Estimating and Testing Cross-Sectional Asset Pricing Models: A Robust IV Econometric Technique

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Abstract  Misspecified models, noisy betas, and weak instruments are well-known problems in finance and can lead to poor test performance. In this paper, we introduce a new technique for estimating and testing cross-sectional asset pricing models that addresses these problems. We apply our technique to three popular cross-sectional asset pricing models: CAPM, the Fama-French three-factor model, and the Campbell-Vuolteenaho "good beta-bad beta" model. The estimates of these three models illustrate several desirable properties of our new technique: 1) The asymptotic size of our tests is correct, so our technique does not overreject; 2) Our tests are robust to many sources of weak identification, including misspecified models, measurement error, and weak instruments; 3) Our technique provides an automatic alert for weak instruments; 4) Our tests have considerable statistical power. For example, the tests have sufficient power to reject the three cross-sectional asset pricing models; 5) Using our technique, model rejections can be informative (i.e., they provide constructive information about what is missing from an asset pricing model). In addition to illustrating our technique, we obtain substantive results that may be of wide interest. First, we find evidence that is consistent with the Lewellen-Nagel-Shanken critique (i.e., that the tight factor structure of size and book-to-market portfolios means that many tests have poor power). Second, we find evidence that returns are linked to both Fama-French betas and Campbell-Vuolteenaho betas. Third, we find that characteristics (e.g., size and momentum) contain additional information about returns that is not fully captured by any of the three asset pricing models.

Keywords: asset pricing models, asset pricing tests, misspecification, measurement error, weak instruments, CAPM, Fama-French three-factor model, Campbell-Vuolteenaho

JEL codes: G12, G11, C12, C13

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1. Introduction

Finance research has shown that important econometric problems can affect tests of asset pricing models. The sources of these problems include misspecified models, weak instruments, and measurement error, all of which can lead to poor test performance. In this paper, we introduce a new technique for estimating and testing cross-sectional asset pricing models that addresses these problems.

Our technique comes out of a broader literature in econometrics on "weak identification." A number of lessons have emerged from this literature. First, standard asymptotic results often provide poor approximations to the distributions of estimators and test statistics. The actual size of a test (probability of rejecting a true model) can be dramatically different than the nominal size (e.g., 0.05). The power of a test can be low, so that the test fails to reject a false model. Both the size and power problems can be serious even when the sample size is large. Second, standard correction techniques – including many bootstraps – may fail.¹

The standard methodology in empirical tests of cross-sectional asset pricing models is based on a two-pass approach. The first step is to estimate the betas. The second step is to estimate a cross-sectional regression of expected returns on the betas. The coefficients on the betas are the estimated risk prices associated with the factors (and may reflect the underlying parameters of the asset pricing model, such as the coefficient of relative risk aversion).

Weak identification (due, e.g., to a misspecified model, weak instruments, or measurement error²) can create serious problems for tests of asset pricing models. Many estimation techniques will yield point estimates with seemingly tight standard errors, even when the data contain little information about the true parameter. The usual

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¹ For insights on these econometric issues and associated research directions, see, e.g., Dufour (1997, 2003), Staiger and Stock (1997), Wand and Zivot (1998), Stock and Wright (2000), Stock, Wright and Yogo (2002), Kleibergen (2002, 2005). The reliability of bootstrapping procedures depends on regularity conditions. These regularity conditions may fail (or hold only weakly) in regressions where the independent variables are generated regressors. For econometric analysis of these issues, see, e.g., Dufour (1997), Andrews (2000), and Bolduc, Khalaf, and Yelou (2007).

confidence intervals may be tightly centered on the wrong value of the parameter. Weak identification can therefore lead to poor coverage. In other words, the estimated confidence interval may not contain (cover) the true value of the parameter. For example, in some versions of the ICAPM, the coefficients on the betas reflect the coefficient of relative risk aversion. In a situation where weak identification leads to poor coverage, the true value of the coefficient of relative risk aversion could be 2, but, if the point estimate is 45 and the standard error is 10, a 5% confidence interval will be approximately (25, 65), which does not cover the true value of 2.

Research on the implications of the weak identification problem for asset pricing tests is limited. Our paper is one of the first that addresses cross-sectional asset pricing models. We introduce a new technique for estimating and testing cross-sectional asset pricing models that is robust to the problems that arise due to weak identification. Our method works as follows. We conduct a statistical assessment of pricing errors viewed as functions of the parameters of the model. (In this respect, our approach is similar to GMM.) On this basis, we define a formal measure of model fit (in the spirit of the J statistic in GMM) that accounts for pre-estimation of the betas. This measure of (the $\hat{p}_{\text{max}}$ statistic) statistic can be interpreted as the p-value for the rejection of the model. We then derive the set of model parameters that are statistically compatible, given the data, with the $\hat{p}_{\text{max}}$ statistic. For example, for the standard 5% size, our technique selects the set of parameters that yield a $\hat{p}_{\text{max}}$ statistic greater than or equal to 0.05. This set is guaranteed to cover the true (unknown) parameters with the desired size.

We apply our approach to three popular asset pricing models -- CAPM, Fama-French three-factor model, and Campbell-Vuolteenaho "good beta-bad beta" model. Because the Campbell-Vuolteenaho model can more easily be related to structural parameters, such as the coefficient of relative risk aversion, we use it as the example in explaining our technique.

Our technique has several desirable properties. The asymptotic size of our tests is correct, so our technique does not overreject. Our tests are robust to many sources of weak identification. Our technique provides an automatic alert for weak instruments. Our

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tests have considerable statistical power. For example, the tests have sufficient power to reject the three cross-sectional asset pricing models. Using our technique, model rejections can be informative (i.e., they provide constructive information about what is missing from an asset pricing model).

The paper is organized as follows. Section 2 describes our econometric procedure. Section 3 briefly describes the salient features of the Campbell-Vuolteenaho model. Section 4 presents empirical results for the Campbell-Vuolteenaho model, Section 5 for the CAPM, and Section 6 for the Fama-French three-factor model. Section 7 summarizes some of the main implications for cross-sectional asset pricing models that emerge from our empirical results.

2. Econometric Procedure

We begin by writing the asset pricing model in a form that separates out the exogenous variables \( \mathcal{X} \) that enter the model linearly:

\[
\mathcal{X}_i' \delta + \mathcal{F}(\mathcal{Y}_i, \vartheta) = \mathcal{U}_i, \quad i = 1, ..., n,
\]

where \( \mathcal{X}_i \) is a \( k \)-dimensional vector of exogenous variables and \( \delta \) is the corresponding vector of parameters, \( \mathcal{F}(\mathcal{Y}_i, \vartheta) \) is a (possibly nonlinear) function of observed variables, \( \mathcal{Y}_i \) is a matrix that can include both endogenous and exogenous variables, \( \vartheta \) is an \( m \times 1 \) vector of unknown parameters of interest, and \( \mathcal{U}_i \) is a disturbance with mean zero.

To illustrate our econometric technique, consider the hypothesis \( \mathcal{H}_0 : \vartheta = \vartheta_0 \).

If (1) holds, then clearly \( \mathcal{X}_i' \delta + \mathcal{F}(\mathcal{Y}_i, \vartheta_0) = \mathcal{U}_i \). Define \( \mathcal{Z}_i \) as a \( k_2 \times 1 \) vector of exogenous or predetermined variables such that \( k_2 \geq m \). Since both \( \mathcal{X}_i \) and \( \mathcal{Z}_i \) are exogenous or predetermined (i.e., orthogonal to \( \mathcal{U}_i \)), if we regress \( \mathcal{F}(\mathcal{Y}_i, \vartheta_0) \) on \( \mathcal{X}_i \) and \( \mathcal{Z}_i \), the coefficients on \( \mathcal{Z}_i \) in the resulting artificial regression

\[
\mathcal{F}(\mathcal{Y}_i, \vartheta_0) = \mathcal{X}_i' \varphi_X + \mathcal{Z}_i' \varphi_Z + \varepsilon_i
\]

should be statistically zero. Hence, \( \mathcal{H}_0 \) can be tested by assessing

\[
\mathcal{H}_0' : \varphi_Z = 0
\]
in regression (2). \( \mathbf{Z}_t \) can be viewed as a vector of extra “instruments,” which may include exogenous variables in \( \mathbf{Y} \). The advantage of reformulating the asset pricing model as we have done in equation (2) is that we can readily deal with the sources of weak identification (such as measurement error) that may be present in equation (1).

This has some important benefits for testing. First, the size of our tests (e.g., 5\%) will be correct even if the underlying asset pricing model is misspecified. In particular, the size of the test will be correct even if the betas that are estimated in the first stage of the usual two-pass procedure are measured with error (or misspecified). Second, the size of the test will not be affected by the quality of the instruments \( \mathbf{Z}_t \). Third, the size of the test will be correct regardless of whether we use all relevant instruments or not.\(^4\) To summarize, we can test the null hypothesis correctly: 1) allowing for possible measurement error in the betas (or misspecification; 2) without assuming that the parameters are identified; and 3) allowing for missing instruments.

We denote the desired size of the test by \( \alpha \); e.g., \( \alpha = .05 \). Using a standard F test statistic, we can calculate the p-value under the null hypothesis \( \mathcal{H}_0' \), which we denote as \( \hat{p}(\theta_0, \mathcal{F}) \). To obtain a confidence set for \( \theta \) with the desirable statistical properties described above, we can invert the test of the null hypothesis \( \mathcal{H}_0' \). In other words, to obtain a joint confidence region of the desired size for the elements of the vector \( \theta \), we need to collect all \( \hat{\theta}_0 \) values that are not rejected by the test at the \( \alpha \) significance level. Then, using projection methods, we can obtain confidence sets for each element of \( \theta \).\(^5\)

The point estimates of the parameters are obtained differently in our approach than in the standard approach. In the standard approach, equation (1) is estimated directly (e.g., by OLS or GMM). In our approach, equation (1) is mapped into the artificial regression (2) for all values of the structural parameters of interest, which allows us to test each of these values. We then choose the parameter vector that yields the

\(^4\) More formally, we can ensure that the nominal size of the test is equal to the asymptotic size. This will also hold even if underlying factors are irrelevant or redundant [see Kan and Zhang (1999a,b)].

\(^5\) More generally, we can obtain confidence sets for any scalar function of the form \( \omega' \theta \) where \( \omega \) is a non-zero \( m \)-dimensional vector. For example, if \( m = 2 \), so the asset pricing model has two parameters, and \( \omega = (1, -1) \), we can obtain a test – and confidence interval – for the null hypothesis that the difference between the two parameters is 0.
highest value of $\hat{p}(\vartheta_0, \mathcal{F})$, as a point estimate. Point estimates so defined correspond to the “least-rejected” models.

In addition to providing confidence intervals that are robust to noisy betas (plus endogeneity and weak instruments), our approach also provides a natural test of the overall fit of the model. In other words, our approach gives us a counterpart to the $R^2$ in a least squares regression or the J statistic in GMM estimation. For a particular value $\vartheta_0$ of the parameter vector $\vartheta$, our procedure provides a p-value $\hat{p}(\vartheta_0, \mathcal{F})$. Define:

$$p_{\text{max}}(\mathcal{F}) = \max_{\vartheta_0} \{ \hat{p}(\vartheta_0, \mathcal{F}) \}. \quad (4)$$

In other words, $p_{\text{max}}(\mathcal{F})$ is the p-value for the parameters of the asset pricing model that best fits the data. It therefore has a natural interpretation as the p-value for the overall rejection of the model.\(^6\)

Asset pricing tests have recently been criticized on statistical and data-related grounds, and specific guidelines to avoid consequential pitfalls have been set forth and are gaining credibility [see Lewellen, Nagel and Shanken (2007), Lewellen and Nagel (2006)]. Our inference method meets most of the recommendations: (i) we propose confidence set estimates for parameters of interest that pass valid statistical tests; (ii) we aim to use test portfolios sorted in various ways to avoid the trivial factor structure critique of Lewellen and Nagel (2006); for the same reason, we will also propose GLS extensions of the our method. Relative to the above described method, GLS will raise further numerical burdens yet conceptually, follows the same principles.

Indeed, testing $\mathcal{H}_0'$ may be interpreted in the risk-return framework as assessing whether additional explanatory variables hold further information on the left hand side returns that are not explained by the betas. Since the regression of the pricing errors on these additional explanatory variables does not suffer from endogeneity, then any test applicable in this regular context may be used; this includes the usual F-test as described above, or its robust versions [as in GMM] if serial and/cross-sectional dependence is

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\(^6\) This description of our econometric procedure is intended to be informal. Technical details are provided in the Appendix.
relevant. The exclusion restrictions in $\mathcal{H}_0'$ may also be viewed as reflecting orthogonality conditions consistent with the GMM principle.

3. Campbell-Vuolteenaho Model: Application of Econometric Procedure

The Campbell-Vuolteenaho model is based on Campbell's (1993) version of the Merton (1973) ICAPM. Campbell (1993) assumes an infinitely-lived investor with Epstein-Zin (1989, 1991) preferences, time discount factor $\delta$, relative risk aversion $\gamma$, and elasticity of intertemporal substitution $\psi$. The model's asset pricing implications are based on the investor's first-order condition. In the main version of the Campbell-Vuolteenaho model, the investor is assumed to hold a portfolio that is fully invested in equity. Campbell (1993) is based on the assumption that asset returns are conditionally lognormal and at the returns on the investor's portfolio (and its two components) are homoskedastic.\(^7\)

In Campbell's approximate, discreet-time version of the ICAPM, the optimality conditions imply the following risk premium on each asset $i$:

$$ E_t[r_{i,t+1}] - r_{f,t+1} + \frac{\sigma^2_{i,t}}{2} = \gamma \text{Cov}_t \left( r_{i,t+1}, r_{p,t+1} - E_t r_{p,t+1} \right) + (1 - \gamma) \text{Cov}_t \left( r_{i,t+1}, -N_{p,DR,t+1} \right) $$

where $r_{f,t+1}$ is the return on the risk-free asset, $p$ is the optimal portfolio that the agent chooses to hold, and $N_{p,DR,t+1} = (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{p,t+j}$ is the discount-rate (or expected-return) news on the portfolio. The left-hand side is the expected excess log return on asset $i$. (The variance term on the left-hand side enters to adjust for Jensen's inequality.) The right-hand side of the equation is a weighted average of two covariances, with the weights based on the coefficient of relative risk aversion $\gamma$. The first covariance is the covariance of the return on asset $i$ with the return on portfolio $p$. This represents the myopic component of asset demand. If $\gamma$ were equal to 1, portfolio choice would be myopic and the model would reduce to the CAPM. The second

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\(^7\) Campbell and Vuolteenaho argue that the empirical evidence shows that changes in volatility are much less persistent than changes in expected returns and therefore generate relatively modest intertemporal hedging effects. In their empirical work, they assume constant variances.
covariance is the covariance of the return on asset $i$ with (the negative of) discount rate news.

To derive the empirical version of their model, Campbell and Vuolteenaho rewrite the previous equation, replacing the first covariance with news covariances. In other words, they use the following equation

$$r_{p,t+1} - E_t r_{p,t+1} = N_{p,CF,t+1} - N_{p,DR,t+1},$$

which decomposes news about the portfolio return into cash flow news and discount rate news.\(^8\) This leads to the following equation:

$$E_t \left[ r_{t+1} \right] - E_t \left[ r_{f,t+1} \right] + \frac{\sigma_{r,td}^2}{2} = \gamma \sigma_{p,t}^2 \beta_{i,CF,t}^i + \sigma_{p,t}^2 \beta_{i,DR,t}^i.$$  

(7)

The main version of the Campbell-Vuolteenaho model, which they refer to as the “two-beta ICAPM,” therefore has three main implications. First, the main risk factors that determine expected returns are an asset’s cash flow beta and discount rate beta. Second, the risk price associated with cash flow beta should be $\gamma$ times greater than the risk price associated with discount rate beta. Third, the risk price associated with discount rate beta should equal the variance of the return on portfolio $p$.

Campbell and Vuolteenaho also consider a second version of their model in which the investor’s portfolio $p$ places a weight $w$ on equity and $(1-w)$ on the risk-free asset. Under this alternative assumption, they obtain:

$$E_t \left[ r_{t+1} \right] - E_t \left[ r_{f,t+1} \right] + \frac{\sigma_{r,td}^2}{2} = \gamma w \sigma_{p,t}^2 \beta_{i,CF,t}^i + w \sigma_{p,t}^2 \beta_{i,DR,t}^i.$$  

(8)

The second version, which they refer to as the “factor model,” has the same first two implications as the main version of the model, but it loosens the restriction that the risk price associated with discount rate beta is equal to $\sigma_{p,t}^2$.\(^9\)

Campbell and Vuolteenaho make several modifications and additional assumptions in taking the model to the data. First, they assume that the equity portfolio return is equal to the return on the value-weighted index and the risk-free return is equal to the real Treasury-bill return and that the latter return is constant. Second, they use

\(^8\) They also multiply and divide the covariance terms by the conditional variance of the return on portfolio $p$ ($\sigma_{p,t}^2$) in order to rewrite the covariances as factor loadings.

\(^9\) We discuss below how the two parameters of the model can be recovered from the estimated risk prices.
simple expected excess returns $E_t[R_{i,t+1} - R_{f,t+1}]$ on the left-hand side, instead of log returns, $E_t[r_{i,t+1}]r_{f,t+1} + \sigma^2_{r_t}/2$. Third, they focus on the unconditional version of the model. Fourth, they use the market portfolio of stocks as the reference portfolio. With these additional assumptions and modifications, the model can be written as

$$E[R_t - R_f] = \gamma \sigma_M^2 \beta_{i,CF} + \sigma_M^2 \beta_{i,DR}.$$  

Campbell and Vuolteenaho estimate two forms of both the two-beta ICAPM and the factor model. The first form is the Sharpe-Lintner form in which the zero-beta rate is restricted to be equal to the Treasury-bill rate. The second form is the Black (1972) form with an unrestricted zero-beta rate.

The cross-sectional regression estimated by Campbell and Vuolteenaho is

$$\bar{R}_i^c = g_0 + g_1 \hat{\beta}_{i,CF} + g_2 \hat{\beta}_{i,DR} + e_i \tag{9}$$

where the bar denotes the time-series mean and $\bar{R}_i^c = \bar{R}_i - \bar{R}_f$ denotes the sample average simple excess return on asset $i$. In the two-beta ICAPM, $g_1 = \gamma \sigma_M^2$ and $g_2 = \sigma_M^2$, and the restriction on $g_2$ is imposed in estimating the model. The coefficient of relative risk aversion can therefore be calculated as $\gamma = g_1/\sigma_M^2$. In the factor model, $g_1 = \gamma w \sigma_M^2$ and $g_2 = w \sigma_M^2$, and both parameters are estimated without restrictions. As in the two-beta ICAPM, the coefficient of relative risk aversion can be calculated as $\gamma = g_1/\sigma_M^2$. The additional parameter of the factor model (the investor’s portfolio weight $w$ on equity) can be calculated as $w = g_2/\sigma_M^2$. Since $\sigma_M^2 > 0$ and $w > 0$, both the two-beta ICAPM and the factor model imply $g_2 > 0$. Campbell and Vuolteenaho do not impose this restriction. In fact, their point estimate of $g_2$ is sometimes negative (although insignificant, based on the reported standard errors). For both the two-beta ICAPM and the factor model, the Sharpe-Lintner form implies $g_0 = 0$, and Campbell and Vuolteenaho impose this restriction. In the Black form, $g_0$ has the interpretation of the difference between the return on the zero-beta asset and the return on the risk-free asset and is estimated without restriction.

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10 See, e.g., the first column of Table 6 in Campbell and Vuolteenaho (2004).
We next describe the application of our econometric procedure to the Campbell-Vuolteenaho model. The intercept term $g_0$ in the regression can be represented by the $\mathcal{X}'\delta$ term in equation (1), since estimation of the intercept involves only exogenous variables and enters the asset pricing model linearly. Specifically:

$$\mathcal{X}'\delta = -g_0$$  \hspace{1cm} (10)

where $\mathcal{X}' = 1$ and $\delta$ is $-g_0$. The pricing error function

$$\mathcal{F}_i(\mathcal{Y}, \vartheta) = \tilde{R}_i - g_1\hat{\beta}_{i,CF} + g_2\hat{\beta}_{i,DR}.$$  \hspace{1cm} (11)

The variables $\mathcal{Y}_i$ are the factor loadings:

$$\mathcal{Y}_i = (\hat{\beta}_{i,CF}, \hat{\beta}_{i,DR})$$  \hspace{1cm} (12)

The parameter vector $\vartheta$ consists of the risk prices:

$$\vartheta = (g_1, g_2).$$  \hspace{1cm} (13)

We consider various choices of the additional variables $Z_i$ used in the artificial regression (2). To avoid potential endogeneity, we use the split-sample approach. For example, two of the additional variables $Z_i$ that we consider are size and book-to-market. We calculate the means of these variables for each portfolio over the first 30% of the sample. Using these means (along with a constant) as the $Z_i$, we use the remaining 70% of the sample to estimate the model.

4. Campbell-Vuolteenaho Model: Empirical Results

For comparability with Campbell and Vuolteenaho (2004), we use the same data (obtained from the web site of the American Economic Review). The additional variables $Z_i$ (specifically size and book-to-market for the 25 size and book-to-market portfolios) were obtained from the web site maintained by Ken French.\(^{11}\)

Empirical results for what Campbell and Vuolteenaho call the "modern sample" (1963:7-2001:12) are presented in Table 1 (which is comparable to Table 7 in their paper). As noted above, the $p_{\text{max}}$ statistic provides an overall test of the model. Using

\(^{11}\) Size and book-to-market are not available from either of these two sources for the 20 additional risk-based portfolios studied by Campbell and Vuolteenaho, so we calculate these from CRSP data.
our procedure, the model is not rejected. In fact, the p-value for the rejection of the Sharpe-Lintner form of the model is about 0.92.

The point estimate of the risk price on cash flow beta \( (g_1) \) is 0.0679 (in monthly terms) for the Sharpe-Lintner form of the model, which translates into a risk price of about 81% per year. This is in the same ballpark as the risk prices for cash flow news for the modern sample obtained by Campbell and Vuolteenaho, which range from 58% to 68% (depending on the precise version of their model). The estimate based on our procedure is much closer to their estimates for the modern sample, for example, than their estimates for their early sample (1929:1-1963:6), which range from 8% to 21%.

Interestingly, though, our estimate of 81% is substantially higher than the Campbell-Vuolteenaho estimate of 58% per year for the Sharpe-Lintner form of the factor model.

As noted above, the size of our confidence interval is robust to the many sources of weak identification, including endogeneity and measurement error in the factor loadings. Interestingly, in this case, the confidence interval from our procedure is tighter than the confidence intervals Campbell and Vuolteenaho obtain from their bootstrap procedures. Their confidence intervals are (-0.0061, 0.1027) and (-0.0363, 0.1329), depending on which bootstrap procedure they use, so they are unable to reject the null hypothesis that the risk price of cash flow news is 0. In contrast, our confidence interval is (0.0241, 0.1287), so our procedure provides stronger evidence for cash flow news as a risk factor.

As noted above, the two-beta ICAPM imposes the restriction that the risk price of discount rate news is equal to the variance of the market portfolio. The factor model loosens this restriction. In the Sharpe-Lintner form of the model, the estimate of \( g_2 \) is 0.0045, compared to the Campbell-Vuolteenaho estimate of 0.0012. Both their estimate and our estimate imply that the risk price of discount rate news is economically small -- less than 6% per year. The confidence interval from our procedure is fairly tight (-0.0005, 0.0081), but it does not reject the null hypothesis that the risk price of discount rate news is equal to the variance of the market return, which is 0.0020.

The Black form of the factor model loosens one further restriction -- that the zero-beta rate is the same as the risk-free rate. Based on our procedure, the difference between the rates is slightly positive but statistically insignificant. This is similar to what
Campbell and Vuolteenaho find. The restriction that the risk price of discount rate news should be equal to the variance of the market return is not rejected.

We consider two robustness checks. First, we estimate the model using the 25 size and book-to-market portfolios that are widely used to evaluate cross-sectional asset pricing models. Second, we consider a different set of additional variables $Z_i$.

The results for the 25 size and book-to-market portfolios, which are presented in Table 2, are broadly similar to the results in Table 1. The risk price associated with cash flow beta is positive and economically substantial. The restriction on the risk price associated with discount rate beta (that it equal the variance of market returns) is approximately true -- and not rejected. In fact, the Campbell-Vuolteenaho model fits the 25 size and book-to-market portfolios particularly well. The $p_{\text{max}}$ statistic is close to 1, implying that the data are consistent with the model. The risk price of cash flow beta is significantly different from zero in both the Sharpe-Lintner and Black forms of the model.

A common source of instrumental variables in IV estimation is lagged values. We are estimating a cross-sectional model, so there is no time dimension, but we can use the split-sample approach to obtain something like lagged values as instruments. Using the first 30% of the sample, we estimate the cash flow and discount rate beta for each portfolio. We then use these own-model betas (along with a constant) as the $Z_i$ in the first column of Table 3. We also consider expanded instrument sets that include (in addition to the own-model betas) the Fama-French betas and the Fama French betas plus a characteristic (book-to-market). The estimated risk prices associated with cash flow and discount rate beta are positive and statistically significant in most of the specifications. Adding the Fama-French betas -- and a characteristic -- leads to tighter confidence intervals, suggesting that these variables contain useful information about returns. Regardless of the choice of instrument set, the model is not rejected, but the $p_{\text{max}}$ statistics become smaller when we add the Fama-French betas and book-to-market as additional instruments.
In Panel B, we explore instrument sets that include two of the three Fama-French betas plus the beta on squared market returns. The point estimates of the risk prices are similar to those in previous tables, but the $p_{\max}$ statistics reject the Black form of the model, and the confidence intervals tighten in the Sharpe-Lintner form of the model.

Lewellen and Nagel (2007) and Lewellen, Nagel, and Shanken (2008) have argued that the tight factor structure of size and book-to-market portfolios means that many common tests have little power to reject false asset pricing models. To test this, we estimate the Campbell-Vuolteenaho model on the 25 size and momentum portfolios, using own-model betas as the instruments. The results are presented in Table 4. They are consistent with the Lewellen-Nagel-Shanken critique. The $p_{\max}$ statistic is about .016 for the Sharpe-Lintner form of the model, implying rejection of the model at the 5% level.

As discussed above, our procedure for determining the confidence interval is based on testing candidate values of a parameter. Candidate values that are not rejected at the 0.05 significance level are included in the confidence interval. If there is no candidate value of a parameter that is not rejected, then the confidence interval is empty. This provides evidence for the rejection of a model. Table 4 shows that the confidence interval for each parameter is the empty set, a result that conveys the same message as the $p_{\max}$ statistic.

When we estimate the Campbell-Vuolteenaho model using the 25 size and momentum portfolios, the point estimates also change dramatically. The risk prices associated with both cash flow and discount rate beta become negative (and significantly less than 0 in the case of cash flow beta$^{13}$).

The Black form of the model is exactly identified when we use own-model betas as instruments (since the Black form has three parameters and the own-model betas -- plus the constant -- provide three instruments), so we cannot calculate the $p_{\max}$ statistic.

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$^{12}$ The beta on squared returns is calculated by regressing excess returns for each portfolio on the square of excess market returns. To ensure that this instrument is predetermined, the beta on squared returns is estimated over the first 30% of the sample, and the remainder of the sample is used for the estimation presented in the table.

$^{13}$ The point estimate of $g_1$ is negative and the confidence interval does not include 0.
However, if we add momentum as an additional instrument, we can calculate the $p_{max}$ statistic for both the Black and Sharpe-Lintner forms. The result is a strong rejection of the Campbell-Vuolteenaho model: the $p_{max}$ statistic is less than 0.001 in each case.

5. CAPM

CAPM was long the leading cross-sectional asset pricing model. Although it has taken many hits in recent years, it may be useful to apply our econometric procedure to this venerable model.

If the CAPM is the true model, the own-model instruments (CAPM beta and a constant) are natural instruments. However, if the CAPM is a poor cross-sectional asset pricing model, the own-model instruments may not have great relevance in explaining returns. We therefore begin by estimating the CAPM using own-model instruments.

Our estimates of the Black form of the model illustrate a desirable property of our technique: it sets off an automatic "warning bell" when the instruments are weak. For the Black form, this is indicated by infinite confidence intervals.

As an alternative, we try instruments from a model that has largely supplanted the CAPM. As Table 5 shows, using the Fama-French three-factor betas as instruments leads to a strong rejection of the CAPM, a result that is consistent with much other empirical evidence against the CAPM.

6. Fama-French Three-Factor Model

The Fama-French three-factor model has been one of the leading cross-sectional asset pricing models for the last 15 years or so. We begin by presenting estimates of the model and then use the model to illustrate some of the properties of our econometric technique.

Table 6 presents initial estimates of the Fama-French three-factor model. As with the CAPM, we begin with a natural set of instruments -- the own-model betas. In contrast to the CAPM, there is no "warning bell" to indicate that the own-model betas are weak instruments. In fact, the own-model betas yield reasonably tight confidence intervals. This provides an initial piece of evidence that the Fama-French three-factor model is a
reasonably good cross-sectional asset pricing model. Since the own-model betas provide only as many instruments as there are parameters in the model, the model is exactly identified with these instruments, and we cannot calculate the $p_{\text{max}}$ statistic. For a more formal, statistical test of the model, we need to consider additional instruments.

As illustrated in the previous section, the effect of instruments on the confidence intervals (and the $p_{\text{max}}$ statistic) can be informative, suggesting elements that may be missing in the asset pricing model that we are testing. With this in mind, we consider three additional instrument sets. The first adds the betas from the Campbell-Vuolteenaho model to the Fama-French betas. The resulting $p_{\text{max}}$ statistic is 0.3980, so this set of instruments does not reject the model. The second adds both the Campbell-Vuolteenaho betas and a characteristic (specifically, momentum) to the own-model betas. The $p_{\text{max}}$ statistic is 0.1943. The third adds the beta on squared market returns to the own-model betas. The $p_{\text{max}}$ statistic is 0.9968. The point estimates of the risk prices are similar across the four instrument sets.

The results in the previous section provide a hint that variables that contain useful information about returns tend to be strong instruments. In turn, strong instruments produce tighter estimates of the parameters. What can we learn by comparing the results for these instrument sets? Table 7 provides a summary of the width of the confidence intervals for each parameter across these four instrument sets. The own-model betas produce reasonably tight confidence intervals. But, when we add the Campbell-Vuolteenaho betas, the confidence intervals become tighter. They become tighter still when we add both the Campbell-Vuolteenaho betas and a characteristic (momentum). This suggests that the Campbell-Vuolteenaho betas and characteristics may contain useful information about returns, above and beyond the information that is contained in the Fama-French betas.

The first three columns of Table 7 might suggest that increasing the number of instruments necessarily leads to tighter confidence intervals. There is no theorem to this effect, however, and it is easy to find counterexamples. One is provided in the last column of Table 7. When we add the beta on squared returns to the own-model betas, the confidence intervals are wider than when the only instruments are the own-model betas.
The Fama-French three-factor model clearly provides a good characterization of cross-sectional returns on the 25 size and book-to-market portfolios. But the results in Tables 6 and 7 suggest that there are other variables with useful information about returns. Is there enough information in these additional variables to lead to a statistical rejection of the Fama-French three-factor model? To explore this further, we consider instrument sets that contain the Campbell-Vuolteenaho betas and combinations of characteristics. Table 8 shows that it is possible to find a variety of instrument sets that reject the Fama-French three-factor model. All of these instrument sets have at least one of the Campbell-Vuolteenaho betas plus two or three characteristics.\textsuperscript{14} This suggests that the Campbell-Vuolteenaho betas and the characteristics reflect risks that are not fully captured by the three Fama-French betas.

We can test whether the Lewellen, Nagel, and Shanken critique is relevant for the Fama-French three-factor model by estimating the model on the 25 size and momentum portfolios. The first column of Table 9 presents results based on own-model betas. The confidence intervals are finite, so the data suggest that the own-model betas are relevant instruments. On the other hand, in comparison with the results for the 25 size and book-to-market portfolios, the confidence intervals are substantially looser. This provides an initial hint that the Fama-French three-factor model is less successful at explaining cross-sectional returns for the 25 size and momentum portfolios than for the 25 size and book-to-market portfolios. Another hint is that none of the Fama-French risk prices has a significant effect on returns.

In the remaining three columns of Table 9, we add characteristics to the baseline instrument set. Adding either size or momentum to the own-model betas leads to tighter confidence intervals. This is a sign that these characteristics contain additional information about cross-sectional returns that is not captured by the Fama-French betas. Another sign is provided by the $p_{\text{max}}$ statistic. Although the model is not rejected, the $p_{\text{max}}$ statistics are 0.0848 when size is added to the instrument set and 0.0605 when momentum is added. The empirical results do not, however, suggest that the Fama-French factors are unrelated to returns. In fact, when momentum is added to the

\textsuperscript{14} All of the instrument sets also include the own-model betas.
instrument set, both the SMB and HML betas have risk prices that are significantly different from zero.

When both characteristics -- size and momentum -- are added to the instrument set, the Fama-French three-factor model is strongly rejected. The $p_{\text{max}}$ statistic is 0.0005.

7. Conclusion

Lewellen and Nagel (2007) and Lewellen, Nagel, and Shanken (2008) have argued that the tight factor structure of the size and book-to-market portfolios means that many tests of asset pricing models have low power. We find evidence that is consistent with the Lewellen, Nagel, and Shanken critique. Our tests lead to rejection of two popular asset pricing models -- the Fama-French three-factor model and the Campbell-Vuolteenaho model -- when these models are estimated on 25 size and momentum portfolios. The models are more easily and more strongly rejected when estimated on size and momentum portfolios than when they are estimated on the usual size and book-to-market portfolios.

Our technique uses instrumental variables. (To ensure that the instruments are predetermined, we use a split-sample approach.) The parameter estimates, confidence intervals, and the $p_{\text{max}}$ statistic provide many clues about what is missing from the asset pricing models. This can best be illustrated with specific examples. When we estimate the Fama-French three-factor model using only the three Fama-French betas (and a constant) as instruments, we obtain reasonably tight confidence intervals for the Fama-French risk prices. However, we obtain tighter confidence intervals when we add Campbell-Vuolteenaho betas and characteristics (such as momentum). This provides a hint that the Campbell-Vuolteenaho betas and characteristics contain additional information about returns that is not fully captured by the Fama-French betas. This is confirmed by the $p_{\text{max}}$ statistics. When combinations of Campbell-Vuolteenaho betas and various characteristics are added to the baseline instrument set (the Fama-French betas), the Fama-French three-factor model is rejected by the data.

15 In other words, we use the first 30% of the sample to calculate the instruments and the remainder of the sample to estimate the model.
We obtain complimentary results for the Campbell-Vuolteenaho model. When we estimate the model on the 25 size and book-to-market portfolios and use own-model betas (i.e., the Campbell-Vuolteenaho betas) as instruments, we obtain point estimates of the risk prices that are similar to those obtained by Campbell and Vuolteenaho (2004). The confidence intervals are reasonably precise and the model is not rejected by the data. However, if we add other potential risk measures (such as Fama-French betas and the beta on squared market returns), the model is rejected by the data.

Our technique is less kind to the CAPM. When we use the most natural instruments (the CAPM beta and a constant), our technique immediately issues a warning that the instruments are weak. When we use the Fama-French betas as instruments, there is no longer any indication of the weak instrument problem, but the $p_{\text{max}}$ statistic strongly rejects the model.

We find evidence that characteristics such as size, book-to-market, and momentum contain useful information about cross-sectional returns. For example, if we use only size and book-to-market as instruments (and not the own-model betas) when we estimate the Campbell-Vuolteenaho model, there is no sign that the instruments are weak. We obtain point estimates of the risk prices that are similar to those obtained by Campbell and Vuolteenaho (2004), and the confidence intervals are reasonably tight. In a similar vein, there is evidence that characteristics contain additional information about cross-sectional returns that is not fully captured by the asset pricing models. For example, if we estimate the Fama-French three-factor model using own-model betas (plus one or two of the Campbell-Vuolteenaho betas) and add characteristics such as size and book-to-market or momentum, the model is rejected by the data. If we use the same instruments (i.e., Fama-French and Campbell-Vuolteenaho betas) but omit characteristics from the instrument set, the $p_{\text{max}}$ statistic fails to reject the model.
References


Table 1
Estimates of the Campbell & Vuolteenaho Model
(45 portfolios)
Instruments: Size and Book-to-Market

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharpe-Lintner form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{sb} - R_{sf} (g_0)$</td>
<td>0.0027</td>
<td>0</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0071, 0.0182)</td>
<td></td>
</tr>
<tr>
<td>$\beta_{CF}$ premium ($g_1$)</td>
<td>0.0555</td>
<td>0.0679</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0266, 0.1322)</td>
<td>(0.0241, 0.1287)</td>
</tr>
<tr>
<td>$\beta_{DR}$ premium ($g_2$)</td>
<td>0.0030</td>
<td>0.0045</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0084, 0.0094)</td>
<td>(-0.0005, 0.0081)</td>
</tr>
<tr>
<td>$p_{max}$</td>
<td>*</td>
<td>0.9157</td>
</tr>
</tbody>
</table>

The statistic $p_{max}$ is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). The Sharpe-Lintner form of each model imposes the restriction that $g_0 = 0$; i.e., that no constant is included in the model. In this case, no confidence interval is reported since $g_0$ is not estimated. The Black form of each model imposes no restriction on the estimate of $g_0$. The additional variables $Z_i$ are a constant and the means of size and book-to-market for each portfolio over the first 30% of the sample.
Table 2  
Estimates of the Campbell & Vuolteenaho Model  
(25 portfolios)  
Instruments: Size and Book-to-Market

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharpe-Lintner form</th>
</tr>
</thead>
</table>
| $R_{jb} - R_{rf}(g_0)$  
(95% confidence interval) | -0.0011 (-0.0151, 0.0103) | 0 |
| $\beta_{CF}$ premium ($g_1$)  
(95% confidence interval) | 0.0737 (0.0292, 0.1373) | 0.0695 (0.0426, 0.1078) |
| $\beta_{DR}$ premium ($g_2$)  
(95% confidence interval) | 0.0033 (-0.0051, 0.0125) | 0.0026 (-0.0008, 0.0050) |
| $p_{max}$ | * | 0.9911 |

The statistic $p_{max}$ is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). The Sharpe-Lintner form of each model imposes the restriction that $g_0 = 0$; i.e., that no constant is included in the model. In this case, no confidence interval is reported since $g_0$ is not estimated. The Black form of each model imposes no restriction on the estimate of $g_0$. The instruments $Z_i$ are a constant and the means of size and book-to-market for each portfolio over the first 30% of the sample.
Table 3
Estimates of the Campbell & Vuolteenaho Model
25 Size and Book-to-Market Portfolios

Panel A: Instrument Sets with Own-Model Betas, Fama-French Betas, and B/M

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharpe-Lintner form</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Instruments</td>
<td>Instruments</td>
</tr>
<tr>
<td></td>
<td>Own-model betas</td>
<td>Own-model betas &amp; FF3 betas</td>
</tr>
<tr>
<td>$R_{zb} - R_{rf}$ ($g_0$) (95% confidence interval)</td>
<td>0.0008 (-0.0134, 0.0148)</td>
<td>-0.0024 (-0.0138, 0.0053)</td>
</tr>
<tr>
<td>$\beta_{CF}$ premium ($g_1$) (95% confidence interval)</td>
<td>0.0590 (-0.0223, 0.1400)</td>
<td>0.0769 (0.0376, 0.1350)</td>
</tr>
<tr>
<td>$\beta_{DR}$ premium ($g_2$) (95% confidence interval)</td>
<td>0.0027 (-0.0042, 0.0098)</td>
<td>0.0043 (-0.0004, 0.0107)</td>
</tr>
<tr>
<td>$p_{\text{max}}$</td>
<td>*</td>
<td>0.3913</td>
</tr>
</tbody>
</table>
The statistic $p_{max}$ is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). The Sharpe-Lintner form of each model imposes the restriction that $g_0 = 0$; i.e., that no constant is included in the model. In this case, no confidence interval is reported since $g_0$ is not estimated. The Black form of each model imposes no restriction on the estimate of $g_0$. 

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Panel B: Instrument Sets That Include $\beta_{ESR}$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharpe-Lintner form</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Own-model betas, $\beta_{FF3}^{EMR}$, $\beta_{FF3}^{SMB}$, $\beta_{ESR}$</td>
<td>Own-model betas, $\beta_{FF3}^{EMR}$, $\beta_{FF3}^{SMB}$, $\beta_{ESR}$</td>
</tr>
<tr>
<td>$R_{zb} - R_{rf} (g_0)$ (95% confidence interval)</td>
<td>-0.0007 ($\Theta$)</td>
<td>-0.0011 ($\Theta$)</td>
</tr>
<tr>
<td>$\beta_{CF}$ premium ($g_1$) (95% confidence interval)</td>
<td>0.0682 ($\Theta$)</td>
<td>0.0698 ($\Theta$)</td>
</tr>
<tr>
<td>$\beta_{DR}$ premium ($g_2$) (95% confidence interval)</td>
<td>0.0033 ($\Theta$)</td>
<td>0.0036 ($\Theta$)</td>
</tr>
<tr>
<td>$p_{max}$</td>
<td>0.0426</td>
<td>0.0333</td>
</tr>
</tbody>
</table>
Table 4
Estimates of the Campbell & Vuolteenaho Model
25 Size and Momentum Portfolios

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharp-Lintner form</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Instruments</td>
<td>Instruments</td>
</tr>
<tr>
<td></td>
<td>Own-model betas</td>
<td>Own-model betas &amp; momentum</td>
</tr>
<tr>
<td>$R_{dB} - R_{of} (g_0)$ (95% confidence interval)</td>
<td>0.0356 (0.0044, 0.0878)</td>
<td>0.0212 ($\emptyset$)</td>
</tr>
<tr>
<td>$\beta_{CF}$ premium ($g_1$) (95% confidence interval)</td>
<td>-0.0312 (-0.0717, -0.0022)</td>
<td>-0.0391 ($\emptyset$)</td>
</tr>
<tr>
<td>$\beta_{DR}$ premium ($g_2$) (95% confidence interval)</td>
<td>-0.0037 (-0.0238, 0.0254)</td>
<td>-0.0249 ($\emptyset$)</td>
</tr>
<tr>
<td>$p_{max}$</td>
<td>*</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

The statistic $p_{max}$ is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). The Sharpe-Lintner form of each model imposes the restriction that $g_0 = 0$; i.e., that no constant is included in the model. In this case, no confidence interval is reported since $g_0$ is not estimated. The Black form of each model imposes no restriction on the estimate of $g_0$. To ensure that the instruments are predetermined, the own-model betas and momentum are calculated over the first 30% of the sample. The remaining 70% of the sample is used for estimation.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Black form</th>
<th>Sharpe-Lintner form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instruments</td>
<td></td>
<td>Instruments</td>
</tr>
<tr>
<td>( R_{zb} - R_{tf} ) ((g_{0})) (95% confidence interval)</td>
<td>( \beta_{EMR_CAPM} )</td>
<td>( \beta_{EMR_FF3}, \beta_{SMB_FF3}, \beta_{HML_FF3} )</td>
</tr>
<tr>
<td>( \beta ) premium ((g_{1})) (95% confidence interval)</td>
<td>0.0056 (-(\infty, +\infty))</td>
<td>-0.0383 ((\emptyset))</td>
</tr>
<tr>
<td>( p_{\text{max}} )</td>
<td>*</td>
<td>0.0105</td>
</tr>
</tbody>
</table>

The statistic \( p_{\text{max}} \) is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). \( \beta_{EMR\_CAPM} \) is estimated from a regression of portfolio returns on a constant and the EMR factor. \( \beta_{EMR\_FF3}, \beta_{SMB\_FF3}, \beta_{HML\_FF3} \) are estimated from a regression of portfolio returns on a constant, EMR, SMB, and HML. A constant is included in each instrument set. To ensure that the instruments are predetermined, the instruments are generated using the first 30\% of the sample. The remaining 70\% of the sample is used for estimation.
Table 6
Estimates of Fama-French Three-Factor Model
25 Size and B/M Portfolios

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Own-model betas</th>
<th>Own-model betas and CV betas</th>
<th>Own-model betas and CV betas, and momentum</th>
<th>Own-model betas and $\beta_{ESR}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{sb} - R_{rf}$ ((g_0)) (95% confidence interval)</td>
<td>0.0192 (-0.0004, 0.0544)</td>
<td>0.0173 (-0.0009, 0.0429)</td>
<td>0.0149 (-0.0015, 0.0332)</td>
<td>0.0203 (0.0005, 0.0616)</td>
</tr>
<tr>
<td>$B_{MKT}$ premium ((g_1)) (95% confidence interval)</td>
<td>-0.0118 (-0.0454, 0.0069)</td>
<td>-0.0100 (-0.0344, 0.0075)</td>
<td>-0.0076 (-0.0251, 0.0081)</td>
<td>-0.0128 (-0.0522, 0.0061)</td>
</tr>
<tr>
<td>$B_{SMB}$ premium ((g_2)) (95% confidence interval)</td>
<td>0.0015 (-0.0008, 0.0034)</td>
<td>0.0015 (-0.0003, 0.0030)</td>
<td>0.0015 (0.0001, 0.0027)</td>
<td>0.0015 (-0.0012, 0.0036)</td>
</tr>
<tr>
<td>$B_{HML}$ premium ((g_2)) (95% confidence interval)</td>
<td>0.0037 (0.0010, 0.0060)</td>
<td>0.0038 (0.0018, 0.0056)</td>
<td>0.0038 (0.0023, 0.0053)</td>
<td>0.0037 (0.0005, 0.0062)</td>
</tr>
<tr>
<td>$p_{max}$</td>
<td>*</td>
<td>0.3980</td>
<td>0.1943</td>
<td>0.9968</td>
</tr>
</tbody>
</table>

The statistic $p_{max}$ is the p-value for the rejection of the model. The formal definition of the statistic is provided in the paper. The statistic is not reported when a model is exactly identified (denoted with an *). The Sharpe-Lintner form of each model imposes the restriction that $g_0 = 0$; i.e., that no constant is included in the model. In this case, no confidence interval is reported since $g_0$ is not estimated. The Black form of each model imposes no restriction on the estimate of $g_0$. To ensure that the instruments are predetermined, the own-model betas and momentum are calculated over the first 30\% of the sample. The remaining 70\% of the sample is used for estimation.
Table 7
Fama-French Three-Factor Model
Width of Confidence Intervals across Instrument Sets

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Own-Model Betas</th>
<th>Own-Model Betas and Campbell-Vuolteenaho betas</th>
<th>Own-Model Betas, Campbell-Vuolteenaho Betas, and Momentum</th>
<th>Own-Model Betas and $\beta_{ESR}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{zb} - R_{rf}$ ($g_0$)</td>
<td>0.0548</td>
<td>0.0438</td>
<td>0.0347</td>
<td>0.0621</td>
</tr>
<tr>
<td>$B_{MKT}$ premium ($g_1$)</td>
<td>0.0523</td>
<td>0.0419</td>
<td>0.0332</td>
<td>0.0583</td>
</tr>
<tr>
<td>$B_{SMB}$ premium ($g_2$)</td>
<td>0.0042</td>
<td>0.0033</td>
<td>0.0026</td>
<td>0.0048</td>
</tr>
<tr>
<td>$B_{HML}$ premium ($g_2$)</td>
<td>0.0050</td>
<td>0.0038</td>
<td>0.0030</td>
<td>0.0057</td>
</tr>
</tbody>
</table>
Table 8
Estimates of Fama-French Three-Factor Model
25 Size and B/M Portfolios
Additional Instrument Sets

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Own-model betas, $\beta_{CF}^{CV}$, size &amp; B/M</th>
<th>Own-model betas, $\beta_{DR}^{CV}$, size &amp; B/M</th>
<th>Own-model betas, $\beta_{CF}^{CV}$, size &amp; momentum</th>
<th>Own-model betas, $\beta_{CF}^{CV}$, $\beta_{DR}^{CV}$, size &amp; B/M</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{cb} - R_{cf}$ ($g_0$) (95% confidence interval)</td>
<td>0.0039 (Ø)</td>
<td>0.0057 (Ø)</td>
<td>0.0103 (Ø)</td>
<td>0.0042 (Ø)</td>
</tr>
<tr>
<td>$\beta_{MKT}$ premium ($g_1$) (95% confidence interval)</td>
<td>0.0028 (Ø)</td>
<td>0.0010 (Ø)</td>
<td>-0.0033 (Ø)</td>
<td>0.0025 (Ø)</td>
</tr>
<tr>
<td>$\beta_{SMB}$ premium ($g_2$) (95% confidence interval)</td>
<td>0.0018 (Ø)</td>
<td>0.0018 (Ø)</td>
<td>0.0016 (Ø)</td>
<td>0.0018 (Ø)</td>
</tr>
<tr>
<td>$\beta_{HML}$ premium ($g_3$) (95% confidence interval)</td>
<td>0.0042 (Ø)</td>
<td>0.0042 (Ø)</td>
<td>0.0042 (Ø)</td>
<td>0.0042 (Ø)</td>
</tr>
<tr>
<td>$p_{max}$</td>
<td>0.0201</td>
<td>0.0346</td>
<td>0.0493</td>
<td>0.0378</td>
</tr>
</tbody>
</table>
Table 9
Estimates of Fama-French Three-Factor Model
25 Size and Momentum Portfolios

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Own-model Betas</th>
<th>Own-model betas &amp; Size</th>
<th>Own-model betas &amp; Momentum</th>
<th>Own-model betas, Size &amp; Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{zb} - R_{pf} (g_0) )</td>
<td>0.0324</td>
<td>-0.0009</td>
<td>0.0615</td>
<td>0.0458</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0462, 0.0761)</td>
<td>(-0.0506, 0.0215)</td>
<td>(0.0516, 0.0731)</td>
<td>(Ø)</td>
</tr>
<tr>
<td>( \beta_{MKT} ) premium ( (g_1) )</td>
<td>-0.0220</td>
<td>0.0081</td>
<td>-0.0481</td>
<td>-0.0338</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0618, 0.0502)</td>
<td>(-0.0126, 0.0535)</td>
<td>(-0.0587, -0.0390)</td>
<td>(Ø)</td>
</tr>
<tr>
<td>( \beta_{SMB} ) premium ( (g_2) )</td>
<td>0.0017</td>
<td>-0.0005</td>
<td>0.0047</td>
<td>0.0055</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0083, 0.0076)</td>
<td>(-0.0065, 0.0027)</td>
<td>(0.0032, 0.0063)</td>
<td>(Ø)</td>
</tr>
<tr>
<td>( \beta_{HML} ) premium ( (g_3) )</td>
<td>-0.0093</td>
<td>0.0045</td>
<td>-0.0240</td>
<td>-0.0211</td>
</tr>
<tr>
<td>(95% confidence interval)</td>
<td>(-0.0286, 0.0239)</td>
<td>(-0.0052, 0.0262)</td>
<td>(-0.0288, -0.0201)</td>
<td>(Ø)</td>
</tr>
<tr>
<td>( p_{max} )</td>
<td>*</td>
<td>0.0848</td>
<td>0.0605</td>
<td>0.0005</td>
</tr>
</tbody>
</table>
Appendix

A  Econometric Methods

Let $R_{it}$ refer to returns on asset $i$ at time $t$, $i = 1, \ldots, n$, $t = 1, \ldots, T$, $R_t = (R_{1t}, R_{2t}, \ldots, R_{nt})'$, and let the $T \times n$ matrix $\bar{R} = [R_1, \ldots, R_T]'$ and the $n \times 1$ vector which includes the time series means of these asset returns over the full sample

$$\bar{R} = (\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_n)', \quad \bar{R}_i = \frac{1}{T} \sum_{t=1}^{T} R_{it}. \quad \text{(A.1)}$$

Denote by $f_{jt}$ the realization of factor $j$ at time $t$, $j = 1, \ldots, s$, $t = 1, \ldots, T$, $f_t = (1, f_{1t}, f_{2t}, \ldots, f_{st})'$, and define $W$ as the $T \times (s + 1)$ matrix

$$W = [f_1, \ldots, f_T]' .$$

Consider the regression of returns on a constant and factor realizations leading to the $n \times s$ matrix of estimated factor betas

$$\hat{\beta} = \left( A (W'W)^{-1} W'R \right)' , \quad \text{(A.2)}$$

where $A$ is the $s \times (s + 1)$ selection matrix

$$A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix} .$$

let $\hat{\beta}_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, s$, refer to the element of $\hat{\beta}$ which corresponds to the estimate for asset $i$ of the factor $j$ beta.

Some of the methods we use rely on a sample split approach. So to avoid confusion, we let $\hat{\beta}_{(1)}^{(1)}$, $\hat{\beta}_{ij}^{(1)}$, $j = 1, \ldots, s$, $W^{(1)}$ and $\bar{R}^{(1)}$ refer to the counterparts of $\hat{\beta}$, $\hat{\beta}_{ij}$, $j = 1, \ldots, s$, $W$ and $\bar{R}$, obtained from the subsample which spans observations 1 to $\bar{T}$. Similarly, we let $\hat{\beta}_{(2)}^{(2)}$, $\hat{\beta}_{ij}^{(2)}$, $j = 1, \ldots, s$, $W^{(2)}$ and $\bar{R}^{(2)}$ refer to their counterparts from the subsample which spans observations $\bar{T} + 1$ to $T$. We assume that $\bar{T}$ and $T - \bar{T}$ are large enough to allow application
of (i) the multivariate regression of $R^{(1)}$ on a $W^{(1)}$ [that is $\tilde{T} - (s + 1) - n > 0$] and, (ii) the univariate regression of $\hat{\beta}^{(2)}$ on a constant and $\tilde{\beta}^{(2)}$ [that is $n > s + 1$]. Subject to these restrictions, typically, $\tilde{T}$ may be set to $\cdot 3T$.

Our aim is to estimate the coefficient $\gamma$ of the regression

$$R = X\gamma + u,$$

$$X = \begin{bmatrix} \tau_n, & \beta \end{bmatrix},$$

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_s)'$$

(A.3)

where $u$ is an error term with mean zero and $\tau_n$ is an $n$-dimensional vector of ones.

Since $\beta$ is unobservable, estimating equations typically consist of the regression of $R$ on $\hat{X} = \begin{bmatrix} \tau_n, & \hat{\beta} \end{bmatrix}$. Typically, the pricing restrictions underlying (A.3) imply

$$E(R) = X\gamma$$

(A.4)

although we allow for model misspecification, mainly by focusing on the estimating equations, as with e.g. GMM, as follows.

Our econometric methods rely on formulating and analyzing pricing error equations based on the asset pricing model under test, of the form:

$$X_i'\delta + F_i(Y, \vartheta) = U_i, \quad i = 1, \ldots, n,$$

(A.5)

where $F_i(\ldots, i = 1, \ldots, n$ are scalar possibly non-linear functions [in-variables and in-parameters] that may have a different form for each observation, $\vartheta$ is an $m \times 1$ vector of unknown parameters of interest, $X_i$ is a $k_1$-dimensional vector of exogenous variables, $Y$ is a matrix of observed [endogenous and exogenous] variables and $U_i$ is a disturbance with mean zero. For instance, in the above context, $\delta = -\gamma_0, \vartheta = (\gamma_1, \ldots, \gamma_s)'$ so $s = m$

$$Y = [R, W], \quad X_i = 1, \quad F_i(Y, \vartheta) = R_i - \sum_{j=1}^{s} \gamma_j \tilde{\beta}_{ij}, \quad i = 1, \ldots, n.$$  

(A.6)

It is worth noting that in (A.6), the $\tilde{\beta}_{ij}$ are functions [refer to (A.2)] of the random return matrix $R$ given $W$. So the series of pricing errors under consideration obtains as

$$U_i = R_i - \gamma_0 - \sum_{j=1}^{s} \gamma_j \tilde{\beta}_{ij}, \quad i = 1, \ldots, n.$$ 

(A.7)

Because the principles underlying our test methodology are not necessarily restricted to cross-sectional factor models, we believe that presenting our test procedure in a general
though tractable form allows one to consider possible extensions for alternative models leading
to pricing error functions of a different structural form. It is worth noting that despite
our focus on GMM-type pricing error analysis, our methods differs from traditional GMM
(as reviewed in e.g. Shanken & Zhou (2007)). Indeed, whereas GMM-motivated pricing
errors [which impose (A.4)] are typically expressed as a function of the unknown betas, our
formulation uses the estimated ones, which allows for model mis-specification (since (A.4) is
not explicitly imposed) and conforms with a two-pass approach. As may become clear from
our presentation, (A.5) - (A.7) may be viewed as an econometric structural specification which
serves to define the risk premia [that is the \( \gamma \) vector] allowing for random risk measures. In
other words, \( \gamma \), our parameter of interest, is defined by (A.5) - (A.7) rather than strictly
via (A.4). Our definition of the risk premia formally embeds the possibility that betas are
noisy, or formally, that the betas are random variates; most importantly, the level of the
confidence set we will propose for the risk premia will not be affected by the quality of the
first pass regression i.e. the model underlying the estimated \( \hat{\beta}_{ij} \). The intuition underlying
our approach may be traced back to Fama & MacBeth (1973), Fama & French (1992) or
Fama & French (1995), though here we formally correct for endogeneity of the pre-estimated
"betas". While our proposed point estimators require that (A.5) - (A.7) uniquely define \( \gamma \),
our confidence sets as defined below are valid [i.e. have the desired level] even when (A.5) -
(A.7) hold for some set of \( \gamma \) values. Point identification is thus not required, and no further
conditions on the acceptable set of \( \gamma \) values are needed. Our framework thus differs in this
respect from available works on two-pass methods that allow for mis-specification; see Kan
et al. (2008), and the references therein.\(^1\)

Aside from these definitional distinctions, we still analyze pricing errors so defined, based
on a GMM-type rationale as follows. For any known \( \hat{\theta} \), if the following hypothesis

\[
H_0 : \theta = \hat{\theta}
\]  \hspace{1cm} (A.8)

holds in the context of (A.5), then clearly \( X_i' \delta + F_i (Y, \hat{\theta}) = U_i \). Thus, if \( Z_i \) is a \( k_2 \times 1 \) vector
of exogenous or predetermined variables such that \( k_2 \geq m \), then the coefficients of \( Z_i \) in the

\(^1\)For instance, for a model that is misspecified [in the sense that \( E(\bar{R}) \neq X\gamma \) regardless of \( \gamma \)], Kan et al.
(2008) define \( \gamma \) as the unique vector that minimizes a quadratic form in \( [E(\bar{R}) - X\gamma] \).
artificial regression

$$\mathcal{F}_i \left( \mathcal{Y}, \hat{\theta} \right) = \mathcal{X}_i' \mathbf{\alpha}_X + \mathcal{Z}_i' \mathbf{\alpha}_Z + \varepsilon_i$$  \hspace{1cm} (A.9)

should be statistically zero. Hence, $\mathcal{H}_0$ can be tested by assessing

$$\mathcal{H}_0' : \mathbf{\alpha}_Z = 0$$  \hspace{1cm} (A.10)

in regression (A.9). $\mathcal{Z}_i$ can be viewed as a vector of instruments, which may include exogenous variables [or exogenous functions of these variables] in $\mathcal{Y}$.

Note that extending the above test to accommodate joint tests on $\delta$ and $\vartheta$ of the form

$$\mathcal{H}_{10} : \delta = \hat{\delta} \text{ and } \vartheta = \hat{\vartheta}$$

is straightforward. Indeed, our method leads to assessing in the context of the artificial regression

$$\mathcal{F}_i \left( \mathcal{Y}, \hat{\theta} \right) + \mathcal{X}_i' \hat{\delta} = \mathcal{X}_i' \mathbf{\alpha}_X + \mathcal{Z}_i' \mathbf{\alpha}_Z + \varepsilon_i$$

the significance of all regressors. We thus see that $\delta$ may be easily partialled out, whereas inference on the components of $\vartheta$ [see below] is non-separable, even when the model is additively separable. So we proceed in what follows with our focus on tests of $\mathcal{H}_0$ and associated confidence regions for components of $\vartheta$.

Whereas usual estimation and testing in the context of (A.5) requires dealing with endogeneity and error-in-variables, translating inference into (A.9) leads to the regular regression framework, since regressors in the latter framework [namely $\mathcal{X}_i'$ and $\mathcal{Z}_i'$] are exogenous. In this way, endogeneity and error-in-variable problems are conveniently dealt with, and [via the definition of the pricing errors $\mathcal{F}_i(\mathcal{Y}, \hat{\vartheta})$] our approach remains structural. Most importantly, and since our approach has an IV-based interpretation, it is worth noting that the validity of inference in the context of (A.9) is not affected by the quality of the instruments $\mathcal{Z}_i$. Furthermore, and this is a crucial observation from our perspective, our proposed test is valid [in terms of size control] whether the econometric model leading to estimating the "betas" is correctly specified [or in other words is "the true model"] or not, or whether we used all relevant instruments or not. To sum up, we can thus test $\mathcal{H}_0$ correctly [formally, ensuring level-control at least asymptotically, as argued below] without assuming that $\vartheta$ is
identified, allowing for possible mis-specification in pre-estimating "bets", and allowing for missing instruments.

Indeed, in the context of (A.9), the F-statistic for $\mathcal{H}_0$ is given by

$$
T\left(\hat{\vartheta}, \mathcal{F}\right) = \frac{\mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right)' (M[\mathcal{X}] - M[\mathcal{W}]) \mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right) / k_2}{\mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right)' (M[\mathcal{W}]) \mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right) / (n - k_1 - k_2)}
$$

(A.11)

where $\mathcal{W} = \begin{bmatrix} \mathcal{X} & \mathcal{Z} \end{bmatrix}$, $M[\mathcal{W}] = I - \mathcal{W} (\mathcal{W}^\prime \mathcal{W})^{-1} \mathcal{W}'$, $M[\mathcal{X}] = I - \mathcal{X} (\mathcal{X}' \mathcal{X})^{-1} \mathcal{X}'$,

$$
\mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right) = [\mathcal{F}_1(\mathcal{Y}, \hat{\vartheta}), \ldots, \mathcal{F}_n(\mathcal{Y}, \hat{\vartheta})]' ,
$$

(A.12)

$$
\mathcal{Z} = \begin{bmatrix} \mathcal{Z}_1 & \mathcal{Z}_2 & \ldots & \mathcal{Z}_n \end{bmatrix}', \quad \mathcal{X} = \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 & \ldots & \mathcal{X}_n \end{bmatrix}'
$$

(A.13)

so $\mathcal{F}\left(\mathcal{Y}, \hat{\vartheta}\right)$ is $n \times 1$, $\mathcal{Z}$ is $n \times k_2$ and $\mathcal{X}$ is $n \times k_1$.

Since (A.9) is a classical exogenous-regressor based regression equation, the statistic $T(\hat{\vartheta}, \mathcal{F})$ follows a central F-distribution with degrees of freedom $k_2$ and $n - k_1 - k_2$ which we denote as $F(k_2, n - k_1 - k_2)$. For our cross-sectional tests, the latter distributional result rests on the following: the structure under test [which defines the risk premia] is (A.5) - (A.7) with (A.8) and the instruments are exogenous; no further restrictions are required, neither on the specification for the model used to estimate the betas nor on the quality of the instruments.² Let $F_{\alpha}(k_2, n - k_1 - k_2)$ refer to the $\alpha\%$ cut-off from the $F(k_2, n - k_1 - k_2)$ distribution. Furthermore, let $\hat{p}(\hat{\vartheta}, \mathcal{F})$ refer to the p-value associated with $T(\hat{\vartheta}, \mathcal{F})$ based on this distribution, i.e. setting $G_{F(k_2, n - k_1 - k_2)}(x) = P[F(k_2, n - k_1 - k_2) \geq x]$, we have

$$
\hat{p}(\hat{\vartheta}, \mathcal{F}) = G_{F(k_2, n - k_1 - k_2)}[T(\hat{\vartheta}, \mathcal{F})].
$$

(A.14)

To obtain a confidence set for $\vartheta$ that enjoys all the above described statistical properties [robustness to the parameter identification status, the quality of instruments, the specification for pre-estimated risk factors and to missing instruments], the above test is inverted. In other words, to obtain a joint confidence region [of level 1-$\alpha$] for the elements of the vector $\vartheta$, we need

²Formally, $T(\vartheta_0, \mathcal{F})$ follows a central F-distribution with degrees of freedom $k_2$ and $n - k_1 - k_2$ when $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T$ are i.i.d. Gaussian, although $\chi^2$ critical points $[k_2 T(\vartheta_0, \mathcal{F}) \sim \chi^2(k_2)$ under $\mathcal{H}_0]\$ may also be used given the regular central limit theory associated with classical regression. GLS corrections may also be considered for non-spherical regression errors.
to collect all \( \hat{\vartheta} \) values that are not rejected by the above described test at the \( \alpha \) significance level. Then using projection methods, we can obtain confidence sets for each element of \( \vartheta \), or more generally for any scalar function of the form \( \omega' \hat{\vartheta} \) where \( \omega \) is a non-zero \( m \)-dimensional vector. Clearly, when \( \omega \) is a selection vector, which consist e.g. of zeros everywhere except at position \( j \), the function \( \omega' \hat{\vartheta} \) yields the \( j \)th components of \( \hat{\vartheta} \). Inverting pivotal statistics is rare in financial application, although Shanken (1996) and Lewellen et al. (2006) argue that proceeding in this may guard against many pitfalls of asset pricing tests.

Projection methods typically require minimizing and maximizing \( \omega' \hat{\vartheta} \) subject to \( \hat{\vartheta} \) being covered by the confidence region. So in general, and depending on the structure of the \( \mathcal{F}_i(Y, \hat{\vartheta}) \) function, the joint confidence region is derived by numerical techniques [for example, grid searches], and its projections require global (constrained) maximizing algorithms. If the structural equation is additively separable and linear in \( \vartheta \), that is if \( \mathcal{F}_i(Y, \hat{\vartheta}) \) in (A.5) takes the from

\[
\mathcal{F}_i \left( Y, \hat{\vartheta} \right) = \mathcal{J}_i(Y) - \mathcal{G}_i(Y)' \hat{\vartheta}
\]

where \( \mathcal{J}_i(., .), i = 1, \ldots, n \) are scalar possibly non-linear functions of the data and \( \mathcal{G}_i(., .), i = 1, \ldots, n \) are \( m \)-dimensional possibly non-linear functions of the data, then we can exploit the analytical solutions from Dufour & Taamouti (2005). The latter were originally proposed for linear simultaneous equation model, yet it can be easily verified [see also Dufour & Taamouti (2007), Dufour et al. (2006), Bolduc et al. (2008)] that the geometrical tools applied by Dufour & Taamouti (2005) also work for an equation of the form (A.5) with (A.15); in the (A.7) case, we have \( \mathcal{J}_i(Y) = \bar{R}_i \) and \( \mathcal{G}_i(Y) = (\hat{\beta}_{i1}, \ldots, \hat{\beta}_{is})' \) and \( \hat{\vartheta} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_s)' \) so \( \mathcal{G}_i(Y)' \hat{\vartheta} = \sum_{j=1}^s \hat{\gamma}_j \hat{\beta}_{ij} \).

Indeed, our paper provides an interesting extension of the geometrical approach in Dufour & Taamouti (2005) to the cross-sectional asset pricing test context. For further reference, let

\[
\mathcal{J}(Y) = [\mathcal{J}_1(Y), \ldots, \mathcal{J}_n(Y)]', \quad \mathcal{G}(Y) = \left[ \begin{array}{ccc}
\mathcal{G}_1(Y) & \mathcal{G}_2(Y) & \mathcal{G}_n(Y)
\end{array} \right]'
\]

\footnote{It is easy to see [refer e.g. to Dufour (2003)] that if (A.5) is a linear Limited Information Simultaneous Equation, then (A.11) corresponds to the test proposed by Anderson & Rubin (1949). Dufour & Taamouti (2006) also show that this statistic corresponds closely to Stock & Wright (2000)'s asymptotic GMM-based test. See also Dufour (1997), Dufour & Jasiak (2001), Dufour & Taamouti (2007), Staiger & Stock (1997), and the surveys by Stock et al. (2002) and Dufour (2003).}
so that $\mathcal{J}(\mathcal{Y})$ is $n \times 1$ and $\mathcal{G}(\mathcal{Y})$ is $n \times m$.

For completion, we provide in what follows the formulas based on Dufour & Taamouti (2005) as applied in the present paper. Inverting the above discussed F-test requires solving the following inequality in $\hat{\vartheta}$:

$$\frac{\mathcal{F}(\mathcal{Y}, \hat{\vartheta})^T (M[\mathcal{X}] - M[\mathcal{W}]) \mathcal{F}(\mathcal{Y}, \hat{\vartheta}) / k_2}{\mathcal{F}(\mathcal{Y}, \hat{\vartheta}) M[\mathcal{W}] \mathcal{F}(\mathcal{Y}, \hat{\vartheta}) / (n - k_1 - k_2)} < F_{\alpha}(k_2, n - k_1 - k_2). \quad (A.16)$$

Inequality (A.16) may be re-expressed as

$$\hat{\vartheta}^T A_{22} \hat{\vartheta} + A_{12}' \hat{\vartheta} + A_{11} < 0 \quad (A.17)$$

where

$$A_{22} = \mathcal{G}(\mathcal{Y})' H \mathcal{G}(\mathcal{Y}), \quad A_{12} = -2 \mathcal{G}(\mathcal{Y})' H \mathcal{J}(\mathcal{Y}), \quad A_{11} = \mathcal{J}(\mathcal{Y})' H \mathcal{J}(\mathcal{Y}),$$

$$H = M[\mathcal{X}] - \left[ 1 + \frac{k_2 F_{\alpha}(k_2, n - k_1 - k_2)}{n - k_1 - k_2} \right] M[\mathcal{W}]$$

so projections based confidence sets for any linear transformation of $\hat{\vartheta}$ of the form $\omega' \hat{\vartheta}$ [denoted $\text{CS}_\alpha(\omega' \hat{\vartheta})$ in what follows] can be obtained as follows. Let $\bar{A} = -\frac{1}{2} A_{22}^{-1} A_{12}, \quad \bar{D} = \frac{1}{4} A_{12}' A_{22}^{-1} A_{12} - A_{11}$. If all the eigenvalues of $A_{22}$ are positive so $A_{22}$ is positive definite then:

$$\text{CS}_\alpha(\omega' \hat{\vartheta}) = \left[ \omega' \bar{A} - \sqrt{\bar{D} (\omega' A_{22}^{-1} \omega)}, \omega' \bar{A} + \sqrt{\bar{D} (\omega' A_{22}^{-1} \omega)} \right], \quad \text{if} \quad \bar{D} \geq 0, \quad (A.18)$$

$$\text{CS}_\alpha(\omega' \hat{\vartheta}) = \emptyset, \quad \text{if} \quad \bar{D} < 0. \quad (A.19)$$

If $A_{22}$ is non-singular and has one negative eigenvalue then: (i) if $\omega' A_{22}^{-1} \omega < 0$ and $\bar{D} < 0$:

$$\text{CS}_\alpha(\omega' \hat{\vartheta}) = \left] -\infty, \omega' \bar{A} - \sqrt{\bar{D} (\omega' A_{22}^{-1} \omega)} \right] \cup \left[ \omega' \bar{A} + \sqrt{\bar{D} (\omega' A_{22}^{-1} \omega)}, +\infty \right]; \quad (A.20)$$

(ii) if $\omega' A_{22}^{-1} \omega > 0$ or if $\omega' A_{22}^{-1} \omega \leq 0$ and $\bar{D} \geq 0$ then:

$$\text{CS}_\alpha(\omega' \hat{\vartheta}) = \mathbb{R}; \quad (A.21)$$

(iii) if $\omega' A_{22}^{-1} \omega = 0$ and $\bar{D} < 0$ then:

$$\text{CS}_\alpha(\omega' \hat{\vartheta}) = \mathbb{R} \setminus \{ \omega' \bar{A} \}. \quad (A.22)$$
The projection is given by (A.21) if $A_{22}$ is non-singular and has at least two negative eigenvalues.

An empty confidence region occurs when

$$T_{\text{min}}(\mathcal{F}) \geq F_\alpha(k_2, T - k_1 - k_2), \quad T_{\text{min}}(\mathcal{F}) = \min_{\hat{\vartheta}} \left\{ T(\hat{\vartheta}, \mathcal{F}) \right\}$$

or alternatively, when

$$p_{\text{max}}(\mathcal{F}) \leq \alpha, \quad p_{\text{max}}(\mathcal{F}) = \max_{\hat{\vartheta}} \left\{ \hat{p}(\hat{\vartheta}, \mathcal{F}) \right\}$$

where $\hat{p}(\hat{\vartheta}, \mathcal{F})$ is the $p$-value defined above [refer to A.14] associated with the $T(\hat{\vartheta}, \mathcal{F})$ statistic. It follows that $T_{\text{min}}(\mathcal{F})$ may be interpreted as a J-type specification test statistic, in which case the $F_\alpha(k_2, n - k_1 - k_2)$ provides an identification-robust bound cut-off point.

Because of the duality between our confidence set and test procedures, and because the cut-off point is invariant to $\hat{\vartheta}$, an empty confidence region [of level 1-$\alpha$] corresponds to a significant J-type test at level $\alpha$.

Analyzing the parameters values associated with the largest $p$-value, which corresponds to the model most compatible with the data, or, alternatively, that is least-rejected provides useful information. This corresponds to the Hodges-Lehman estimation principle [see Hodges & Lehmann (1963), Hodges & Lehmann (1983) and Dufour et al. (2006)]. Interestingly, when the structural equation is additively separable and linear in $\vartheta$, i.e. when the model takes the form (A.5) with (A.15), then the value of $\hat{\vartheta}$ which minimizes $T(\hat{\vartheta}, \mathcal{F})$ can be obtained using eigenvalue eigenvector solutions. Let $\hat{\lambda}$ refer to the eigenvalue associated with the following determinantal equation

$$\begin{bmatrix} \mathcal{J}(\mathcal{Y}) & \mathcal{G}(\mathcal{Y}) \end{bmatrix}' M[\mathcal{X}] \begin{bmatrix} \mathcal{J}(\mathcal{Y}) & \mathcal{G}(\mathcal{Y}) \end{bmatrix} - \hat{\lambda} \left[ \mathcal{J}(\mathcal{Y}) \mathcal{G}(\mathcal{Y}) \right]' M[\mathcal{W}] \left[ \mathcal{J}(\mathcal{Y}) \mathcal{G}(\mathcal{Y}) \right] = 0$$

then the Hodges-Lehman point estimates [for convenience, we provide the formula for $\hat{\vartheta}$ and $\hat{\delta}$] obtain as follows:

$$\begin{bmatrix} \hat{\vartheta} \\ \hat{\delta} \end{bmatrix} = \begin{bmatrix} \mathcal{G}(\mathcal{Y})' \mathcal{G}(\mathcal{Y}) - \hat{\lambda} \mathcal{G}(\mathcal{Y})' M[\mathcal{W}] \mathcal{G}(\mathcal{Y}) & \mathcal{G}(\mathcal{Y})' \mathcal{X}' \\ \mathcal{X}' \mathcal{G}(\mathcal{Y}) & \mathcal{X}' \mathcal{X} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{G}(\mathcal{Y})' - \hat{\lambda} \mathcal{G}(\mathcal{Y})' M[\mathcal{W}] \mathcal{G}(\mathcal{Y}) & \mathcal{G}(\mathcal{Y})' \mathcal{X}' \\ \mathcal{X}' \mathcal{G}(\mathcal{Y}) & \mathcal{X}' \mathcal{X} \end{bmatrix} \mathcal{J}(\mathcal{Y}).$$

(A.23)
These formulae correspond to LIML estimation in linear Simultaneous Equations models although of course their K-class (rather than the likelihood-based) interpretation [see, e.g. (Davidson & MacKinnon 1993, Chapter 18) or Wang & Zivot (1998)] is valid in our setting. Here again, the fact that the econometric specification leading to the "beta" estimates is non-linear does not invalidate this solution.

The above discussion assumed that exogenous instruments exist. To address this problem empirically, we proceed following recommendations from the weak instruments literature and rely on a split-sample method [see e.g. Dufour & Jasiak (2001) or Angrist & Krueger (1994)]. Specifically, we construct instruments using available data from time 1 to \( \tilde{T} \). We then construct pricing errors from the remaining samples

\[
\mathcal{U}_{i}^{(2)} = \tilde{R}_{i}^{(2)} - \gamma_{0} - \sum_{j=1}^{s} \gamma_{j} \hat{\beta}_{ij}^{(2)}, \quad i = 1, \ldots, n, \tag{A.24}
\]

and proceed by test inversion as above. Interestingly, and in contrast with traditional split sample methods including Dufour & Jasiak (2001), splitting the sample here does not cost degrees-of-freedom, since the \( F_{\alpha}(k_{2}, n - k_{1} - k_{2}) \) distributional result we use is not altered by the size of the underlying time series regression. For an OLS-based split sample cross-sectional regression method, see also Beaulieu et al. (2008).

The above defined GMM-type procedure is valid for any exogenous instrument set even if its explanatory power is weak, in the sense that test rejections are compelling (are not spurious), or alternatively, the values of \( \gamma \) not retained in our confidence set can be safely rejected at the considered significance level. Nevertheless, the power of the inverted test and the tightness of the associated confidence set depend on the choice of instruments and on their explanatory power. In our empirical setting, estimates of the betas from the first subsample [formally, the \( \hat{\beta}_{ij}^{(1)} \)] provide natural instruments. We also add instruments that may capture missing risk measures. As originally argued in Fama & MacBeth (1973), this would provide useful clues about the direction in which a given asset pricing model should be modified.
References


