Three Easy Pieces:
Applications of Vector Expected Utility
PRELIMINARY AND INCOMPLETE!

Marciano Siniscalchi
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Abstract

I illustrate the analytical and numerical tractability of VEU preferences (Siniscalchi [24] by means of two sample applications from financial economics.

I first show that a simple VEU specification of ambiguity-averse preferences with CRRA utility and plausible risk-aversion parameters can match both the observed equity premia for risky securities, and the observed risk-free rates. This addresses the so-called equity premium/risk-free rate puzzle.

I then consider a multiperiod specification of VEU preferences to analyze the term structure of interest rates. Standard specifications of CRRA utility and consumption growth are unable to generate upward-sloping yield curves. I show that VEU preferences instead can match this feature of the data.

Both applications rely upon a novel, equivalent characterization of VEU preferences, In the original formulation, ambiguity is captured via the integral of an act’s utility profile with respect to a signed measure. In the proposed alternative, integration with respect to a signed measure is replaced with a standard expectation operator, but the integrand is distorted so as to reflect perceived ambiguity.

1 Introduction

Siniscalchi [24] introduces a model of choice under uncertainty, deemed VEU (vector expected utility), according to which the act $f : \Omega \rightarrow X$ is evaluated via the functional

$$V(f) = \int_{\Omega} u \circ f\, dp + A \left( \int_{\Omega} u \circ f\, dm \right).$$

(1)

The main ingredients of the VEU representation are:

- a von Neumann-Morgenstern utility function $u : \rightarrow \mathbb{R}$, with the usual interpretation;
- a baseline probability measure $p$;
- an adjustment vector $\int_{\Omega} u \circ f\, dm \in \mathbb{R}^n$ (where $0 \leq n \leq \infty$) whose $i$-th component is the Lebesgue integral $\int_{\Omega} u \circ f\, dm_i$ of the real function $u \circ f$ with respect to a signed measure $m_i$ on $\Omega$;
- a function $A$, defined on a suitable subset of $\mathbb{R}^n$ and symmetric about zero: $A(\varphi) = A(-\varphi)$. 


Siniscalchi [24] argues that the VEU model is appealing on methodological grounds. The baseline probability $p$ is uniquely identified from preferences, independently of the other elements of the VEU representation. The representation in Eq. (1) suggests an “anchoring and adjustment” interpretation of choice behavior under uncertainty, where the baseline EU evaluation $\int u \circ f dp$ serves as anchor. The adjustment term $A\left(\int u \circ f dm\right)$ is axiomatically related to the variability, or dispersion, of an act’s utility profile $u \circ f$ around its baseline EU evaluation $\int u \circ f dp$. The signed measures $m_i$ translate key complementarity intuitions about ambiguity in a direct way (see below for an example based on the three-color urn version of the Ellsberg paradox). The functional $A$ allows for a broad range of attitudes toward ambiguity, from aversion à la Gilboa-Schmeidler [10] to appeal, and even encompasses mixed and possibly stake-dependent ambiguity attitudes (see [24] for an example).

This paper has a complementary objective: it demonstrates that the VEU model can also lead to tractable specifications of ambiguity-sensitive preferences, via judicious choices of the signed measures $m_i$ and the functional $A$. This cannot be convincingly illustrated with the kind of simple examples that space constraints allow in a mainly theoretical paper such as [24]. Putting the VEU model to work in actual economic applications provides a better indication of its potential and, of course, limitations. This is precisely what the present paper does.

Specifically, I present two “medium-sized” applications from financial economics. Sec. 3 analyzes the equity premium/risk-free rate puzzle; I show that a particularly simple specification of ambiguity-averse VEU preferences, featuring CRRA utility and a low value of the relative risk-aversion parameter, can match both observed risk premia and real risk-free rates. Sec. 4 concerns the term structure of interest rates. I extend the VEU preference specification of Sec. 3 to a multi-period setting, and show that it can generate upward-sloping real yield curves, while maintaining standard (even “textbook”) assumptions on utility and consumption growth, with plausible parameter values.

Two caveats are in order. First, the analysis in Sec. 3 and 4 uses models and assumptions that are close to those found in comparable papers in the literature, not “dumbed-down” versions thereof; also, some closely related contributions are discussed. However, at the present time, there is no attempt to consider more general and/or flexible specifications to match currently unexplored features of the data, or to perform robustness checks and detailed comparisons with the existing literature. Yet, the analysis presented in this paper suggests that exploring these directions is possible within the VEU preference model.

Second, I wish to demonstrate that VEU representations can be analytically and numerically tractable. However, I certainly do not wish to claim that they are the only tractable specifications of ambiguity-sensitive preferences! For instance, “epsilon-contamination” (Kopylov [17]) and mean-variance preferences (Grant and Kaji [12]) represent tractable specifications of the maxmin-expected utility (MEU) decision model (Gilboa and Schmeidler [10]); the “multiplier preferences” of Hansen and Sargent [13] may be viewed as a tractable specification of Maccheroni, Marinacci and Rustichini [18]’s variational preferences. Rather, I wish to suggest that VEU preferences can be a useful addition to the toolbox of applied economists who are concerned with ambiguity.
Finally, both applications rely crucially upon one novel theoretical contribution, which is the first “easy piece” that the title of this paper refers to. To motivate it, refer to Eq. (1) above. An essential feature of the signed measures $m_i$ that appear in it is the normalization condition $m_i(\Omega) = m_i(\emptyset) = 0$; this intuitively reflects the fact that the certain event, and the impossible event, are not affected by ambiguity. One way to construct signed measures $m_i$ that satisfy this condition is to consider random variables $\zeta_i : \Omega \to \mathbb{R}$ such that $\int_\Omega \zeta_i \, dp = 0$, and then define

$$m_i(E) = \int_E \zeta_i \, dp$$

for every event $E$ (I am abstracting from technicalities such as boundedness and countable additivity, which will be dealt with in due course). In this case, and assuming for notational simplicity that only finitely many measures are employed, Eq. (1) can be rewritten as follows:

$$V(f) = E_p[u \circ f] + A[E_p[\zeta_1 u \circ f], \ldots, E_p[\zeta_n u \circ f]]$$

(2)

where $E_p[\cdot]$ represents expectation with respect to the baseline prior $p$. Thus, integrals with respect to signed measures are replaced with standard expectations, except that the integrand $u \circ f$ is multiplicatively distorted via the random variables $\zeta_i$.

In addition to presenting the VEU model with a more “familiar” face, the formulation in Eq. (2) directly suggests that standard manipulations of random variables and expectations may be employed in applications of VEU preferences. Sections 3 and 4 demonstrate this point.

Furthermore, the Radon-Nikodym Theorem (e.g. [1], Chap. 12) suggests that it should always be possible to rewrite Eq. (1) in the form of Eq. (2). Proposition 2.1 takes care of some required technicalities and shows that this is indeed the case: every VEU preference admits a representation in a form similar to Eq. (2). Moreover, Proposition 2.2 shows that the updating rule for signed measures proposed and axiomatized in [24] has a direct and informative translation in terms of the “distortions” $\zeta_i$.

2 The two faces of VEU preferences

I begin by reviewing the formal definition of a VEU representation of preferences. Then, the alternative Radon-Nikodym representation is developed. Finally, updating is discussed.

2.1 VEU representation with signed measures

Consider a set $\Omega$ (the state space) and a sigma-algebra $\Sigma$ of subsets of $\Omega$ (events). Adopt the following conventional notation: for any interval $\Gamma \subset \mathbb{R}$, $B_0(\Sigma, \Gamma)$ is the set of bounded, $\Sigma$-measurable simple functions on $\Omega$ taking values in $\Gamma$, and $B(\Sigma, \Gamma)$ is its sup-norm closure; if $\Gamma = \mathbb{R}$, these sets will be denoted simply as $B_0(\Sigma)$ and $B(\Sigma)$. The collection of bounded, countably additive measures on $\Sigma$, is denoted by $ca(\Sigma)$, whereas $ca_1(\Sigma)$ indicates the set of countably additive probability measures on $(\Omega, \Sigma)$.

Turning to the decision setting, consider a convex set $X$ of consequences prizes and let $L_0$ be the set of simple acts on the state space $(\Omega, \Sigma)$, i.e. the family of $\Sigma$-measurable functions from $\Omega$ to $X$ with finite
range. With the usual abuse of notation, denote by $x$ the constant act assigning the consequence $x \in X$ to each $\omega \in \Omega$. The main object of interest is a preference relation $\succ$ on $L_0$.

The VEU representation employs collections of signed measures to encode adjustments to the baseline EU evaluation of acts. Such “adjustment measures” are normalized so as to reflect the fact that the empty event $\emptyset$ and the certain event $\Omega$ are not subject to ambiguity. These collections can be finite or countably infinite; in the latter case, adjustment measures are also required to be uniformly bounded and uniformly continuous. Each adjustment measure is then used to compute the integral of the function to be evaluated, and an aggregator, or “adjustment functional,” is employed to obtain a scalar adjustment value. This functional satisfies continuity and symmetry requirements.

The following definition provides the details. Some additional notation is useful: for a function $u : X \rightarrow \mathbb{R}$, $u(X) =\{u(x) : x \in X\}$; also, $0_n$ denotes the zero vector in $\mathbb{R}^n$ ($0 \leq n \leq \infty$). For any collection $m = (m_i)_{0 \leq i < n}$ of measures on $(\Sigma, \Omega)$, and for any $a \in B(\Sigma)$, let $\int a \, dm = \left(\int a \, dm_i\right)_{0 \leq i < n}$ if $n \neq 0$, and $\int a \, dm = 0$ otherwise. Finally, for any collection $m = (m_i)_{0 \leq i < n} \subset \mathcal{C}(\Sigma)$ and any interval $\Gamma \subset \mathbb{R}$, the range of $m$ and $\Gamma$ is the set $R_0(m, \Gamma) = \left\{\int a \, dm : a \in B_0(\Sigma, \Gamma)\right\}$.

**Definition 1** A tuple $(u, p, n, m, A)$ is a VEU representation of a preference relation $\succ$ on $L_0$ if

1. $u : X \rightarrow \mathbb{R}$ is non-constant and affine, $p \in \mathcal{C}_1(\Sigma)$, $n \in \mathbb{Z}_+ \cup \{\infty\}$;
2. $m = (m_i)_{0 \leq i < n} \subset \mathcal{C}(\Sigma)$ satisfies
   (a) $m_i(\Omega) = m_i(\emptyset) = 0$ for $0 \leq i < n$;
   (b) for every $E \in \Sigma$ there exists $N(E) \in \mathbb{R}$ such that $|m_i(E)| < N(E)$ for $0 \leq i < n$; and
   (c) for all sequences $(E_k)_{k \geq 0} \subset \Sigma$ with $E_k \supset E_{k+1}$ for all $k$ and $\bigcap_k E_k = \emptyset$, $\sup_{0 \leq i < n} |m_i(E_k)| \rightarrow 0$.
3. $A : R_0(m, u(X)) \rightarrow \mathbb{R}$ satisfies
   (a) for all sequences $(\varphi^k)_{k \geq 0} \subset R_0(m, u(X))$ such that $\sup_{0 \leq i < n} |\varphi^k| \rightarrow 0$, $A(\varphi^k) \rightarrow 0$;
   (b) for all $\varphi \in R_0(m, u(X))$, $A(\varphi) = A(-\varphi)$;
4. for all $a, b \in B_0(\Sigma, u(X))$, $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$ implies $\int a \, dp + A(\int a \, dm) \geq \int b \, dp + A(\int b \, dm)$;

and, for every pair of acts $f, g \in L_0$,

$$f \succ g \iff \int_{\Omega} u \circ f \, dp + A\left(\int_{\Omega} u \circ f \, dm\right) \geq \int_{\Omega} u \circ g \, dp + A\left(\int_{\Omega} u \circ g \, dm\right).$$

(3)

Condition 2 reflects the normalization, uniform boundedness, and uniform continuity assumptions discussed above. Condition 3(a) implies the normalization $A(0_n) = 0$ (take $\varphi^k = 0_n$ for all $k$): if all ambiguity about an act cancels out, then there is no adjustment to the baseline evaluation. Therefore, for general sequences converging to $0_n$, this condition imposes supnorm-continuity at the origin. Condition 3(b) is a central symmetry assumption, discussed at length in [24]. Condition 4 ensures monotonicity of the VEU representation; in many cases of interest, easy-to-check necessary and sufficient conditions can be provided: see the Appendix in [24] for details.
Finally, observe that, by Theorem 1 in [24], if the state space \( \Omega \) is finite, then VEU preferences can always be represented using finitely many adjustment measures. In this case, the uniform boundedness and uniform continuity conditions 2(b) and 2(c) are automatically satisfied.

### 2.2 VEU representation with Radon-Nikodym densities

Consider a probability measure \( p \) on \( (\Omega, \Sigma) \); let \( E_p : B(\Sigma) \to \mathbb{R} \) denote expectation with respect to \( p \).

The discussion in the Introduction indicates that every \( \Sigma \)-measurable function \( \zeta : \Omega \to \mathbb{R} \) such that \( E_p[\zeta] = 0 \) defines a signed measure \( m_\zeta : \Sigma \to \mathbb{R} \) via the relation

\[
\forall E \in \Sigma, \quad m_\zeta(E) = \int_E \zeta(\omega) \, dp(\omega) = E_p[\zeta 1_E],
\]

where \( 1_E \) denotes the indicator function of the event \( E \). In particular, \( m_\zeta \) satisfies \( m_\zeta(\Omega) = 0 \); if, furthermore, \( |\zeta| \) is \( p \)-integrable, i.e. \( \zeta \in L_1(p) \), then \( m_\zeta \) has bounded total variation, so \( m_\zeta \in ca(\Sigma) \).

This suggests the following counterpart to Def. 1. First, additional notation, mirroring the one introduced before Def. 1, is needed. Given a collection \( \zeta = (\zeta_i)_{0 \leq i < n} \) of \( \Sigma \)-measurable functions, and \( a \in B(\Sigma) \), let \( E_p[\zeta a] = (E_p[\zeta_i a])_{0 \leq i < n} \) if \( n \neq 0 \), and \( E_p[\zeta a] = 0 \) otherwise. Also, given \( \zeta = (\zeta_i)_{0 \leq i < n} \) and an interval \( \Gamma \subset \mathbb{R} \), let \( R_0(\zeta, \Gamma) = \left\{ E_p[\zeta a] : a \in B_0(\Sigma, \Gamma) \right\} \).

**Definition 2** A tuple \((u, p, n, \zeta, A)\) is a VEU representation with Radon-Nikodym adjustments (a VEU-RN representation for short) of a preference relation \( \succcurlyeq \) on \( L_0 \) if

1. \( u : X \to \mathbb{R} \) is non-constant and affine, \( p \in ca_1(\Sigma), n \in \mathbb{Z}_+ \cup \{\infty\} \);
2. \( \zeta = (\zeta_i)_{0 \leq i < n} \) is a collection of random variables in \( L_1(p) \) that satisfies
   (a) for every \( i \) with \( 0 \leq i < n \), \( E_p[\zeta_i] = 0 \);
   (b) for every \( E \in \Sigma \) with \( p(E) \neq 0 \), there is \( N(E) < \infty \) such that \( \sup_{0 \leq i < n} |E_p[\zeta_i 1_E]| < N(E) \);
   (c) for all sequences \((E_k)_{k \geq 0} \subset \Sigma \) with \( E_k \supset E_{k+1} \) for all \( k \) and \( \bigcap_k E_k = \emptyset \), \( \sup_{0 \leq i < n} |E_p(\zeta_i 1_{E_k})| \to 0 \).
3. \( A : R_0(m, u(X)) \to \mathbb{R} \) satisfies
   (a) for all sequences \((\varphi_k)_{k \geq 0} \subset R_0(\zeta, u(X)) \) such that \( \sup_{0 \leq i < n} |\varphi_k| \to 0 \), \( A(\varphi^k) \to 0 \);
   (b) for all \( \varphi \in R_0(\zeta, u(X)) \), \( A(\varphi) = A(-\varphi) \);

4. for all \( a, b \in B_0(\Sigma, u(X)) \), \( a(\omega) \geq b(\omega) \) for all \( \omega \in \Omega \) implies \( E_p[a] + A(E_p[a]) \geq E_p[b] + A(E_p[b]) \);

and, for every pair of acts \( f, g \in L_0 \),

\[
f \succcurlyeq g \iff E_p[u \circ f] + A\left(E_p[\zeta u \circ f] \right) \geq E_p[u \circ g] + A\left(E_p[\zeta u \circ g] \right).
\]

We then have:

**Proposition 2.1** A preference \( \succcurlyeq \) on \( L_0 \) admits a VEU representation if and only if it admits a VEU-RN representation.
The (easy) proof can be found in the Appendix.

The advantage of the representation in Eq. (5) is the exclusive use of standard expectation operators. As the following examples demonstrates, this gives access to many standard manipulations of random variables even in the expanded setting of VEU preferences.

Furthermore, recall that the signed measures in the VEU representation are intended to capture patterns of “complementarity” among ambiguous events. Radon-Nikodym derivatives can reflect the same considerations. For instance, consider the three-color urn version of the Ellsberg [8] paradox. Let \( \Omega = \{r, g, b\} \), where each state represents a possible draw from the urn. The individual is only told that the urn contains 30 red balls and 60 green and/or blue balls. To reflect this, let the baseline probability \( p \) be uniform and define a random variable \( \zeta \) on \( \Omega \) by letting

\[
\zeta(r) = 0, \quad \zeta(g) = 1, \quad \zeta(b) = -1.
\]

Finally, let the adjustment function \( A : \mathbb{R} \to \mathbb{R} \) be defined by \( A(\varphi) = -|\varphi| \) and, without loss of generality, assume that utility \( u \) is linear. It is immediate to verify that the VEU-RN specification \( (u, p, \zeta, 1, A) \) is consistent with the modal preferences in this setting: the agent prefers betting on red than on green, but prefers betting on green or blue rather than on red or blue. Assuming that a bet yields 1 dollar in case of a correct guess and zero otherwise, so that acts can represented by indicator functions,

\[
E_p[1_{r}] - |E_p[\zeta 1_{r}]| = \frac{1}{3} \quad \text{and} \quad E_p[1_{g}] - |E_p[\zeta 1_{g}]| = \frac{1}{3} - \left| \frac{1}{3} \right| = 0
\]

whereas

\[
E_p[1_{r,b}] - |E_p[\zeta 1_{r,b}]| = \frac{2}{3} - \left| \frac{1}{3} \right| = \frac{1}{3} \quad \text{and} \quad E_p[1_{g,b}] - |E_p[\zeta 1_{g,b}]| = \frac{2}{3}.
\]

TODO elaborate interpretation

### 2.3 Updating

Siniscalchi [24] also proposes and axiomatizes an updating rule for VEU preferences. Given a VEU representation \( (u, p, n, m, A) \) for the prior preference \( \succcurlyeq \), and an event \( E \in \Sigma \) for which \( p(E) > 0 \), the conditional preference \( \succcurlyeq_E \) has a VEU representation \( (u, p(\cdot|E), n, m_E, A) \); here, \( p(\cdot|E) \) is the usual Bayesian update of \( p \) and \( m_E = (m_{i|E})_{0 \leq i < n} \), where, for all \( 0 \leq i < n \) and \( F \in \Sigma \),

\[
m_{i|E}(F) = m_i(E \cap F) - p(F|E)m_i(E) = m_i(E \cap F) + p(F|E)m_i(\Omega \setminus E). \tag{6}
\]

Proposition 2.1 implies that the conditional VEU representation \( (u, p(\cdot|E), n, m_E, A) \) will also have a VEU-RN representation. The following result shows that this can be obtained directly from the VEU-RN representation of the prior preference.

**Proposition 2.2** Consider a preference \( \succcurlyeq \) with VEU representation \( (u, p, n, m, A) \) and VEU-RN representation \( (u, p, n, \zeta, A) \). Suppose that \( m_i(F) = E_p[\zeta 1_F] \) for all \( 0 \leq i < n \) and \( F \in \Sigma \).

If its update \( \succcurlyeq_E \) has a VEU representation \( (u, p(\cdot|E), n, m_E, A) \), where \( m_E \) is obtained from \( m \) via Eq. (6), then it also has a VEU-RN representation \( (u, p(\cdot|E), n, \zeta_E, A) \) where, for \( 0 \leq i < n \),

\[
\zeta_{i|E} = p(E)\left( \zeta_{i|E} - E_p[\zeta_{i|E}|E] \right). \tag{7}
\]
Conversely, if $\succcurlyeq_F$ has a VEU-RN representation $(u, p(\cdot|E), n, \zeta_E, A)$ where, for $0 \leq i < n$, Eq. (7) holds, then it also has a VEU representation $(u, p(\cdot|E), n, m_E, A)$ where, for $0 \leq i < n$, Eq. (6) holds.

The updating rule in Eq. (7) can be interpreted as “recentering and rescaling” the Radon-Nikodym derivative $\zeta_j$. TODO elaborate

3 The equity premium / Risk-free rate puzzle

This section considers a textbook example of asset pricing with VEU preferences. Here, “textbook” should be taken literally: I closely follow Chapter 1 of Cochrane [5]. This is intentional: the main message will be that, for the purposes of pricing, VEU preferences may be seen as yielding a stochastic discount factor that differs from the usual one, but can be manipulated in essentially the same way. For instance, the ratio of its standard deviation to its mean still provides a bound on the slope of the mean-variance frontier; simple calculations show that even “moderate” degrees of ambiguity aversion can accommodate a high equity premium for “acceptable” values of the relative risk aversion coefficient.

3.1 Stochastic discount factor for VEU preferences

Consider a one-period investment decision: at time $t$, the agent can purchase one unit of Asset $j$ $(j = 1, \ldots, J)$ for a price $p_j^t$, and at time $t$, she will obtain a random payoff equal to $x_j^{t+1}$. The agent's endowment in each time period is denoted by $e_t, e_{t+1}$. The agent must choose how many units $\xi^j$ of asset $j$ to buy, subject to the budget constraint.

Assume that the agent’s preferences are VEU, with per-period utility $u(\cdot)$, adjustment function $A(\cdot)$, and a single signed adjustment measure represented by the zero-mean random variable $\zeta_{t+1}$. Hence, the agent solves

$$\max_{\xi^1, \ldots, \xi^J} u(c_t) + \beta \left\{ E_t[u(c_{t+1})] + A \left( E_t[\zeta_{t+1} u(c_{t+1})] \right) \right\} \text{ s.t. } c_t = e_t - \sum_{j=1}^J p_j^t \xi^j, \quad c_{t+1} = e_{t+1} + \sum_{j=1}^J x_j^{t+1} \xi^j. $$

Substituting the constraints into the objective function and assuming that, at the optimum, $A$ is differentiable at $E_t[u(c_{t+1})\zeta_{t+1}]$ (see below) yields, for each $j = 1, \ldots, J$, the first-order condition

$$-u'(c_t)p_j^t + \beta \left\{ E_t[u'(c_{t+1})x_j^{t+1}] + A'(E_t[\zeta_{t+1} u(c_{t+1})]) E_t[\zeta_{t+1} u'(c_{t+1})x_j^{t+1}] \right\} = 0.$$  

We can then solve for $p_j^t$:

$$p_j^t = E_t \left[ \frac{\beta u'(c_{t+1})[1 + \zeta_{t+1} A'(E_t[\zeta_{t+1} u(c_{t+1})])]}{u'(c_t)} x_j^{t+1} \right] \equiv E_t[\alpha_{t+1} x_j^{t+1}].$$  

In other words, one obtains a standard pricing relationship in terms of a stochastic discount factor $m_{t+1}$ (henceforth, SDF). Relative to the usual EU setting, $m_{t+1}$ is distorted so as to reflect perceived ambiguity; however, all standard algebraic manipulations of $m_{t+1}$ apply verbatim.
3.2 Sharpe ratio and risk-free rate

As is customary, Eq. (8) can also be interpreted as the condition characterizing a no-trade equilibrium in an “endowment economy.” That is, we assume that the observed consumption process $c_t, c_{t+1}$ arises as an equilibrium of an economy with a representative agent whose endowment coincides with $c_t, c_{t+1}$. In this case, Eq. (8) indicates how prices must adjust so as to ensure that the representative agent is happy to consume her endowment. The equity premium/risk-free rate puzzle is the statement that this pricing relation cannot account for the historical risk-free rate and for the historical equity premium and at the same time.

Furthermore, for “plausible” specification of the agent’s utility function, the predicted equity premium is significantly smaller than the actual one. The expression “equity premium puzzle” is sometimes used with reference to this fact alone. However, the inability to match both the equity premium and the risk-free rate, even allowing for “extreme” specifications of the utility function, is arguably a more severe shortcoming of the standard model, and poses a more intriguing puzzle.

To elaborate, I first use Eq. (8) to derive the risk-free rate in the economy under consideration. If $p_t^1 = 1$ and $x_{t+1}$ is the constant number $R_f$, the basic pricing equation reduces to $1 = E_t[m_{t+1}]R_f$, so $R_f = \frac{1}{E_t[m_{t+1}]}$; in other words, the well-known result for EU preferences holds verbatim in the present setting, because it follows from manipulations of the SDF $m_{t+1}$. The only caveat is that the adjustment function $A$ must be differentiable at the consumer’s optimum—or, since we are considering an endowment economy, at the endowment point $c_t, c_{t+1}$. I will verify that this is the case below.

Similarly, recall that, in an economy with SDF $m_{t+1}$, the absolute value of the Sharpe ratio of any portfolio of securities (i.e. the ratio of mean return in excess of the risk-free rate to its standard deviation) cannot exceed $\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]}$, the ratio of the standard deviation of the SDF to its expected value; in particular, efficient portfolios attain this bound. To see this (cf. [5, §1.4]), consider a portfolio that promises the payoff $x_{t+1}$ and has a price of $p_t$, and let $R_{t+1} = \frac{x_{t+1}}{p_t}$ be its return (a random variable). Then, from Eq. (8), $1 = E_t[m_{t+1}R_{t+1}] = \text{Corr}_t(m_{t+1}, R_{t+1})\sigma_t(m_{t+1})\sigma_t(R_{t+1}) + E_t[m_{t+1}]E_t[R_{t+1}]$, or, multiplying both sides by $R_f = (E_t[m_{t+1}])^{-1}$ and rearranging, $E_t[R_{t+1}] − R_f = −\text{Corr}_t(m_{t+1}, R_{t+1})\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]}\sigma_t(R_{t+1})$. This implies that, as claimed, $E_t[R_{t+1}] − R_f \leq \frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]}$. Again, since this follows from manipulations of the SDF, it applies equally well to EU and VEU preferences (with the same caveat as in the preceding paragraph).

If asset markets are complete, then in equilibrium the market portfolio will be efficient, and hence its Sharpe ratio will attain the bound $\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]}$. This provides one illustration of the equity-premium puzzle: I continue to follow Cochrane [5], pp. 28–29.

To proceed, adopt the two following, standard assumptions: per-period utility is of the power, or CRRA, form, so $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, and consumption growth is log-normal, so $\Delta \ln c_{t+1} \equiv \ln c_{t+1} − \ln c_t \sim N(\mu, \sigma^2)$. With EU preferences, the adjustment function $A$ is identically zero, and standard calculations show that $\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]} \approx \gamma \sigma$. Assume that log consumption growth has a mean and standard deviation equal to 1%: thus, $\mu = \sigma = 0.01$; with EU preferences, and letting $\gamma = 2$ (values considered reasonable range from 1 to 3) yields $\frac{\sigma_t(m_{t+1})}{E_t[m_{t+1}]} \approx 0.02$. However, in US post-war data, the Sharpe ratio of the market portfolio equals about 0.50!
This could be accommodated by taking $\gamma = 50$, even though such a value is generally considered unreasonable. However, with the above specification of preferences and consumption, and assuming that $\beta = 0.99$, this leads to a risk-free rate of 1.46, i.e. 46%! The actual real rate for US post-war data is approximately 1.01, or 1%. For future reference, note that, even with $\gamma = 2$, one obtains a real risk-free rate of approximately 3%, still much higher than the historical average. Decreasing the value of $\gamma$ so as to match the real risk-free rate of course leads to even more severe violations of the bound on Sharpe ratios.

3.3 Analysis with VEU preferences

I now consider the following simple specification of VEU preferences. First, the assumption that consumption growth is lognormally distributed is now interpreted as an assumption on the representative agent’s baseline prior. Next, a single adjustment measure, represented by a Radon-Nikodym derivative as in Sec. 2, will be employed. Let the adjustment function be $A(\varphi) = -\theta |\varphi|$, where the parameter $\theta \in [0, 1]$ serves as a measure of ambiguity aversion; also, let the random variable $\zeta_{t+1}$ be defined by $\zeta_{t+1} = \text{sign}(\Delta \ln c_{t+1} - \mu)$, where $\text{sign}(k)$ equals 1 if $k > 0$, $-1$ if $k < 0$, and 0 if $k = 0$.

The interpretation is that the representative agent does not perceive any ambiguity as to the distribution of the absolute value of deviations of consumption growth from its mean $\mu$; however, she perceives ambiguity about their sign. To make this more concrete, an individual with these preferences is willing to make statements such as, “consumption growth next period will be within 5% of its mean with probability 0.2”; however, she will be unwilling to say that “consumption growth next period is equally likely to be at most 5% higher, or at most 5% lower than its mean.” Moreover, this individual does not perceive any ambiguity as to the variance of consumption growth, but does consider its mean to be ambiguous.

Note that the resulting VEU preferences are also consistent with the maxmin expected-utility model of Gilboa and Schmeidler [10], in the following sense. Let $\Omega = \mathbb{R}$ and assume that the baseline prior is normal with mean $\mu$ and variance $\sigma^2$. Then the VEU functional associating with each measurable $a : \Omega \to \mathbb{R}$ the utility index $E_t[a] - \theta E_t[|\zeta_{t+1} a|]$ also admits a multiple-priors representation à la Gilboa and Schmeidler; in particular, it is monotonic.

I can now compute the stochastic discount factor and its relevant statistics:

**Proposition 3.1** Under the assumptions just mentioned, $m_{t+1} = \beta e^{-\gamma \ln c_{t+1} (1 - \theta \zeta_{t+1})}$; furthermore,

$$E_t[m_{t+1}] = \beta e^{-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2} \{1 + \theta \{2\Phi(\gamma \sigma) - 1\}\}$$ and $\sigma[m_{t+1}] = E_t[m_{t+1}] \left(\frac{1 + \theta^2 - 2\theta [1 - 2\Phi(2\gamma \sigma)]}{1 + \theta [2\Phi(\gamma \sigma) - 1]}\right)^{1/2} - e^{\gamma^2 \sigma^2} - 1$,

where $\Phi(\cdot)$ denotes the c.d.f. of a standard normal random variable.

Before we prove this result, verify that the proposed preference specification helps address concerns regarding high equity premia. Adopt the parameterization proposed in the preceding subsection, and let $\theta = 0.5$. Recall that we are maintaining the “reasonable” value $\gamma = 2$ for the relative risk-aversion coefficient.

These choices yield a risk-free rate of about 2.2% (a better approximation to the actual average) and, most importantly, $\frac{\sigma[m_{t+1}]}{E_t[m_{t+1}]} \approx 0.512$. Indeed, increasing $\theta$ to its maximum value of 1 yields a risk-free rate
of 1.4% and $\frac{\sigma(m_{t+1})}{\mu(m_{t+1})} \approx 1$. This may help accommodate the fact that the cited figure of 0.50 is most likely a lower bound for the Sharpe ratio of an efficient portfolio.  

It is worth investigating the source of the sizable discrepancy between the results for the VEU and EU specifications of preferences. Given the above parameterization, consumption growth is not very volatile; hence, if $\theta = 0$, $m_{t+1}$ has too little volatility to match the recorded Sharpe ratios. However, $\zeta_{t+1}$ is quite a bit more volatile: its standard deviation equals 1. Since $m_{t+1} = \beta e^{-\gamma \Delta \ln c_{t+1}} (1 - \theta \zeta_{t+1})$, if $\theta > 0$, volatility in $\zeta_{t+1}$ translates into higher volatility of the stochastic discount factor. At the same time, there is a small but positive effect of $\zeta_{t+1}$ on $E_t[m_{t+1}]$, which allows for a closer approximation to the average recorded risk-free rate.

The proof of Proposition 3.1 is simple enough, and highlights the main message of this section: under the VEU specification of preferences, the analysis involves fairly standard techniques. Thus, I will provide it in the main text.

We first show that $E_t[\zeta_{t+1} u(c_{t+1})] > 0$, so in particular $A = -\theta |\cdot|$ is differentiable at the optimum. If $\gamma = 1$, then note that $E_t[\zeta_{t+1} \Delta \ln c_{t+1}] = E_t[\zeta_{t+1} \Delta \ln c_{t+1}]$, as $c_{t+1}$ is constant and $E_t[\zeta_{t+1} \gamma] = 0$ for any constant $\gamma$. By the same argument, $E_t[\zeta_{t+1} \Delta \ln c_{t+1}] = E_t[\zeta_{t+1} \Delta \ln c_{t+1} - \mu] = E_t[\Delta \ln c_{t+1} - \mu] > 0$, as claimed. If $\gamma \neq 1$, consider the following simple result:

**Remark 3.1** Let $X \sim N(\mu, \sigma^2)$. Then, for every $t \in \mathbb{R}$, $E_t[\text{sign}(X - \mu)e^{tX}] = e^{t\mu + \frac{1}{2}t^2\sigma^2}[1 - 2\Phi(-t\sigma)]$, where $\Phi(\cdot)$ is the standard normal cdf.

**Proof:** Denote the pdf and cdf of a normal r.v. with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$ by $\phi(\cdot; \hat{\mu}, \hat{\sigma}^2)$ and $\Phi(\cdot; \hat{\mu}, \hat{\sigma}^2)$ respectively. Then, from standard calculations,

$$
\int_{-\infty}^{\infty} e^{tx} \phi(x; \mu, \sigma^2) \, dx = e^{t\mu + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \phi(x; \mu + t\sigma^2, \sigma^2) \, dx = e^{t\mu + \frac{1}{2}t^2\sigma^2}[1 - \Phi(\mu + t\sigma^2, \sigma^2)] = e^{t\mu + \frac{1}{2}t^2\sigma^2}[1 - \Phi(-t\sigma)].
$$

Similarly,

$$
\int_{-\infty}^{\mu} e^{tx} \phi(x; \mu, \sigma^2) \, dx = e^{t\mu + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\mu} \phi(x; \mu + t\sigma^2, \sigma^2) \, dx = e^{t\mu + \frac{1}{2}t^2\sigma^2}\Phi(\mu; \mu + t\sigma^2, \sigma^2)] = e^{t\mu + \frac{1}{2}t^2\sigma^2}\Phi(-t\sigma).
$$

The result now follows. □

From Remark 3.1, 

$$
E_t[\zeta_{t+1} u(c_{t+1})] = u(c_t)E \left[ \frac{u(c_{t+1})}{u(c_t)} \right] = u(c_t)E \left[ \zeta_{t+1} e^{(1-\gamma)\Delta \ln c_{t+1}} \right] = u(c_t)e^{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2}[1 - 2\Phi(-(1-\gamma)\sigma)].
$$

Since $\gamma \neq 1$ and $c_{t+1} > 0$ at the optimum, we verify that indeed $E_t[\zeta_{t+1} u(c_{t+1})] \neq 0$, so $A$ is differentiable at the optimum. Furthermore, if $\gamma > 1$, then the term in square brackets in the right-hand side is negative, $u(c_{t+1}) = \frac{c_{t+1}^\gamma}{\gamma}$ is also negative, and therefore the entire expression is positive. If, on the other hand, $\gamma < 1$, then the term in square brackets is positive and so is $u(c_{t+1})$.

\[1\] For instance, if markets are incomplete, there is no guarantee that the market portfolio will be efficient.
To compute $\sigma$ as claimed.

Again invoking Remark 3.1 and standard calculations,

$$E_t[m_{t+1}] = \beta \left(e^{-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2} - \theta e^{-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2} [1 - 2\Phi(\gamma \sigma)]\right) = \beta e^{-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2} \{1 + \theta [2\Phi(\gamma \sigma) - 1]\}, \quad (10)$$

To compute $\sigma[m_{t+1}]$, note first that

$$E \left[ (e^{-\gamma \ln c_{t+1}}(1 - \theta \zeta_{t+1}))^2 \right] = E_t[e^{-2\gamma \ln c_{t+1}}] + \theta^2 E_t[e^{\frac{1}{2}e^{-2\gamma \ln c_{t+1}}} - 2\theta E_t[e^{\frac{1}{2}e^{-2\gamma \ln c_{t+1}}} \zeta_{t+1}] = e^{-2\gamma \mu + 2\gamma^2 \sigma^2} + \theta^2 e^{-2\gamma \mu + 2\gamma^2 \sigma^2} - 2\theta e^{-2\gamma \mu + 2\gamma^2 \sigma^2} [1 - 2\Phi(2\gamma \sigma)] = \{1 + \theta^2 - 2\theta [1 - 2\Phi(2\gamma \sigma)]\} e^{-2\gamma \mu + 2\gamma^2 \sigma^2}$$

(recall that $\zeta_{t+1}^2 = 1$) and

$$E_t[m_{t+1}]^2 = \beta^2 e^{-2\gamma \mu + \gamma^2 \sigma^2} \{1 + \theta [2\Phi(\gamma \sigma) - 1]\}^2;$$

therefore, the standard deviation $\sigma[m_{t+1}]$ of $m_{t+1}$ equals

$$\sigma[m_{t+1}] = \beta \sqrt{\{1 + \theta^2 - 2\theta [1 - 2\Phi(2\gamma \sigma)]\} e^{-2\gamma \mu + 2\gamma^2 \sigma^2} - e^{-2\gamma \mu + \gamma^2 \sigma^2} \{1 + \theta [2\Phi(\gamma \sigma) - 1]\}^2} = E_t[m_{t+1}] \sqrt{\frac{1 + \theta^2 - 2\theta [1 - 2\Phi(2\gamma \sigma)]}{\{1 + \theta [2\Phi(\gamma \sigma) - 1]\}^2} e^{\gamma^2 \sigma^2} - 1} = E_t[m_{t+1}] \sqrt{\frac{1 + \theta^2 - 2\theta [1 - 2\Phi(2\gamma \sigma)]}{\{1 + \theta [2\Phi(\gamma \sigma) - 1]\}^2} e^{\gamma^2 \sigma^2} - 1} \quad (11)$$

as claimed. ■

TODO other forms of the equity-premium puzzle

### 3.4 Comments

As noted above, the preferences constructed in this section are consistent with the Gilboa-Schmeidler [10] MEU model. In particular, the corresponding set of priors is the convex hull of two points—each a mixture of the Gaussian baseline $p$ with the corresponding “half-normal” probability law (i.e. the distribution of the absolute value of a normal variate with mean zero). However, the MEU representation of these preferences does not yield additional insight, and indeed partially obfuscates the reason why the equity premium/risk-free rate puzzle can be “resolved”—i.e. the fact that the random variable $\zeta_{t+1}$ has a much higher variance than $\Delta \ln c_{t+1}$, which is inherited by the SDF $m_{t+1}$.

A very recent working paper by Alonso and Prado [2] obtains similar results to the ones provided here, employing MEU preferences in a model that is directly related to Mehra and Prescott [20]. Numerically, their formulation requires higher values of the CRRA parameter, even with high levels of ambiguity aversion, to match historical equity premia; for instance, from their Table 1, taking $\gamma = 2$ yields at most an equity premium of approximately 4%, and a risk-free return of about 2.5%.

Maenhout [19] adopts the “multiplier” preferences of Hansen and Sargent [13] to analyze the equity premium puzzle. He emphasizes that the portfolio choices of a Hansen-Sargent decision-maker are indistinguishable from those of an individual who has recursive preferences as in Duffie and Epstein [7],
with a higher risk-aversion coefficient. Strzalecki [25] provides an axiomatic characterization of multiplier preferences that emphasizes the connection with recursive expected-utility models.

Finally, Epstein and Wang [9] analyze asset pricing in an economy with MEU preferences. Their focus is on the indeterminacy of equilibrium, which they relate to increased volatility relative to the benchmark EU model. Interestingly, one of their results shows that equilibrium prices can be viewed as the distorted expectation of the product of the SDF and the asset’s payoffs, where (1) the expectation is taken with respect to the “real” probability measure, and (2) the distortion is the Radon-Nikodym derivative of one of the (extremal) elements of the representative agent’s set of priors with respect to the “real” probability. The similarity with Eq. (8) is only superficial, however: for instance, the Radon-Nikodym derivative in [9] has an expectation of one, not zero, and has an entirely different interpretation. Moreover, these authors explicitly acknowledge (cf. their footnote 23) that their results do not address the equity-premium puzzle.

4 Term structure of interest rates

I now illustrate how the simple specification of adjustments proposed in the preceding section can be extended to settings in which ambiguity concerns beliefs about the realization of a sequence of random variables, i.e. a stochastic process. I will use another “textbook” example: computing the yield curve from the equilibrium conditions in a standard endowment economy where consumption occurs at more than two dates.

4.1 Notation and background

A zero-coupon bond with maturity \( n \) promises the payment of one unit of account \( n \) periods into the future. Denote its price at time \( t \) by \( P^{(n)}_t \). The yield at time \( Y^{(n)}_t \) of a zero-coupon bond with maturity \( n \) is the implied per-period rate of return from holding the bond to maturity, i.e. buying it at time \( t \) and redeeming it at time \( t + n \):

\[
P^{(n)}_t = \frac{1}{[Y^{(n)}_t]^n}.
\]

At any given time \( t \), the yield curve is the collection of yields on bonds of different maturities.

While, in any given period, the yield curve can exhibit different shapes, its historical average yield curve has been upward-sloping. However, it turns out that a simple, multi-period extension of the “textbook” CRRA model in Sec. 3, with standard EU preferences, cannot generate this pattern: see e.g. Chap. 19 in Cochrane [5].

This section will show that VEU preferences can deliver an upward-sloping yield curve for “plausible” choice of the preference parameters \( \gamma \) and \( \theta \), and “calibrated” choices of the parameters governing the stochastic evolution of consumption growth. In particular, this can be achieved with a single adjustment measure, generated by the discounted sum of distortion variables \( \zeta_{t+1}, \zeta_{t+2}, \ldots \) defined as in the previous Section.

It should be emphasized that, strictly speaking, upward-sloping yield curves do not pose a “puzzle” in the same way as historical equity premia and risk-free rates do. For instance, they can be obtained in endowment economies with non-CRRA utility functions and suitably rich consumption-growth processes.
(cf. Gollier [11]), or with recursive-utility preferences (Piazzesi and Schneider [22]). Indeed, the classic Cox-Ingersoll-Ross [6] model, which is derived from equilibrium in an economy with production, can also produce upward-sloping yield curves.\(^2\) Finally, a variety of term-structure models that are based on no-arbitrage, rather than equilibrium considerations, can match a variety of patterns exhibited by yield curves.

The contribution of this section is to indicate how a simple modification of the standard CRRA model can account for upward-sloping yield curves.

### 4.2 Baseline EU model

Again, I literally adhere to a textbook approach, following §19.3 in [5]. If the representative agent is an EU maximizer, then assume that, at time \( t \), she evaluates the consumption process \( \{c_\tau\}_{\tau \geq t} \) via the functional

\[
E_t \left[ \sum_{\tau \geq t} \beta^{\tau-t} \frac{1}{\gamma} c_{t+\tau}^{1-\gamma} \right],
\]

where the expectation is taken with respect to information available at time \( t \). Calculations analogous to the ones performed in Section 3 show that, in this case, the price \( p_t \) at time \( t \) of a security that delivers the stream of payoffs \( \{x_\tau\}_{\tau > t} \) is

\[
p_t = E_t \left[ \sum_{\tau > t} m_\tau x_\tau \right], \quad \text{where} \quad m_\tau = \beta^{\tau-t} \left( \frac{c_{t+\tau}}{c_t} \right)^{-\gamma}.
\]

In the special case of a zero-coupon bond with maturity \( n \), we then get

\[
P_t^{(n)} = E_t [m_{t+n}] \quad \text{and} \quad y_t^{(n)} = \ln Y_t^{(n)} = -\frac{1}{n} \ln \frac{1}{E_t [m_{t+n}]} (12)
\]

(it is common in the literature to consider log yields).

Recall that, in the analysis of Section 2, it was necessary at this point to specify the distribution of consumption growth \( \Delta \ln c_{t+1} \). Similarly, it is now necessary to specify the entire stochastic process of consumption growth \( \{\Delta \ln c_\tau\}_{\tau > t} \), where \( \Delta \ln c_\tau \equiv \ln c_\tau - \ln c_{\tau-1} \). Again in the interest of simplicity, consider an AR(1) specification:

\[
\Delta \ln c_{\tau+1} = (1 - \rho) \mu + \rho \Delta \ln c_\tau + \epsilon_\tau, \quad \epsilon_\tau \sim N(0, (1 - \rho^2) \sigma^2) \text{ i.i.d.} \quad (13)
\]

This formulation ensures that the unconditional mean and standard deviation of consumption growth at any time \( \tau \) are \( \mu \) and \( \sigma \) respectively. We then get

\[
y_t^{(n)} = -\frac{1}{n} \ln E_t \left[ \beta^n e^{-\gamma \sum_{\tau=t+1}^{t+n} \Delta \ln c_\tau} \right]. (14)
\]

The expectation in Eq. (14) can be computed explicitly; furthermore, it can be shown that \( y_t^{(n)} \) is decreasing in \( n \); see [5], pp. 358-9.

\(^2\)In the CIR model, the yield curve does not depend upon the consumption process, but is fully determined by the parameters governing the production technology.
4.3 A multi-period Radon-Nikodym distortion

Turning now to VEU preferences, continue to assume that a single adjustment measure is used, and that the adjustment function is of the form \( A(x) = -\theta |x| \). Extending the intuition underlying the analysis in Sec. 3, assume that the representative agent perceives ambiguity about the evolution of the consumption-growth process \( \{c_\tau\}_{\tau \geq t} \) in Eq. (13); for \( \tau > t \), define

\[
\zeta_\tau = \text{sign} (\Delta c_\tau - E_{t-1}[\Delta c_\tau]) = \text{sign} \epsilon_\tau.
\]  

(15)

To complete the specification of preferences, the distortion of expectation with respect to the baseline prior will be defined as a normalized, geometrically discounted sum of the random variables \( \zeta_{t+1}, \zeta_{t+2}, \ldots \). Formally, assume that the representative agent values the consumption process \( \{c_\tau\}_{\tau \geq t} \) according to the functional

\[
E_t \left[ \sum_{\tau \geq t} \beta^{\tau-t} \frac{c_\tau^{1-\gamma}}{1-\gamma} \right] - \theta E_t \left[ (1 - \alpha) \sum_{\tau > t} \alpha^{\tau-t-1} \zeta_\tau \sum_{\tau \geq t} \beta^{\tau-t} \frac{c_\tau^{1-\gamma}}{1-\gamma} \right].
\]  

(16)

Given the choice of adjustment functional, monotonicity requires that the Radon-Nikodym distortion be at most equal to 1 in absolute value; the above discounting and normalization ensures that this will be the case in every state.

Intuitively, this formulation implies that perceived ambiguity increases in the time horizon, but does so at a decreasing rate. It also implies that, loosely speaking, average ambiguity per time period is decreasing in the horizon. For instance, investors may expect the Fed to take some policy action over the following few months, but may be uncertain as to the exact timing of the intervention; at short time horizons, this uncertainty may loom large, but its impact over longer horizons may be diminished.

4.4 Analysis

We first show that, under the maintained assumptions about consumption growth and adjustments, the expectation in the adjustment term is positive. Thus, at the no-trade equilibrium in the corresponding endowment economy, the VEU preference functional is differentiable at the optimum. The basic intuition is similar to the one developed in Sec. 3, but a somewhat more subtle and algebraically intensive argument is required; for this reason, the proof is relegated to the Appendix.

**Lemma 4.1** Under the assumptions in Eq. (13) and (15),

\[
E_t \left[ (1 - \alpha) \sum_{\tau > t} \alpha^{\tau-t-1} \zeta_\tau \sum_{\tau \geq t} \beta^{\tau-t} \frac{c_\tau^{1-\gamma}}{1-\gamma} \right] > 0.
\]

I now use the conclusion of Lemma 4.1 to derive the basic pricing relation. As in the analysis of Section 3, if the endowment coincides with the optimal consumption process, then the price \( p_t \) of a security that delivers the payoff process \( \{x_\tau\}_{\tau > t} \) is

\[
p_t = E_t \left[ \left( 1 - \theta (1 - \alpha) \sum_{\tau > t} \alpha^{\tau-t-1} \zeta_\tau \right) \sum_{\tau > t} \beta^{\tau-t} \left( \frac{c_\tau}{c_t} \right)^{-\gamma} x_\tau \right] \equiv E_t \left[ \sum_{\tau > t} m_\tau x_\tau \right].
\]
In particular, for a zero-coupon bond with maturity $n$,
\[
P_t^{(n)} = E_t \left[ \left( 1 - \theta (1 - \alpha) \sum_{\tau > t} \alpha^{\tau - t - 1} \xi \right) \beta^n \left( \frac{c_{t+n}}{c_t} \right)^{-\gamma} \right] = E_t \left[ \left( 1 - \theta (1 - \alpha) \sum_{\tau = t+1}^{t+n} \alpha^{\tau - t - 1} \xi \right) \beta^n \left( \frac{c_{t+n}}{c_t} \right)^{-\gamma} \right].
\]

The second equality follows from the law of iterated expectations, noting that $E_{t+n}[\xi] = E_{t+n}[\text{sign}(\epsilon_{\tau})] = 0$ for all $\tau > t + n$.

It is possible to obtain an explicit solution for $P_t^{(n)}$, and hence for the entire yield curve: the main ingredients can be gleaned from the proof of Lemma 4.1. However, it is easier, and equally instructive, to proceed numerically. To evaluate Eq. (17), stipulate that one period corresponds to one quarter, and fix a horizon of interest—say, 30 years, or 120 quarters. Next, draw a sufficiently large sample of sequences $\epsilon_{t+1}, \ldots, \epsilon_{t+120}$ of normal variates, with mean zero and standard deviation $(1 - \rho)\sigma$, and use it to construct a corresponding sample path from the AR(1) process for consumption growth in Eq. (13). For each such sample path, the argument of the expectation in Eq. (17) can be readily computed. Averaging over all sample paths yields an estimate of $P_t^{(n)}$.

I choose the same parameters as in Sec. 3 where applicable: thus, $\theta = 0.5$, $\gamma = 2$, $\mu = 0.01$ and $\sigma = 0.01$. The yearly discount factor is kept at 0.99. The key new parameters are $\rho$, the autocorrelation coefficient for consumption growth, and $\alpha$, the “discount factor” for the Radon-Nikodym distortions.

I set $\rho = 0.3$, which seems to be consistent with values reported in the literature. Based on quarterly data of US consumption of non-durables from 1953 to 2000, Piazzesi [21] reports an autocorrelation coefficient slightly larger than 0.3. For a similar sample, Ang, Piazzesi and Wei [3, Table 3] report an autocorrelation coefficient equal to 0.27. Santos and Veronesi [23] report a coefficient equal to 0.21. Piazzesi and Schneider [22] report a coefficient of 0.36.

Fig. 1 summarizes the results of the calculations. The solid red line (at the top of the graph) corresponds to the benchmark EU model, whereas the finely dotted black line (at the bottom) represents actual average real yields from US Treasury Inflation-Protected Securities (TIPS) at various maturities, for the period 2000-2007, at quarterly frequency; this is based on data collected and interpolated by J. Houston McCulloch at Ohio State University. Consistently with what was found in Sec. 3, the benchmark EU model overestimates the risk-free rate, regardless of the maturity. Furthermore, the yield curve it generates is essentially flat for the parameter values under consideration (upon close inspection, it is actually slightly downward-sloping). By way of contrast, the actual yield curve is initially steep, and then flattens out.

The dashed green line (second from the top) corresponds to an “ambiguity discount” parameter $\alpha = 0.5$, in addition to the parameter choices indicated above. The main observation is that the yield curve is upward sloping and lies below the EU benchmark. Thus, as claimed, the simple parameterization of VEU preferences considered in this section can indeed capture the curvature of the actual yield curve.

---

3 The MATLAB code is available upon request. The only complication is the so-called “initial-value problem”: how should the sequence of consumption growth be started off? The code adheres to the common practice of iterating Eq. (13) with new values of $\epsilon_{t+1}$ until the resulting sample means and standard deviations match the required values.

The dotted blue line corresponds to a different parameterization, aimed at approximating the actual yield curve more closely. Specifically, I set $\theta = 1.0$, $\gamma = 1.75$ and $\alpha = 0.5$. Within the constraints of the simple formulation considered here, this yields a better approximation.

4.5 Comments

Two recent working papers use versions of Hansen and Sargent's multiplier preferences to address the term structure of interest rates.

Brevik [4] adopts the Hansen-Sargent [14, 15] robustness approach with uncertainty about "hidden states", and also delivers upward-sloping yield curves. In [4], the drift of the consumption process is given by a second, unobserved process; the agent tries to infer the current drift from data, but perceives ambiguity about the hidden process that drives it.

Kleshchelski and Vincent [16] combine multiplier preferences with stochastic volatility of output growth. They show that a concern for model misspecification amplifies the effects of conditional heteroskedasticity, and can generate both a considerable equity premium and an upward-sloping yield curve.

By way of comparison, the simple preference specification adopted in this section generates upward-sloping yield curves without relying on the agent's beliefs about hidden states (which, by definition, cannot be elicited from preferences) and retaining a very simple stochastic structure for consumption
growth. It must be noted, however, that both Brevik [4] and Kleshchelski and Vincent [16] provide additional, insightful analysis of other empirical features of yield curves. Given the more limited objectives of the present paper, attempts to provide such analysis within the VEU preference model are left to future work.

A Proofs

A.1 Proof of Proposition 2.1

It is clear that a VEU-RN representation determines a VEU representation by defining the measures \((m_i)_{0 \leq i < n}\) as in Eq. (4); in particular, Conditions 2–4 in Def. 2 map directly to Conditions 2–4 in Def. 1, and the fact that \(\zeta_i \in L_1(p)\) for \(0 \leq i < n\) ensures that the corresponding measure \(m_i\) is of bounded variation, and hence an element of \(ca(\Sigma)\).

In the opposite direction, let \((u', p', n', m', A')\) be a VEU representation of \(\succeq\). It may in general be impossible to invoke the Radon-Nikodym theorem on the components of \(m'\), because these are not required to be absolutely continuous with respect to \(p\). So, a slightly indirect route must be followed.

By Theorem 1 in [24], \(\succeq\) satisfies the axioms for VEU preferences and therefore it has a (non-sharp) VEU representation \((u, p, n, m, A)\) as constructed in §B.5.2 of [24]. In particular, for every \(0 \leq i < n\), \(m_i\) is of bounded (total) variation, and absolutely continuous w.r.t. the finite measure \(p\). Specifically, by Lemma B.14 in [24], \(p(E) = 0\) implies \(m_i(E) = 0\); since \(m_i\) is countably additive and its total variation is finite, it is absolutely continuous with respect to \(p\) (cf. [1], Lemma 9.59). Hence, \(m_i\) admits a Radon-Nikodym derivative \(\zeta_i \in L_1(p)\) with respect to \(p\) ([1], Theorem 12.18). Furthermore, \(E_t[\zeta_i] = \int_\Omega \zeta_i \, d\pi(\omega) = m_i(\Omega) = 0\). Conditions 2–4 in Def. 1 then directly translate to the corresponding conditions in Def. 2.

A.2 Proof of Proposition 2.2

Observe first that, for every \(0 \leq i < n\) and \(F \in \Sigma,\)

\[
E_p \left[ p(E) \left( \zeta_i 1_E - E_p[\zeta_i 1_E | E] \right) 1_F | E \right] = E_p \left[ \zeta_i 1_E 1_F p(E) - E_p[\zeta_i 1_E] 1_F | E \right] =
\]

\[
= E_p \left[ 1_{E \cap F} \zeta_i | E \right] p(E) - E_p[\zeta_i 1_E] p(E | F) =
\]

\[
= E_p \left[ 1_{E \cap F} \zeta_i - m_i(E) p(F | E) =
\right.
\]

\[
= m_i(E \cap F) - m_i(E) p(F | E).
\]

Now assume that \(\succeq_E\) has a VEU-RN representation \((u, p(\cdot | E), n, \zeta_E, A)\), with \(\zeta_E\) as in Eq. (7). Define \(m_E\) by letting \(m_i(E) = E_p[\zeta_i 1_E | F] | E\) for all \(i\). Since the first expression in the above chain of equalities is precisely \(E_p[\zeta_i 1_E | F] | E\), it follows that \(m_{i,E}\) is related to \(m_i\) via Eq. (6).

Conversely, assume \(\succeq_E\) has a VEU representation \((u, p(\cdot | E), n, m_E, A)\), with \(m_E\) as in Eq. (6). Then each \(m_{i,E}\) is related to \(m_i\) via Eq. (6), and the above argument shows that \(p(E) \left( \zeta_i 1_E - E_p[\zeta_i 1_E | E] \right)\) is the Radon-Nikodym derivative of \(m_{i,E}\). But this is precisely \(\zeta_{i,E}\) as defined in Eq. (7).
A.3  Proof of Lemma 4.1

I show that, for all $\tau > t$, $E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right] > 0$; this clearly implies the claim. Note first that

$$E_t [\xi_s] = E_t [\epsilon_s] = 0$$

for all $s > \tau$; hence, the preceding expectation equals $E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right]$. Now note that, for any $s > t$,

$$\Delta \ln \xi_s = \kappa_t + \rho^{s-t} \Delta \ln \xi_t + \epsilon_s + \rho \epsilon_{s-1} + \ldots + \rho^{s-t-1} \epsilon_{t+1}$$

where $\kappa_t$ is a constant that depends on $\rho$ and $\mu$, and so

$$\Delta \ln \xi_t + \ldots + \Delta \ln \xi_{t+1} = \kappa + \rho \Delta \ln \xi_t + \sum_{s=t+1}^{\tau} \eta_s \epsilon_s$$

where $\kappa$, $\rho$ and $\delta_{t+1}, \ldots, \delta_{\tau}$ are suitable constants. Note that, if $\rho \geq 0$, then all such constants are non-negative, and the $\eta_s$'s are strictly positive.

Consider first the case $\gamma = 1$, i.e. log utility. Then

$$E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right] = E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right] \Delta \ln \xi_t + \ldots + \Delta \ln \xi_{t+1}$$

the first equality follows from the observation that $\ln \xi_t$ is known (i.e. constant) at time $t$, and $E_t [\xi_s] = 0$ for all $s > t$ by the law of iterated expectations; the second equality follows from $E_t [\xi_s] = 0$ and the fact that $\kappa$, $\rho$ and, again, $\Delta \ln \xi_t$ are constant (at $t$). Since $\xi_s = \text{sign}(\epsilon_s)$ and $\epsilon_{t+1}, \ldots, \epsilon_{\tau}$ are mutually independent, the expectation in the last line above reduces to

$$E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \eta_s \epsilon_s \right] = E_t \left[ \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \eta_s \epsilon_s \right] > 0.$$

Next, consider the case $\gamma \neq 1$. We then have

$$E_t \left[ \left( \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right)^{1-\gamma} \xi_t^{1-\gamma} \right] = \frac{\xi_t^{1-\gamma}}{1-\gamma} E_t \left[ \left( \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right) e^{(1-\gamma)(\Delta \ln \xi_t + \ldots + \Delta \ln \xi_{t+1})} \right]$$

$$= \frac{\xi_t^{1-\gamma}}{1-\gamma} E_t \left[ \left( \sum_{s=t+1}^{\tau} \alpha^{s-t-1} \xi_s \right) e^{(1-\gamma)\sum_{u=t+1}^{\tau} \eta_u \epsilon_u} \right].$$

Let $s \in \{t + 1, \ldots, \tau\}$. Then $\xi_s = \text{sign}(\epsilon_s)$ is independent of $\epsilon_u$ for $u \neq s$ in $\{t + 1, \ldots, \tau\}$: consequently,

$$E_t \left[ \prod_{u=t+1}^{\tau} e^{(1-\gamma)\eta_u \epsilon_u} \right] = E_t \left[ \xi_s e^{(1-\gamma)\eta_s \epsilon_s} \right] E_t \left[ e^{\sum_{u=t+1}^{\tau} \eta_u \epsilon_u} \right];$$

in the right-hand side, the second expectation is strictly positive, and the first expectation equals $e^{\frac{1}{2}(1-\gamma)^2(1-\rho)^2 \sigma^2 [1-2\Phi(-\gamma)(1-\rho)\sigma]}$. If $0 \leq \gamma < 1$, the term in square brackets is positive, so the entire expectation is positive; and if $\gamma > 1$, the term in square brackets is negative, and so is the entire expectation.
It follows that either all terms $\alpha^s-t-1 \xi_s e^{(1-\gamma)\sum^s_{i=t+1} \eta_i \epsilon_s}$ are strictly positive (if $\gamma < 1$), or they are all strictly negative (if $\gamma > 1$). Hence, $E_t \left[ \sum^s_{i=t+1} \alpha^s-t-1 \xi_s e^{(1-\gamma)\sum^s_{i=t+1} \eta_i \epsilon_s} \right]$ is strictly positive if $\gamma < 1$, in which case also $\frac{\gamma}{1-\gamma} > 0$, or it is strictly negative if $\gamma > 1$, in which case $\frac{\gamma}{1-\gamma} > 0$. It follows that $E_t \left[ \left( \sum^s_{i=t+1} \alpha^s-t-1 \xi_s \right) \frac{\gamma}{1-\gamma} \right] > 0$, as claimed.

References


