\delta}$ and $\frac{\Delta L}{r+\delta} + G_L(p)$ for $p \in [0, \underline{p}]$.\footnote{This comes from the fact that
$G_L(p) = k_L(\frac{p}{1-p})^{\alpha_L-1}(\frac{\alpha_L-p}{1-p})$ is increasing in $p$.}

If $s_H^2 \geq s_L^2$, then
\[
Z_L'(p) \geq \Delta_H - \frac{s_H^2}{s_L^2} \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2} \frac{\Delta L}{r+\delta} = \Delta_H - \Delta_L > 0;
\]

if $s_H^2 < s_L^2$, then
\[
Z_L'(p) \geq \Delta_H - \frac{s_H^2}{s_L^2} \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2} \frac{\Delta L}{r+\delta} \big(\frac{\Delta H}{r+\delta} + G_L(p)\big)
= \Delta_H - \frac{s_H^2}{s_L^2} \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2} \frac{\Delta H}{r+\delta} \big(\frac{\Delta H}{r+\delta} + G_L(p)\big)
= \frac{s_H^2}{s_L^2} (\Delta_H - \Delta_L) + \frac{s_H^2 - s_L^2}{s_L^2} (r+\delta)G_L(p) > 0.
\]

Therefore, we conclude that $Z_L'(p) > 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases, which implies that $Z_L(p) < 0$ for all $p < \underline{p}$ and hence there is no profitable one-shot deviation as $dt$ is sufficiently small.

For $p > \underline{p}$, similarly define:
\[
Z_H(p) = \lim_{dt \to 0} \frac{\tilde{W}_L(p) - W_H(p)}{dt} = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W''_H(p). \tag{46}
\]

Under PAM equilibrium, we have $Z_H(p) = 0$ from Lemma 5. Secondly, notice that
\[
Z_H(p) = w_L(p) - w_H(p) + [\Sigma_L(p) - \Sigma_H(p)]W''_H(p) = w_L(p) - w_H(p) + \frac{s_H^2 - s_L^2}{s_H^2} \big((r+\delta)W_H(p) - w_H(p)\big)
\]

with $W'_H(p)$ lies between $\frac{\Delta h}{r+\delta} + G_H(p)$ and $\frac{\Delta h}{r+\delta}$ for $p \in [\underline{p}, 1]$.\footnote{This comes from the fact that
$G_H(p) = k_H\left(\frac{1-p}{p}\right)^{\alpha_H-1}\left(\frac{1-\alpha_H-p}{1-p}\right)$ is decreasing in $p$.}

Hence, if $s_H^2 > s_L^2$
\[
Z_H'(p) \leq \Delta_L - \Delta_H < 0;
\]

and if $s_L^2 < s_H^2$
\[
Z_H'(p) \leq \Delta_L - \frac{s_H^2}{s_L^2} \Delta_H + \frac{s_L^2 - s_H^2}{s_H^2} \frac{\Delta L}{r+\delta} \big(\frac{\Delta H}{r+\delta} + G_L(p)\big) < 0.
\]

Therefore, $Z_H'(p) < 0$ for both $s_H \geq s_L$ and $s_H < s_L$ cases and hence $Z_H(p) < 0$ for all $p > \underline{p}$.

Second, since there is no one-shot deviation for any $p$, obviously there will be no any other deviation for any $p$. Consider any deviation starting at $p$. Then the above result says it is better not to deviate for at least $dt$ time. Suppose after $dt$, we achieve a new $p'$. Similarly, there should
be no deviation for at least $dt'$ time. Iterate using the same logic and we can see that any deviation is not profitable. ■
References


