Duration-Based Volatility Estimation

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Abstract

We develop a novel approach to the estimation of the integrated volatility of a general jump-diffusion with stochastic volatility. Our approach exploits the fundamental duality between the speed (distance traveled per fixed unit time) and passage time (time taken to travel a fixed distance) of the Brownian motion. The new class of IV estimators derived in this paper are shown to be robust to both jumps and market micro structure noise. Moreover, their asymptotic properties are superior to those of existing robust IV estimators.

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1 Introduction

The importance of return volatility for asset pricing theory as well as practical financial management has led to a rich literature on volatility estimation, mostly based on time series observations of asset prices. Information regarding asset values historically has been available to the research community only through selective transaction and settlement prices such as daily opening and closing prices and recorded high and low prices during the trading day. However, within the last decade tick-by-tick data for many liquid financial markets, detailing all transactions and quotes with corresponding time stamps, have become commonplace. Hence, the price history is available as a sequence of observations on a two-dimensional vector whose coordinates take values on a price and time grid dictated by the institutional set-up of the market.\footnote{Of course, additional information may also be available, such as the depth of the order book and size of transactions, etc. We do not consider procedures which draw on such auxiliary information. The expanded information set can be analyzed via so-called marked point processes, see, e.g., the surveys by Engle and Russell (2004), Bauwens and Hautsch (2008) and Pacurar (2008).} For example, for stocks traded on the New York Stock Exchange (NYSE) prices are recorded in increments of one cent and time in increments of one second.

Hence, nowadays, the task of volatility estimation is to convert the sequence of recorded high-frequency price-time data into a measure of return variation for a given period of interest such as a trading day. The main complication is that return volatility is stochastically evolving throughout the trading day: it displays a pronounced intraday pattern (diurnal effect), is subject to large short-lived shocks (news release effect), and features a strong degree of temporal persistence at the inter-daily horizon (volatility clustering effect). This raises a number of questions regarding the underlying concept of volatility. First, should we think of volatility in continuous time or as a discrete-time process updated at equidistant points (in calendar time) or for each new transaction or quote (in tick time)? Second, do we estimate volatility at each point in time within the trading day and then subsequently aggregate to obtain a measure covering the interval of interest, or should we aim more directly
for an overall daily measure in one step? Third, how do we formally interpret the volatility measure if it represents return variation over a period where the underlying high-frequency volatility potentially is fluctuating strongly?

We adopt the coherent framework for addressing these issues which has been developed within the recent literature on realized volatility. We posit an underlying continuous-time setting and a frictionless market where expected asset returns are finite and risk-free arbitrage opportunities are absent. This implies that the asset price process constitutes a special semi-martingale for which the relevant asset return variation is given by the associated quadratic variation process, see, e.g., Back (1991) and Andersen, Bollerslev, and Diebold (2008) for discussion. A general representation of a continuous-time semimartingale in financial economics is the stochastic volatility (SV) jump-diffusion. The standard “plain vanilla” realized volatility (RV) estimator is given by the cumulative sum of equidistant high-frequency intraday squared returns and, under the given (ideal) conditions, it is consistent for the underlying quadratic return variation as the intraday return horizon shrinks to zero. What makes this non-parametric procedure so attractive is that it remains valid for a very general class of underlying processes while the implementation is entirely model independent. In practice, however, one must contend with the fact that market imperfections introduce significant microstructure noise in ultra high-frequency data. Consequently, most implementations of RV estimators are limited to exploit the data at much lower frequencies such as 2-5 minute returns.

A common trait of these realized return variation procedures is that the high-frequency price-time data are used to directly measure the size of the price changes over an exogenously given time grid, either in the form of equidistant sampling (calendar time) or over a certain number of transactions or quotes (event or tick time). Such measures readily map into corresponding volatility measures expressed in terms of the observed price variation per time unit. However, a natural dual approach exists which focuses on measuring the time duration per unit price change, the so-called price duration. In this paper, we adopt this dual
approach as a method of robust RV estimation through the measurement of price durations. In doing so, we exploit the theory of durations for Brownian excursions. While duration-based volatility estimation is not new to financial econometrics, it has to our knowledge not been used for RV estimation in models of stochastic volatility in a systematic way, nor have its robustness properties been fully explored. The closest precursor to our work is arguably Cho and Frees (1988) who also consider duration based volatility estimation. However, they limit attention to the unduly restrictive case of constant volatility and do not consider the very significant effects induced by finite sample biases and duration censoring. This early application of the price duration approach to volatility estimation appears to have generated little, if any, subsequent research.\textsuperscript{2,3}

Much of the recent volatility estimation literature has been dedicated to developing techniques that render the measurement and inference of volatility robust to market microstructure noise at the very highest sampling frequencies. We show that our duration-based estimators have excellent asymptotic efficiency relative to existing comparable noise-robust RV estimators. When price jumps are present, the standard RV measures include the squared jump components. A number of procedures have been developed to estimate the jumps and continuous components of volatility separately. In this context, we show that our estimator is robust to jumps as it is consistent for the continuous part of the quadratic variation. These properties are confirmed through extensive Monte Carlo simulation studies and fur-

\textsuperscript{2}We have identified one actual application, namely Park (1993), while less directly related papers, which fail to cite Cho and Frees (1988), include Zumbach (1998) who considers the scaling properties of price durations across different thresholds and Kamstra and Milevsky (2005) who develop a test for the geometric Brownian motion with drift for the S&P 500 index based on the closed-form expression for the first hitting time distribution.

\textsuperscript{3}A more fertile literature with applications exploiting price durations is the discrete-time parametric approach to high-frequency trade dynamics using point processes. These are extensions of the Autoregressive Conditional Duration (ACD) models, pioneered by Engle and Russell (1998). Later work develops Stochastic Duration and Stochastic Intensity models, but all rely on discrete time ARCH or SV style specifications, see e.g., the survey by Bauwens and Hautsch (2008)). The conceptual differences between our approach and this literature mirror the distinction between the realized volatility measures and high-frequency intraday ARCH and SV models. The realized volatility approach is nonparametric and model-free, based on a general continuous-time setting and focused on ex-post volatility measurement over a non-trivial interval, while ACD type models are parametric, cast in discrete time and focus on estimation of the spot volatility or price intensity at each transaction or quote, that is, an ex-ante forecast of return variation over the next instant. As such, they represent very different methodologies, even if they explore similar quantities.
ther investigated using actual data. Analyzing tick-by-tick data for the sample of 30 Dow Jones stocks from January 2005 through May 2007, we find that our estimator performs well compared with leading sub-sampled RV and bi-power variations estimators of integrated volatility. Moreover, duration based estimators are not substantially harder to implement and are not sensitive to the choice of threshold in a range above 3-4 bid-ask spreads. Since the duration approach extracts information from the same underlying data as RV estimators, but does so in a very different manner, the two procedures are naturally complements, even ignoring the rather compelling efficiency properties of the duration-based measures.

The remainder of the paper is organized as follows. Section 2 briefly reviews the related introduces the basic theory of Brownian excursions in the constant volatility case. Section 3 develops an estimator of RV based on a localization argument and derives the asymptotic properties of the estimator. Section 4 provides a Monte Carlo study ... Section 5 applies the estimation to the set of 30 Dow Jones stocks... Section 6 concludes.

2 Related Realized Volatility Literature

In the RV literature, the evolution of the log price of a single financial asset is given in reduced form by a semi-martingale characterized as the unique (weak) solution to a stochastic differential equation (SDE)

\[ ds_t = \mu_t \, dt + \sigma_t \, dW_t + \Delta X_t, \quad s_0 \equiv s \]  

(1)

The solution to (1) is known as a jump-diffusion process with instantaneous drift \( \mu_t \), instantaneous volatility \( \sigma_t \) and jumps given by the finite activity jump process \( X_t \). Two key objects of interest are the integrated volatility (IV) and quadratic variation (QV) of the log
price process\textsuperscript{4}

\[ IV_t = \int_0^t \sigma_u^2 \, du, \quad QV_t = IV_t + \sum_{0 < u \leq t} \Delta X_u^2 \quad (2) \]

The quadratic variation has two separate components, namely, first, the return variation generated by the diffusive volatility term and, second, the cumulative sum of squared jump sizes. The latter component is important for a variety of reasons, including the interpretation of news and practical risk management. However, the former is often of key importance for volatility forecasting as the continuous or diffusive component of return variation is the main driver of the pronounced persistence in return volatility. As such, methods have been developed to separately identify these two components, as we discuss below.

Although the volatility process of asset returns is itself inherently unobservable, the volatility path \( \{\sigma_t\} \) and jumps \( \{\Delta X_t\} \) are de facto observable in the hypothetical scenario where the econometrician observes the log price process \( \{s_t\} \) in continuous time, since

\[ \forall t : \lim_{m \to \infty} \left( \int_0^t \sigma_u^2 \, du + \sum_{0 < u \leq t} \Delta X_u^2 - \sum_{k=0}^{m} \left[ s_{kt+1} - s_{kt} \right]^2 \right) = 0 \quad (3) \]

(see e.g. Karatzas and Shreve (1991, Theorem 1.5.8), Protter (2004)). In practice, however, the choice of sampling frequency is limited by the liquidity of the given asset as well as the significant market microstructure noise found in the data from most financial markets. The presence of such noise in high-frequency returns implies that the log price process itself is latent and the limit (3) cannot be expected to hold in practice. Nonetheless, (3) motivates

\textsuperscript{4}In the absence of jumps Hull and White (1987), for instance, show that the Black-Scholes option pricing formula can be generalized to the stochastic volatility case by replacing the constant volatility by the average volatility \( \left( \frac{1}{t} IV_t \right) \) and averaging out over the distribution of integrated volatility.
the use of \textit{realized volatility} (RV), as an estimator of the quadratic variation:\footnote{The term ‘\textit{realized volatility}’ was used by Fung and Hsieh (1991) to describe their ad-hoc daily volatility estimate calculated as the sum of squared 15 min returns across a number of markets, including S&P500 futures, Bond futures and Currency futures. Such measures were only later put in a proper probabilistic context as an approximation of the quadratic variation of a semi-martingale, see, e.g. Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2002)).}

\begin{equation}
RV_t = \sum_{i=1}^{N} \left[ s_{t_i} - s_{t_{i-1}} \right]^2
\end{equation}

In (4) the sampling frequency of the $N$ observations $(t_i - t_{i-1})$ is often deliberately chosen to be lower than the actual frequency of the data in order to ameliorate the potential bias due to the aforementioned market microstructure effects. Improving the robustness and efficiency of the basic RV estimator (4) to market micro structure noise and as well as devising methods for estimating integrated volatility (i.e., filtering out jumps) has been an extremely active area of recent research and lead to the development of a new generation of volatility estimators. These include

- \textit{Subsampled RV} estimators in which (4) is applied to relatively low frequency intraday returns but overlapping subsamples are used to produce a sequence of robust, albeit individually inefficient, estimators of QV which can be averaged to produce the subsampled QV estimate, see, e.g., Zhang, Mykland, and Ait Sahalia (2005). The resulting estimator gains in robustness to market microstructure noise but it will estimate the full QV and does not involve an attempt to correct for jumps.\footnote{Further refinements along these lines include the \textit{Two-scale} and \textit{Multiscale RV} estimators of Zhang, Mykland, and Ait-Sahalia (2005) and Zhang (2006) and the class of \textit{Kernel based RV} estimators of Barndorff-Nielsen, Hansen, Lunde, and Shephard (2007).}

- \textit{Bi-power variation} estimators exploit the finite activity of large jumps to produce a jump-robust estimator of IV. Rather than squaring returns, Bi-power estimators cumulate products of adjacent absolute returns, thereby ensuring that isolated jumps will cease to matter asymptotically (see e.g. Barndorff-Nielsen and Shephard (2004)).\footnote{Extensions of this approach include alternative transformations of the bipower variation statistic and different lag choices between the nearby absolute returns whose product is cumulated. In addition, bipower

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{Column 1} & \textbf{Column 2} & \textbf{Column 3} & \textbf{Column 4} & \textbf{Column 5} & \textbf{Column 6} & \textbf{Column 7} \\
\hline
\hline
\end{tabular}
\caption{Table Caption}
\end{table}
- Range based RV estimators differ from the aforementioned methods which all conceptually rely on the relationship between squared returns and volatility (4). Instead range based RV estimators, originally proposed by Parkinson (1980), exploit the distribution of the maximum and minimum of the log price process. Recent contributions include Christensen and Podolskij (2007b), Christensen and Podolskij (2007a) and Dobrev (2007).

In this paper we take a conceptually different approach to robust RV estimation by relying instead on the theory of durations for Brownian excursions. While duration based volatility estimation is not new to financial econometrics, it has to our knowledge not been used for RV estimation in models of stochastic volatility in a systematic way nor have its robustness properties been fully explored. We develop the duration analogues to the sub-sampled RV, bi-power and range based estimators and show that these have better asymptotic properties in the hypothetical no-noise continuous time limit and argue that they conceptually should exhibit similar robustness properties to both market microstructure noise and jumps as the existing estimators. These conjectures are substantiated by our Monte Carlo simulations and seemingly collaborated by our analysis of the tick-by-tick data of Dow Jones 30 index stocks.

3 Constant Volatility Estimators Based on Passage Times

A great number of classic results exist on the properties of Brownian passage times. Although these results are limited to the constant drift and volatility case, they can be extended to a more general class of diffusions by a suitable time-change or localization argument. In this section we, for simplicity, introduce our estimators under the assumption of zero drift and constant volatility, deferring the more general stochastic volatility case to Section 4 below.

variation has been extended to the concept of multi-power variation, see, e.g., Barndorff-Nielsen, Graversen, Jacod, and Shephard (2006) and Barndorff-Nielsen, Shephard, and Winkel (2006). In addition, many alternative jump detection procedures have also been proposed. For a recent overview of these developments, see Andersen and Todorov (2007).
3.1 Duality between transition times and the magnitude of Brownian Increments

Consider a Brownian motion \( \{B_t\} \) with constant instantaneous volatility \( \sigma \) and zero drift

\[
B_t = \sigma \int_0^t dW(t), \ t \geq 0, \ B_0 \equiv 0
\]

If the object of interest is to obtain an estimate of the instantaneous variance of the process, one can rely on the fact that the increments \( h_t = (B_{t_0+t} - B_{t_0}) \) over a given time interval \( t \) satisfy \( h_t \sim N(0, \sigma^2 t) \) and so we have the moment condition

\[
\mathbb{E} \left[ h_t^2 \right] = \sigma^2 t \tag{5}
\]

from which \( \sigma^2 \) can be estimated e.g. based on a sample of equispaced returns. This is essentially the logic underlying the RV estimator (4). Rather than measure the (squared) size of Brownian increments over a fixed time interval, one can instead measure the time it takes the Brownian motion to travel a given distance. Denoting the fixed distance \( h \), we can measure time in several alternative ways, each giving rise to a distinct estimator:

\[
\begin{align*}
\text{first exit time} & : \tau_h = \inf_{s > 0} \{ |B_s| > h \} \\
\text{first hitting time} & : \tau_h = \inf_{s > 0} \{ B_s = h \} \tag{6} \\
\text{first range time} & : \tau_h = \min_{\theta} \left\{ \left( \sup_{s \in [0,\theta]} B_s - \inf_{s \in [0,\theta]} B_s \right) = h \right\}
\end{align*}
\]

In the sequel, we use the term “passage time” to denote either of these three measures of duration. As we shall see below, estimators based on the expected first exit time is the natural analogues to RV estimators based on (5) while the estimators based on the expected first range time are the natural analogues to the intraday range estimator of Christensen and Podolskij (2007b) and the generalized range proposed by Dobrev (2007).

The moment generating functions for the passage times \( \tau_h \) are well known but the prob-
ability densities (shown in Figure 1) are only known in closed form for the first hitting time (see e.g. Borodin and Salminen (2002)). From the simple form of the moment generating functions given in Table 2, it is immediately clear (see Lemma A.1) that

$$\tau_h \overset{D}{=} \frac{h^2}{\sigma^2} \tilde{\tau}_1$$

where \(\tilde{\tau}_1\) is the passage time of a standard Brownian motion with respect to the threshold \(h = 1\). This implies that the case of a general volatility \(\sigma^2\) and threshold \(h\) can always be reduced to the baseline case of a standard Brownian motion with threshold 1 through scaling of all passage times by the factor \(\frac{\sigma^2}{h^2}\). This fact will be extremely convenient since it allows us to derive all relevant passage time properties from the baseline case.

### 3.2 Estimation based on passage time moments

The expected passage times can be retrieved from the moment generating functions for the first exit time, the first range time and the first hitting time given in Table 2:

$$\mathbb{E}[\tau_h] = \begin{cases} 
\frac{h^2}{\sigma^2} & \text{(first exit time)} \\
\frac{1}{2} \frac{h^2}{\sigma^2} & \text{(first range time)} \\
\infty & \text{(first hitting time)}
\end{cases}$$

(8)

A method of moments estimator of volatility can then be estimated based on an observed sample of passage times, except in the case of the first hitting time which does not have a first moment. Comparing (5) and (8), the duality between fixed-threshold passage times and fixed-time interval Brownian increments is clear and gives rise to two separate yet closely related approaches to variance estimation. The moment condition for the first exit time was used by Cho and Frees (1988) who studied the effect of price discretization on volatility estimation and considered (8) as a more robust alternative to the existing ARCH/GARCH estimation procedures. Kamstra and Milevsky (2005) used the moment for the first hitting
time, but in the case of a Brownian motion with drift so that the expected hitting time is finite. An important, but often overlooked, issue that arises when applying the moment conditions of the form (8) to estimate $\sigma^2$ is that they can suffer from quite severe small sample biases induced by Jensen’s inequality. This is because the expected passage time is inversely proportional to the instantaneous variance. For a given sample of passage times of size $N$, we show in Proposition A.2, that

$$E \left[ \frac{h^2}{\frac{1}{N} \sum_{i=1}^{N} \tau_{i,h}} \right] = c_N \sigma^2$$ (9)

where

$$c_N = \begin{cases} \int_0^\infty \frac{1}{[\cosh(\sqrt{2}N)]^2} d\lambda & \approx 1 + \frac{2}{3N} + O(\frac{1}{N^2}) \quad \text{(first exit time)} \\ \int_0^\infty \frac{1}{[\cosh(\sqrt{3}/2)]^{2N}} d\lambda & \approx 1 + \frac{1}{3N} + O(\frac{1}{N^2}) \quad \text{(first range time)} \\ \frac{1}{N^2} & \quad \text{(first hitting time)} \end{cases}$$

The Jensen correction factors $c_N$ can be calculated explicitly for each value of $N$. Table 1 shows that the Jensen effect is quite substantial for small $N$ and that it is well approximated by $1 + \frac{2}{3N}$ for $N$ greater than 20.

In applications it is often more convenient to work with an unbiased estimator for which no sample size dependent correction is necessary. A particularly convenient small sample unbiased estimator can be based on the first moment of the reciprocal passage times derived in Corollary A.3:

$$E \left[ \frac{1}{\tau_h} \right] = \begin{cases} 2\beta(2) \frac{\sigma^2}{h^2} & \quad \text{(first exit time)} \\ (4 \log 2) \frac{\sigma^2}{h^2} & \quad \text{(first range time)} \\ \frac{\sigma^2}{h^2} & \quad \text{(first hitting time)} \end{cases}$$ (10)

where $\beta(\cdot)$ is the Dirichlet Beta function.$^8$ The moment conditions (10) have, to our knowl-

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$^8$The Dirichlet Beta function is given by $\beta(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\pi}$. 

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edge, not previously applied to the problem of volatility estimation. Moreover, we see that while the expected hitting time is infinite, the expected reciprocal hitting time is well behaved. In Corollary A.6 we show that the volatility estimator, \( \hat{\sigma}_i = \frac{1}{c_1} \frac{h^2}{\tau_h} \), based on a single observed transition, satisfies

\[
\mathbb{E}[\hat{\sigma}_i^2] = \sigma^2 \quad \text{and} \quad \mathbb{V}[\hat{\sigma}_i^2] \approx \begin{cases} 
0.7681 \, \sigma^4 & \text{(first exit time)} \\
0.4073 \, \sigma^4 & \text{(first range time)} \\
2.0000 \, \sigma^4 & \text{(first hitting time)}
\end{cases}
\]

Note that the range time based estimator is the most efficient since it exploits more information about the joint infimum and supremum of the process than do the first exit time or first hitting time. The variance of the estimator depends on the fourth moment (the so called quarticity) of the log price process but does not depend on the threshold \( h \).

When applying (8) or (10) to the estimation of volatility in markets with periodic market closures, one faces the familiar issue of right censoring. This implies that the distribution of the observed passage times \( \tau_h \) will depend on the amount of time remaining until markets close since it is a draw from a conditional distribution where the conditioning event is that the transition took place prior to end-of-trading. For simplicity, assume that the length of a trading day is normalized to one and that the current time is \( t = 1 - T \in [0;1) \), then the appropriate moment associated with a transition \( |B_{1-T+\tau_h} - B_{1-T}| = h \) is \( E \left[ \frac{1}{\tau_h} | \tau_h \leq T \right] \). The conditional expectation is only known in closed form for the first hitting time, but can conveniently be represented using rapidly converging series expansions as shown in Proposition A.7. The effect of censoring, illustrated in Figure 2 for the baseline case \( (\sigma = h = 1) \), is quite extreme for the hitting time but much less pronounced for the exit time and especially the first range time. To see the effect for an arbitrary \( (\sigma, h) \), the distributional identity (7) implies that the time axis should be multiplied by \( \sigma^2/h^2 \). Thus we see that the conditioning will tend to have a large impact whenever \( T \frac{\sigma^2}{h^2} \) is small, i.e.

\[\text{This is analogous to the RV moment condition (5) as well except there the factor multiplying } \sigma^4 \text{ is 2, i.e. the same as for our least efficient estimator.}\]
when the (unconditional) expected passage time represents a large fraction of the remaining
time until the end of a trading day \(T\). As a practical matter, we find that the conditioning
can safely be ignored when there is more than 3 (first exit time) or 2 (first range time)
expected passage times until end of trading. For the first hitting time on the other hand, the
conditional expectation should always be used. It is important to note, that in situations
when censoring cannot be ignored, the relationship between the expected reciprocal passage
times and volatility is no longer linear and one must again explicitly account for Jensen effect
in small samples.

3.3 Discretization

A final concern that must be addressed is time discretization, i.e. the fact that prices are
observed on a discretely spaced time grid. This implies that passage times must be corrected
for the fact that the supremum (infimum) of the unobserved continuous time Brownian
motion is greater (smaller) than that of the observed random walk. This exact question was
addressed by Rogers and Satchell (1991) who develop a discretization correction which we
state in Lemma A.8.

In order to be feasible, the proposed estimators must be corrected for the discrepancy
between the first passage times of the unobserved process and those measured on an observed
finite sample of points. In particular, the first passage times of a random walk overstate the
Brownian first passage times. This is because the maxima of a random walk understate the
maxima of a Brownian motion, so that it takes somewhat longer on average for the random
walk to break a given threshold level. The larger is the discretization step, the larger is this
discrepancy.

Rogers and Satchell (1991) develop an exact correction for the amount by which the
maximum \(\tilde{M}\) or the range \(\tilde{R}\) of a random walk based on \(k\) equidistant observations in a
finite interval understate the Brownian maximum \(M\) and range \(R\) in the same interval (see
appendix). They have used this to make appropriate finite sample adjustments of various
range-based volatility estimators, while here we adapt the same results to correct the passage
time estimators proposed above. Our main insight stemming from Rogers and Satchell
(1991), is that knowledge of the moments of the discrepancy due to discretization allows
us to approximate the ratio between moments of the Brownian maximum/range and the
random walk maximum/range as a function of the number of sample points \( k \). In particular,
we show that

\[
\mathbb{E} [M^2] \approx f_M(k) \times \mathbb{E} \left[ \tilde{M}^2 \right] \\
\mathbb{E} [R^2] \approx f_R(k) \times \mathbb{E} \left[ \tilde{R}^2 \right] 
\]

where the scaling factors \( f_M(k) \) and \( f_R(k) \) approach 1 from above as \( k \) gets large (see
appendix).

From this it follows that for any given threshold \( h \) one needs to scale up the finite sample
volatility estimators based on the first exit time and the first range time by a factor of \( f_M(k) \)
and \( f_R(k) \) respectively, where \( k \) is the (varying) number of observations between the start
and end point of each \( h \)-excursion. By aggregating the resulting estimates at each sample
point on a given grid we obtain a novel realized duration-based volatility estimator (DV) dual
to RV in the sense that it is based on measuring passage times rather than price increments.
The next section lays out a unified theoretical framework for DV based on first exit times
and DV based on first range times.

4 Stochastic Volatility Estimators Based on Passage
Times

When volatility is randomly varying throughout the trading day, we can no longer rely on
observing an i.i.d. passage times. This leads us to consider instead local volatility estimation
as proposed by Foster and Nelson (1996) and more recently Mykland (2006). Based on
consistent estimators of local volatility, an IV estimator can then be constructed by simple integration.

Consider then a fixed time grid \(0 < t_1 < t_2 < \cdots < t_N < T\) consisting of \(N\) equispaced points with \(\Delta_N = t_i - t_{i-1}\) denoting the mesh size. Our assumption will be that the volatility process is locally constant, i.e. it can be approximated well by a piecewise constant function as the mesh size tends to zero. At a given grid point \(t_i\) we wish to construct an estimator of the local volatility. Given a threshold size \(h\), we can, starting from \(t_i\), look for the next passage time \(\tau_h^+(t_i)\) and use the estimator \(\frac{1}{c_1} \frac{h^2}{\tau_h^+(t_i)}\). However, we can do better. Since the Brownian motion is time reversible, we can also, starting from \(t_i\), look back in time for the next passage time \(\tau_h^-(t_i)\) and construct the independent estimator \(\frac{1}{c_1} \frac{h^2}{\tau_h^-(t_i)}\).

This leads to the optimal bi-directional estimator:

\[
\hat{\sigma}_h^2(t_i) = \frac{1}{2c_1} \left[ \frac{h^2}{\tau_h^+(t_i)} + \frac{h^2}{\tau_h^-(t_i)} \right]
\]  

(11)

which (by independence) satisfies

\[
\mathbb{V}[\hat{\sigma}_h^2(t_i)] \approx \begin{cases} 
0.38405 \sigma^4 & \text{(first exit time)} \\
0.20367 \sigma^4 & \text{(first range time)} \\
1.00000 \sigma^4 & \text{(first hitting time)} 
\end{cases}
\]

(12)

We thus have a sequence of local volatility estimates \(\{\hat{\sigma}_h^2(t_i)\}_{i=1,\ldots,N}\) from which we can
construct the estimate of integrated volatility on \([0; T]\) as

\[
\hat{I}_T = \frac{T}{N} \sum_{i=1}^{N} \hat{\sigma}_h^2(t_i)
\]

The distribution of \((13)\) will depend on \(h\) although the distribution of each individual estimate \(\hat{\sigma}_h^2(t_i)\) does not. This is because the covariance between neighboring volatility estimates, i.e. \(\text{Cov}(\hat{\sigma}_h^2(t_i), \hat{\sigma}_h^2(t_{i-1}))\), will likely involve overlapping transitions when \(h\) is large relative to the mesh size \(\Delta_N\). However, the threshold \(h\) can be chosen independently of \(N\) and we can consider letting \(h\) tend to zero so that terms in \((13)\) have arbitrarily low correlation. In Proposition A.11, we show that the resulting estimator satisfies

\[
\sqrt{N} \left( \hat{I}_T - IV_T \right) \sim \text{Normal} \left( 0, \nu \int_0^T \sigma_u^4 \, du \right)
\]

where \(\nu\) is the constant in \((12)\). Note that the asymptotic distribution depends on the unknown integrated quarticity. However, it is easy to show that the same localization argument can be used to consistently estimate other integrated moments of interest such as the integrated quarticity itself, thus enabling the calculation of IV confidence intervals.

5 Monte Carlo Experiments

In this section we document the performance of the proposed realized duration-based volatility estimators on finite samples of simulated data from various jump-diffusion models. For each model specification, we generate 23,400 intraday observations, corresponding to one observation per second during a typical trading day from 9:30 am to 4:00 pm. We analyze the performance of the IV estimators based on the first range time and first exit time introduced above by simulating 2,500 days and, for each model specification, produce the signature plots of the mean and RMSE of the obtained integrated variance estimates as a function of the threshold level. For comparison we also display the subsampled 2-minute RV
and subsampled 2-minute BV estimates as a realistic benchmark\textsuperscript{10}.

In the Monte Carlo experiments we deliberately calibrate the threshold levels for calculating the passage time based estimators in such a way that the obtained conclusions are directly comparable to the results we obtain using the Dow Jones 30 stock data in Section 6. To this end, we exploit the empirically stable relationship between the average log-spread and daily volatility (the ratio typically falls in the range [0.02, 0.04] with a mean of 0.03, as shown in the table accompanying the signature plots in Appendix B). We therefore express the thresholds in units of the mean log-spread and further assume that the log-spread equals 0.03 times the daily volatility in our simulations. In this way, we ensure sufficient consistency between our plots on real and simulated data as a function of the threshold expressed in units of the mean log-spread. Furthermore, we focus on thresholds in the range from 1 to 10 times the mean log-spread, corresponding to the range from 0.03 to 0.3 times the daily volatility. This choice is very sensible since it ensures a mean duration of the first exit times ranging from about 20 seconds for the smallest threshold to about 30 minutes for the largest threshold and mean duration of the first range times ranging from about 10 seconds to about 15 minutes.

5.1 The Pure Diffusion Case

We consider the following specifications for the evolution of the log-price process (see appendix).

1. Constant volatility model (M0)

2. One-factor affine stochastic volatility model (M1)

In the constant volatility model the annualized volatility is set to 20%, so that the daily variance is $1.59 \times 10^{-4}$ (assuming 252 trading days per year). Figure 3a shows that there is

\textsuperscript{10}Subsampling of RV with important extensions to optimal two-scale and multi-scale RV estimators in the presence of microstructure noise has been advocated by Zhang, Mykland, and Ait-Sahalia (2005) and Zhang (2006). Subsampling of BV has been empirically documented to have similar merits by Dobrev (2007).
virtually no bias of our first exit time DV estimates for thresholds larger than 2 log-spreads, while the first range time DV estimates are unbiased for thresholds larger than 4 log-spreads. The observed mild downward bias for smaller thresholds is due to the loss in accuracy of our finite sample correction factors for passage times that tend to span a very small number of sample points. Overall, Figure 3a indicates that the first exit time is slightly more reliable than the first range time in finite samples.

The corresponding signature plots of the standard deviations of the estimators are shown on Figure 4a. The efficiency of both DV estimators increases as the threshold decreases since there is now less overlap between observed passage times starting from neighboring grid points. Importantly, the first exit time DV for threshold levels below 6 log-spreads and the first range time DV for threshold levels below 8 log-spreads are more efficient than the subsampled 2min RV and the subsampled 2min BV.

The overall conclusion from Figures 3a and 4a is that there is a wide range of realistic threshold levels (from 2 log-spreads to 6 log-spreads for the first exit time DV and from 4 log-spreads to 8 log-spreads for the first range time DV) offering substantial efficiency gains from using our DV measures of the daily integrated variance. As it turns out, roughly the same conclusion remains valid also if there is substantial intraday volatility variation in the form of a realistic U-shaped pattern as well as finite activity jumps.

5.2 The Pure Diffusion Case with U-Shaped Intraday Volatility Pattern

It is well known that intraday stock returns exhibit a pronounced U-shaped pattern of intraday volatility along with other activity-related variables such as trading volume, number of transactions, and quoted spread. On average, the variance in the first hour of trading is often three times larger than the variance in the middle of the day and peaks again to a somewhat lower level in the last hour of trading.

Although the asymptotics of our DV estimators is not affected by such (non-explosive)
intraday variation in volatility, it is important to investigate the extent to which there might be a detrimental impact on the finite sample performance documented in the previous section. Therefore, we repeat the same analysis after augmenting our baseline diffusive volatility specifications with a deterministic time-of-the-day volatility multiplier having a U-shaped pattern. In particular, our time-of-the-day multiplier takes an additive form of exponentials popular in the market microstructure literature (see details in the appendix).

Figures 3b and 4b compared to figures 3a and 3b show that the U-shaped intraday volatility pattern has a negligible impact on the quality of the first exit time and first range time DV estimates, especially for the range of threshold levels identified as most attractive (from 2 log-spreads to 6 log-spreads for the first exit time DV and from 4 log-spreads to 8 log-spreads for the first range time DV). This finding should not be surprising given that for small enough thresholds volatility is locally constant to a very good first order of approximation, which as detailed in our theory section, is the main requirement for our DV estimators to work well in practice.

5.3 The Jump-Diffusion Case with U-Shaped Intraday Volatility Pattern

An important feature of the first exit time and first range time DV estimators established in the theory section is their asymptotic robustness to jumps. Therefore, we further extend our Monte Carlo experiments by adding finite activity jumps to the pure diffusive specifications with a U-shaped intraday volatility pattern. In particular, we simulate one jump on each day with a 25% average contribution to the daily diffusive variance and carry out the same analysis based on signature plots. Figures 3c and 4c for this case reveal that our DV estimators are remarkably robust to jumps for threshold levels in the most desirable range from 2 log-spreads to 6 log-spreads for the first exit time DV and from 4 log-spreads to 8 log-spreads for the first range time DV. Note that our estimators tend to fall slightly below the subsampled BV estimator. This is to be expected since it is well known that the BV
estimator does not fully eliminate the impact of jumps and in fact has an upward bias of roughly 5.4% in our simulations.

The robustness to jumps of passage time based estimators is not surprising since jumps are automatically truncated at the chosen threshold thereby limiting their impact on the estimator. This leads to a trade-off in the threshold choice between robustness to jumps and robustness to market microstructure noise; a fact which is borne out by the simulations shown in Figure 3c: For the largest thresholds, the jump truncation is insufficient leading to a significant upwards bias in the passage time estimators. For the very smallest thresholds, on the other hand, the effect of jumps is eliminated but the threshold is now insufficient for eliminating the impact of the noise, leading to a slight downwards bias. This leaves us with the intermediate range for which the estimator has good robustness properties with respect to both microstructure noise and jumps.

Overall, the documented finite sample performance of our DV estimators under different model specifications provides solid initial evidence that they should perform reasonably well on real data as well. This is the main task of the next section.

6 A Realized Volatility Analysis of the Dow Jones 30

In recent years the Dow Jones 30 stocks have an increasingly denser (nearly second-by-second) record of intraday transaction prices and quotes. This naturally provides an excellent laboratory for testing the empirical performance of our passage time based volatility estimators on real data. For our analysis we use NYSE TAQ data for the period January 1, 2005 to May 31, 2007 (more recent data has not been made available yet). Table 3 summarizes basic descriptive statistics for the Dow Jones components, including companies that have been included in or excluded from the index during this period. Ignoring short trading days around major holidays we obtain a sample of 601 regular trading days of nearly second-by-second intraday data for each stock from 9:30 am to 4:00 pm as in our simulation
study.

In order to avoid distortions due to the bid-ask bounce, we apply our first exit time and first range time estimators separately on the series of ask quotes and the series of bid quotes after filtering out spread outliers to eliminate possible data errors. Not surprisingly, the obtained estimates based on the ask series are nearly identical to those based on the bid series, so we report only the former. Microstructure noise will still distort measured returns and, hence, measured transition times to some extent, but when considering large enough thresholds (e.g. 3-5 spreads) the transition time distortions will relatively speaking be small and the quality of our first exit time and first range time DV estimates should be comparable to the quality of subsampled RV and BV estimates at modest sampling frequencies such as 2 to 5 min. Being agnostic about the noise process, the robustness argument is analogous to the one that justifies the use of, say, 2-min subsampled RV or BV.

The signature plots for each stock are given in Appendix B. The first plot represents a nice summary of our main findings by plotting the cross-sectional average of our DV estimates for threshold levels expressed in units of the average log-spread (consistently with our simulation study). As evident from this plot, our first exit time DV estimates on average agree with the 2min BV estimates, while the first range time DV estimates are slightly lower (quite similar to the jump scenario in the simulations). What is even more striking, though, is that essentially the same pattern of the signature plots emerges also for each individual stock. Moreover, the average correlation between our DV estimates and the BV estimates is as high as 0.85-0.95 (not only for 2min BV but also for all BV estimates at frequencies in the range 30 sec to 5 min), as shown in the second figure in Appendix B. At the same time, much like in the simulations, for threshold levels from 3 log-spreads to 6 log-spreads, the standard deviation of our DV estimates is lower than the one of subsampled 2min BV. Most notably, robustly across all stocks we obtain a similar mean level as BV, very high correlation with BV but markedly lower variance.

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7 Conclusion

In this paper we have proposed a radically different approach to realized volatility and integrated volatility estimation based on a localization argument and the theory of Brownian passage times. The estimators are asymptotically superior to standard RV estimators in the infeasible case where continuous observations are available and no jumps present. Moreover, passage time based estimators perform extremely well in finite samples in both Monte Carlo experiments as well as in our analysis of the Dow Jones 30 stock sample. The rationale behind this performance is quite intuitive: As long as the threshold is chosen large enough, the market microstructure noise distortion of the observed passage times will be relatively small. On the other hand, as long as the threshold is small enough, individual large jumps will automatically be truncated and will only affect the local volatility estimate at a single grid point. As we have demonstrated in the Monte Carlo experiments (and seen in the real data) this leaves us with an intermediate range of thresholds ranging from about 3 to 6 log-spreads for which our estimators have good robustness properties with respect to both jumps and micro structure noise. Importantly, the estimates are relatively stable with respect to the exact choice of threshold within this range and practical implementation will therefore not need to involve any delicate first step estimation of an ‘optimal’ threshold.

Overall, these quite remarkable results for the Dow Jones 30 stocks present an overwhelming evidence for the superior finite sample efficiency of the introduced realized duration-based volatility estimators. At the same time, certain aspects of the behavior of our DV estimators on real data call for further investigation. In particular, future research is needed to shed more light on what type of noise structures are consistent with the relative behavior of our first exit time and first range time based estimators over the range of thresholds considered.
References


Andersen, T. G. and V. Todorov (forthcoming 2007). Realized Volatility and Multipower Variation. Wiley. in Encyclopedia of Quantitative Finance, Rama Cont (Eds.).


A Technical Appendix

A.1 Monte Carlo Experiment: Benchmark models

Constant volatility model (M0):

\[ dp(t) = \sigma dW(t) \]

One-factor affine stochastic volatility model (M1):

\[ dp(t) = \sigma(t) dW_1(t) \]
\[ d\sigma^2(t) = \eta(\theta - \sigma^2(t)) dt + \nu \sigma(t) dW_2(t) \]
\[ \rho = \text{corr}(dW_1(t), dW_2(t)) \]

We use the parameter estimates in Andersen, Benzoni, and Lund (2002).

A.2 Passage time results

**Lemma A.1** Let \( \tau_h \) be the first exit time, the first hitting time or the first range time as given in (6) for a driftless Brownian motion with constant volatility \( \sigma^2 \), then

\[ \tau_h \overset{D}{=} \frac{h^2}{\sigma^2} \tilde{\tau}_1 \]

where \( \tilde{\tau}_1 \) is the corresponding passage time of a standard Brownian motion with threshold \( h = 1 \).

**Proof.** This is clear from the moment generating functions given in Table 2:

\[ \mathbb{E}\left[ e^{-\alpha (\frac{h^2}{\sigma^2}) \tilde{\tau}_1} \right] = \mathbb{E}\left[ e^{-\alpha (\frac{h^2}{\sigma^2})} \tilde{\tau}_1 \right] = \mathcal{L}_{\tau_h}(\alpha) \]

** Proposition A.2** If \( \{\tau_{i,h}\} \) is an i.i.d sample of \( N \) first exit times from the interval \([-h; h]\) of a driftless Brownian motion with \( B_0 = 0 \) and constant volatility \( \sigma \), then the expected value of the volatility estimator based on the first exit time moment is given by

\[ \mathbb{E}\left[ \frac{h^2}{\sum_{i=1}^{N} \tau_{i,h}} \right] = N \ c_N \ \sigma^2 \]

where

\[ c_N = \int_{0}^{\infty} \frac{1}{[\cosh(\sqrt{2} \lambda)]} \ d\lambda \quad \text{(first exit time)} \]
\[ c_N = \int_{0}^{\infty} \frac{1}{[\cosh(\sqrt{\lambda/2})]^N} \ d\lambda \quad \text{(first range time)} \]
\[ c_N = \frac{1}{N^2} \quad \text{(first hitting time)} \]
Moreover, asymptotically as $N \to \infty$

\[
\begin{align*}
N c_N &= 1 + \frac{2}{3N} + O\left(\frac{1}{N^2}\right) \quad \text{(first exit time)} \\
N c_N &= 1 + \frac{1}{3N} + O\left(\frac{1}{N^2}\right) \quad \text{(first range time)}
\end{align*}
\]

**Proof.** By independence the moment generating function of the convolution $\tau_{1,h} + \cdots + \tau_{N,h}$ is given by $[\mathcal{L}_{\tau_h}(\alpha)]^N$. We therefore have

\[
\mathbb{E} \left[ \frac{h^2}{N \sum_{i=1}^{N} \tau_i} \right] = h^2 N \int_0^{\infty} [\mathcal{L}_{\tau_h}(\alpha)]^N d\alpha
\]

The result then follows from the change of variables $\lambda = \frac{ah^2}{\sigma^2}$. The asymptotic approximation can be derived for the first exit time and the first range time via a mean value expansion:

\[
\mathbb{E} \left[ \frac{h^2}{N \sum_{i=1}^{N} \tau_i} \right] = \mathbb{E} \left[ \frac{h^2}{\mathbb{E} \tau + \frac{1}{N} \sum_{i=1}^{N} (\tau_i - \mathbb{E} \tau)} \right] = \sigma^2 \times \mathbb{E} \left[ \frac{1}{1 + \frac{1}{N} \sum_{i=1}^{N} (\tau_i - \mathbb{E} \tau)} \right]
\]

A second order Taylor expansion of $1/(1+x)$ around 1 yields:

\[
\approx \sigma^2 \times \mathbb{E} \left[ 1 - \frac{1}{N} \sum_{i=1}^{N} (\tau_i - \mathbb{E} \tau) + \left( \frac{1}{N} \sum_{i=1}^{N} (\tau_i - \mathbb{E} \tau) \right)^2 - \cdots \right]
\approx \sigma^2 \times \mathbb{E} \left[ 1 + \frac{\mathbb{E} \tau}{(\mathbb{E} \tau)^2} \sum_{i=1}^{N} (\tau_i - \mathbb{E} \tau)^2 + O(1/N^2) \right] = \sigma^2 \left[ 1 + \frac{1}{N} \frac{\mathbb{V} \tau}{(\mathbb{E} \tau)^2} + O(1/N^2) \right]
\]

from which the result follows by calculation of the first two moments of the relevant passage time using the moment generating function. ■

**Corollary A.3** The expectation of the reciprocal passage time satisfies

\[
\mathbb{E} \left[ \frac{h^2}{\tau_{h}} \right] = \left\{ \begin{array}{ll}
2C \sigma^2 & \text{(first exit time)} \\
(4 \log 2) \sigma^2 & \text{(first range time)} \\
\sigma^2 & \text{(first hitting time)}
\end{array} \right.
\]

**Proof.** Apply Proposition A.2 with $N = 1$. ■

**Proposition A.4** If $\{\tau_{i,h}\}$ is an i.i.d sample of $N$ first exit times from the interval $[-h;h]$ of a driftless Brownian motion with $B_0 = 0$ and constant volatility $\sigma$, then the second moment of the volatility estimator based on the first exit time moment is given by

\[
\mathbb{E} \left[ \frac{h^4}{\left(\frac{1}{N} \sum_{i=1}^{N} \tau_{i,h}\right)^2} \right] = N^2 C_N \sigma^4
\]

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where

\[
C_N = \int_0^\infty \int_\Lambda^\infty \frac{1}{[\cosh(\sqrt{2\lambda})]^N} \, d\lambda \, d\Lambda \quad \text{(first exit time)}
\]

\[
C_N = \int_0^\infty \int_\Lambda^\infty \frac{1}{[\cosh(\sqrt{\lambda}/2)]^{2N}} \, d\lambda \, d\Lambda \quad \text{(first range time)}
\]

\[
C_N = \frac{3}{N^4} \quad \text{(first hitting time)}
\]

**Proof.** By independence the moment generating function of the convolution \(\tau_{1,h} + \cdots + \tau_{N,h}\) is given by \([\mathcal{L}_{\tau_h}(\alpha)]^N\). We therefore have

\[
E\left[\frac{1}{\left(\sum_{i=1}^N \tau_{i,h}\right)^2}\right] = \int_0^\infty \int_\Lambda^\infty [\mathcal{L}_{\tau_h}(\alpha)]^N \, d\alpha \, d\Lambda = \frac{\sigma^4}{h^4} \int_0^\infty \int_\Lambda^\infty \frac{1}{[\cosh(\sqrt{2\lambda})]^N} \, d\lambda \, d\Lambda
\]

\]

**Corollary A.5** The second moment of the reciprocal passage time satisfies

\[
\mathbb{E}\left[\frac{h^4}{\tau_h^2}\right] = \begin{cases} 
6 \beta(4) \sigma^4 \quad &\text{(first exit time)} \\
9 \zeta(3) \sigma^4 \quad &\text{(first range time)} \\
3\sigma^4 \quad &\text{(first hitting time)}
\end{cases}
\]

**Proof.** Apply Proposition A.4 with \(N = 1\).

**Corollary A.6** Let \(\hat{\sigma}_i^2 = \frac{1}{c_1} \frac{h^2}{\tau_{i,h}}\) be the volatility estimate based on a single passage time observation \(\tau_{i,h}\), then \(\mathbb{E}[\hat{\sigma}_i^2] = \sigma^2\) and

\[
\mathbb{V}[\hat{\sigma}_i^2] = \left(\frac{C_1}{c_1^2} - 1\right) \sigma^4 = \begin{cases} 
\left(\frac{6 \beta(4)}{4\beta(2)^2} - 1\right) \sigma^4 \approx 0.7681 \sigma^4 \quad &\text{(first exit time)} \\
\left(\frac{9 \zeta(3)}{4\log(2)^2} - 1\right) \sigma^4 \approx 0.4073 \sigma^4 \quad &\text{(first range time)} \\
\left(\frac{3}{2\pi} - 1\right) \sigma^4 \approx 2.000 \sigma^4 \quad &\text{(first hitting time)}
\end{cases}
\]

**Proof.** Apply Corollary A.3 and A.5.
A.3 Censoring

Proposition A.7 Let $\bar{T} = T \frac{\sigma^2}{\sigma^2}$ be the time remaining until end of trading measured in multiples of expected exit times $(\frac{\sigma^2}{\sigma^2})$, then the censored reciprocal moments are given by

$$
\mathbb{E} \left[ \frac{1}{T_h} | T_h \leq \bar{T} \right] = \begin{cases} 
\frac{\sigma^2}{\bar{T}^2} \sum_{k=-\infty}^{\infty} \frac{\text{sign}(4k+1)}{2^k} \left( \frac{2 \exp\left(-\frac{(4k+1)^2}{2 \bar{T}}\right)}{\sqrt{2\pi}} + \frac{\text{erfc}\left(\frac{4k+1}{\sqrt{2 \bar{T}}}\right)}{4k+1} \right) 
(\text{first exit time}) 

\frac{\sigma^2}{\bar{T}^2} \sum_{k=-\infty}^{\infty} \text{sign}(4k+1) \text{erfc}\left(\frac{4k+1}{\sqrt{2 \bar{T}}}\right) 
(\text{first range time}) 

\frac{\sigma^2}{\bar{T}^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{2 \exp\left(-\frac{1}{2 \bar{T}}\right)}{\sqrt{2\pi}} + \frac{\text{erfc}\left(\frac{1}{\sqrt{2 \bar{T}}}\right)}{1} 
(\text{first hitting time}) 
\end{cases}
$$

Proof. In the case of the first hitting time, the censored moments can be based on the known pdf given in Table 2, for the first exit time and first range time the result follows from Borodin and Salminen (2002) □

A.4 Discretization bias correction

We state the multiplicative discretization bias corrections derived based on the additive corrections derived in Rogers and Satchell (1991).

Lemma A.8 Let $B_t$ be a driftless Brownian motion on $[0; 1]$ with volatility $\sigma^2$ and let $X_t = B_{\frac{tN}{N}}$ be the embedded (discretely observed) random walk observed at $N$ points. Define $\Delta = \sup_{t \in [0; 1]} B_t - \sup_{t \in [0; 1]} X_t$ then

$$
\mathbb{E}[\Delta] = \sqrt{2\pi} \left[ \frac{1}{4} - \frac{\sqrt{\pi}}{6} \right] \frac{\sigma}{\sqrt{N}} 
$$

$$
\mathbb{E}[\Delta^2] = \left[ \frac{1+3\pi/4}{12} \right] \frac{\sigma^2}{N} 
$$

Lemma A.9 The following relationships hold approximately for the maximum $\widetilde{M}$ and the range $\widetilde{R}$ of a random walk based on $K$ equidistant observations in a finite interval and the maximum $M$ and the range $R$ of a Brownian motion in the same interval:

$$
\mathbb{E}[M^2] \approx f_M(K) \times \mathbb{E}\left[ \tilde{M}^2 \right] 
$$

$$
\mathbb{E}[R^2] \approx f_R(K) \times \mathbb{E}\left[ \tilde{R}^2 \right] 
$$
where

\[ f_M(k) = \frac{48k}{48k - 4(20 - 8\sqrt{2})\sqrt{k} + 3\pi + 4} \]

\[ f_R(k) = \frac{48k \ln 2}{48k \ln 2 - 4(20 - 8\sqrt{2})\sqrt{k} + 3\pi + 4} \]

**Proof.** Let \( \Delta = M - \tilde{M} \). Then from Rogers and Satchell (1991) it follows:

\[ \mathbb{E} \left[ \tilde{M}^2 \right] \approx \mathbb{E} \left[ M^2 \right] - 2\mathbb{E} \left[ M \right] \mathbb{E} \left[ \Delta \right] + \mathbb{E} \left[ \Delta^2 \right] \]

\[ \mathbb{E} \left[ \tilde{R}^2 \right] \approx \mathbb{E} \left[ R^2 \right] - 4\mathbb{E} \left[ R \right] \mathbb{E} \left[ \Delta \right] + 4\mathbb{E} \left[ \Delta^2 \right] \]

Then

\[ \frac{\mathbb{E} \left[ \tilde{M}^2 \right]}{\mathbb{E} \left[ M^2 \right]} \approx 1 - \frac{2\mathbb{E} \left[ M \right] \mathbb{E} \left[ \Delta \right]}{\mathbb{E} \left[ M^2 \right]} + \frac{\mathbb{E} \left[ \Delta^2 \right]}{\mathbb{E} \left[ M^2 \right]} \]

\[ \frac{\mathbb{E} \left[ \tilde{R}^2 \right]}{\mathbb{E} \left[ R^2 \right]} \approx 1 - 4\frac{\mathbb{E} \left[ R \right] \mathbb{E} \left[ \Delta \right]}{\mathbb{E} \left[ R^2 \right]} + \frac{4\mathbb{E} \left[ \Delta^2 \right]}{\mathbb{E} \left[ R^2 \right]} \]

and the scaling factors are obtained by rearranging the expressions after substituting \( \mathbb{E} \left[ M \right] = \sqrt{\frac{2}{\pi}} \times \sigma \sqrt{T} \), \( \mathbb{E} \left[ M^2 \right] = \sigma^2 T \), \( \mathbb{E} \left[ R \right] = \sqrt{\frac{8}{\pi}} \times \sigma \sqrt{T} \), \( \mathbb{E} \left[ R^2 \right] = 4 \ln 2 \times \sigma^2 T \), \( \mathbb{E} \left[ \Delta \right] = \sqrt{\frac{2}{\pi}} \left( \frac{1}{4} - \frac{\sqrt{2} - 1}{6} \right) \times \sigma \sqrt{\frac{T}{k}} \), \( \mathbb{E} \left[ \Delta^2 \right] = \frac{1}{12} \left( 1 + \frac{3\pi}{4} \right) \times \sigma^2 \frac{T}{k} \).

**A.5 Stochastic volatility**

We focus on the volatility estimators based the (reciprocal) first exit time and first range times.

**Lemma A.10** Let \( t_i \) and \( t_{i+1} \) be two neighboring grid points with \( \Delta = |t_{i+1} - t_i| \), and let \( \tau^+(t_i) \) and \( \tau^-(t_{i+1}) \) be the forward passage time starting from \( t_i \) and the time reversed passage time starting from \( t_{i+1} \) respectively corresponding to the threshold \( h \). Then

\[ \forall \varepsilon > 0 \exists \delta > 0 : \left| \mathbb{E} \left[ \frac{1}{\tau^+(t_i)} \frac{1}{\tau^-(t_{i+1})} \right] - \mathbb{E} \left[ \frac{1}{\tau^+(t_i)} \right] \mathbb{E} \left[ \frac{1}{\tau^-(t_{i+1})} \right] \right| < \varepsilon \]

i.e. the reciprocal passage times are asymptotically uncorrelated as \( h \searrow 0 \) for a fixed \( \Delta \).

**Proof.** (Sketch)

\[ \mathbb{E} \left[ \frac{1}{\tau^+(t_i)} \frac{1}{\tau^-(t_{i+1})} \right] = \mathbb{E} \left[ \frac{1}{\tau^+(t_i)} \frac{1}{\tau^-(t_{i+1})} \tau^+(t_i) < \frac{\Delta}{2}, \tau^-(t_{i+1}) < \frac{\Delta}{2} \right] \text{Pr} \left\{ \tau^+(t_i) < \frac{\Delta}{2}, \tau^-(t_{i+1}) < \frac{\Delta}{2} \right\} \]
\[
\mathbb{E}\left[\frac{1}{\tau^+_h(t_i)} \left\{ \tau^+_h(t_i) < \frac{\Delta}{2} \right\}\right] + \mathbb{E}\left[\frac{1}{\tau^+_h(t_i)} \frac{1}{\tau^-_h(t_{i+1})} \left\{ \tau^+_h(t_i) \geq \frac{\Delta}{2} \right\} \cup \left\{ \tau^-_h(t_{i+1}) \geq \frac{\Delta}{2} \right\}\right] \Pr\left\{ \left\{ \tau^+_h(t_i) < \frac{\Delta}{2} \right\} \cap \left\{ \tau^-_h(t_{i+1}) < \frac{\Delta}{2} \right\} \right\}
\]

Now the result follows from the fact that the reciprocal passage times are uniformly integrable and

\[
\Pr\left\{ \left\{ \tau^+_h(t_i) > \frac{\Delta}{2} \right\} \right\} \quad \text{and} \quad \Pr\left\{ \left\{ \tau^-_h(t_{i+1}) > \frac{\Delta}{2} \right\} \right\} \to 0 \quad \text{as} \quad h \searrow 0
\]

and the Cauchy-Schwartz inequality

\[
\mathbb{E}\left[\frac{1}{\tau^+_h(t_i)} \frac{1}{\tau^-_h(t_{i+1})}\right] \leq \sqrt{\mathbb{E}\left[\left(\frac{1}{\tau^+_h(t_i)}\right)^2\right] \mathbb{E}\left[\left(\frac{1}{\tau^-_h(t_{i+1})}\right)^2\right]}
\]

\[\blacksquare\]

**Proposition A.11** Let \(\{t_1, \ldots, t_N\}\) be a (non-random) time grid with mesh size \(\Delta\) on \([0; T]\) and let \(\hat{\sigma}^2_i(t_i)\) be the bidirectional volatility estimator given in (11). Define \(\hat{IV}_{T,h} = T \frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}^2_i(t_i)\), then, assuming the paths of the volatility process \(\sigma^2_t\) are Lipschitz, as \(N \to \infty, h \to 0\)

\[
\sqrt{N} \left(\hat{IV}_{T,h} - IV_T\right) \sim \text{Normal} \left(0, \nu \int_0^T \sigma^4_t \, dt\right)
\]

where \(\nu\) is given in (12).

**Proof.** (Sketch) Note that \(T/N = \Delta\), then

\[
\sum_{i=1}^{N} \left(\int_{t_{i-1} - \frac{\Delta}{2}}^{t_{i+1} + \frac{\Delta}{2}} \sigma^2_u \, du - \Delta \hat{\sigma}^2_i(t_i)\right) = \sum_{i=1}^{N} \left(\int_{t_{i-1} - \frac{\Delta}{2}}^{t_{i} + \frac{\Delta}{2}} \sigma^2_u \, du - \Delta \sigma^2_{t_i}\right) + \Delta \sum_{i=1}^{N} (\sigma^2_{t_i} - \hat{\sigma}^2_i(t_i))
\]

The first term is the approximation error of a standard Riemannian sum and will tend to zero if \(\sigma^2_t\) is bounded and continuous almost everywhere as \(N \to \infty\). Consider next each term, \(\sigma^2_{t_i} - \hat{\sigma}^2_h(t_i)\), in the second sum. Since \(\sigma^2_t\) is Lipschitz, there exists a constant \(\kappa\) such that \(\forall u : |\sigma_{t+u} - \sigma_t| < \kappa |u|\).

Now let’s first focus on the first exit time we measure:

\[
\tau^+_h(t_i) = \inf_s \left\{ \left| \int_0^s \sigma_{t_i + u} B_u - B_{t_i} \right| = h \right\}
\]

and compare it to the exit time one would measure if volatility was locally constant in a neighborhood of \(t_i:\)

\[
\tau_h = \inf_s \left\{ \sigma^2_{t_i} |B_t - B_{t_i}| = h \right\}
\]

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The worst that could happen is if \( \sigma_{t_i + u} = \sigma_{t_i} + \kappa u \) which implies that

\[
\left| \int_0^t \sigma_{t_i + u} \, dB_u \right| = \left| \sigma_{t_i} (B_t - B_{t_i}) + \kappa \int_0^t u \, dB_u \right| \leq \sigma_{t_i}^2 |B_t - B_{t_i}| + O \left( t^{3/2} \right)
\]

The worst that can happen is that our effective threshold is off by a term of order \( O(t^{3/2}) = O(h^{9/4}) \). Thus

\[
\mathbb{E} \left[ \hat{\sigma}^2_h(t_i) \right] = \frac{h^2}{c_1} \mathbb{E} \left[ \frac{1}{\tau^+_h(t_i)} \right] = \frac{h^2}{c_1} \mathbb{E} \left[ \frac{1}{\tau_h} \right] + o(h^4) = \sigma_{t_i}^2 + o(h^4)
\]

For small enough \( h \), we have that the terms \( \sigma_{t_i}^2 - \hat{\sigma}^2_h(t_i) \) are uncorrelated across grid points \( i \) (at least arbitrarily low correlation for small \( h \)) with mean zero \( o(h^4) \) and variance given by (12) from which the result follows form the CLT. ■
### Tables and Figures

Table 1: The column $c_N$ contains the exact Jensen correction terms for the estimator (8) based on the first moment of the first exit time (where $\mathcal{C}$ is the Catalan constant). For large $N$ it can be approximated by $c_N \approx 1 + \frac{2}{3N}$.

<table>
<thead>
<tr>
<th>#Obs</th>
<th>First Exit Time Jensen Correction $c_N$</th>
<th>Approximation $1 + \frac{2}{3N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2\mathcal{C}$</td>
<td>$1.8319$</td>
</tr>
<tr>
<td>2</td>
<td>$\log(4)$</td>
<td>$1.3863$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{2}(2\mathcal{C} - 1)$</td>
<td>$1.2479$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{3} (\log(16) - 1)$</td>
<td>$1.1817$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5}{24} (18\mathcal{C} - 11)$</td>
<td>$1.1432$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{1}{756} (1536 \log(4) - 1321)$</td>
<td>$1.0692$</td>
</tr>
<tr>
<td>20</td>
<td>$\frac{(82575360\log(4) - 84385621)}{29099070}$</td>
<td>$1.0339$</td>
</tr>
</tbody>
</table>
Table 2: Summary of the properties of the first hitting time, the first exit time and the first range time estimators.

<table>
<thead>
<tr>
<th></th>
<th>First Hitting Time for Threshold $h$</th>
<th>First Exit Time for Threshold $h &gt; 0$</th>
<th>First Range Time for Threshold $h &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition</strong></td>
<td>$\tau^+<em>{h}(t) = \min{\theta : W</em>{t+\theta} - W_t = h}$</td>
<td>$\tau^+_{h}(t) = \min{\theta :</td>
<td>W_{t+\theta} - W_t</td>
</tr>
<tr>
<td></td>
<td>$\tau^-<em>{h}(t) = \min{\theta : W</em>{t-\theta} - W_t = h}$</td>
<td>$\tau^-_{h}(t) = \min{\theta :</td>
<td>W_{t-\theta} - W_t</td>
</tr>
<tr>
<td><strong>Laplace Transform</strong></td>
<td>$\mathcal{L}_{\tau_h}(\alpha) = e^{-\frac{1}{2} \sqrt{2\alpha}}$</td>
<td>$\mathcal{L}_{\tau_h}(\alpha) = \frac{1}{\cosh\left(\frac{1}{2} \sqrt{2\alpha}\right)}$</td>
<td>$\mathcal{L}_{\tau_h}(\alpha) = \frac{1}{\cosh^2\left(\frac{1}{2} \sqrt{2\alpha}\right)}$</td>
</tr>
<tr>
<td><strong>PDF</strong></td>
<td>$p_{\tau_h}(t) = \frac{h}{\sigma \sqrt{2\pi t^3/2}} e^{-\frac{h^2}{2\sigma^2}}$</td>
<td>$p_{\tau_h}(t) = \sum_{k=0}^{\infty} \frac{2h(1+4k)}{\sigma \sqrt{2\pi t^5/2}} e^{-\frac{(1+4k)^2h^2}{2\sigma^4}}$</td>
<td>$p_{\tau_h}(t) = \sum_{k=1}^{\infty} \frac{4h(-1)^k h^2}{\sigma \sqrt{2\pi t^5/2}} e^{-\frac{k^2h^2}{2\sigma^4}}$</td>
</tr>
<tr>
<td><strong>Reciprocal Moments</strong></td>
<td>$m_1 = \mathbb{E}\left(\tau_h\right)^{-1} = 1 \times \frac{\sigma^2}{h^2}$</td>
<td>$m_1 = \mathbb{E}\left(\tau_h\right)^{-1} = 2\beta(2) \times \frac{\sigma^2}{h^2}$</td>
<td>$m_1 = \mathbb{E}\left(\tau_h\right)^{-1} = 4 \ln 2 \times \frac{\sigma^2}{h^2}$</td>
</tr>
<tr>
<td></td>
<td>$m_2 = \mathbb{E}\left(\tau_h\right)^{-2} = 3 \times \frac{\sigma^4}{h^4}$</td>
<td>$m_2 = \mathbb{E}\left(\tau_h\right)^{-2} = 6\beta(4) \times \frac{\sigma^4}{h^4}$</td>
<td>$m_2 = \mathbb{E}\left(\tau_h\right)^{-2} = 9\zeta(3) \times \frac{\sigma^4}{h^4}$</td>
</tr>
<tr>
<td><strong>Scaling Factors</strong></td>
<td>$\mu_1 = 1$</td>
<td>$\mu_1 = 2\beta(2) \approx 1.83193$</td>
<td>$\mu_1 = 4 \ln 2 \approx 2.77259$</td>
</tr>
<tr>
<td></td>
<td>$\mu_2 = 3$</td>
<td>$\mu_2 = 6\beta(4) \approx 5.93367$</td>
<td>$\mu_2 = 9\zeta(3) \approx 10.81851$</td>
</tr>
<tr>
<td><strong>Unidirectional Estimators</strong></td>
<td>$\frac{h^2}{\mu_1 \tau_h(t)}$ or $\frac{h^2}{\mu_1 \tau_h(t)}$</td>
<td>$\frac{h^2}{\mu_1 \tau_h(t)}$ or $\frac{h^2}{\mu_1 \tau_h(t)}$</td>
<td>$\frac{h^2}{\mu_1 \tau_h(t)}$ or $\frac{h^2}{\mu_1 \tau_h(t)}$</td>
</tr>
<tr>
<td><strong>Bidirectional Estimator</strong></td>
<td>$\frac{1}{2} \left[ \frac{h^2}{\mu_1 \tau_h(t)} + \frac{h^2}{\mu_1 \tau_h(t)} \right]$</td>
<td>$\frac{1}{2} \left[ \frac{h^2}{\mu_1 \tau_h(t)} + \frac{h^2}{\mu_1 \tau_h(t)} \right]$</td>
<td>$\frac{1}{2} \left[ \frac{h^2}{\mu_1 \tau_h(t)} + \frac{h^2}{\mu_1 \tau_h(t)} \right]$</td>
</tr>
<tr>
<td><strong>Unidirectional Variance</strong></td>
<td>$\left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 = 2 \times \sigma^4$</td>
<td>$\left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 \approx 0.76809 \times \sigma^4$</td>
<td>$\left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 \approx 0.40733 \times \sigma^4$</td>
</tr>
<tr>
<td><strong>Bidirectional Variance</strong></td>
<td>$\frac{1}{2} \left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 = 1 \times \sigma^4$</td>
<td>$\frac{1}{2} \left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 \approx 0.38405 \times \sigma^4$</td>
<td>$\frac{1}{2} \left(\frac{\mu_2}{\mu_1} - 1\right) \times \sigma^4 \approx 0.20367 \times \sigma^4$</td>
</tr>
</tbody>
</table>
Table 3: Descriptive statistics for our Dow Jones 30 stock sample.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Mean $ Price</th>
<th>Mean Daily 2min RV</th>
<th>Mean Daily 2min BV</th>
<th>Mean LogSpread</th>
<th>Mean LogSpread/Sigma</th>
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<td>AA</td>
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<td>0.00022</td>
<td>0.00021</td>
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<td>0.00013</td>
<td>0.00023</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Mean ALL: 47 0.00012 0.00011 0.00035 0.03
Median ALL: 48 0.00011 0.00010 0.00033 0.03
Max ALL: 86 0.00037 0.00035 0.00055 0.05
Min ALL: 22 0.00005 0.00005 0.00023 0.02
Figure 1: The passage time densities for the baseline case \( \left( \frac{h}{\sigma} = 1 \right) \). The first range time (thin line), the first exit time (dashed line) and the first hitting time (thick line).

Figure 2: Conditional expectation of reciprocal passage times for the baseline case \( \sigma = h = 1 \). The expected reciprocal first range time (thin line), the expected reciprocal first exit time (dashed line). The unconditional expectations are indicated by thin grey lines.
Figure 3: Signature plots based on Monte Carlo experiments described in Section 5. The figure shows the mean of the estimators as a function of the threshold. The true value is 0.000159.
Figure 4: Signature plots based on Monte Carlo experiments described in Section 5. The figure shows the standard deviation of the estimators as a function of the threshold.
B Signature plots

This section provides the signature plots for the 33 stocks which were part of the Dow Jones 30 index at any point during the period January 2005 through May 2007. The benchmark estimators are the 2 minute subsampled bi-power estimator (BV) and the 2 minute subsampled realized volatility estimator (RV). The two passage time based estimators (denoted DV) are based on the first exit time and first range time. Table 3 above provides some descriptive statistics for the stock sample.

The first two figures show the cross sectional average signature plot for our estimators and their correlation with the 2 minute subsampled BV estimator across threshold choices. The following pages contain the signature plots for each of the 33 individual stocks.