Macroeconomic Learning and the Propagation of Technology Shocks*

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Abstract

We extend the standard dynamic stochastic general equilibrium (DSGE) model to include learning about the underlying fundamentals of productivity. Agents in the model do not observe the source of the productivity shock but are assumed instead to observe an aggregate innovation to productivity that they are unable to accurately decompose into its individual temporary, trend and level components. Using the Kalman filter to conduct signal extraction exercises, agents are allowed to update their beliefs about the source of the productivity shock. We examine the impulse responses following a typical permanent real business cycle shock within this environment of learning. We find that the introduction of this particular learning mechanism into the DSGE framework adds little. That is, impulse responses to a permanent shock to the level of productivity are qualitatively similar under full information and learning environments. For the Kalman filter to have any qualitative impact on dynamic behavior we need to assume a counterfactually high relative variance for the trend component of productivity; one rejected by the data. We propose an alternative learning mechanism in which technology shocks diffuse slowly through the economy. We find this alternative form of learning to be more successful at generating a variety of impulse responses with reasonable speeds of learning and with acceptable variances of the technology components.

Keywords: Kalman Filter, Impulse Responses, Learning, Technology Diffusion

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1 Introduction

A recent strand of the macroeconomic literature has empirically tested the real business cycle (hereafter RBC) hypothesis by examining how a permanent shock to the level of productivity (TFP or labor productivity) propagates over time, see for example Gali (1999), Basu, Fernald and Kimball (2006, hereafter BFK), Francis and Ramey (2005) and Christiano, Eichenbaum and Vigfusson (2003). Even to a casual reader of this literature what is glaringly obvious is the lack of consistency in the qualitative nature of the impulse responses, especially for the labor input, generated by this shock, labeled a technology shock. Some researchers, for instance Christiano, Eichenbaum and Vigfusson (2003), find technology shocks to be expansionary for the labor input, while others like Gali (1999) and Francis and Ramey (2005), find technology shocks contractionary.

In an attempt to explain their findings researchers on both sides of the debate have proposed theoretical models with various bells and whistles to mimic their empirical impulse responses. The standard model of Kydland and Prescott (1982) provides the benchmark for the RBC paradigm and has labor rising after a positive shock to technology. In order to generate a fall in labor several adjustments have to be made to the standard theory. Gali (1999) and BFK both suggest adding nominal rigidities coupled with a monetary authority that is not too accommodative. On the other hand, Francis and Ramey (2005) suggest a model with the real rigidities of habit in consumption and investment adjustment costs or an economy with a Leontief production technology. The nature of the real rigidities serves to make the income effects from the technology shock dominate any substitution effects leading households to consume more leisure. Boldrin, Christiano and Fisher (2001) also find these rigidities to be key in matching empirical asset pricing facts.

Another strand of macroeconomics has examined the effects of learning on the transmission of technology shocks through RBC models of the macroeconomy, see the recent contributions of Bullard and Duffy (2004) and Edge, Laubach and Williams (2007, hereafter ELW). Bullard and Duffy analyze the effects of learning under parameter uncertainty, while ELW analyze the effects of Kalman filter learning under uncertainty of the source of the productivity shock.

Our aim in this paper is to relate the two literatures. As motivation for our approach we cite an excerpt from a recent speech that Federal Reserve Chairman Ben Bernanke made to Congress in July 2007 on the U.S. productivity slowdown. In his speech he pointed to difficulties in determining the source(s) of productivity changes. According to Bernanke’s testimony, “The cooling of productivity growth in recent quarters is likely the result of cyclical or other temporary

\[1\] Gali and Rabanal (2004) provides a comprehensive review of this literature.
factors, but the underlying pace of productivity gains may also have slowed somewhat (Bernanke 2007).”

The first goal of our paper is to look critically at the resulting dynamic responses to various technological disturbances in a learning environment similar to that of ELW in which agents cannot directly observe the underlying components of the productivity process. Specifically, we begin with the assumption that agents in these models have to learn the source of the technology shock hitting the economy by using the Kalman filter to update their forecasts of these underlying components. Therefore, households and firms in these models make allocation decisions based on incomplete information regarding stochastic processes. We find that in the standard RBC model, under reasonable parameterizations of a Kalman filter learning process, impulse responses to a shock having permanent effects on the level of productivity are immune to the learning process – in particular we have to make certain unreasonable moment assumptions regarding the trend component in order to generate the Gali-type negative labor impulses in response to permanent technology shocks. We view this as a shortcoming of the Kalman filtering process because it rules out cases in which agents take more leisure in response to a level increase in productivity.

Our second goal is to propose an alternative learning process where technology shocks are characterized by slow diffusion in the spirit of Rotemberg (2003). We interpret the initial periods of slow technological dissemination as the time necessary for agents to decipher the nature and extent of the technological shock and/or the time it takes firms to figure out how best to incorporate the new process. We view learning in this environment as capturing the fact that the full potential of any new innovation is never realized on impact: households need time to figure it out and firms need time to incorporate it into their existing production process or to acquire a new vintage of capital that adopts, or more accurately reflects, the new technology. Additionally, it is usually the case that there are later periods of adjustments to any new technology which would diminish further the extent of the initial impact. Our inspiration for this type of learning comes from another branch of the macroeconomic literature that has focused on the lag between a technological innovation and the ultimate gains in aggregate productivity. For example, Sichel and Oliner (1994, 2000) investigate the particular case of the delayed effect of computer technology. According to Sichel and Oliner, this technology was introduced in the 1980s but did not have a significant impact on productivity.

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2 ELW provided other motivation for incorporating learning by appealing to the fact that real time empirical estimates of the U.S. rate of trend productivity growth as reported by the Council of Economic Advisors or the Congressional Budget Office are often significantly revised over time as more data is observed and past data is updated. This reflects a macroeconomic reality in which economic agents are unable to observe the components of productivity and instead must rely on forecasts of productivity components that they update dynamically.
until the 1990s. Modeling technology as a slowly diffusing process we are able to recover the gamut of impulse responses found in the literature by simply altering the speed of diffusion. In particular our model is able to generate a negative labor response following permanent shocks to the level of productivity under reasonable speeds of diffusion.

As a third and final goal we test the need for the kinds of rigidities commonly employed in the RBC literature in the presence of learning. We examine learning in the RBC model with and without the real rigidities of habit persistence and investment adjustment costs and compare the resulting dynamic responses to various technology disturbances. Our results using the ELW Kalman filtering approach are the same with or without these rigidities. However, the slow diffusion learning environment clearly impact the results. We view these findings as suggesting that our slow diffusion learning story is a viable alternative to these rigidities.

The rest of the paper proceeds as follows. Section 2 presents the basic RBC model under full information and reports the impulse responses to various productivity shocks in the full information environment. Section 3 introduces uncertainty in the productivity series, presents the Kalman filtering approach and reports the impulse responses under uncertainty. Section 4 introduces a slowly diffusing technology process into the basic RBC framework and reports impulse responses in the resulting model. Section 5 discusses and concludes.

2 The Model

We analyze a real business cycle model with technology growth, modified with habit persistence in consumption and capital adjustment costs (see Boldrin, Christiano and Fisher (2001)). The household is assumed to maximize

$$\sum_{t=0}^{\infty} \beta^t \left\{ \ln(C_t - hC_{t-1}) + \theta \ln(1 - \tilde{l}_t) \right\}$$

by choice of consumption $C_t$ and hours $\tilde{l}_t$ at each time $t$ subject to the following constraints:

$$C_t + I_t \leq Y_t$$

$$K_{t+1} = (1 - \delta)K_t + \left( 1 - S \left( \frac{I_t}{I_{t-1}} \right) \right) I_t$$

$$Y_t = X_t^{1-\alpha} K_t^{\alpha} \tilde{l}_t^{1-\alpha}$$
Output, $Y_t$, is produced using labor and capital, $K_t$, and is divided between consumption and investment, $I_t$. The evolution of capital is governed by equation (3). The household is endowed with one unit of time to divide between labor and leisure. At each time period the household can observe the composite labor-augmenting technology process $X_t$, but not its individual components $Z_t$ and $S_t$. The nonstationary component $Z_t$ follows the unit root process described in equation (6). The stationary component $S_t$ represents a temporary disturbance to productivity and follows the process given by (7). Equation (8) describes the process $\gamma_{z,t}$, which is the trend process for both the $X_t$ and $Z_t$ processes. The white noise disturbance terms $\epsilon^s_t$, $\epsilon^z_t$ and $\epsilon^\gamma_t$ are assumed to be orthogonal to one another with respective variances $\sigma^2_s$, $\sigma^2_z$ and $\sigma^2_\gamma$. The parameter $\beta \in (0,1)$ represents the discount factor for the household, $h \in (0,1)$ measures the degree of habit persistence, $\theta > 0$ measures the relative value of leisure to the household, $\delta \in (0,1)$ measures the rate of capital depreciation, $\alpha \in (0,1)$ measures the capital share of output, $\rho_s \in (0,1)$ measures the persistence of the temporary shock, $\rho_\gamma \in (0,1)$ measures the persistence of the trend shock, and the function $S(\cdot)$ governs the degree of investment adjustment costs in the model.}\(^3\)

In the absence of distortions, the social planner problem and the decentralized competitive equilibrium are equivalent for this model. Here, we express the equilibrium using the solution to the social planner’s problem. The respective first-order conditions for consumption, labor, capital and investment are given as follows:

$$
\frac{1}{C_t - \bar{h}C_{t-1}} = \Lambda_t + \frac{\beta h}{E_t(C_{t+1}) - \bar{h}C_t}
$$

\(^3\)On the balanced growth path, $S(\cdot) = S'(\cdot) = 0$. 

\[ X_t = e^{\theta_t} Z_t \]  
\[ Z_t = Z_{t-1} e^{\gamma_{z,t}} e^{\epsilon^z_t} \]  
\[ s_t = \rho_s s_{t-1} + \epsilon^s_t, \]  

\[ \gamma_{z,t} = (1 - \rho_\gamma) \gamma_z + \rho_\gamma \gamma_{z,t-1} + \epsilon^\gamma_t \]
\[
\frac{\theta}{1 - \lambda_t} = \Lambda_t \left( (1 - \alpha)K_t^\alpha \pi_t^{1-\alpha} (e^{s_t} Z_t)^{1-\alpha} \right)
\]

\[
E_t \left( \Lambda_{t+1} \beta \left( \alpha K_{t+1}^{-\alpha} (e^{s_{t+1}} Z_{t+1} \pi_{t+1})^{1-\alpha} \right) \right) + E_t \Xi_{t+1} \beta (1 - \delta) = \Xi_t
\]

\[
\Xi_t \left( \left( 1 - S \left( \frac{I_t}{I_{t-1}} \right) \right) - S' \left( \frac{I_t}{I_{t-1}} \right) \frac{I_t}{I_{t-1}} \right) + \beta E_t \left( \Xi_{t+1} S' \left( \frac{I_{t+1}}{I_t} \right) \frac{I_{t+1}^2}{I_t^2} \right) = \Lambda_t
\]

The standard approach to transform the model into a stationary form, used by works such as King, Plosser and Rebelo (1988) and Del Negro et al. (2004), is to divide or multiply the model variables by the technology shock \(Z_t\). Thus in the standard approach, the following variables represent stationary transformations:

\[
\ddot{\pi}_t = \frac{K_t}{Z_t}, \dot{\pi}_t = \frac{C_t}{Z_t}, \ddot{\iota}_t = \frac{I_t}{Z_t}, \dot{\lambda}_t = \Lambda_t Z_t, \ddot{\xi}_t = \Xi_t Z_t
\]

We are interested in analyzing the system under learning about the sources of productivity shocks. Under this type of learning we assume that individuals do not know the trend \(Z_t\) of the economy, but must form a forecast and update it through time. This leads to some potential difficulty with detrending the model. We deal with this difficulty by exploiting the fact that although agents do not observe \(Z_t\), they do observe \(X_t = e^{s_t} Z_t\). Furthermore, \(X_t\) and \(Z_t\) share the same trend, so we detrend the model by dividing through by \(X_t\). Using this detrending approach the following variables are stationary:\footnote{The relationship between \(\ddot{\pi}_t\) and \(\ddot{\pi}_t^*\) is given by the expression \(\ddot{\pi}_t = \ddot{\pi}_t e^{s_t}\). Similarly, the relationship between \(\dot{\lambda}_t\) and \(\dot{\lambda}_t^*\) is given by the expression \(\dot{\lambda}_t = \dot{\lambda}_t e^{s_t}\). In the impulse response analysis later in this paper, we report all responses in terms of the deviations of \(\log(C_t)\) from its unshocked path.}

\[
\dddot{k}_t = \frac{K_t}{X_t}, \dddot{c}_t = \frac{C_t}{X_t}, \dddot{i}_t = \frac{I_t}{X_t}, \\
\dddot{\lambda}_t = \Lambda_t X_t, \dddot{\xi}_t = \Xi_t X_t
\]

The stationary forms of equations(9) – (12), (2) and (3), in this case, are given respectively as:
\[
\frac{1}{\tilde{c}_t^* - h\tilde{c}_{t-1}^* \left( \frac{X_t}{X_{t-1}} \right)^{-1}} = \tilde{\lambda}_t^* + \frac{\beta h}{E_t \left( \tilde{c}_{t+1}^* \left( \frac{X_{t+1}}{X_t} \right) \right) - h\tilde{c}_t^*}
\] (15)

\[
\frac{\theta}{1 - \tilde{l}_t} = \tilde{\lambda}_t^* \left( 1 - \alpha \tilde{k}^*_t \tilde{l}^{-\alpha} \right)
\] (16)

\[
E_t \left( \frac{X_{t+1}}{X_t} \right)^{-1} \tilde{\lambda}_{t+1}^* \beta \left( \alpha \tilde{k}^*_t \tilde{l}^{-1} \tilde{l}_{t+1}^{-1} \right) + E_t \left( \frac{X_{t+1}}{X_t} \right)^{-1} \tilde{\xi}_{t+1}^* \beta \left( 1 - \delta \right) = \tilde{\zeta}_t^*
\] (17)

\[
\tilde{k}_t^* \tilde{l}^{-\alpha} = \tilde{c}_t^* + \tilde{t}_t^*
\] (19)

The steady state version of the above system of equations is reported in the appendix.

### 2.1 Log-Linearized Model

The log-linearized system is given by:\(^5\)

\[
\tilde{c}_t^* = \mu_1 \tilde{c}_{t-1}^* + \mu_2 E_t \tilde{c}_{t+1}^* + \mu_3 \tilde{\lambda}_t^* + \mu_4 \Delta \hat{x}_t + \mu_5 E_t \Delta \hat{x}_{t+1}
\] (21)

\[
\mu_1 = \frac{h e^{\gamma z}}{(e^{\gamma z})^2 + \beta h^2}, \mu_2 = \beta \mu_1, \mu_3 = -\frac{(e^{\gamma z} - h)(e^{\gamma z} - \beta h)}{(e^{\gamma z})^2 + \beta h^2}, \\
\mu_4 = -\mu_1, \mu_5 = \mu_2
\]

\[
\hat{l}_t = \lambda_1 \tilde{\lambda}_t^* + \lambda_2 \tilde{k}_t^*
\] (22)

\(^5\)Here, \(\tilde{x}_t^* = \log(\bar{x}_t^*) - \log(\bar{x})\).
$\lambda_1 = \frac{1 - \tilde{l}}{(1 - \alpha)l + \alpha}, \lambda_2 = \alpha \lambda_1$

$$\xi_t^* = \phi_1 E_t \lambda_{t+1} + \phi_2 E_t \hat{k}_{t+1} + \phi_3 E_t \hat{\xi}_{t+1} + \phi_4 E_t \hat{\xi}_{t+1} + \phi_5 E_t \Delta \hat{x}_{t+1}$$ (23)

$$\phi_1 = \frac{e^{\gamma z} - \beta (1 - \delta)}{e^{\gamma z}}, \phi_2 = -(1 - \alpha) \phi_1,$$
$$\phi_3 = -\phi_2, \phi_4 = \frac{\beta (1 - \delta)}{e^{\gamma z}}, \phi_5 = -1$$

$$\lambda_t^* = \delta_1 \tilde{c}_t^* + \delta_2 \tilde{c}_{t-1}^* + \delta_3 E_t \tilde{c}_{t+1} + \delta_4 \tilde{\xi}_t + \delta_5 \Delta \hat{x}_t + \delta_6 E_t \Delta \hat{x}_{t+1}$$ (24)

$$\delta_1 = -(1 + \beta) S'' e^{2\gamma z}, \delta_2 = S'' e^{2\gamma z}, \delta_3 = \beta S'' e^{\gamma z}.$$  
$$\delta_4 = 1, \delta_5 = -\delta_2, \delta_6 = \delta_3$$

$$\tilde{\iota}_t^* = \eta_1 \tilde{\iota}_t^* + \eta_2 \tilde{\kappa}_t^* + \eta_3 \tilde{\iota}_t$$ (25)

$$\eta_1 = -\tilde{\iota}_z, \eta_2 = \alpha \tilde{y}_z, \eta_3 = (1 - \alpha) \tilde{y}_z$$

$$\tilde{\kappa}_t^* = \rho_1 \tilde{\kappa}_{t-1}^* + \rho_2 \tilde{c}_{t-1}^* + \rho_3 \Delta \hat{x}_t$$ (26)

$$\rho_1 = e^{-\gamma z} (1 - \delta), \rho_2 = e^{-\gamma z} \frac{\tilde{\iota}}{\tilde{k}}, \rho_3 = -1$$

$$s_t = \rho_s s_{t-1} + \epsilon_t^s$$ (27)

$$\tilde{\gamma}_{z,t} = \rho_{\tilde{\gamma}} \tilde{\gamma}_{z,t-1} + \epsilon_t^\gamma$$ (28)

$$\Delta \hat{x}_t = \log \left( \frac{X_t}{X_{t-1}} \right) - \log \left( \frac{X_t}{X_{t-1}} \right)^{ss} = s_t - s_{t-1} + \tilde{\gamma}_{z,t} + \epsilon_t^z$$ (29)

Here, $\log \left( \frac{X_t}{X_{t-1}} \right)^{ss} = \gamma_z$ refers to the steady state of $\log \left( \frac{X_t}{X_{t-1}} \right)$. 
2.2 Impulse Response Functions with Full Information

We first analyze impulse responses to each type of shock in the absence of learning in the model. In this section our underlying assumption is that individuals directly observe the source of the shock, an assumption we will relax in the next section. In the baseline parameterization of the model, we set \( \alpha = 0.4, \beta = 0.987, \delta = 0.012, \theta = 0.78, \gamma_z = \log(1.00300) \), standard values in the macroeconomic literature and values used by ELW. We set the coefficients \( \rho_s \) and \( \rho_\gamma \) governing the duration of the temporary and trend shocks, respectively, both to 0.7. From here on we follow Boldrin, Christiano and Fisher (2001) and Francis and Ramey (2005) and refer to the model with the rigidities of habit persistence and investment adjustment costs as the modified model and the model without these rigidities as the standard model. To obtain the standard model without we set \( h = 0 \) and \( S'' = 0 \), and to obtain the modified model we set \( h = 0.75 \) and \( S'' = 5.16 \). Absent learning we present two sets of impulse responses, with and without the rigidities of habit persistence in consumption and investment adjustment costs, in Figures 1 – 3. We present impulse responses in the levels of the variables by adding back the change in the trend from the technological processes. Notice that a temporary technology shock will not change the underlying trend, thus, its detrended and trended impulse responses are identical.

We begin with the impulse responses to a temporary technology shock in both versions of the model. This type of temporary shock is common in the theoretical literature, see for example Kydland and Prescott (1982), King, Plosser and Rebelo (1988) and Cooper and Johri (2002). The temporary shock, however, does not have a well-defined counterpart in the empirical literature, since the two primary means to identify technology shocks, the long-run restrictions of the structural VAR approach used by works such as Gali (1999) and Francis and Ramey (2005) and the calculations of Solow residuals used by works such as Basu and Kimball (1997) and Burnside, Eichenbaum and Rebelo (1995) only identify permanent technology shocks. The theoretical temporary technology shock, here, works to initially raise the level of output for a given amount of capital and labor. Ultimately, the effect of this productivity shock should disappear with the speed of return governed by the autoregressive parameter \( \rho_s \). Thus, an empirical temporary shock would necessarily be a shock that generates positive productivity growth followed by negative productivity growth that returns productivity to its original level for a given level of factor input. The standard techniques of empirical technology shock identification are unable to identify this type of shock. Despite this shortcoming, we include responses to the temporary shock for several reasons. First, the temporary shock is a theoretically familiar shock that can aid comparisons with previous work.
Second, a temporary shock helps uncover and highlight some of the dynamic mechanisms in play in the RBC model. Finally, the temporary shock provides another dimension in the learning problem of the economic agents in the model and thus another possible friction to help explain some of the empirically observed responses to permanent technology shocks found in the literature.

Figure 1 shows that in response to the temporary technology shock productivity, output, consumption, investment and capital increase, while the labor response varies across models. In the rigid-free world labor input rises since households take advantage of the temporarily higher productivity by dramatically increasing investment. This increase in investment limits the initial increase in consumption, which follows a steady path showing the underlying desire to smooth consumption. Thus, the real wage rises initially because of the technology shock and continues to remain high because of the increase in capital and the persistence of the temporary shock. At the same time, the limited rise in consumption keeps the marginal value of consumption from falling too much and limits the negative impact on hours. The modified model, on the other hand, restricts the range of consumption and investment responses. In particular, the initial increase in investment is limited, causing capital formation to be smaller in the modified model and leading households to increase consumption by a larger amount than in the model without rigidities. The marginal value of consumption falls, putting downward pressure on the labor response of households. The increase in the real wage in this case is constrained since the rise in capital is restricted by the investment adjustment costs. Ultimately, the negative effect on hours from the fall in the value of consumption dominates the positive pressure on hours from the rise in the real wage. Therefore, even with a temporary shock to technology the consumption and investment rigidities are enough to cause a negative labor response.\footnote{Investment adjustment cost is the more important of the two rigidities in delivering a negative response in hours to a temporary technology shock. We analyze the case with only the investment adjustment friction activated (not reported) and were able to still recover the negative response of hours in this case.}

Figure 2 shows the responses to a shock to the trend in productivity.\footnote{The initial magnitude of the trend and level shocks in this paper have been constrained so that the level of productivity permanently rises in the long run by one percent.} This shock acts to temporarily increase the growth rate of productivity, with \( \rho_\gamma \) governing the duration of higher growth.\footnote{When \( \rho_\gamma = 0 \), the trend shock is the same as the permanent level shock we analyze below.} In this situation, agents realize that productivity has increased and will grow even higher in the future due to the nature of the trend shock. In the rigid-free world, agents desire to smooth their consumption profile, consistent with the permanent income hypothesis. Realizing that their consumption opportunities are now higher in the future, agents increase consumption and decrease the level of investment, driving down the capital stock slightly. The increase in consumption lowers
the utility value of working in addition to limiting the initial growth in capital, while the increase in productivity increases the incentive to work. Initially, the negative effects on hours dominate causing hours to fall. Eventually, the capital stock begins to grow causing labor to increase. In the modified model the presence of investment adjustment costs keeps investment from falling and along with the presence of habit persistence limits the initial rise in consumption. In the modified model, agents again value consumption smoothing, but habit persistence significantly alters the utility value associated with a given consumption profile. In particular, despite the initial subdued response of consumption, the utility value associated with additional consumption falls and, just as in the rigid-free model, this fall outweighs the productivity gains to the shock causing hours to initially fall. Over time, the capital stock rises as it did in the rigid-free model and the response of labor becomes positive before returning to its steady state. Notice that the two sets of impulse responses result in the same ultimate change in the model variables, it just takes longer for the shock in the modified world to fully work itself out.

Figure 3 shows the impulse responses to a shock to the level of productivity. The recent debate about the plausibility of the technology-driven RBC model explaining economic fluctuations has focused on the impulse responses to this shock. This shock is similar in nature to the trend shock in productivity in that it drives up permanently the level of labor productivity. Unlike the trend shock, however, there is no extended growth in the technology series. Following the level shock the unit root process \( X_t \) permanently shifts up in a one-time increase. Thus any differences in the responses of the model variables to the trend and level shocks must be due solely to the presence of \( \rho \). In the standard model, following the onset of the shock agents realize that the level of technology has increased and unlike a trend shock will not continue to grow over time. In this case agents no longer see as high a value in foregoing investment for consumption today, since productivity will not grow further in the future. Agents thus respond by immediately increasing investment and consumption. In this case, we obtain the standard results found elsewhere in the literature that positive innovation to technology in a standard RBC model leads the increase in productivity to outweigh the fall in the utility value of consumption causing output, all components of output, and all factor inputs to rise. As highlighted in Francis and Ramey (2005), however, the addition of habit formation and investment adjustment costs to the standard model causes the labor input to fall on impact in response to this shock. This occurs because investment adjustment

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9 For the sake of space we do not report the responses of \( \lambda_t \), the shadow price of consumption.

10 We also analyzed the model responses to smaller values of \( \rho \) and found qualitatively similar responses to the ones reported here with values as small as 0.5.
costs limit the increase in investment and thus capital. Habit formation also limits the utility value associated with additional consumption. Thus, following the level shock agents no longer desire to use hours to produce more output for increased consumption and investment. Thus agents respond by consuming more leisure and lowering the amount they work.

In looking at the responses of the standard model together in Figures 1 – 3, we see that individuals respond differently to each of the three shocks. In particular, in the standard model hours increase in response to a temporary and permanent shock and decrease in response to a trend shock. Thus, it seems possible that even in a rigid-free model if agents are unsure of the source of the shock when they experience a permanent shock, they might possibly confuse this shock with a trend shock and respond by decreasing hours. We view this story of learning as an intuitively appealing alternative of the modified model’s mechanism of investment adjustment costs and habit persistence. We add signal extraction to the model in the next section and investigate the plausibility of this story of learning.

3 The System Under Uncertainty and Learning

3.1 Signal Extraction

Next we analyze the case where agents do not directly observe the source of the productivity shock and instead must form a belief and update this belief about the source of the productivity shock. In particular, agents observe the composite productivity shock \( \Delta x_t \) when forming forecasts of the underlying processes \( s_t, s_{t-1} \) and \( \hat{\gamma}_{z,t} \).\(^{11}\)

Kalman Filter

We can map the signal extraction problem into the Kalman filter framework explained in Hamilton (1994). We maintain the standard assumption of ELW (2007) that individuals observe the aggregate shock process with a lag. We discuss the implications of this assumption later. The state vector is thus given as:

\[
\psi_t = \begin{pmatrix} s_t \\ \hat{\gamma}_{z,t} \\ s_{t-1} \end{pmatrix}
\]

\(^{11}\)In ELW agents can only observe the aggregate productivity series with a lag. We assume here that agents can observe the contemporaneous productivity series. Given that agents learn and form expectations within the model detrended by \( X_t \), in order to be consistent, \( X_t \) must be in their information set. We explore further this idea of model-consistent learning and examine alternative timing assumptions using the Kalman filter in Fout and Francis (2008).
The state equation can thus be expressed as

$$
\psi_{t+1} = F\psi_t + v_{t+1},
$$

(31)

where

$$
F = \begin{pmatrix}
\rho_S & 0 & 0 \\
0 & \rho_\gamma & 0 \\
1 & 0 & 0
\end{pmatrix}
$$

(32)

and

$$
v_t = \begin{pmatrix}
\epsilon_i^t \\
\epsilon_i^7 \\
0
\end{pmatrix}.
$$

(33)

The observation equation is given as

$$
\Delta x_t = H'\psi_t + w_t,
$$

(34)

where

$$
H' = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}
$$

(35)

and

$$
w_t = \epsilon_i^t.
$$

(36)

The error terms $v_t$ and $w_t$ are each serially uncorrelated and uncorrelated with one another at all leads and lags:

$$
E(v_t v_{\tau}) = \begin{cases}
Q & \text{for } t = \tau \\
0 & \text{otherwise}
\end{cases}
$$

(37)

$$
E(w_t w_{\tau}) = \begin{cases}
R & \text{for } t = \tau \\
0 & \text{otherwise}
\end{cases}
$$

(38)

$$
E(v_t w_{\tau}) = 0 \text{ for all } t \text{ and } \tau,
$$

(39)

where
Once the signal extraction problem is mapped into the above state space representation, the Kalman filter can be used to produce recursive linear least squares forecast of the unknown state vector \( \psi_t \). The recursion is started with an initial forecast of \( \hat{\psi}_{t|0} \) and an initial mean square error of the forecast \( P_{t|0} \), where

\[
\hat{\psi}_{t|t-1} = E(\psi_t | \Delta x_{t-1}, \Delta x_{t-2}, \ldots, \Delta x_1)
\]

and

\[
P_{t|t-1} = E[(\psi_t - \hat{\psi}_{t|t-1})(\psi_t - \hat{\psi}_{t|t-1})'].
\]

The initial forecast and its initial MSE are given respectively as

\[
\hat{\psi}_{1|0} = E(\psi_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

and

\[
P_{1|0} = E[(\psi_1 - \hat{\psi}_{1|0})(\psi_1 - \hat{\psi}_{1|0})'] = \begin{pmatrix}
\frac{\sigma^2}{1-\rho_s^2} & \frac{\rho \sigma^2}{1-\rho_s^2} & 0 \\
\frac{\rho \sigma^2}{1-\rho_s^2} & \frac{\sigma^2}{1-\rho_s^2} & 0 \\
0 & 0 & \frac{\sigma^2}{1-\rho_s^2}
\end{pmatrix}
\]

The recursion follows the processes given below:\(^{12}\)

\[
\hat{\psi}_{t+1|t} = F\hat{\psi}_{t|t-1} + FP_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}(\Delta x_t - H'\hat{\psi}_{t|t-1})
\]

\[
P_{t+1|t} = FP_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}F' + Q
\]

The Kalman gain is thus given by

\(^{12}\)See Hamilton(1994).
We can also restrict the above model in order to compare the results of this paper to previous work in the literature. For instance, ELW analyze the case of signal extraction with only a permanent and trend component in the productivity series.\(^{13}\) In order to analyze this case as well as the other two cases of signal extraction with two shocks, we can write out case-specific versions of the state equation (31), the updating equation (46) and the observation equation (34). We report these values in the technical appendix.

The actual system of the model includes the linearized resource and capital evolution constraints (25) and (26), the linearized labor-leisure condition (22), the linearized technology processes (27)–(29), the Kalman filter equations (44)–(47) and modified versions of equations (21),(23) and (24). The latter three equations must be modified to include forecasts of the future aggregate productivity shock \(\Delta x_{t+1}\) formed with the Kalman filter. These modified equations are reported in the technical appendix. The model now consists of two types of equations – the capital, resource and technology equations, which depend only on the true observed values of \(\hat{x}_t\) and the household’s decisions, which also depend on forecasted values of the future aggregate productivity series and the underlying unobserved components of productivity. In solving the model, we allow agents to observe the state vector which includes only \(\Delta \hat{x}_t, \hat{c}_{t-1}, \hat{i}_{t-1}\) and \(\hat{k}_{t-1}\). The actual underlying shock processes given in equations (27)–(29) affect the model only through the determination of the aggregate shock \(\Delta \hat{x}_t\).

### 3.2 Impulse Response Functions with Signal Extraction

We now analyze the impulse response functions in the case of learning in an environment of uncertainty about the source of the productivity shock, when the productivity shock consists of three components; a temporary component, a trend component and a permanent component.\(^{14}\) Agents cannot directly observe the source of the technology shock and must use signal extraction to disentangle any aggregate shock. For the remainder of the paper, we omit the responses associated with

\[
K_t \equiv F P_{t|t-1} H (H' P_{t|t-1} H + R)^{-1}.
\]

\(^{13}\)ELW do not actually follow a dynamic Kalman filtering approach as we do and instead restrict the gain parameter to a constant value.

\(^{14}\)As mentioned above, it is also possible to analyze the impulse responses when uncertainty comes from only two of these three shocks. For instance, we can calculate the impulse responses in the case where productivity consists of only a level and trend component. Edge et al. (2007) analyze impulse responses in a similar environment of learning. We find that the responses with three shocks are similar to the responses with only two shocks and thus we omit the two shock exercise.
a temporary productivity shock and focus on the responses to the trend and level shocks. These responses are shown in Figures 4 and 5. We set the ratio of variances $\frac{\sigma_z}{\sigma_s}$ and $\frac{\sigma_z}{\sigma_t}$ to 0.17 and 1, respectively. The former is the signal-to-noise ratio reported by ELW for postwar US data.

Our paper departs from ELW in very important ways. Our informational assumptions are fundamentally different. Whereas we assume agents observe that there is a composite time $t$ shock and thus detrend by that aggregate, ELW assumes that the agents detrend by the time $t$ aggregate but only observe the time $t-1$ aggregate when updating forecasts of the aggregate productivity series and its unobserved components. This timing difference is enough to cause major variances in the nature of the impulse responses. In particular, under the ELW assumption the Kalman filter learning process set out in this paper is unable to generate an initial fall in the labor input no matter the assumptions of the model (rigid-free or not) and no matter which variant of the technology shock hitting the economy.\(^{15}\) We also find in results below, under reasonable parameterization, our informational assumption does little to change the full information results for the level shock case, the most famous of the three shocks. That is, in the standard model with learning we will be able generate a fall in labor input to a level shock only if:

1. real rigidities are present or,
2. we assume a counterfactually high variance for the trend shock (resulting in a high signal to noise ratio) – one that is not supported by the data.\(^{16}\)

First, we analyze the responses following a shock to the trend of productivity displayed in Figure 4. ELW report impulse responses to a permanent increase to the growth rate of productivity. In the context of the model of the previous section, this would correspond to the case where $\rho_\gamma$, the persistence parameter for the trend shock, is equal to unity. Our model, like other standard models, including the model of ELW, is linearized around the steady state. Since $\gamma_z$, the unconditional expectation of the growth rate of productivity, is a parameter of this steady state, it is impossible to examine a permanent shock to the trend growth rate to productivity in our model and other standard linearized models of the business cycle, since the trend would no longer have an unconditional expectation and the model would thus have no steady state. We thus analyze the model with $\rho_\gamma$ bounded away from one – set to 0.7 in our case.

\(^{15}\) These results are reported in Fout and Francis (2008).
\(^{16}\) This is related to the pile-up problem of Stock and Watson (1998) and addressed in both ELW and Roberts (2001).
A comparison of the impulse responses following a shock to trend productivity with learning and under full information, Figure 4 and 2, respectively, highlights some additional features of imperfect information. In this case, given the signal-to-noise ratio, agents put considerably higher weight on a productivity disturbance being either temporary or permanent as compared with being a trend shock. Thus, following a trend shock with no habit or investment adjustment costs, agents also put some weight on the shock being temporary or permanent (in levels) and respond by raising consumption and investment, similar to the responses to these two shocks in the full information cases. In this rigid-free case, individuals increase their hours worked, the opposite result from the full-information case. Output also increases and by more than the rise in hours causing productivity to rise. The responses under the case of habit and investment adjustment costs look similar to the modified version of the full information case.

We carry out a final set of comparisons between the impulse responses to a shock to the level of productivity under the full information and learning cases of Figures 3 and 5, respectively. The impulse responses are similar under both the standard and modified models. Other than affecting the shapes of the impulse responses, the addition of learning does little to change the dynamics in the standard RBC model – consumption, investment and factor inputs rise. Thus, while we find that this type of learning can potentially change the response to the trend shock in the standard model, without the need to resort to rigidities, this is not so in the case of a level shock. The reason is that under realistic parameterizations of the learning process, agents will believe any shock they face will have most of its effect on impact and respond by increasing both consumption and output. Only when a significantly high weight is put on the probability of a trend shock, a parameterization not supported by the data, will agents respond to a permanent shock by reducing hours. Specifically, we would need the variance of the trend shock to be counterfactually high relative to the other components of technology to affect the full information results. For instance, if the signal-to-noise ratio is set equal to unity – agents believe shocks to the trend and shocks to the level of technology are equally likely – the standard model can deliver a negative hours response following a technology shock. We investigate and discuss alternative parameterizations and timing assumption with the Kalman filter further in Fout and Francis (2008).

In the next section we propose a type of learning that is able to deliver the immediate fall in labor following both a shock to the trend and level of productivity in both the modified and standard models. This new specification of learning models technology as a slowly diffusing process as suggested by Rotemberg (2003). However, our technology process will be more in line with the
way we have modeled technology in this paper. We will interpret the initial slow diffusion periods as the time it takes for agents to decipher the type and magnitude of the technological change.

4 Slow Technology Diffusion

Next we explore another learning mechanism. We introduce a technology process that is slow to diffuse through the economy.\textsuperscript{17} The process we have in mind consists of an underlying innovation process $\tilde{X}_t$ that is slowly adapted into the technology process $X_t$. We assume that the process $\tilde{X}_t$ embodies the technological change or innovation, while the process $X_t$ gradually incorporates the change. Over time, more of the change is absorbed, and in the limit $X_t$ converges to $\tilde{X}_t$.\textsuperscript{18} The equations for $\tilde{X}_t$ and $X_t$ are as follows:

\begin{align*}
\tilde{X}_t &= e^{st} Z_t \quad (49) \\
X_t &= \sum_{i=0}^{\infty} (\omega e^{\gamma z})^i (1 - \omega) \tilde{X}_{t-i} \quad (50)
\end{align*}

The parameter $\omega$ governs the speed at which technological innovations diffuse through the economy. High values of $\omega$ mean slower diffusion, while a value of zero corresponds with the full information case. The $X_t$ process still affects production the same way as specified in equation (6), and the log-linearized system still consists of equations (21) – (28). Equation (29), however, needs to be replaced with the following log-linearized version of (49):

\begin{align*}
\Delta \tilde{x}_t &= \Delta s_t + \tilde{\gamma}_{zt} + \epsilon^2_t \\
\quad (51)
\end{align*}

We next need to derive a log-linearized version of (50). To do this we first divide the equation by $\tilde{X}_t$ to make it stationary, yielding:

\begin{align*}
\frac{X_t}{\tilde{X}_t} &= \sum_{i=0}^{\infty} (\omega e^{\gamma z})^i (1 - \omega) \left( \prod_{j=0}^{i-1} e^{\gamma z t-j} e^{\epsilon^2 t-j} \right)^{-1} e^{s_t-i-s_t} \quad (52)
\end{align*}

The log-linearized version of (52) is given by:\textsuperscript{19}

\textsuperscript{17}The type of information diffusion we explore here is similar in spirit to Mankiw and Reis (2001), who also assume that information disseminates slowly through the economy. Mankiw and Reis, however, model the information diffusion as a staggered, random update to heterogeneous agents’ information sets.

\textsuperscript{18}We present graphical evidence of the convergence of $X_t$ to $\tilde{X}_t$ in Figure 8 in the technical appendix following shocks to the temporary, trend and permanent components.

\textsuperscript{19}Since $X_t$ converges to $\tilde{X}_t$, in the steady state $\left( \frac{X_t}{\tilde{X}_t} \right)^{ss} = 1.$
\[
\log \left( \frac{X_t}{X_t} \right) = \sum_{i=1}^{\infty} \omega^i s_{t-i} - \omega s_t = \sum_{i=0}^{\infty} \omega^i (\gamma_{z,t-i} + \epsilon^z_{t-i}) \quad (53)
\]

Lagging equation (53) yields:
\[
\log \left( \frac{X_{t-1}}{X_{t-1}} \right) = \sum_{i=1}^{\infty} \omega^i s_{t-1-i} - \omega s_{t-1} = \sum_{i=0}^{\infty} \omega^i (\gamma_{z,t-1-i} + \epsilon^z_{t-1-i}) \quad (54)
\]

Taking the difference between equations (53) and (54) and rearranging gives the following log-linearized expression for \( \Delta x_t \):
\[
\Delta x_t = \Delta x_t - \sum_{i=1}^{N} \omega^i \Delta s_{t-i} - \omega \Delta s_t - \sum_{i=0}^{N} \omega^i (\Delta \gamma_{z,t-i} + \Delta \epsilon^z_{t-i}) \quad (55)
\]

In the above equation, we write a finite version of the sum, since we are interested in impulse response functions and thus we only care about the terms from the date of impact of a shock and later. In other words as long as \( N \) exceeds the time frame of the impulse response analysis, the infinite and the finite cases are equivalent for our purposes. The log-linearized system for this form of learning consists of (21) – (28), (51) and (55). When \( \omega = 0 \) the system without learning is recovered. The impulse response functions for the case where \( \omega = 0.75 \) are reported in Figures 6 – 7. This corresponds to the case where 25 percent of the innovation is immediately embodied in the aggregate production function.

Figure 6 shows the responses to a trend shock in this environment of learning. In both the standard and modified models the shock causes consumption to rise by much more than output. Here, agents realize that the effect of the shock will increase over time for two reasons, (1) the technology shock will diffuse fully over time and, (2) a shock to trend also means higher levels of productivity in the future, explaining the relatively large initial rise in consumption. In the standard version of the model, with no habit or investment adjustment costs, investment falls relative to its unshocked path to accommodate this increase in consumption. This limits the rise in capital which, when coupled with the fall in the marginal value of consumption, causes hours to fall, a finding similar to that in the full information case. The modified version of the model follows a similar logic, except that the presence of investment adjustment costs and habit persistence keeps investment from falling and consumption from rising as much as each does in the standard model. This also limits the initial response of hours.

Next, we examine the effects of slow diffusing technology when there is a permanent shock to the level of productivity. This is demonstrated in Figure 7. We find that labor responds negatively
to a permanent shock to technology under both models. In this case, the basic reason for this is that the slow diffusion of technology allows the level shock to mimic the trend shock. This can be verified by comparing Figure 7 with Figure 2; the latter being the impulse responses to a trend shock under full information. These figures are remarkably similar. With the slow diffusion of technology, agents now respond to a permanent shock as they would to a trend shock in the full information case. In other words, they respond by immediately raising consumption, since they know productivity will be even higher in the future. In the standard model, investment falls and when coupled with the fall in the marginal utility of consumption allows the rigid-free model to deliver a negative labor response. The rigid-free version of the model can also deliver positive hours responses following a permanent technology shock with smaller values of $\omega$. For instance, in the extreme case when $\omega$ is set to zero we recover the full information responses that appear in Figure 3. Thus by varying the degree of learning $\omega$ we are able mimic a variety of responses to a level shock found in the empirical literature.

5 Conclusion

In this paper, we have included two models of learning in the RBC framework. In the first model of learning, agents in the model are unable to observe the source of the productivity shock. They are assumed instead to observe an aggregate productivity series that they use to form forecasts of the underlying temporary, trend and level components. Agents use signal extraction to update these beliefs about the source of the productivity shock. We find that this method of signal extraction has little impact on the results when there is a permanent technological innovation. We introduce an alternative model of learning into the standard RBC framework that assumes that technology diffuses slowly through the economy. The assumption here is that it takes time for agents to learn about the underlying technology shock and implement it into production. We find that this alternative form of learning is able to produce a variety of impulse responses by simply adjusting the speed of technological diffusion. We thus find this story of learning a compelling alternative to the use of the real rigidities of habit persistence and investment adjustment costs in the RBC model.
References


Figure 1: Impulse Responses to a Temporary Shock in Productivity with Full Information

Note: The solid line represents the impulse responses following a temporary shock to productivity of the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the model without these frictions.
Figure 2: Impulse Responses to a Trend Shock in Productivity with Full Information

Note: The solid line represents the impulse responses following a trend shock to productivity of the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the model without these frictions.
Figure 3: Impulse Responses to a Permanent Shock in Productivity with Full Information

Note: The solid line represents the impulse responses following a permanent shock to productivity of the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the model without these frictions.
Figure 4: Impulse Responses to a Trend Shock in Productivity with Kalman Filtering

Note: The solid line represents the impulse responses following a trend shock to productivity of the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the model without these frictions. In this case, agents observe a composite productivity shock that they use to create forecasts of the underlying temporary, trend and permanent components and the future composite productivity shock.
Figure 5: Impulse Responses to a Permanent Shock in Productivity with Kalman Filtering

Note: The solid line represents the impulse responses following a permanent shock to productivity of the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the model without these frictions. In this case, agents observe a composite productivity shock that they use to create forecasts of the underlying temporary, trend and permanent components and the future composite productivity shock.
Figure 6: Impulse Responses to a Trend Shock with Slow Information Diffusion ($\omega = 0.75$)

Note: The solid line represents the impulse responses to a trend shock to productivity generated by the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the modified model without these frictions. In this case, technological innovations diffuse slowly through the economy.
Figure 7: Impulse Responses to a Permanent Shock with Slow Information Diffusion ($\omega = 0.75$)

Note: The solid line represents the impulse responses to a permanent shock to productivity generated by the modified model with habit persistence and investment adjustment costs, while the dashed line represents the impulse responses of the modified model without these frictions. In this case, technological innovations diffuse slowly through the economy.
Technical Appendix

Stationary Equations when Detrending by $Z_t$

The stationary forms of equations (9) – (12), (2) and (3) when detrending by $Z_t$ are given respectively as follows:

\[
\frac{1}{\tilde{c}_t - h\tilde{c}_{t-1}e^{-\gamma_s t}e^{-\epsilon_t^2}} = \lambda_t + \frac{\beta h}{E_t (\tilde{c}_{t+1}e^\gamma s_{t+1}e^{\epsilon_{t+1}} - h\tilde{c}_t)} \tag{56}
\]

\[
\frac{\theta}{1 - l_t} = \tilde{\lambda}_t \left( (1 - \alpha)(e^{\theta t})^{1-\alpha} \hat{k}_t^\alpha (\tilde{l}_t)^{-\alpha} \right) \tag{57}
\]

\[
E_t \left( e^{-\gamma_s t+1}e^{-\epsilon_{t+1}}\tilde{\lambda}_{t+1}\beta \left( \alpha \hat{k}_{t+1}^\alpha (e^{\theta_{t+1} t}\tilde{l}_{t+1})^{1-\alpha} \right) \right) + E_t \left( e^{-\gamma_s t+1}e^{-\epsilon_{t+1}}\tilde{\xi}_{t+1}\beta (1 - \delta) \right) = \tilde{\xi}_t \tag{58}
\]

\[
\tilde{\lambda}_t = \tilde{\xi}_t \left( 1 - S \left( \frac{\tilde{c}_t e^\gamma s_t e^{\epsilon_t}}{\tilde{l}_{t-1}} \right) - S' \left( \frac{\tilde{c}_t e^\gamma s_t e^{\epsilon_t}}{\tilde{l}_{t-1}} \right) \right) + \beta E_t \left( \tilde{\xi}_{t+1} S' \left( \frac{\tilde{c}_t e^\gamma s_t e^{\epsilon_t}}{\tilde{l}_{t-1}} \right) \right) \tag{59}
\]

\[
\tilde{k}_t^\alpha (e^{\theta t} \tilde{l}_t)^{1-\alpha} = \tilde{c}_t + \tilde{l}_t \tag{60}
\]

\[
E_t \left( \tilde{k}_{t+1} e^\gamma s_{t+1} e^{\epsilon_{t+1}} \right) = (1 - \delta) \tilde{k}_t + \left( 1 - S \left( \frac{\tilde{c}_t e^\gamma s_t e^{\epsilon_t}}{\tilde{l}_{t-1}} \right) \right) \tilde{l}_t \tag{61}
\]

Steady State Solution

The variables $\tilde{c}_t$ and $\tilde{c}_t^\alpha$ share the same steady state denoted $\tilde{c}$ as do $\tilde{\lambda}_t$ and $\tilde{\lambda}_t^\alpha$ denoted $\tilde{\lambda}$. The steady state versions of equations (56) – (61) are given as follows:

\[
\frac{1}{\tilde{c} - h\tilde{c}e^{-\gamma_s e^{\epsilon_t^2}}} = \tilde{\lambda} + \frac{\beta h}{\tilde{c} e^{\gamma s} - h\tilde{c}} \tag{62}
\]

\[
\frac{\theta}{1 - l} = \tilde{\lambda}(1 - \alpha)\tilde{k}_t^\alpha \tilde{l}^{-\alpha} \tag{63}
\]

\[
\tilde{\lambda} \beta \alpha \tilde{k}_{t+1}^{\alpha - 1} \tilde{l}^{1-\alpha} + \tilde{\xi} \beta (1 - \delta) = \tilde{\xi} e^\gamma s \tag{64}
\]
The steady state for both the version of the model detrended by $Z_t$ and the version detrended by $X_t$ is given by equations (62)–(67), which can be manipulated to yield the following steady state ratios:

\[ \tilde{k} = \tilde{\lambda} \]  
\[ \tilde{k}^{\alpha \tilde{t}^{1-\alpha}} = \tilde{c} + \tilde{i} \]  
\[ \tilde{k} e^{\gamma z} = (1 - \delta)\tilde{k} + \tilde{i} \]  

Equation (62) yields the following expression for the steady state value $\tilde{\lambda}$:

\[ \tilde{\lambda} = \frac{e^{\gamma z} - \beta h}{\tilde{c}e^{\gamma z} - \tilde{h}\tilde{c}} \]  

Combining (71) with (16), (68) and (70) yields the following expression for the steady state value $\tilde{k}$:

\[ \tilde{k} = \left( \frac{\theta (e^{\gamma z} - h)}{(1 - \alpha) (e^{\gamma z} - \beta h)} \right) \phi_k^{2} \left( \frac{e^{\gamma z}(1 - \alpha \beta) - (1 - \alpha)\beta(1 - \delta)}{\alpha \beta} \right) + \phi_k \right)^{-1}, \]  

with

\[ \phi_k = \left( \frac{e^{\gamma z} - \beta(1 - \delta)}{\alpha \beta} \right)^{\frac{1}{1-\alpha}}. \]
Log-Linearized Equations when Detrending by $Z_t$

The log-linearized model when detrended by $Z_t$ is given by the following equations:

\[
\dot{c}_t = \mu_1 \dot{c}_{t-1} + \mu_2 E_t \dot{c}_{t+1} + \mu_3 \dot{\lambda}_t + \mu_6 \dot{\gamma}_{z,t} + \mu_7 \epsilon_t^z + \mu_8 E_t \dot{\gamma}_{z,t+1} + \mu_9 E_t \epsilon_{t+1}^z
\]

(73)

\[
\mu_6 = -\mu_1, \mu_7 = \mu_6, \mu_8 = \mu_2, \mu_9 = \mu_8
\]

\[
\dot{l}_t = \lambda_1 \dot{\lambda}_t + \lambda_2 \dot{k}_t + \lambda_3 s_t
\]

(74)

\[
\lambda_3 = (1 - \alpha) \lambda_1
\]

\[
\dot{x}_t = \phi_1 \dot{\lambda}_{t+1} + \phi_2 E_t \dot{k}_{t+1} + \phi_3 E_t \dot{l}_{t+1} + \phi_4 E_t \dot{x}_{t+1} + \phi_5 E_t \dot{t}_{t+1} + \phi_6 E_t \dot{s}_{t+1} + \phi_7 E_t \dot{z}_{t+1} + \phi_8 E_t \epsilon_{t+1}^z
\]

(75)

\[
\phi_6 = \phi_3, \phi_7 = \phi_8 = -1
\]

\[
\dot{\lambda}_t = \delta_1 \dot{t}_t + \delta_2 \dot{\lambda}_{t-1} + \delta_3 E_t \dot{\lambda}_{t+1} + \delta_4 \dot{\xi}_t + \delta_5 \dot{\gamma}_{z,t} + \delta_6 \epsilon_{t+1}^z + \delta_7 E_t \dot{\gamma}_{z,t+1} + \delta_8 E_t \epsilon_{t+1}^z,
\]

(76)

\[
\delta_7 = -\delta_2, \delta_8 = \delta_7, \delta_9 = \delta_3, \delta_{10} = \delta_9
\]

\[
\dot{\xi}_t = \eta_1 \dot{\xi}_t + \eta_2 \dot{k}_t + \eta_3 \dot{l}_t + \eta_4 s_t
\]

(77)

\[
\eta_4 = \eta_3
\]

\[
\dot{k}_t = \rho_1 \dot{k}_{t-1} + \rho_2 \dot{t}_{t-1} + \rho_3 \dot{\lambda}_{t-1} + \dot{\gamma}_{z,t} + \rho_5 \epsilon_t^z
\]

(78)

\[
\rho_4 = \rho_5 = -1
\]

The model also includes the exogenous processes specified by (27) and (28).

---

\(^{20}\)Here, $\dot{x}_t = \log(\dot{x}_t) - \log(\dot{x})$. 

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Log-Linearized Equations with Forecasted Values

The versions of equations (21),(23) and (24) modified to include forecasts of the future aggregate productivity shock \( \Delta x_{t+1} \) formed with the Kalman filter look as follows:

\[
\begin{align*}
\dot{c}_t^* &= \mu_1 \dot{c}_{t-1}^* + \mu_2 E_t \hat{c}_{t+1}^* + \mu_3 \hat{\lambda}_t^* + \mu_4 \Delta x_t + \mu_5 \Delta \tilde{x}_{t+1|t} & (79) \\
\dot{\xi}_t^* &= \phi_1 \hat{\lambda}_{t+1}^* + \phi_2 E_t \delta_{t+1}^* + \phi_3 E_t \hat{\xi}_{t+1}^* + \phi_4 E_t \hat{\lambda}_{t+1}^* + \phi_5 E_t \dot{\xi}_{t+1}^* + \phi_6 \Delta \tilde{x}_{t+1|t} & (80) \\
\hat{\lambda}_t^* &= \delta_1 \hat{\lambda}_t^* + \delta_2 \hat{\xi}_{t-1}^* + \delta_3 E_t \hat{\xi}_{t+1}^* + \delta_4 \dot{\xi}_t^* + \delta_5 \Delta x_t + \delta_6 \Delta \tilde{x}_{t+1|t} & (81)
\end{align*}
\]

Here, \( \Delta \tilde{x}_{t+1|t} \) represent the forecast of \( \Delta x_{t+1} \) based on observation of \( \Delta x_t \).

5.1 Case Specific Kalman Filtering

The Kalman Filtering approach in this paper can be restricted so that uncertainty only stems from two shocks at a time. We can write out case-specific versions of the state equation (31), the updating equation (46) and the observation equation (34) as follows:

\[
\begin{align*}
\psi_{t+1} &= F_j \psi_t + v_{t+1,j}, \quad (82) \\
\hat{\psi}_{t+1|t} &= F_j \hat{\psi}_{t|t-1} + K_{t,j}(\Delta x_t - H_j' \hat{\psi}_{t|t-1}) \quad (83) \\
\Delta x_t &= H_j' \psi_t + w_{t,j} \quad (84)
\end{align*}
\]

Thus, the parameter matrices \( F_j, K_j \) and \( H_j \) and the error vectors \( v_{t,j} \) and \( w_{t,j} \) take on case specific values for each of the cases indexed by \( j = 1, 2, 3 \). The covariance matrices \( Q_j \) and \( R_j \) also take on case specific values. The cases and the corresponding parameter values are listed below:

Case 1: Signal Extraction with Uncertainty of Trend and Permanent Shocks

\[
F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (85)
\]

\[
H_1' = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad (86)
\]
\[ vt,1 = \begin{pmatrix} 0 \\ \epsilon_t^7 \\ 0 \end{pmatrix} \] (87)

\[ wt,1 = \epsilon_t^7 \] (88)

\[ Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2_\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (89)

\[ R_1 = \sigma^2_z \] (90)

Case 2: Signal Extraction with Uncertainty of Temporary and Trend Shocks

\[ F_2 = \begin{pmatrix} \rho_s & 0 & 0 \\ 0 & \rho_\gamma & 0 \\ 1 & 0 & 0 \end{pmatrix} \] (91)

\[ H'_2 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \] (92)

\[ vt,2 = \begin{pmatrix} \epsilon_t^7 \\ \epsilon_t^7 \\ 0 \end{pmatrix} \] (93)

\[ wt,2 = 0 \] (94)

\[ Q_2 = \begin{pmatrix} \sigma^2_s & 0 & 0 \\ 0 & \sigma^2_\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (95)

\[ R_2 = 0 \] (96)

Case 3: Signal Extraction with Uncertainty of Temporary and Permanent Shocks

\[ F_3 = \begin{pmatrix} \rho_s & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \] (97)

\[ H'_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \] (98)

\[ vt,3 = \begin{pmatrix} \epsilon_t^7 \\ 0 \\ 0 \end{pmatrix} \] (99)
\[ w_{t,3} = \epsilon_t^z. \]  

(100)

\[ Q_3 = \begin{pmatrix} \sigma_s^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]  

(101)
Figure 8: Impulse Responses of the Underlying and Implemented Technology Processes to Temporary, Trend and Permanent Shocks in the Case of Slow Diffusion of Technology ($\omega = 0.5$)

Note: The solid line represents the impulse response function of the underlying technology process $X_t$, and the dashed line represents the impulse response function of the implemented technology process $Y_t$ which converges to the underlying technology process as the technology shock diffuses through the economy.