A New Test for Jumps in Asset Prices

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September, 2005

Abstract

This paper proposes a new test for the presence of jumps in asset prices. The test is derived from a direct application of Itô’s lemma to the semi-martingale process of asset prices. Intuitively, the proposed test measures the impact of jumps on the third and higher order return moments and is also directly related to the profit/loss function of a variance swap replication strategy. We derive its asymptotic distribution and perform extensive simulations to examine the finite sample properties. Compared to Barndorff-Nielsen and Shephard’s bi-power variation test, the test proposed in this paper has a faster rate of convergence to its asymptotic distribution and is more powerful in detecting jumps. Moreover, in the presence of i.i.d. market microstructure noise, we show that the jump test remains valid with a modified asymptotic variance that is derived in closed form.

Keywords: semi-martingale process; random jumps; swap variance; bi-power variation; market microstructure noise

JEL Classifications: C14, C22, G12

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1 Introduction

Discontinuities or “jumps” are believed to be an essential component of financial asset prices. The arrival of unanticipated news may result in significant shift in market valuation of certain financial assets. As pointed out by Aıt-Sahalia (2004), relative to continuous price changes which are often modeled as a diffusive process, jumps have distinctly different implications for the valuation of derivatives (e.g. Merton, 1976a,b), risk measurement and management (e.g. Duffie and Pan, 2001), as well as asset allocation (e.g. Jarrow and Rosenfeld, 1984). The importance of jumps is also clear from the empirical literature on asset return modeling where the focus is often on decomposing the total asset return variation into a continuous diffusive component and a discontinuous pure jump component (see, for instance, Andersen, Benzoni, and Lund, 2002; Andersen, Bollerslev, and Diebold, 2003; Bates, 2000; Chernov, Gallant, Ghysels, and Tauchen, 2003; Das, 2002; Eraker, Johannes, and Polson, 2003; Garcia, Ghysels, and Renault, 2004; Ho, Perraudin, and Sørensen, 1996; Maheu and McCurdy, 2004; Pan, 2002; Schaumburg, 2004).

Identifying jumps is therefore important and a number of statistical tests have recently been developed in the literature specifically for this purpose. For instance, Aıt-Sahalia (2002) exploits the transition density implied from diffusion processes to test the presence of jumps using discrete financial data. Carr and Wu (2003) examine the impact of jumps on option prices and use the decay of time-value with respect to option maturity to test the existence of jumps. Johannes (2004) proposes non-parametric tests of jumps in a time-homogeneous jump diffusion process. Mancini (2003) introduces a jump threshold to estimate the characteristics of jumps in a general Poisson-diffusion process. Other tests include the parametric MCMC filtering approach of Johannes, Polson, and Stroud (2004a,b) and the wavelet approach of Wang (1995).

The availability of high-frequency data, while still observed at discrete time intervals, leads to the possibility of detecting jumps along the realized sampling path of asset prices over a short time interval. In a recent paper, Barndorff-Nielsen and Shephard (2006) propose an approach based on bi-power variation (BPV) to test for jumps in high frequency returns. The method builds on their earlier work (i.e. Barndorff-Nielsen and Shephard, 2004b) where $BPV$ is used to separate the integrated (diffusive) variance from the total variance. The basic idea is that while both realized variance (RV) and $BPV$ estimate the integrated variance, the latter is robust to the presence
of jumps so that the difference between RV and BPV can be used to measure the contribution of jumps to the total asset return variation. The asymptotic distribution of the BPV jump test is derived by Barndorff-Nielsen and Shephard (2006). Huang and Tauchen (2005) examine the finite sample properties of the BPV jump test via extensive Monte Carlo simulations. Empirical applications of BPV can be found in Andersen, Bollerslev, and Diebold (2003), Forsberg and Ghysels (2004), Ghysels, Santa-Clara, and Valkanov (2004), Huang and Tauchen (2005), and Tauchen and Zhou (2005).

This paper contributes to the existing literature by developing a new test for the presence of jumps in the asset price process. The test is derived from the application of Itô’s lemma to the semi-martingale process of asset prices. Intuitively, the test measures the impact of jumps on the third and higher order return moments and is also directly related to the profit/loss function of a variance swap replication strategy using a log contract. For this reason, we refer to our approach as the “swap variance” test.

While the swap variance jump test statistic is similar to the BPV test in that it is easy to compute and valid for very general specifications of the price process, it has a number of important advantages. First, the swap variance test converges at a faster rate to its asymptotic distribution which, as confirmed by simulations, leads to better finite sample properties. Second, since the swap variance test exploits higher order return moments, such as skewness and kurtosis, which are more sensitive to the presence of jumps, the test is generally more powerful. The distinctive impact of stochastic volatility and random jumps on the higher order return moments helps to disentangle jumps from diffusive price changes. Finally, the swap variance test remains valid in the presence of i.i.d. market microstructure noise. This feature is particularly important since the jump tests are typically implemented using high frequency data (see, e.g., Barndorff-Nielsen and Shephard, 2006; Huang and Tauchen, 2005). We show that the test can be implemented with a modified asymptotic variance that is derived in closed form. Because the feasible swap variance test requires robust estimates of the integrated variance, quarticity, and sexticity, we also present some new analytical results regarding the impact of i.i.d. noise on bi-power, quad-power, and six-power variation. These results can then be used to correct the bias of the power variation measures and improve the performance of the feasible swap variance test in the presence of noise.

The remainder of this paper is structured as follows. Section 2 reviews the BPV jump test of Barndorff-Nielsen and Shephard (2004b) and then outlines the new swap variance jump test. Extensive simulations are
performed to investigate the finite sample properties of the test. Section 3 presents both theoretical and simulation results for the jump test in the presence of i.i.d. market microstructure noise. Section 4 contains an empirical illustration using IBM high frequency data. Section 5 concludes.

2 Testing for jumps in asset returns

In this section, we develop a new model-free test for the presence of jumps in asset returns under general specification of the asset price process. Because our test is closely related to Barndorff-Nielsen and Shephard’s bi-power variation test (Barndorff-Nielsen and Shephard, 2004b, 2006), we also provide a brief review of their approach and discuss the key differences between the two tests.

Let the logarithmic asset price at time $t \in [0, T]$, i.e. $y_t = \ln S_t$, be specified as a general semi-martingale process on the probability space $(\Omega, \mathcal{F}, P)$ with an information filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$:

$$dy_t = (\alpha_t - \lambda_t \eta_t - \frac{1}{2} V_t)dt + \sqrt{V_t}dW_t + J_t dq_t,$$

where $\alpha_t$ is the instantaneous drift, $V_t$ is the instantaneous variance when there is no random jump, $W_t$ is a standard Brownian motion, $q_t$ is a counting process with finite instantaneous intensity $0 \leq \lambda_t < \infty$, and $J_t$ is a non-zero random variable representing the jump in price with instantaneous mean of $\exp(J_t) - 1 = \eta_t$.

Note that since the demeaned asset price process is a local martingale, it can be decomposed canonically into two orthogonal components, namely a purely continuous martingale and a purely discontinuous martingale (see Jacod and Shiryaev, 2003, Theorem 4.18). As such, the jump diffusion model in Eq. (1) is a very general representation of the asset return process. Moreover, we impose no additional structure on the drift, stochastic volatility, or jump component except the following (regularity) conditions:

**Assumption (A):** $\alpha_t$ is a predictable process of locally bounded variation and $V_t$ is a strictly positive càdlàg process with $\int_0^T V_t dt < +\infty, \forall T > 0$.

As detailed in Andersen, Bollerslev, Diebold, and Labys (2003), for the price process in Eq. (1) to be arbitrage free, one also needs to impose the (necessary) condition that $\Pr[\text{sign} (\alpha_t) = -\text{sign} (\alpha_t + J_t dq_t)] > 0$. This ensures that predictable jumps in the drift do not constitute an arbitrage opportunity since they are accompanied by random jumps of comparable magnitude with positive probability.
Throughout the paper, we assume that the price process \( y_t \) is observed at regular time intervals \( \delta = 1/M \) over the period \([0, T]\), i.e. \( \{y_{i\delta}\}_{i=0}^{N} \), where \( N = MT \). In this setting, realized variance (RV) is defined as:

\[
RV_M(T) = \sum_{i=1}^{N} r_{\delta,i}^2,
\]

where \( r_{\delta,i} = y_{i\delta} - y_{(i-1)\delta} \), see Andersen, Bollerslev, Diebold, and Labys (2003); Barndorff-Nielsen and Shephard (2004a). It is well known (see, for instance, Jacod and Shiryaev, 1987) that:

\[
\lim_{M \to \infty} RV_M(T) = V(0, T) + \int_0^T J_u^2 \, du,
\]

where \( V(0, t) \equiv \int_0^t V_u \, du \). In words, RV is a consistent estimator (as \( M \to \infty \)) of the total variance of the price process, i.e. the quadratic variation of both the continuous diffusive component and the discontinuous jump component. In a recent paper, Barndorff-Nielsen and Shephard (2004b) introduce a so-called bi-power variation (BPV) measure which is defined in normalized form as:

\[
BPV_M(T) = \frac{1}{\mu_p^2} \sum_{i=1}^{T-1} |r_{\delta,i+1}| |r_{\delta,i}|,
\]

where \( \mu_p = 2^{p/2} \Gamma \left( \frac{p+1}{2} \right) / \sqrt{\pi} \) for \( p > 0 \). They show that for the asset return process in Eq. (1):

\[
\lim_{M \to \infty} BPV_M(T) = V(0,T).
\]

That is, BPV is a consistent estimator (as \( M \to \infty \)) of the integrated variance of the price process. Intuitively, with only a finite number of jumps (i.e. \( q_T < \infty \)), the probability of jumps occurring in any two consecutive intervals tends to zero when the sampling frequency \( (M) \) increases so that the probability limit of BPV is unaffected by the presence of jumps.

It is clear that the difference between \( RV \) and \( BPV \) can be used to (i) measure the contribution of the jump component to the total variance of the process and (ii) devise a statistical test for the presence of jumps in the sample paths of the price process. This first issue is the primary focus of Barndorff-Nielsen and Shephard (2004b), while the second is the focus of Barndorff-Nielsen and Shephard (2006). In particular, for the price process as specified in Eq. (1) and under assumption (A), Barndorff-Nielsen and Shephard (2006) show that under the null hypothesis of no jump, i.e. \( \lambda_t = 0 \) for \( t \in [0,T] \), we have the following one-sided test statistic:

\[
\frac{\sqrt{N}}{\sqrt{\Omega_{BPV}}} \left( RV_M(T) - BPV_M(T) \right) \overset{d}{\to} \mathcal{N}(0,1),
\]
where \( \Omega_{BPV} = \left( \frac{\pi^2}{4} + \pi - 5 \right) Q_{(0,T)} \) and \( Q_{(0,T)} = \int_0^T V_u^2 du \) denotes the integrated quarticity. It is noted that in the original development of the bi-power variation jump test, Barndorff-Nielsen and Shephard (2006) required the joint process \((\alpha_t, V_t)\) to be independent from \( W_t \) which would rule out any “leverage effect”. However, in a recent paper by Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005), this assumption is relaxed and the only conditions imposed are the ones stated in Assumption (A).

Because \( Q_{(0,T)} \) is latent in practice, the test in Eq. (2) is infeasible. Nevertheless, a feasible test can be constructed by simply replacing \( \Omega_{BPV} \) with a consistent estimate that is computed from observed returns. In doing so, care should be taken to mitigate the impact of jumps on estimates of \( Q_{(0,T)} \) in order to avoid a deterioration of power under the alternative. Barndorff-Nielsen and Shephard (2004b) propose the following consistent estimator that is robust to the presence of jumps for appropriate integer values of \( p \), i.e.

\[
\hat{\Omega}^{(p)}_{BPV} = \left( \frac{\pi^2}{4} + \pi - 5 \right) \frac{N^2 p^{-p}}{N - p + 1} \sum_{j=0}^{N-p} \prod_{k=1}^{p} |r_{\delta,j+k}|^{4/p}.
\]  

Barndorff-Nielsen and Shephard (2006) set \( p = 4 \) (i.e. quad-power variation) while Andersen, Bollerslev, and Diebold (2003) favor \( p = 3 \) (i.e. tri-power variation). For a detailed discussion of multi-power variation and its properties, see Barndorff-Nielsen, Graversen, Jacod, and Shephard (2006); Barndorff-Nielsen, Shephard, and Winkel (2005).

Simulation results reported by Barndorff-Nielsen and Shephard (2006) and Huang and Tauchen (2005) suggest that the “difference test” in Eq. (2) has poor finite sample properties. Instead, the corresponding logarithmic or ratio-test often has better performance, i.e.

\[
\frac{V_{(0,T)}}{\sqrt{\Omega_{BPV}}} \left( \ln RV_M (T) - \ln BPV_M (T) \right) \xrightarrow{d} \mathcal{N}(0,1),
\]  

and

\[
\frac{V_{(0,T)}}{\sqrt{\Omega_{BPV}}} \left( 1 - \frac{BPV_M (T)}{RV_M (T)} \right) \xrightarrow{d} \mathcal{N}(0,1).
\]

Similarly, feasible versions of the above tests can be obtained by replacing \( \Omega_{BPV} \) and \( V_{(0,T)} \) with robust and consistent estimates, for instance, \( \hat{\Omega}^{(4)}_{BPV} \) and \( BPV_M (T) \) respectively.
2.1 The “swap variance” jump test

Below, we develop a new model-free test for the presence of jumps and derive its asymptotic distribution. For the logarithmic price process $y_t = \ln S_t$ in Eq. (1), a direct application of Itô’s lemma leads to the following semi-martingale process for asset prices:

$$dS_t/S_t = (\alpha_t - \lambda_t \eta_t) \, dt + \sqrt{V_t} \, dW_t + (\exp (J_t) - 1) \, dq_t.$$  \hspace{1cm} (6)

Taking the difference between Eq. (1) and Eq. (6) gives:

$$2 (dS_t/S_t - dy_t) = V_t dt + (\exp (J_t) - J_t - 1) \, dq_t,$$

or, equivalently, in integral representation:

$$2 \int_0^T (dS_t/S_t - dy_t) = V_{(0,T)} + 2 \int_0^T (\exp (J_t) - J_t - 1) \, dq_t. \hspace{1cm} (7)$$

From a finance standpoint, the left-hand-side of Eq. (7) can be interpreted as the cumulative delta-hedged gains (under continuous re-balancing) of two short log contracts.\(^1\) Hence, in the absence of jumps, a delta hedged log contract can be used to perfectly replicate a variance swap. This key insight is due to Neuberger (1990, 1994) (see also Dupire, 1993; Carr and Madan, 1998; Demeterfi, Derman, Kamal, and Zou, 1999; Carr, 2000). Clearly, when there are discontinuities in the price process, this replication strategy will be subject to a stochastic and unhedgeable error which can be quantified as:

$$P&L \text{ due to jumps} = 2 \int_0^T (\exp (J_t) - J_t - 1) \, dq_t.$$ 

It is exactly this property that forms the basis of our jump test. Specifically, we introduce a so-called “swap variance” ($SwV$) measure which is defined as the discretized version of the left-hand side of Eq. (7), i.e.

$$SwV_M (T) = 2 \frac{1}{N} \sum_{j=1}^{N} (R_{\delta,j} - r_{\delta,j}) = 2 \sum_{j=1}^{N} R_{\delta,j} - 2 \ln (S_T/S_0), \hspace{1cm} (8)$$

where $R_{\delta,j} = (S_{j\delta} - S_{(j-1)\delta})/S_{(j-1)\delta}$. By construction we have that:

$$\text{plim}_{M \to \infty} (SwV_M (T) - RV_M (T)) = \begin{cases} 0 & \text{if no jumps in } [0, T] \\ 2 \int_0^T (\exp (J_t) - \frac{1}{2} J_t^2 - J_t - 1) \, dq_t & \text{if jumps in } [0, T] \end{cases}$$

\(^1\)A log contract initiated at $t = 0$ with maturity $T$ has a terminal payoff equal to $\ln(S_T/S_0)$. Because the delta of such a contract is equal to the reciprocal of the asset price, it follows that $dS_u/S_u$ is equal to the instantaneous gain associated with the delta-position in the underlying asset.
Therefore the difference between $SwV$ and $RV$ can be used to test for the presence of jumps in an analogous fashion to the bi-power variation test of Barndorff-Nielsen and Shephard (2004b, 2006). The key difference between the two tests is that the bi-power variation test attempts to detect jumps by measuring the quadratic variation of the purely discontinuous jump component whereas the swap variance test does so by measuring the cumulative delta-hedged gains of a variance swap replication strategy involving a log contract. To offer further intuition about the swap variance test, consider the following Taylor series expansion:

$$SwV_M(T) - RV_M(T) = \frac{1}{3} \sum_{i=1}^{N} r_{\delta,i}^3 + \frac{1}{12} \sum_{i=1}^{N} r_{\delta,i}^4 + \ldots.$$  

From this it is clear that the swap variance test exploits the impact of jumps on the third and higher order moments of asset returns. This is consistent with Bandi and Nguyen (2003) who propose non-parametric estimates of jump terms in a time-homogeneous jump-diffusion model using higher order moments of instantaneous returns, and Johannes (2004) who proposes a non-parametric test of jumps in a time-homogeneous jump-diffusion process. Since $SwV_M(T) - RV_M(T) = \frac{1}{3} \sum_{i=1}^{N} c^{\delta,i} r_{\delta,i}^3$ where $c_{\delta,i}$ is between 0 and $r_{\delta,i}$, the difference between the swap variance and realized variance is likely to be positive (negative) if there is a single large positive (negative) jump in the testing interval. From the above, it is also clear that, as opposed to the $BPV$ test, the $SwV$ jump test is a two-sided test. The following theorem provides the formal swap variance jump test statistics as well as their asymptotic distributions.

**Theorem 2.1 (Swap variance jump tests)** For the price process specified in Eq. (1) with assumption (A), under the null hypothesis of no jumps, i.e. $H_0: \lambda_t = 0$ for $t \in [0, T]$, we have

(i) the difference test:

$$\frac{N}{\sqrt{\Omega_{SwV}}} (SwV_M(T) - RV_M(T)) \xrightarrow{d} \mathcal{N}(0, 1) \quad (9)$$

(ii) the logarithmic test:

$$\frac{V_{(0,T)} N}{\sqrt{\Omega_{SwV}}} (\ln SwV_M(T) - \ln RV_M(T)) \xrightarrow{d} \mathcal{N}(0, 1) \quad (10)$$

(iii) the ratio test:

$$\frac{V_{(0,T)} N}{\sqrt{\Omega_{SwV}}} \left(1 - \frac{RV_M(T)}{SwV_M(T)}\right) \xrightarrow{d} \mathcal{N}(0, 1) \quad (11)$$

where $\Omega_{SwV} = \frac{1}{3} \mu_6 X_{(0,T)}$ and $X_{(0,T)} = \int_0^T V^3 u \, du$. 

7
Proof See Appendix A.

It is clear from Theorem 2.1 that the convergence rate of the \(SwV\) test statistics in Eqs. (9)–(11) is of order \(N\), compared to \(\sqrt{N}\) for the tests based on \(BPV\). It can thus be expected that the finite sample properties of the \(SwV\) test are more favorable relative to those of the \(BPV\) test. Further, it is important to emphasize that the above tests are designed to test for the presence of jumps over a fixed time interval. However, once the null hypothesis of no jumps is rejected over a particular time interval, a sequential testing procedure can be implemented to identify the exact timing and size of any subsequent jumps.

Analogous to the \(BPV\) tests, feasible \(SwV\) tests can be obtained by replacing \(V_{(0,T)}\) and \(\Omega_{SwV}\) with consistent and robust estimators. As before, \(V_{(0,T)}\) can be estimated using \(BPV_{M}(T)\) while estimates of \(\Omega_{SwV}\) can be obtained as:

\[
\hat{\Omega}^{(p)}_{SwV} = \frac{\mu_{6}}{9} \frac{N^{3} \mu_{6}/p}{N - p + 1} \sum_{j=0}^{N-p} \prod_{k=1}^{p} |r_{\delta,j+k}|^{6/p}
\]

for \(p \in \{1, 2, \ldots\}\). Clearly, \(\hat{\Omega}^{(4)}_{SwV}\) and \(\hat{\Omega}^{(6)}_{SwV}\) are the obvious candidates for the robust estimation of \(\Omega_{SwV}\).

2.2 Simulations: the finite sample properties of the swap variance jump tests

In this section, we investigate the finite sample properties of the proposed swap variance jump tests using simulations. When implementing the test statistics, we consider both the infeasible test as well as various feasible versions. The infeasible test uses the simulated stochastic volatility path (which is unobserved in practice) to compute the asymptotic variance of the test statistics, whereas feasible test relies only on the returns (which are observed in practice) to compute the estimates of asymptotic variance. Hence, comparing the performance of the feasible test to the infeasible one isolates the impact of asymptotic variance estimation. To simulate the price process in Eq. (1), we use an Euler discretization scheme and specify the stochastic variance (SV) component as a standard mean-reverting square-root process, i.e.

\[
dV_t = \beta (\alpha - V_t) dt + \sigma_v \sqrt{V_t} dW^v_t
\]

with \(dW^s_t dW^v_t = \rho dt\). In the initial setup, the SV parameters are set equal to \(\alpha = 0.04/250, \beta = 0.08, \sigma_v = 0.003,\) and \(\rho = 0\), corresponding to an unconditional annualized return volatility of 20%. The impact of
a leverage effect \((\rho \neq 0)\) and alternative specifications of the variance process are investigated in the robustness analysis. All simulation results reported in this paper are based on 100,000 replications.

### 2.2.1 Properties under the null

To examine the size of the \(SwV\) test, we simulate the price process in Eqs. (1) and (13) as described above with \(J_t = 0\) for \(t \in [0, T]\). In order to gauge the impact of the sampling frequency \(M\), sampling horizon \(T\), as well as the sample size \(N = MT\), we consider four scenarios, namely:

(i): \(\{M, T\} = \{26, 1\}\) corresponding to 1-day at 15-minute frequency and \(N = 26\),

(ii): \(\{M, T\} = \{78, 1\}\) corresponding to 1-day at 5-minute frequency and \(N = 78\),

(iii): \(\{M, T\} = \{78, 5\}\) corresponding to 5-day at 5-minute frequency and \(N = 390\), and

(iv): \(\{M, T\} = \{390, 1\}\) corresponding to 1-day at 1-minute frequency and \(N = 390\).

Figure 1 reports the QQ plots for the three versions of the \(SwV\) jump tests given by Eqs. (9–11) in columns and for each of the above four scenarios in rows. For comparison purposes, Figure 2 draws the corresponding QQ plots for the \(BPV\) tests. The results are based on the infeasible asymptotic variance, computed from the simulated volatility path. Because the results for the feasible tests are qualitatively similar (with only minor deterioration in finite sample properties for all tests), they are not reported to conserve space.

Comparison of the QQ plots in Figure 1 within each row indicates that the finite sample distribution of the difference test is fat tailed whereas those of the logarithmic- and ratio-tests are somewhat thin tailed. Nevertheless, with an increase in \(N\) – either through a higher sampling frequency or a longer sampling horizon – the finite sample distribution of all test statistics rapidly converges to their asymptotic counterparts. Further, from the results reported in Figure 2 it can be seen that the distribution of the \(SwV\) tests compare favorably to those of the \(BPV\) tests. Consistent with the simulation results of Huang and Tauchen (2005) for the \(BPV\) test, we also find that the ratio test has better finite sample properties. Therefore, throughout the remainder of this paper we will exclusively focus on the ratio test.

To get a better idea of the magnitude of size distortion in finite sample, Figure 3 plots the 1% and 5% size of the \(SwV\) and \(BPV\) ratio-tests for sampling frequencies between 5 seconds (i.e. \(M = 4680\)) and 15 minutes (i.e.
Keeping the horizon fixed at 1 day (i.e. \( T = 1 \)). Consistent with the QQ plots, we find that the \( BPV \) test over-rejects the null hypothesis due to a heavy right tail, while the \( SwV \) test under-rejects the null hypothesis as a result of thin tails on both sides. Nevertheless, at a 2 minute frequency the size distortion of the \( SwV \) test is largely eliminated. On the other hand, for the \( BPV \) test to attain comparable size the sampling frequency would need to increase beyond 15 seconds which can be explained by its slower rate of convergence to the asymptotic distribution (i.e. \( \sqrt{N} \) as opposed to \( N \) for the \( SwV \) tests).

2.2.2 Properties under the alternative

To examine the properties of the \( SwV \) ratio-test under the alternative hypothesis we simulate the price process as before (with \( M = 390 \) and \( T = 1 \)) but now add jumps to the simulated sample path in the following ways:

(i): a single jump at the end of the sample path, i.e. \( J_t = \gamma \) for \( t = T \) and \( J_t = 0 \) otherwise

(ii): a single jump in the middle of the sample path, i.e. \( J_t = \gamma \) for \( t = T/2 \) and \( J_t = 0 \) otherwise

(iii): two consecutive off-setting jumps, or an instantaneous price reversal, in the middle of the sample path, i.e.

\[
J_t = -J_{t+1} = \gamma \text{ for } t = T/2 \text{ and } J_t = 0 \text{ otherwise}
\]

(iv): random jumps, i.e. \( J_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_j) \) and \( dq_t \sim \text{i.i.d. Poisson}(\lambda dt) \)

For cases (i)–(iii), the jump size \( \gamma \) is varied between 15 basis points (bp) and 75bp with an increment of 15bp. For case (iv) the following (daily) parameter configurations are considered \( \{\lambda; \sigma_j\} = \{1; 50\text{bp}\}, \{2; 50\text{bp}\}, \{5; 50\text{bp}\}, \{2; 100\text{bp}\}, \text{ and } \{2; 200\text{bp}\} \). We perform both \( SwV \) and \( BPV \) tests, with the distributions of both tests under the null and alternative as illustrated in Fig. 4 for varying jump sizes. Figure 5 plots a representative price and return sample path for each of the above jump scenarios.

In the presence of jumps, robust estimation of the asymptotic variance is key: the conventional integrated sexticity (or quarticity for \( BPV \)) estimator involves sixth powers of returns and is thus biased upward which, in turn, can lead to a reduction in power of the jump tests. In order to gauge the performance of alternative estimators we consider the following four methods for computing the asymptotic variance of the \( SwV \) and \( BPV \) jump test statistics:
ASV–I: $\Omega_{SwV}, \Omega_{BPV}$, and $V_{(0,T)}$, i.e. valid but infeasible

ASV–II: $\hat{\Omega}_{SwV}^{(1)}$ and $\hat{\Omega}_{BPV}^{(1)}$ and $RV_M(T)$, i.e. feasible but not robust to jumps

ASV–III: $\hat{\Omega}_{SwV}^{(4)}$, $\hat{\Omega}_{BPV}^{(3)}$, and $BPV_M(T)$, i.e. feasible and robust to jumps

ASV–IV: $\hat{\Omega}_{SwV}^{(6)}$, $\hat{\Omega}_{BPV}^{(4)}$, and $BPV_M(T)$, i.e. feasible and robust to jumps

Here, ASV–I serves as a benchmark against which the performance of the robust estimators III and IV is judged. ASV–II is included to measure the deterioration in power due to the use of a non-robust estimator.

Table 1 reports the power of the $BPV$ and $SwV$ ratio-tests for the different jump scenarios (in Panels A–D) and methods of asymptotic variance calculation (in columns I–IV). The results can be summarized as follows. First, as anticipated, robust estimation of the asymptotic variance is crucial. The power of both the $BPV$ and $SwV$ tests is close to zero when the invalid estimator of the asymptotic variance (ASV-II) is used. In contrast, the power of the robust estimators (ASV-III and IV) is generally very close to that of the infeasible benchmark case (ASV-I). This suggests that the deterioration in power for the feasible test is minimal as long as robust estimators are used. Second, for jump scenarios (i), (ii), and (iii) the power of the $SwV$ test is uniformly higher than that of the $BPV$ test. The difference in performance is often substantial. For instance, a single jump of 45bp in the middle of the sample is picked up by the $BPV$ test about half the time while the $SwV$ test picks up nearly 3 out of every 4 such jumps. With two consecutive off-setting jumps in the sample path (Panel C), the difference in power between the two tests is even more striking. The reason that the $BPV$ test performs so poorly here is clearly due to the nature of the jump: while bi-power variation is robust to rare jumps, it is not robust to the two consecutive jumps. Third, with random jumps (Panel D), the $SwV$ test has better (similar) power when the jump volatility and arrival frequency is relatively low (high). Only when the jump intensity increases beyond 1250 jumps/year ($\lambda = 5$ per day) does the $BPV$ test have slightly better power than its $SwV$ counterpart. It is important to keep in mind here that, by construction of the simulation design, the reported power does not converge to unity simply because not every sample path has a jump. In fact, in the present setup, the power of the tests should converge to $1 - e^{-\lambda}$ ($\approx 0.865$ when $\lambda = 2$ as in Panel D of Figure 6) when the sampling frequency increases.
Additional (unreported) simulations indicate that the above results remain qualitatively unchanged when we sample at a lower frequency of 5 minutes over a longer horizon of 5 days. Also when keeping the horizon fixed while changing the sampling frequency, the relative performance of the tests remains stable. This is illustrated in Figure 6 where we plot the power of the (infeasible) $BPV$ and $SwV$ ratio-tests across a range of sampling frequencies between 5 seconds (i.e. $M = 4680$) and 15 minutes (i.e. $M = 26$) with $T = 1$. From here it is clear that the $SwV$ test is generally more powerful than the corresponding $BPV$ test.

2.2.3 Robustness to the dynamics of the volatility process

To conclude this section we briefly investigate the robustness of the above results to the specification of the variance process. In particular, we consider the case where we have (i) positive and negative leverage, (ii) higher persistence of the variance process, (iii) constant variance, and (iv) jumps in the variance process. Table 2 reports the power of the $SwV$ test with $BPV$ as a benchmark. To conserve space, we only report the infeasible test (ASV–I) as a benchmark and the best performing feasible test (ASV–IV).

As can be seen from Panel B in Table 2, leverage leads to some distortion in the power of the $SwV$ test when jumps are small. In particular, with positive leverage (i.e. $\rho = 0.50$) the power to detect small negative jumps increases while it decreases for small positive jumps (and vice versa for negative leverage with $\rho = -0.50$). Keeping in mind that the $SwV$ test exploits third and higher order return moments to detect jumps, this pattern can be explained by the skewness induced by leverage. However, because the distortion is roughly symmetric, the overall power to detect jumps is largely unaffected. For larger jumps, the power-distortion disappears indicating that the performance of the test statistic is now robust to leverage. Regarding the $BPV$ test, our simulation results are consistent with those of Huang and Tauchen (2005) in that leverage does not have any noticeable impact on the power of the test. Clearly, this is no longer surprising in lieu of the recent results in Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) who relax the restriction on leverage effect in deriving the properties of $BPV$.

The impact of a change in the speed of mean reversion of the variance process is consistent for both tests.

---

2While Panel A of Figure 6 seems to indicate that at low sampling frequencies the $BPV$ test is slightly more powerful than the $SwV$ test, this pattern disappears after correcting for the size distortion of the two tests (see Figure 3).
namely higher persistence or weaker mean reversion leads to higher power to detect small jumps but a slightly lower power to detect large jumps. The intuition is as follows: with a more persistent volatility process, episodes of low volatility are more likely which makes the detection of small jumps easier. At the same time, during episodes of high volatility jumps may be masked by volatile returns which makes them harder to detect. The opposite case when volatility is constant ($\sigma_v = 0$) reinforces this point: here the power to detect large jumps increases (due to the lack of extended episodes of high volatility) whereas the power to detect small jumps decreases (due to the lack of extended episodes of low volatility).

Finally, with jumps in the variance process, the BPV and SwV tests remain valid in theory since we only require the variance process to be càdlàg. Nevertheless, it is expected that their power will decrease in finite sample. In the simulations we consider the case where the jump in price occurs simultaneously with a $3\alpha$ jump in the instantaneous variance (both in the middle of the sample). Figure 7 plots representative sample paths of the variance and return process for such scenario. The last column of Table 2 confirms that with a jump in the variance process, the power of both tests is reduced. As opposed to the benchmark case where virtually all large jumps were detected, the SwV test now only picks up about 1 of every 3 large jumps whereas the BPV test flags only 1 out of every 5 large jumps.

3 Swap variance jump test in the presence of market microstructure noise

In practice, a major issue involved in the use of high frequency data for the purpose of realized variance calculation, or indeed jump identification, is the market microstructure noise. Niederhoffer and Osborne (1966) is one of the first studies to recognize that the existence of a bid-ask spread leads to a negative first order serial correlation in observed returns (see also Roll, 1984). The impact that these and other effects have on realized variance has recently been studied in detail and is now well understood (see for instance Aït-Sahalia, Mykland, and Zhang, 2005; Bandi and Russell, 2004a,b; Hansen and Lunde, 2006; Oomen, 2005, 2006; Zhang, Mykland, and Aït-Sahalia, 2005). However, the impact of market microstructure noise on the BPV jump test is, as pointed out by Barndorff-Nielsen and Shephard (2006), currently an open question (see Huang and Tauchen, 2005, for some exploratory analysis of this issue). Below, we contribute to the literature by deriving the asymptotic distribution of
the \( SwV \) jump tests in the presence of i.i.d. market microstructure noise. Since practical implementation of the jump test requires robust estimates of the integrated variance, quarticity, and sexticity, we also derive analytically the impact of i.i.d. noise on bi-power, quad-power, and six-power variation.

Regarding the noise specification, we follow Bandi and Russell (2004b); Aıt-Sahalia, Mykland, and Zhang (2005) and consider the case where the observed price \( y^*_t \) can be decomposed into an “efficient” price component \( y_t \) and an i.i.d. market microstructure noise component \( \varepsilon_i \), i.e.

\[
y^*_i = y_i + \varepsilon_i,
\]

for \( i = 1, 2, \ldots, N \) and \( \varepsilon_i \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2_{\varepsilon}) \). Consistent with the presence of a bid-ask spread, Eq. (14) implies an MA(1) dependence structure on observed returns:

\[
r^*_{\delta,i} = r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1},
\]

where \( r^*_{\delta,i} = y^*_i - y^*_i \). It is noted that while the i.i.d. assumption on \( \varepsilon_i \) is restrictive, it has been used in main market microstructure literature and provides a reasonable approximation in many situations (see Hansen and Lunde, 2006, for a discussion).

**Theorem 3.1 (Swap variance test in the presence of i.i.d market microstructure noise)** For the price process specified in Eq. (1) with assumption (A), in the presence of i.i.d. market microstructure noise as in Eq. (14), and under the null hypothesis of no jumps, i.e. \( H_0 : \lambda_t = 0 \) for \( t \in [0, T] \), we have

(i) the difference test:

\[
\frac{N}{\sqrt{\Omega^*_{SwV}}} (SwV^*_M (T) - RV^*_M (T)) \overset{d}{\rightarrow} \mathcal{N}(0, 1)
\]

(ii) the logarithmic test:

\[
\frac{(V_{(0,T)} + 2N\sigma^2_{\varepsilon})N}{\sqrt{\Omega^*_{SwV}}} (\ln SwV^*_M (T) - \ln RV^*_M (T)) \overset{d}{\rightarrow} \mathcal{N}(0, 1)
\]

(iii) the ratio test:

\[
\frac{(V_{(0,T)} + 2N\sigma^2_{\varepsilon})N}{\sqrt{\Omega^*_{SwV}}} \left( 1 - \frac{RV^*_M (T)}{SwV^*_M (T)} \right) \overset{d}{\rightarrow} \mathcal{N}(0, 1)
\]

where \( \Omega^*_{SwV} = \frac{1}{T} (15X_{(0,T)} + 72N\sigma^2_{\varepsilon}Q_{(0,T)} + 108N^2\sigma^4_{\varepsilon}V_{(0,T)} + N^2 (36N + 84) \sigma^6_{\varepsilon}) \), and \( SwV^*_M (T) \) and \( RV^*_M (T) \) are computed using the contaminated prices \( y^* \).
Proof  See Appendix A. ■

It is interesting to note that market microstructure noise slows down the rate of convergence of all jump tests to $\sqrt{N}$ (keeping in mind that for the difference test, $SwV$ and $RV$ diverge to infinity at rate $N$). Nevertheless, simulations indicate that the finite sample performance of the infeasible $SwV$ jump tests remains satisfactory. The critical issue here is the implementation of the feasible tests which require the robust estimation of the asymptotic variance. This will therefore be the focus of the remainder of this section.

The challenge in constructing a feasible test is to obtain a good estimate of $\Omega^*_{SwV}$, i.e. one that is robust to jumps and incorporates the impact of market microstructure noise at the same time. A natural way of estimating $\Omega^*_{SwV}$ is to estimate each of its components separately, i.e. $\sigma^2_{\varepsilon}$, $V_{(0,T)}$, $Q_{(0,T)}$, and $X_{(0,T)}$. Estimate of the market microstructure noise variance $\sigma^2_{\varepsilon}$ can be easily obtained in practice as the negative of the first order serial covariance of returns (see for instance Roll, 1984). Here, returns at the highest sampling frequency can be used so as to maximize estimation accuracy. Unreported simulation results indicate that these estimates are robust to the presence of jumps. Computing robust but accurate estimates of the integrated variance $V_{(0,T)}$ (and $Q_{(0,T)}$, and $X_{(0,T)}$ alike) is more challenging because we need to avoid, or correct for, the impact of jumps as well as market microstructure noise. In a related context, Bandi and Russell (2004b) suggest the use of lower frequency data over the horizon of interest to compute realized variance. The only difference here is that since we require robustness to jumps, the bi-power variation can be computed from lower frequency returns. In this paper, we propose an alternative approach as follows. We first compute the bi-power variation (denoted by $BPV^*_M (T)$) using data at high frequency to get an estimate of $V_{(0,T)}$. This estimate of $V_{(0,T)}$ is robust to jumps but remains biased as it is based on market microstructure noise contaminated returns. We then correct the bias induced by the presence of market microstructure noise in $BPV^*_M (T)$. The bias correction is based on the analytical results on the impact of market microstructure noise on $BPV^*_M (T)$, as stated in the following proposition:

**Proposition 3.2 (Bias correction for BPV in the presence of i.i.d market microstructure noise)** Under the conditions as specified in Theorem 3.1, and constant return variance $\overline{\nu}$ over the interval $[0, T]$, we have:

$$E \left[ BPV^*_M (T) \mid \overline{\nu}, \gamma \right] = E \left[ BPV^*_M (T) \mid \overline{\nu} \right] ,$$
where $BPV_M^*(T) = BPV_M^*(T)/(1 + c_b(\gamma))$ and

$$
c_b(\gamma) = (1 + \gamma)\sqrt{\frac{1 + \gamma}{1 + 3\gamma}} + \gamma \frac{\pi}{2} - 1 + \frac{\gamma}{(1 + \lambda)\sqrt{2\lambda + 1}} + 2\gamma \pi \kappa(\lambda),$$

(18)

with $\gamma = \frac{\sigma^2}{\sqrt{\delta V}}$, $\lambda = \frac{\gamma}{1 + \gamma}$, $\kappa(\lambda) = \int_{-\infty}^{\infty} x^2 \Phi(x\sqrt{\lambda})(\Phi(x\sqrt{\lambda}) - 1)\phi(x) \, dx$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of the standard normal respectively.

Proof See Appendix A. ■

The parameter $\gamma$ is referred to as the noise ratio, since it measures the magnitude of the market microstructure noise relative to the efficient price innovation. Note that when the sampling frequency decreases, $\gamma$ tends to zero so that at low frequencies the impact of noise is negligible. The function $c_b(\gamma)$ in Eq. (18) measures the impact of i.i.d. market microstructure noise on $BPV$ and, as such, provides the bias correction for the bi-power variation calculated from market microstructure noise contaminated returns. The properties of this function are illustrated in Figure 8 with the plots of $c_b(\gamma)$ and $\partial c_b(\gamma)/\partial \gamma$. It is interesting to note that for small but realistic values of $\gamma$, $BPV$ “behaves” like $RV$ since the slope of $c_b(\gamma)$ is close to 2. On the other hand, when the sampling frequency increases and the noise ratio grows, we have:

$$
\lim_{M \to \infty} \frac{BPV_M^*(T)}{\gamma} = \frac{2}{\sqrt{3}} + \frac{\pi}{2} + 2\pi \kappa(1) \approx 2.2556,
$$

compared to

$$
\lim_{M \to \infty} \frac{RV_M^*(T)}{\gamma} = 2.
$$

This illustrates that $BPV$ is slightly more sensitive to i.i.d. market microstructure noise\(^3\) than $RV$.

For the estimation of $Q_{(0,T)}$ and $X_{(0,T)}$ using quad-power variation (i.e. $QPV_M(T) \propto N \sum_{i=4}^{N} \prod_{i=0}^{3} |r_{\delta,i-k}|$) and six-power variation (i.e. $SPV_M(T) \propto N^2 \sum_{i=0}^{N} \prod_{i=0}^{5} |r_{\delta,i-k}|$), a similar bias correction can be implemented. Specifically, under the assumptions specified in Proposition 3.2 it is easy to show that:

$$
E \left[ QPV_M^*(T) | \mathcal{V}, \gamma \right] = (1 + c_q(\gamma)) E_{\sigma} \left[ QPV_N(T) | \mathcal{V} \right]
$$

$$
E \left[ SPV_M^*(T) | \mathcal{V}, \gamma \right] = (1 + c_s(\gamma)) E_{\sigma} \left[ SPV_N(T) | \mathcal{V} \right]
$$

\(^3\)To mitigate the impact of noise on $BPV$, Andersen, Bollerslev, and Diebold (2003) and Huang and Tauchen (2005) have suggested to use staggering of returns, i.e. $\sum |r_i^*| |r^-_{i-2}|$. Based on the results presented here it is easy to see that with this construction of $BPV$, the bias due to i.i.d. noise is equal to $1 + 2\gamma$, i.e. the same as for $RV$. This may in part explain why the results for the $BPV$ ratio test are better when returns are staggered this way.
with \( c_q(\gamma) \approx 5.46648\gamma^2 + 4\gamma \) and \( c_x(\gamma) \approx 13.2968\gamma^3 + 14.4255\gamma^2 + 6\gamma \).

It is noted that the above results rely on the assumption that the return variance (and hence the noise ratio) is constant over the interval of interest. It is obvious from the proof that the results can be generalized to the case where the noise ratio is constant but both return variance and noise are time varying. However, when return variance varies over time and the noise is constant, the impact is more complicated and the bias correction becomes much more cumbersome. Instead, we use simulations to gauge the effectiveness of the bias correction under alternative scenarios, i.e. under the null hypothesis and the alternative, at low and high sampling frequency, with and without time varying return variance. For comparison purposes, we also report results where no bias correction is applied.

Table 3 reports the results for the estimation of \( V_{(0,T)} \), \( Q_{(0,T)} \), and \( X_{(0,T)} \) with \( BPV \), \( QPV \) and \( SPV \) respectively. Here the price process is simulated at 1 and 15 minute frequency with market microstructure noise added according to Eq. (14) with \( \sigma_e^2 = 0.2\alpha/156 \), corresponding to a 20% bias in realized variance, or roughly a 45% bias in realized volatility, calculated from 5 minute returns. To obtain estimates of \( \gamma \), we use a long data sample and estimate \( \sigma_e^2 \) as the negative of the first order autocovariance of returns at high frequency and the return variance \( \nabla \) using \( BPV \) with low frequency data. The scenarios considered in Panels A and B are consistent with the assumptions in Proposition 3.2. For these cases, it is clear that the bias correction for all measures works well. For instance, at a one minute frequency, the bias of \( BPV \) due to noise is 4.255 leading to a MSE of 18.63. In contrast, with the bias correction in place, the bias disappears entirely leading to a MSE of 0.123, a 150-fold reduction! The use of a lower sampling frequency does improve matters for the uncorrected measures, but at both 1-minute and 15-minute frequency the bias correction leads to superior results. For the \( QPV \) and \( SPV \) measures the results are qualitatively the same. Here, however, an interesting pattern appears which suggests that the squared and cubed bias corrected \( BPV \) leads to somewhat biased but more efficient estimates of \( Q_{(0,T)} \) and \( X_{(0,T)} \) respectively. Intuitively, the advantage of using powers of \( BPV \) instead of higher order return moment is that the latter are relatively unstable and erratic in the presence of jumps. Panel C examines the impact of time varying return variance, where the assumption of constant volatility in Proposition 3.2 is violated. It is therefore not surprising to find that the quality of the bias correction deteriorates, particularly for \( QPV \) and \( SPV \). Nevertheless, compared to using lower frequency returns, the use of high frequency data in conjunction
with the bias correction still leads to better results as measured by MSE. Similar patterns are observed for Panel D, where jumps are added to the return series.

To further investigate the performance of the $SwV$ tests in the presence of i.i.d. market microstructure noise, we perform additional simulations using the above results to guide the choice of asymptotic variance estimator. In particular, we consider the following four methods for calculating the asymptotic variance of the $SwV$ jump tests:

ASV–I: as defined above, infeasible and invalid with i.i.d. noise

ASV–IV: as defined above, feasible but invalid with i.i.d. noise

ASV–V: $\Omega_{SwV}^*$ and $(V_{0,T} + 2N\sigma_\varepsilon^2)$, i.e. infeasible but valid with i.i.d. noise

ASV–VI: $\Omega_{SwV}^*$ and $(V_{0,T} + 2N\hat{\sigma}_\varepsilon^2)$ computed with $V_{0,T}$, $Q_{0,T}$, and $X_{0,T}$ replaced by $BPV_M^\gamma(T)$, $BPV_M^\gamma(T)^2$, $BPV_M^\gamma(T)^3$, i.e. feasible and (approximately) valid with i.i.d. noise.

Table 4 reports the size (Panel A) and power (Panel B) of the $SwV$ tests in the presence of i.i.d. market microstructure noise. From Panel A, it is clear that there is a severe size distortion when one ignores the presence of market microstructure noise, with the size for ASV-I tending to 1 and that of ASV-IV tending to 0 when the sampling frequency increases. Fortunately, the size of the $SwV$ test can be restored when using ASV-V or VI. This also indicates that the feasible asymptotic variance estimator with bias correction works reasonably well.

Next, we examine the power of the test. First, as expected, the presence of market microstructure noise leads to a reduction in power of the jump test. For instance, for the feasible test (VI), the power at the 1 minute frequency decreases from nearly 75% in the absence of noise, to about 38% in the presence of noise. Note that the high power of the (invalid) test based on ASV-I is largely due to the size distortion. Second, in line with the results for the size, the feasible test based on ASV-VI works well in that it closely matches the power of its infeasible counterpart. Third, interestingly there appears to be a nonlinear relation between power and sampling frequency. Within the setting of our simulation experiment, the power of the noise corrected tests peaks at around a 30-second sampling frequency. This pattern clearly reflects the trade-off between an increase of the market structure noise in asset prices and an increase of the power of the jump test in the absence of market microstructure noise.
when the sampling frequency increases.

4 An empirical application: detecting jumps in IBM data

As an illustration of our proposed swap variance jump test, we present an empirical application using IBM data. The data is extracted from the TAQ database and consist of all transactions for the IBM stock that took place between 9.45 and 16.00 over the period of January 1, 2000 through 30 June 2004 (1,128 days). Returns are sampled in calendar time every 15, 60, and 300 seconds and in transaction time every 4, 16, and 80 transactions. Overnight returns are removed. Since there are about 6,200 transactions per day on average, the two sampling schemes yield roughly the same number of observations. The motivation of using the transaction time sampling scheme is that the corresponding return process is believed to be more homogenous and not subject to various calendar time induced return characteristics such as diurnal patterns, surges in market activities, etc (see Oomen, 2006, for further discussion). Using these return series over each day in the sample, we then apply the swap variance ratio test with and without market microstructure noise correction (denoted by $SwV^\gamma$ and $SwV^\gamma$ respectively), as well as the bi-power variation ratio test (denoted by $BPV^*$) for the purpose of comparison. For the bias correction, $\sigma_\varepsilon^2$ and $\Gamma$ are estimated each month using the negative of the first order auto-covariance of transaction returns and the bi-power variation of 15-minute data respectively.

Table 5 reports the number of days ($N_J$) that are identified as having jumps based on particular jump test (using a 1% critical value). The table also reports various percentiles of the maximum absolute intra-day return of those days where a specific test has or has not identified a jump. For instance, with 5-minute returns, the market microstructure noise corrected $SwV^\gamma$ test identifies 109 days with jumps, whereas the $BPV^*$ test identifies 74 days with jumps. Since the simulations suggest that the BPV test tends to over-reject the null hypothesis relative to the $SwV^\gamma$ test in finite sample, the larger number of jumps identified by the $SwV^\gamma$ test relative to the $BPV^*$ test is likely due to the power difference between the two tests. To verify this, we examine the days that are identified as having jumps by one test but not the other. The results in the first panel of Table 5 suggests that among the 109 days identified by $SwV^\gamma$, 86 of them are not identified by $BPV^*$, whereas among the 74 days identified by $BPV^*$, 51 of them are not identified by $SwV^\gamma$. Comparison of the percentiles of maximum absolute
returns for these days suggests that days that are identified as having jumps by \( SwV^\gamma \) but not by \( BPV^* \) have higher returns (in absolute value) than those days identified as having jumps by \( BPV^* \) but not by \( SwV^\gamma \). The difference seems to suggest that the \( SwV^\gamma \) test is more powerful than \( BPV^* \) as it picks up the larger returns or more genuine jumps. The comparison between \( SwV^\gamma \) and \( SwV^* \) illustrates the effect of adjusting market microstructure noise in the swap variance test. Further, the comparison of the return characteristics indicates that, without adjusting for the market microstructure noise, \( SwV^* \) tends to pick up some spurious jumps.

When applied to higher frequency data (i.e. 15 seconds and 1 minute), all tests identify more jumps which is, of course, partly due to increased power. Interestingly, however, \( BPV^* \) now identifies many more jumps than \( SwV^\gamma \). For instance, at a 15-second frequency, \( BPV^* \) identifies nearly every day in the sample as having a jump whereas the \( SwV^\gamma \) test only identifies a more realistic 1 in every 3 days as having a jump. At such a high sampling frequency, this substantial difference is likely due to a combination of size distortion and power properties, further complicated by the market microstructure noise effects which are not corrected for in the \( BPV \) test. The return summary statistics further confirm that the days identified as having jumps by \( SwV^\gamma \) but not \( BPV^* \) have higher (intra-day absolute) returns than those days identified by \( BPV^* \) but not \( SwV^\gamma \). For instance, at a 15-second frequency, \( BPV^* \) appears to pick up returns of rather small magnitude as jumps and fails to detect some seemingly obvious large jumps. This is an indication that there is a severe size distortion for the \( BPV^* \) test when applied to very high frequency data without adjusting for the market microstructure effect. On the other hand, the \( SwV^* \) test picks fewer jumps than \( SwV^\gamma \). In other words, the market microstructure noise has a lesser effect on the size property of the swap variance test. However, the difference of return characteristics between \( SwV^\gamma \) and \( SwV^* \) suggests that \( SwV^* \) is less powerful. These results are consistent with the simulations in Table 4.

The results for returns sampled in transaction time are in general consistent with those based on returns sampled in calendar time. One apparent difference is that at low sampling frequency fewer jumps are detected in transaction time than in calendar time. This is an indication that the calendar time returns are indeed more erratic, and the transaction time returns are more homogenous. When the sampling frequency increases and the market microstructure contamination becomes move severe, this patterns diminishes. Further, in transaction time a change in the sampling frequency appears to have a more monotonic effect on the testing results.
Overall, the empirical application illustrates the robust performance of the proposed $SwV$ jump test and underlines the importance of correcting for market microstructure noise when data observed at very high frequency are used.

5 Conclusion

In this paper, we propose a new test for the presence of jumps in observed asset returns. The test is motivated by the hedging error or profit/loss function of a variance swap replication strategy using a log contract. For this reason, we refer to our approach as the “swap variance” test. While the focus of this paper is on the statistical properties of the test, it also has direct implications for the pricing and hedging of variance derivatives. We perform extensive simulations to examine the finite sample properties of the proposed test. Compared to Barndorff-Nielsen and Shephard’s bi-power variation test, the swap variance test has a faster convergence rate to its asymptotic distribution. In addition, since the proposed test measures the impact of jumps on the third and higher order return moments, it is also more powerful in detecting jumps. Interestingly, in the presence of i.i.d. market microstructure noise, it turns out that the swap variance jump test remains valid. This feature is particularly important since the jump tests are typically implemented using high frequency data (see, e.g., Barndorff-Nielsen and Shephard, 2006; Huang and Tauchen, 2005). Simulation results confirm that with a robust estimate of the modified asymptotic variance, the swap variance test maintains reasonable size and power properties. For the robust estimation of the asymptotic variance, we also derive analytically the bias correction to bi-power, quad-power, and six-power variation in the presence of i.i.d. market microstructure noise. An empirical application using the IBM high frequency data suggests that the swap variance test exhibits nice power properties in detecting large returns. It also illustrates the importance of correcting for market microstructure noise when applying the test to high frequency data.
References


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A Proofs

Proof of Theorem 2.1. We first show that under the null hypothesis of no jumps in the price path, the difference between the \( SwV \) and the \( RV \) converges to zero in probability, i.e.

\[
\lim_{\delta \to 0} (SwV_M(T) - RV_M(T)) = 0
\]

(19)

From the definition of swap variance in Eq. (8), we have

\[
\lim_{\delta \to 0} SwV_M(T) = 2 \int_0^T \frac{dS_u}{S_u} dy_u = V(0,T).
\]

This result only requires the application of Itô’s lemma (i.e. see Eq. 6). Further, under regularity conditions as specified in for instance Andersen, Bollerslev, Diebold, and Labys (2003), Barndorff-Nielsen and Shephard (2004a), or Protter (1990), it also follows that \( \lim_{\delta \to 0} RV_M(T) = V(0,T) \) for continuous semimartingales. It is emphasized that the convergence of the \( SwV \) measure is non-stochastic and that convergence of \( RV \) only requires the absence of jumps and no restrictions on the variance process or the correlation between variance and return processes, such as the leverage effect.

To derive the asymptotic variance of the \( SwV \) test, we use a Taylor series expansion to obtain:

\[
N(SwV_M(T) - RV_M(T)) = \frac{N}{3} \sum_{i=1}^{N} \left[ r_{\delta,i}^3 + O(r_{\delta,i}^4) \right],
\]

(20)

where \( r_{\delta,j} = y_{j\delta} - y_{(j-1)\delta} \). Since continuity of the sampling path implies that \( |r_{\delta,j}| \to 0 \) a.s. as \( \delta \to 0 \), it is sufficient to only consider the leading term in Eq. (20). Thus, the asymptotic variance can be derived from the expectation of the following term:

\[
\left( \frac{N}{3} \sum_{i=1}^{N} r_{\delta,i}^3 \right)^2 = \frac{N^2}{9} \sum_{i=1}^{N} r_{\delta,i}^6 + \frac{2N^2}{9} \sum_{j<i}^{N} \sum_{i}^{N} r_{\delta,i}^3 r_{\delta,j}^3
\]

(21)

where the second term can be shown to have zero expectation as \( \delta \to 0 \). Note that the following Milstein scheme discretization of the process in Eq. (1) with jump intensity \( \lambda_t = 0 \) has almost sure convergence to the continuous sampling path (see Talay, 1996):

\[
r_{\delta,i} = (\alpha_{(i-1)\delta} - \frac{1}{2} V_{(i-1)\delta} \delta) + \sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}) + \frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta)
\]

as \( \delta \to 0 \). Expanding the second term of (21), it is easy to verify that its expectation converges to zero. Now, for the price process defined in Eq. (1) with assumption (A), it follows directly from Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005, Theorem 2.2) that:

\[
\lim_{\delta \to 0} \left[ N(SwV_M(T) - RV_M(T)) \right]^2 = \frac{\mu_6}{9} \int_0^T V_u^3 du.
\]

(22)

Together with Eq. (19), we have the required result.
The asymptotic distribution of the logarithmic test can be derived using the following expansion:

\[
\ln S_w V_M(T) - \ln RV_M(T) \approx \frac{S_w V_M(T) - RV_M(T)}{S_w V_M(T)},
\]

where the right hand size is exactly the ratio test. Clearly, we have that:

\[
\lim_{\delta \rightarrow 0} \frac{S_w V_M(T) - RV_M(T)}{S_w V_M(T)} = 0.
\]

The asymptotic Gaussianity follows directly from the standard results by Lipster and Shiryaev (see, e.g., Shiryaev (1981)) regarding the central limit properties of the martingale sequences. To derive the asymptotic variance of the above ratio test, we explore the insight of the Hausman (1978) test following the idea of Huang and Tauchen (2005). Compared to the swap variance, realized variance only converges to the integrated variance in probability under the assumption of no jumps and is thus a less efficient estimator. Under the null hypothesis of no jumps, \(S_w V_M(T) - RV_M(T)\) is asymptotically independent of \(S_w V_M(T)\) conditional on the volatility path. Thus the ratio test is asymptotically the ratio of two conditionally independent random variables. As a result, the asymptotic variance can be derived straightforwardly and is as given in Theorem 2.1. ■

**Proof of Theorem 3.1.** Analogous to Eq. (20), we have:

\[
N (S_w V_M^* (T) - RV_M^* (T)) = \frac{N}{3} \sum_{i=1}^{N} [r_{\delta,i}^3 + O(r_{\delta,i}^4)],
\]

where \( r_{\delta,j}^* = y_{j\delta}^* - y_{(j-1)\delta}^* \). While the second and higher order terms are not vanishing asymptotically if there is continuous transaction with i.i.d. noise over any sampling interval, simulation results show that the higher order terms are negligible at fixed sampling interval with reasonable assumption for the magnitude of the market microstructure noise. We thus concentrate on the leading term. Firstly, we can see that the probability limit of the leading term in the swap variance test is unaffected by i.i.d. market microstructure noise, i.e.

\[
\lim_{\delta \rightarrow 0} N \sum_{i=1}^{N} r_{\delta,i}^3 = \lim_{\delta \rightarrow 0} \frac{N}{3} \sum_{i=1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^3 = \lim_{\delta \rightarrow 0} \frac{N}{3} \sum_{i=1}^{N} r_{\delta,i}^3 = 0.
\]

However, this is not the case for the variance of the swap variance test as can be seen from the following:

\[
\left( \sum_{i=1}^{N} r_{\delta,i}^3 \right)^2 = \sum_{i=1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^6 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^3 (r_{\delta,j} + \varepsilon_j - \varepsilon_{j-1})^3
\]

\[
= \sum_{i=1}^{N} r_{\delta,i}^6 + 30 \sigma^2 \sum_{i=1}^{N} r_{\delta,i}^4 - 18 \sigma^2 \sum_{i=1}^{N-1} r_{\delta,i}^2 r_{\delta,i+1}^2 + 108 \sigma^4 \sum_{i=1}^{N} r_{\delta,i}^2 + (36N + 84) \sigma^6 + \omega.
\]

where \( \omega \) collects all terms with zero expectation such as \( \varepsilon_i, r_{\delta,i}^2 \varepsilon_i \), etc. Next, for the price process in Eq. (1) and assumption (A), it follows from Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005) that
\[
\frac{N^2}{15} \sum_{i=1}^{N} r_{\delta,i}^6 \to X(0,T), \quad \frac{N}{2} \sum_{i=1}^{N} r_{\delta,i}^4 \to Q(0,T), \quad N \sum_{i=1}^{N-1} r_{\delta,i}^2 r_{\delta,i+1}^2 \to Q(0,T), \text{ and } \sum_{i=1}^{N} r_{\delta,i}^2 \to V(0,T) \]
which leads to the required result. For the logarithmic- and ratio-tests we additionally have that:

\[
RV_{M}^* (T) = \sum_{i=1}^{N} r_{\delta,i}^2 = \sum_{i=1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^2 \xrightarrow{P} V(0,T) + 2N\sigma_{\varepsilon}^2.
\]

The test can also incorporate both the third and fourth moments, namely \( \left( \sum_{i=1}^{N} r_{\delta,i}^3 + \sum_{i=1}^{N} r_{\delta,i}^4 \right)^2 \). It is easy to see that the only extra terms with non-zero expectations are from the following expansion

\[
\left( \sum_{i=1}^{N} r_{\delta,i}^4 \right)^2 = \sum_{i=1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^8 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (r_{\delta,i} + \varepsilon_i - \varepsilon_{i-1})^4 (r_{\delta,j} + \varepsilon_j - \varepsilon_{j-1})^4
\]
which can be derived similarly as the third moment. We note that these terms are of smaller magnitude and negligible in the setting of our simulations. ■

**Proof of Proposition 3.2.** Under the assumptions specified in the proposition, we have that \( r_{\delta,i} \sim \text{i.i.d. } \mathcal{N}(0, \delta V) \) so that \( BPV \) can be expressed as:

\[
BPV^*_N (T) = \delta V \pi / 2 \frac{NT}{NT - 1} \sum_{i=2}^{NT} |\overline{r}_{\delta,i} + \sqrt{\gamma} (\varepsilon_i - \varepsilon_{i-1})| |\overline{r}_{\delta,i-1} + \sqrt{\gamma} (\varepsilon_{i-1} - \varepsilon_{i-2})|,
\]
where \( \overline{r}_{\delta,i} = r_{\delta,i}/\sqrt{\delta V} \sim \text{i.i.d. } \mathcal{N}(0, 1), \varepsilon_i \sim \text{i.i.d. } \mathcal{N}(0, 1), \) and \( \gamma = \sigma_{\varepsilon}^2 / (\delta V) \). Consequently, we can write:

\[
E[BPV^*_N (T)] = T \delta V \pi / 2 E |\overline{r}_{\delta,2} + \sqrt{\gamma} (\varepsilon_2 - \varepsilon_1)| |\overline{r}_{\delta,1} + \sqrt{\gamma} (\varepsilon_1 - \varepsilon_0)|.
\]

Conditional on \( \varepsilon_1 \), we define the function:

\[
c_b (\gamma | \varepsilon_1) = \int_{-\infty}^{\infty} c_b (\varepsilon_1 | x) \phi (x) dx = \int_{-\infty}^{\infty} c_b (\varepsilon_1 | x) \phi (x) dx - 1
\]
with

\[
\xi(a, \sigma) = E |x - a| = 2\phi (a/\sigma) \sigma + 2a\Phi (a/\sigma) - a,
\]
where \( a \) is a constant, \( x \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2) \) and \( \phi (\cdot) \) and \( \Phi (\cdot) \) denote the standard normal density and distribution respectively. The expression for \( c_b (\gamma) \) is then obtained by integrating \( \varepsilon_1 \) out of \( c_b (\gamma | \varepsilon_1) \) and subtracting 1, i.e.

\[
c_b (\gamma) = \int_{-\infty}^{\infty} c_b (\varepsilon_1 | x) \phi (x) dx - 1
\]

\[
= (1 + \gamma) \sqrt{\frac{1 + \gamma}{1 + 3 \gamma}} + \frac{\pi}{2} - 1 + 2 \frac{\gamma}{(1 + \lambda) \sqrt{2 \lambda + 1}} + 2 \gamma \pi \kappa (\lambda)
\]
with \( \lambda \) and \( \kappa (\lambda) \) as defined in the Proposition. ■
Table 1: Power of $\text{BPV}$ and $SwV$ jump ratio-tests

<table>
<thead>
<tr>
<th></th>
<th>Bi-Power Variation Test</th>
<th></th>
<th>Swap Variance Test</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>IV</td>
</tr>
<tr>
<td><strong>Panel A: single jump at end of sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_T = 0$ (size)</td>
<td>1.20</td>
<td>0.40</td>
<td>1.36</td>
<td>1.46</td>
</tr>
<tr>
<td>$J_T = 15\text{bp}$</td>
<td>4.03</td>
<td>0.92</td>
<td>4.33</td>
<td>4.51</td>
</tr>
<tr>
<td>$J_T = 30\text{bp}$</td>
<td>25.3</td>
<td>1.46</td>
<td>25.8</td>
<td>26.3</td>
</tr>
<tr>
<td>$J_T = 45\text{bp}$</td>
<td>60.2</td>
<td>0.99</td>
<td>60.5</td>
<td>61.1</td>
</tr>
<tr>
<td>$J_T = 60\text{bp}$</td>
<td>84.9</td>
<td>0.37</td>
<td>85.0</td>
<td>85.3</td>
</tr>
<tr>
<td>$J_T = 75\text{bp}$</td>
<td>95.6</td>
<td>0.10</td>
<td>95.6</td>
<td>95.8</td>
</tr>
<tr>
<td><strong>Panel B: single jump in middle of sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_T = 15\text{bp}$</td>
<td>2.79</td>
<td>0.61</td>
<td>3.00</td>
<td>3.16</td>
</tr>
<tr>
<td>$J_T = 30\text{bp}$</td>
<td>19.0</td>
<td>0.75</td>
<td>19.1</td>
<td>19.6</td>
</tr>
<tr>
<td>$J_T = 45\text{bp}$</td>
<td>50.3</td>
<td>0.46</td>
<td>49.8</td>
<td>50.8</td>
</tr>
<tr>
<td>$J_T = 60\text{bp}$</td>
<td>77.8</td>
<td>0.18</td>
<td>77.0</td>
<td>77.9</td>
</tr>
<tr>
<td>$J_T = 75\text{bp}$</td>
<td>92.2</td>
<td>0.04</td>
<td>91.6</td>
<td>92.2</td>
</tr>
<tr>
<td><strong>Panel C: two consecutive offsetting jumps in middle of sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_t = -J_{t+1} = 15\text{bp}$</td>
<td>1.12</td>
<td>0.27</td>
<td>1.23</td>
<td>1.39</td>
</tr>
<tr>
<td>$J_t = -J_{t+1} = 30\text{bp}$</td>
<td>1.61</td>
<td>0.06</td>
<td>1.56</td>
<td>2.70</td>
</tr>
<tr>
<td>$J_t = -J_{t+1} = 45\text{bp}$</td>
<td>4.51</td>
<td>0.01</td>
<td>3.42</td>
<td>7.99</td>
</tr>
<tr>
<td>$J_t = -J_{t+1} = 60\text{bp}$</td>
<td>11.3</td>
<td>0.00</td>
<td>7.04</td>
<td>18.2</td>
</tr>
<tr>
<td>$J_t = -J_{t+1} = 75\text{bp}$</td>
<td>22.3</td>
<td>0.00</td>
<td>12.1</td>
<td>31.6</td>
</tr>
<tr>
<td><strong>Panel D: random jumps</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda = 1, \sigma_J = 50\text{bp}$</td>
<td>32.6</td>
<td>0.49</td>
<td>32.5</td>
<td>32.8</td>
</tr>
<tr>
<td>$\lambda = 2, \sigma_J = 50\text{bp}$</td>
<td>53.8</td>
<td>0.59</td>
<td>53.5</td>
<td>54.0</td>
</tr>
<tr>
<td>$\lambda = 5, \sigma_J = 50\text{bp}$</td>
<td>85.4</td>
<td>1.57</td>
<td>85.0</td>
<td>85.5</td>
</tr>
<tr>
<td>$\lambda = 2, \sigma_J = 100\text{bp}$</td>
<td>72.7</td>
<td>0.24</td>
<td>72.5</td>
<td>72.8</td>
</tr>
<tr>
<td>$\lambda = 2, \sigma_J = 200\text{bp}$</td>
<td>80.7</td>
<td>0.13</td>
<td>80.6</td>
<td>80.7</td>
</tr>
</tbody>
</table>

This table reports the power of the $\text{BPV}$ and $SwV$ tests at the 1% critical level (i.e. size of 1% under the null). The method for calculating the asymptotic variance (ASV) is denoted in columns (i.e. I–IV). The results are based on 100,000 replications with 1 day of 1 minute data (i.e. $M = 390$ and $T = 1$) and SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$. 

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Table 2: Robustness to the dynamics of the volatility process

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>$\rho = +0.5$</th>
<th>$\rho = -0.5$</th>
<th>$\beta = 0.04$</th>
<th>$\sigma_v = 0$</th>
<th>$V_t + 3\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Bi-power variation ratio-test</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_t = -60\text{bp}$</td>
<td>77.6</td>
<td>77.7</td>
<td>77.6</td>
<td>77.7</td>
<td>76.9</td>
<td>77.1</td>
</tr>
<tr>
<td>$J_t = -45\text{bp}$</td>
<td>50.5</td>
<td>51.0</td>
<td>50.5</td>
<td>51.0</td>
<td>56.1</td>
<td>56.5</td>
</tr>
<tr>
<td>$J_t = -30\text{bp}$</td>
<td>19.0</td>
<td>19.7</td>
<td>19.1</td>
<td>19.8</td>
<td>29.3</td>
<td>29.8</td>
</tr>
<tr>
<td>$J_t = 0$ (size)</td>
<td>1.20</td>
<td>1.46</td>
<td>1.20</td>
<td>1.47</td>
<td>1.20</td>
<td>1.44</td>
</tr>
<tr>
<td>$J_t = +30\text{bp}$</td>
<td>19.0</td>
<td>19.6</td>
<td>19.0</td>
<td>19.7</td>
<td>19.2</td>
<td>19.9</td>
</tr>
<tr>
<td>$J_t = +45\text{bp}$</td>
<td>50.5</td>
<td>51.1</td>
<td>50.6</td>
<td>51.2</td>
<td>56.0</td>
<td>56.5</td>
</tr>
<tr>
<td>$J_t = +60\text{bp}$</td>
<td>77.6</td>
<td>77.8</td>
<td>77.7</td>
<td>77.8</td>
<td>77.1</td>
<td>77.2</td>
</tr>
</tbody>
</table>

|                  |           |               |               |               |                |                 |
| **Panel B: Swap variance ratio-test** |           |               |               |               |                |                 |
| $J_t = -60\text{bp}$ | 94.6      | 94.5          | 94.3          | 94.2          | 95.0           | 94.9            | 91.8           | 91.8           | 99.8           | 99.8           | 33.0           | 36.6           |
| $J_t = -45\text{bp}$ | 76.2      | 76.3          | 76.2          | 76.3          | 76.2           | 76.3            | 75.5           | 75.7           | 81.5           | 81.1           | 9.20           | 12.2           |
| $J_t = -30\text{bp}$ | 35.6      | 36.2          | 37.2          | 38.0          | 33.6           | 34.4            | 44.3           | 45.1           | 17.8           | 19.1           | 2.09           | 3.63           |
| $J_t = 0$ (size)   | 0.92      | 1.39          | 0.93          | 1.48          | 0.95           | 1.52            | 0.91           | 1.53           | 0.90           | 1.40           | 0.87           | 1.90           |
| $J_t = +30\text{bp}$ | 35.7      | 36.6          | 33.7          | 34.6          | 37.3           | 38.3            | 44.4           | 45.1           | 17.4           | 18.7           | 2.05           | 3.59           |
| $J_t = +45\text{bp}$ | 76.2      | 76.2          | 76.2          | 76.4          | 76.2           | 76.2            | 75.5           | 75.6           | 81.2           | 80.7           | 9.26           | 12.3           |
| $J_t = +60\text{bp}$ | 94.6      | 94.5          | 94.9          | 94.8          | 94.3           | 94.2            | 91.5           | 91.5           | 99.8           | 99.7           | 33.0           | 36.7           |

This table reports the power of the $BPV$ and $SwV$ ratio tests at the 1% critical level, with the asymptotic variance estimated according to method I (infeasible) and IV (feasible). The results are based on 100,000 replications with 1 day of 1 minute data (i.e $M = 390$ and $T = 1$) and SV parameters for the benchmark case equal to $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$. For robustness, the following scenarios are considered: (i) leverage, i.e. $\rho = +0.5$ and $\rho = -0.5$, (ii) low mean reversion / high persistence, i.e. $\beta = 0.04$, (iii) constant volatility, i.e. $\sigma_v = 0$, and (iv) $3\alpha$ jump in middle of variance path, i.e. $V_t + 3\alpha$ for $t = T/2$. 
Table 3: Alternative estimates of integrated variance, quarticity, sexticity in the presence of market microstructure noise

<table>
<thead>
<tr>
<th>Panel A: no noise, no jumps, constant variance</th>
<th>Without noise correction</th>
<th>With noise correction $c(\gamma)$</th>
<th>$BPV^2$</th>
<th>$BPV^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 min frequency</td>
<td>15 min frequency</td>
<td>1 min</td>
<td>15 min</td>
</tr>
<tr>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
<td>$SPV^*$</td>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
</tr>
<tr>
<td>Mean</td>
<td>4.002</td>
<td>1.602</td>
<td>0.641</td>
<td>3.997</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.327</td>
<td>0.314</td>
<td>0.233</td>
<td>1.283</td>
</tr>
<tr>
<td>Bias</td>
<td>0.002</td>
<td>0.002</td>
<td>0.001</td>
<td>-0.003</td>
</tr>
<tr>
<td>MSE</td>
<td>0.107</td>
<td>0.099</td>
<td>0.054</td>
<td>1.646</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: i.i.d. noise</th>
<th>Without noise correction</th>
<th>With noise correction $c(\gamma)$</th>
<th>$BPV^2$</th>
<th>$BPV^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 min frequency</td>
<td>15 min frequency</td>
<td>1 min</td>
<td>15 min</td>
</tr>
<tr>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
<td>$SPV^*$</td>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
</tr>
<tr>
<td>Mean</td>
<td>8.255</td>
<td>7.006</td>
<td>5.943</td>
<td>4.266</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.722</td>
<td>1.506</td>
<td>2.448</td>
<td>1.372</td>
</tr>
<tr>
<td>Bias</td>
<td>4.255</td>
<td>5.406</td>
<td>5.303</td>
<td>0.266</td>
</tr>
<tr>
<td>MSE</td>
<td>18.63</td>
<td>31.50</td>
<td>34.12</td>
<td>1.953</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: i.i.d. noise + SV</th>
<th>Without noise correction</th>
<th>With noise correction $c(\gamma)$</th>
<th>$BPV^2$</th>
<th>$BPV^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 min frequency</td>
<td>15 min frequency</td>
<td>1 min</td>
<td>15 min</td>
</tr>
<tr>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
<td>$SPV^*$</td>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
</tr>
<tr>
<td>Mean</td>
<td>8.278</td>
<td>7.574</td>
<td>7.480</td>
<td>4.282</td>
</tr>
<tr>
<td>Bias</td>
<td>4.272</td>
<td>5.406</td>
<td>6.002</td>
<td>0.275</td>
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</table>

<table>
<thead>
<tr>
<th>Panel D: i.i.d. noise + SV + jumps</th>
<th>Without noise correction</th>
<th>With noise correction $c(\gamma)$</th>
<th>$BPV^2$</th>
<th>$BPV^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 min frequency</td>
<td>15 min frequency</td>
<td>1 min</td>
<td>15 min</td>
</tr>
<tr>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
<td>$SPV^*$</td>
<td>$BPV^*$</td>
<td>$QPV^*$</td>
</tr>
<tr>
<td>Mean</td>
<td>8.658</td>
<td>8.236</td>
<td>8.419</td>
<td>5.151</td>
</tr>
<tr>
<td>Bias</td>
<td>4.652</td>
<td>6.068</td>
<td>6.941</td>
<td>1.144</td>
</tr>
<tr>
<td>MSE</td>
<td>22.27</td>
<td>46.23</td>
<td>107.2</td>
<td>4.558</td>
</tr>
</tbody>
</table>

This table reports summary statistics for $BPV$, $QPV$, and $SPV$ as estimates of $V_{(0,T)}$, $Q_{(0,T)}$, and $X_{(0,T)}$. In panel B, the i.i.d. market microstructure noise is added following Eq. (14) with $\sigma^2 = 0.2\alpha/156$. In panel C, the SV parameters are set as $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$. In Panel D, a jump of the size 75bp with random sign is added in middle of sample path. The last two columns use squared as cubed noise corrected $BPV$ calculated from 1 min data. All $BPV$, $QPV$, and $SPV$ measures are scaled by a factor $250 \times 10^2$, $250^2 \times 10^3$, and $250^3 \times 10^4$ respectively. All results are based on 100,000 replications.
Table 4: Size and power of swap variance ratio-test in presence of market microstructure noise

<table>
<thead>
<tr>
<th></th>
<th>no noise</th>
<th></th>
<th></th>
<th>i.i.d. noise</th>
<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td></td>
<td>I</td>
<td>IV</td>
<td>V</td>
<td>VI</td>
<td>I</td>
<td>IV</td>
</tr>
<tr>
<td>5 sec</td>
<td>1.00</td>
<td>1.05</td>
<td>1.00</td>
<td>1.18</td>
<td>27.0</td>
<td>0.00</td>
</tr>
<tr>
<td>15 sec</td>
<td>0.95</td>
<td>1.09</td>
<td>0.95</td>
<td>1.11</td>
<td>11.9</td>
<td>0.01</td>
</tr>
<tr>
<td>30 sec</td>
<td>0.95</td>
<td>1.26</td>
<td>0.95</td>
<td>1.14</td>
<td>6.39</td>
<td>0.09</td>
</tr>
<tr>
<td>1 min</td>
<td>0.93</td>
<td>1.52</td>
<td>0.93</td>
<td>1.14</td>
<td>3.39</td>
<td>0.32</td>
</tr>
<tr>
<td>2 min</td>
<td>0.88</td>
<td>1.97</td>
<td>0.88</td>
<td>1.12</td>
<td>2.01</td>
<td>0.88</td>
</tr>
<tr>
<td>5 min</td>
<td>0.67</td>
<td>3.15</td>
<td>0.67</td>
<td>0.95</td>
<td>1.12</td>
<td>2.46</td>
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</table>

Panel B: Power of SwV test against 45bp jump in middle of sample

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th>i.i.d. noise</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>IV</td>
<td>V</td>
<td>VI</td>
<td>I</td>
<td>IV</td>
</tr>
<tr>
<td>5 sec</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>74.4</td>
<td>2.50</td>
</tr>
<tr>
<td>15 sec</td>
<td>99.1</td>
<td>99.1</td>
<td>99.1</td>
<td>99.1</td>
<td>69.9</td>
<td>15.1</td>
</tr>
<tr>
<td>30 sec</td>
<td>93.0</td>
<td>93.0</td>
<td>93.0</td>
<td>92.9</td>
<td>62.0</td>
<td>23.5</td>
</tr>
<tr>
<td>1 min</td>
<td>76.0</td>
<td>76.1</td>
<td>76.0</td>
<td>75.8</td>
<td>49.4</td>
<td>26.6</td>
</tr>
<tr>
<td>2 min</td>
<td>51.1</td>
<td>52.5</td>
<td>51.1</td>
<td>50.7</td>
<td>34.6</td>
<td>23.4</td>
</tr>
<tr>
<td>5 min</td>
<td>22.4</td>
<td>26.7</td>
<td>22.4</td>
<td>21.7</td>
<td>17.1</td>
<td>16.7</td>
</tr>
</tbody>
</table>

This table reports the size and power of various feasible and infeasible SwV tests at a 99% confidence level (i.e. size of 1% under the null). The method for calculating the asymptotic variance (ASV) is denoted in columns (i.e. I, IV, V, and VI). The results are based on 100,000 simulation replications with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$ and with $T = 1$ and $M$ varying between 4680 (i.e. 5 seconds) and 78 (i.e. 5 minutes). The the i.i.d. market microstructure noise is added following Eq. (14) with $\sigma^2 = 0.2\alpha/156$. 

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Table 5: Testing for jumps in IBM returns (January 2000 – June 2004)

<table>
<thead>
<tr>
<th></th>
<th>$N_J$</th>
<th>50%</th>
<th>75%</th>
<th>85%</th>
<th>95%</th>
<th>max</th>
<th>$N_J$</th>
<th>50%</th>
<th>75%</th>
<th>85%</th>
<th>95%</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>5 min</td>
<td></td>
<td></td>
<td></td>
<td>80 trans</td>
<td></td>
<td>1 min</td>
<td></td>
<td></td>
<td></td>
<td>16 trans</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects</td>
<td>109</td>
<td>0.72</td>
<td>1.03</td>
<td>1.28</td>
<td>2.01</td>
<td>6.11</td>
<td>56</td>
<td>0.73</td>
<td>1.10</td>
<td>1.58</td>
<td>1.95</td>
<td>4.65</td>
</tr>
<tr>
<td>$BPV^*$ detects</td>
<td>74</td>
<td>0.66</td>
<td>0.91</td>
<td>1.05</td>
<td>1.82</td>
<td>6.11</td>
<td>51</td>
<td>0.71</td>
<td>0.94</td>
<td>1.13</td>
<td>1.57</td>
<td>2.33</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $BPV^*$ does not</td>
<td>86</td>
<td>0.70</td>
<td>0.98</td>
<td>1.18</td>
<td>1.93</td>
<td>2.42</td>
<td>43</td>
<td>0.72</td>
<td>1.11</td>
<td>1.68</td>
<td>2.43</td>
<td>4.65</td>
</tr>
<tr>
<td>$BPV^*$ detects, $SwV^\gamma$ does not</td>
<td>51</td>
<td>0.62</td>
<td>0.75</td>
<td>0.93</td>
<td>1.16</td>
<td>1.41</td>
<td>38</td>
<td>0.71</td>
<td>0.86</td>
<td>1.11</td>
<td>1.49</td>
<td>2.33</td>
</tr>
<tr>
<td>$SwV^*$ detects</td>
<td>127</td>
<td>0.64</td>
<td>0.88</td>
<td>1.03</td>
<td>1.95</td>
<td>6.11</td>
<td>92</td>
<td>0.58</td>
<td>0.84</td>
<td>1.10</td>
<td>1.83</td>
<td>4.65</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $SwV^*$ does not</td>
<td>40</td>
<td>0.69</td>
<td>1.14</td>
<td>1.33</td>
<td>1.93</td>
<td>2.00</td>
<td>16</td>
<td>0.54</td>
<td>1.32</td>
<td>1.69</td>
<td>1.90</td>
<td>1.96</td>
</tr>
<tr>
<td>$SwV^*$ detects, $SwV^\gamma$ does not</td>
<td>58</td>
<td>0.54</td>
<td>0.76</td>
<td>0.86</td>
<td>1.03</td>
<td>2.11</td>
<td>52</td>
<td>0.52</td>
<td>0.69</td>
<td>0.75</td>
<td>1.37</td>
<td>3.17</td>
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<tr>
<td>$SwV^\gamma$ detects</td>
<td>205</td>
<td>0.46</td>
<td>0.65</td>
<td>0.85</td>
<td>1.16</td>
<td>3.08</td>
<td>158</td>
<td>0.45</td>
<td>0.70</td>
<td>0.92</td>
<td>1.25</td>
<td>4.35</td>
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<tr>
<td>$BPV^*$ detects</td>
<td>295</td>
<td>0.42</td>
<td>0.56</td>
<td>0.64</td>
<td>0.88</td>
<td>3.08</td>
<td>353</td>
<td>0.37</td>
<td>0.52</td>
<td>0.66</td>
<td>0.98</td>
<td>4.35</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $BPV^*$ does not</td>
<td>132</td>
<td>0.43</td>
<td>0.64</td>
<td>0.84</td>
<td>1.25</td>
<td>2.82</td>
<td>84</td>
<td>0.40</td>
<td>0.63</td>
<td>0.77</td>
<td>1.11</td>
<td>2.82</td>
</tr>
<tr>
<td>$BPV^*$ detects, $SwV^\gamma$ does not</td>
<td>222</td>
<td>0.39</td>
<td>0.53</td>
<td>0.59</td>
<td>0.73</td>
<td>1.74</td>
<td>279</td>
<td>0.35</td>
<td>0.47</td>
<td>0.59</td>
<td>0.78</td>
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<tr>
<td>$SwV^*$ detects</td>
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<td>0.46</td>
<td>0.65</td>
<td>0.82</td>
<td>1.31</td>
<td>3.08</td>
<td>137</td>
<td>0.42</td>
<td>0.62</td>
<td>0.85</td>
<td>1.21</td>
<td>4.35</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $SwV^*$ does not</td>
<td>77</td>
<td>0.41</td>
<td>0.63</td>
<td>0.86</td>
<td>1.08</td>
<td>2.33</td>
<td>54</td>
<td>0.41</td>
<td>0.70</td>
<td>0.86</td>
<td>1.13</td>
<td>2.82</td>
</tr>
<tr>
<td>$SwV^*$ detects, $SwV^\gamma$ does not</td>
<td>15</td>
<td>0.43</td>
<td>0.47</td>
<td>0.58</td>
<td>0.73</td>
<td>0.74</td>
<td>33</td>
<td>0.32</td>
<td>0.44</td>
<td>0.65</td>
<td>0.95</td>
<td>1.08</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects</td>
<td>355</td>
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<td>0.57</td>
<td>0.79</td>
<td>1.26</td>
<td>4.35</td>
<td>349</td>
<td>0.39</td>
<td>0.64</td>
<td>0.81</td>
<td>1.20</td>
<td>3.65</td>
</tr>
<tr>
<td>$BPV^*$ detects</td>
<td>1105</td>
<td>0.30</td>
<td>0.42</td>
<td>0.52</td>
<td>0.82</td>
<td>2.67</td>
<td>1036</td>
<td>0.31</td>
<td>0.45</td>
<td>0.56</td>
<td>0.89</td>
<td>3.65</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $BPV^*$ does not</td>
<td>12</td>
<td>0.76</td>
<td>1.52</td>
<td>2.49</td>
<td>4.20</td>
<td>4.35</td>
<td>37</td>
<td>0.62</td>
<td>0.94</td>
<td>1.20</td>
<td>1.60</td>
<td>1.96</td>
</tr>
<tr>
<td>$BPV^*$ detects, $SwV^\gamma$ does not</td>
<td>762</td>
<td>0.28</td>
<td>0.39</td>
<td>0.46</td>
<td>0.61</td>
<td>1.68</td>
<td>724</td>
<td>0.30</td>
<td>0.40</td>
<td>0.47</td>
<td>0.63</td>
<td>1.14</td>
</tr>
<tr>
<td>$SwV^*$ detects</td>
<td>274</td>
<td>0.35</td>
<td>0.56</td>
<td>0.78</td>
<td>1.17</td>
<td>4.35</td>
<td>217</td>
<td>0.37</td>
<td>0.62</td>
<td>0.80</td>
<td>1.12</td>
<td>3.65</td>
</tr>
<tr>
<td>$SwV^\gamma$ detects, $SwV^*$ does not</td>
<td>123</td>
<td>0.36</td>
<td>0.52</td>
<td>0.63</td>
<td>1.28</td>
<td>2.82</td>
<td>169</td>
<td>0.38</td>
<td>0.58</td>
<td>0.76</td>
<td>1.26</td>
<td>2.30</td>
</tr>
<tr>
<td>$SwV^*$ detects, $SwV^\gamma$ does not</td>
<td>42</td>
<td>0.34</td>
<td>0.40</td>
<td>0.46</td>
<td>0.61</td>
<td>0.91</td>
<td>37</td>
<td>0.31</td>
<td>0.44</td>
<td>0.56</td>
<td>0.63</td>
<td>0.68</td>
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</tbody>
</table>

This table reports the number of jumps ($N_J$) identified for IBM stock prices during the period of January 1, 2000 through 30 June 2004 (1,128 days) according to different tests. The prices are extracted from the TAQ transaction data. Various percentiles of the maximum absolute return on days identified with jumps are also reported. $SwV^\gamma$ ($SwV^*$) denotes the $SwV$ ratio test with (without) market microstructure noise correction. In the bias correction, $\sigma^2_\varepsilon$ is estimated as the negative of the first order auto-covariance of transaction returns over a month, the noise and return variance ratio is based on the daily return variance computed from 1 month of 15 minute data. $BPV^*$ denotes the $BPV$ ratio test.
Figure 1: QQ-plots of the swap variance jump test statistics

QQ-plots (with normal percentiles on the horizontal axis) for the infeasible $SwV$ difference-, logarithmic-, and ratio-tests (Eqs. 9–11) under the null hypothesis of no jumps. The results are based on 100,000 simulation replications with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, and $\rho = 0$ and varying sampling frequency $M$ and horizon $T$. 

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Figure 2: QQ-plots of the bi-power variation jump test statistics

QQ-plots (with normal percentiles on the horizontal axis) for the infeasible BPV difference-, logarithmic-, and ratio-tests (Eqs. 2, 4, 5) under the null hypothesis of no jumps. The results are based on 100,000 simulation replications with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, and $\rho = 0$ and varying sampling frequency $M$ and horizon $T$. 

1 minute over 1 day
5 minutes over 1 day
5 minutes over 5 days
1 minute over 1 day
Figure 3: Size of jump ratio-tests as a function of the sampling frequency

This figure plots the 1% and 5% size of the infeasible BPV (dashed line) and SwV (solid line) ratio-tests as a function of the sampling frequency. The results are based on 100,000 simulation replications with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$ and with $T = 1$ and $M$ varying between 4680 (i.e. 5 seconds) and 26 (i.e. 15 minutes).

Figure 4: Distribution of jump ratio-test statistics under null and alternative

Distribution of the infeasible BPV (Panel A) and SwV (Panel B) ratio-tests under the null hypothesis and the alternative. Under the alternative, jumps of fixed size (i.e. 30bp and 45bp) are added in the middle of the simulated sample path. Density estimates calculated using the Epanechnikov kernel with Silverman optimal bandwidth. The results are based on 100,000 simulation replications with 1 day of 1 minute data (i.e $M = 390$ and $T = 1$) and SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$. 
This figure plots a simulated price path (left column) and corresponding return series (right column) for four different jump scenarios (in rows). For the simulation we use $M = 26$ and $T = 5$ with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$. For the random jumps, we use $\lambda = 100$ and $\sigma_J = 2\%$. 

$\frac{\alpha}{250}$
Figure 6: Power of jump ratio-tests as a function of the sampling frequency

Panel A: single jump at end ($J_T = 45\text{bp}$)

Panel B: single jump in middle ($J_{T/2} = 45\text{bp}$)

Panel C: 2 jumps in middle ($J_{T/2} = -J_{T/2+1} = 45\text{bp}$)

Panel D: random jumps ($\lambda = 2, \sigma_J = 100\text{bp}$)

This figure plots the power (at 99% confidence level) of the infeasible $BPV$ (dashed line) and $SwV$ (solid line) ratio-tests as a function of the sampling frequency. Panels A–D distinguish among the four different jump scenarios. The results are based on 100,000 simulation replications with SV parameters $\alpha = 0.04/250, \beta = 0.08, \sigma_v = 0.003, \rho = 0$ and with $T = 1$ and $M$ varying between 4680 (i.e. 5 seconds) and 26 (i.e. 15 minutes).
This figure plots a representative volatility (Panel A) and return (Panel B) path when a $3\alpha$ jump is added to the middle of the variance process. For the simulation we use $M = 78$ and $T = 120$ with SV parameters $\alpha = 0.04/250$, $\beta = 0.08$, $\sigma_v = 0.003$, $\rho = 0$.

This figure plots $c_b(\gamma)$ (Panel A) and $\partial c_b(\gamma) / \partial \gamma$ (Panel B) as a function of $\gamma$ (horizontal axis). The expression for $c_b(\gamma)$ is given in Eq. (18). For comparison purpose, the dotted line indicates the impact of noise on RV (i.e. $2\gamma$ in Panel A and 2 in Panel B).