OPTIMAL CHOICE OF MONETARY INSTRUMENTS IN AN ECONOMY WITH REAL AND LIQUIDITY SHOCKS

JOYDEEP BHATTACHARYA AND RAJESH SINGH

Iowa State University

Abstract. Poole (QJE, 1970) using a stochastic IS-LM model presented the first formal treatment of the classic question: how should a monetary authority decide whether to use the money stock or the interest rate as the policy instrument. We update the seminal work of Poole in a microfounded flexible-price general equilibrium model of money using explicit welfare criteria. Specifically, we study the optimal choice of monetary policy instruments in an overlapping-generations economy where limited communication and stochastic relocation creates an endogenous transactions role for fiat money. We characterize stationary welfare maximizing monetary and inflation targets for settings in which the economy is separately hit with i.i.d endowment and liquidity shocks. Our analysis suggests that the central insight of Poole survives: when the shocks are real (nominal and not large), welfare is higher under money growth (inflation rate) targeting.
1. Introduction

The optimal conduct of monetary policy, whether to target the money growth rate or the inflation rate, has survived as one of the most contentious issues in monetary economics.\(^1\) Popular until recently, Milton Friedman’s “mechanical monetarism” advised central banks to stop setting interest rates and instead set the money growth rate permanently at the estimated growth rate of the real economy. In the 80s, however, the dominant paradigm in the practice of monetary policy shifted, bringing with it a renewed “dedication to price stability” via the direct control of inflation via interest rate targeting.\(^2\)

Poole (1970) presented the first formal treatment of the larger question: how should a monetary authority decide whether to use the money stock or the interest rate as the policy instrument. The debate at the time, as summarized by Poole, took the following shape: while some argued that “monetary policy should set the money stock while letting the interest rate fluctuate as it will”, others believed that monetary authorities should “push interest rates up in times of boom, and down in times of recession, while the money supply is allowed to fluctuate as it will.”\(^3\) Using a stochastic IS-LM model, with reduction in variability of aggregate output as the yardstick, Poole reached the conclusion that if “disturbances originated primarily in the IS function that summarized the real sectors of the economy [...]”, the money stock is the proper control instrument. But if the LM function, representing the monetary sector, is the source of the disturbances, the interest rate is the

\(^1\)Smith (1988) presents a discussion of this issue in the context of the Currency versus the Banking school debates in 19th century England.

\(^2\)The paradigm shift had a lot to do with central bankers’ perceptions that inflation targeting was “...a way to prevent the wild swings in monetary policy that were responsible for, or at least complicit in, many of the macroeconomic mistakes of the past. A central bank committed to inflation targeting would likely have avoided both the big deflation during the Great Depression of the 1930s and the accelerating inflation of the 1970s (and thus the deep disinflationary recession that followed).” Mankiw and Reis (2003)

\(^3\)Poole (2000) looks back at this debate: “To a considerable extent, then, the argument over the monetary aggregates 30 years ago was really an argument over the importance of the goal of low inflation and the responsibility of the central bank for the realized rate of inflation on the average over a period of several years. The debate arose from the fact that in that era many – perhaps most – economists believed that inflation was substantially independent of money growth and that central banks could not control inflation. Cost-push inflation was the result of rising labor and material costs. Demand-pull inflation was a consequence of excess aggregate demand. Money growth, many argued, played at best a limited role in creating, or controlling, excess aggregate demand. Fiscal policy was king.”
proper control variable” (Poole and Lieberman, 1972). The bottom-line advice was clear and extremely influential: when the shocks are real in nature, fix the money supply; if the shocks are monetary, fix the interest rate.

This paper takes up Poole’s “instrument problem” within the context of a modern “optimal policy-making framework” as described by Stern and Miller (2004). The setting is a two-period lived pure-exchange overlapping generations model in the tradition of Townsend (1987), Schreft and Smith (1997) and Champ, Smith, and Williamson (1997) where limited communication and stochastic relocation create an endogenous transactions role for fiat money. More specifically, at the end of each period a fraction (deterministic or random) of agents is relocated (the “movers”) to a location different from the one they were born in and the only asset they can use to “communicate” with their past is fiat money. This allows money to be held even when dominated in rate of return. The other asset is a commonly available linear storage technology with a fixed real return. The “stochastic relocations” act like shocks to agents’ portfolio preferences and, in particular, trigger liquidations of some assets at potential losses. They have the same consequences as “liquidity preference shocks” in Diamond and Dybvig (1983), and motivate a role for banks that take deposits, hold cash reserves, and make other less liquid investments. Depending on agents’ risk aversion, the banks’ cash reserves are sensitive/insensitive to the real return on money.

We study two variants of this model, one in which there are real shocks (the young-age good endowment of the agents is stochastic), and one where the fraction of agents relocating is itself random (liquidity preference or monetary shocks). In either case, banks can promise a real return to only the non-movers. For the movers, the banks can promise an amount of money (paid out of the bank’s reserve holdings) but not the real return on it. To see this, consider the case of endowment shocks. Here, the bank this period cares about next period’s endowment because the latter will potentially influence that period’s money demand, hence the price level and thus the return on money between this period and the next. But next period’s money demand depends on the following period’s endowment, and so on. We assume that all agents know the distributions of the real or monetary shocks

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4Stern and Miller (2004) argue that questions regarding optimal monetary policy are best conducted in dynamic, stochastic general equilibrium models of money that incorporate a rationale for why money is held even when dominated in return by assets of similar risk profile. Poole (1970) satisfies all these desiderata except for the return dominance issue and the fact that his criterion for optimality is not agents’ welfare.

5The random relocation with limited communication model has been used to investigate monetary policy issues in Paal and Smith (2000), Smith (2002), Antinolfi, Huybens, and Keister (2001), among others.
and form expectations about the return on money conditional on these distributions, and in a rational expectations equilibrium, these expectations are correct. We focus solely on stationary equilibria.

Our goal is identical to that of Poole (1970): can we use the model to tell us if inflation targeting is superior in a stationary welfare sense to money growth targeting, and when? As a benchmark, we start by studying the deterministic case. Here, as noted by Poole, “it obviously makes no difference whatsoever whether the policy prescription is in terms of setting the interest rate or in terms of setting the money stock...”. The best policy, as discussed in Bhattacharya, Haslag, and Russell (2005) is to hold the money stock fixed (zero inflation). Here the social opportunity cost of using money is the lost return from storage while the private opportunity cost of money is the nominal interest rate; these are equalized when the net inflation rate is zero. In the deterministic case, since every period is exactly the same, the government faces a static problem and hence cares only about this intratemporal margin. With shocks, however, the government’s problem is generically no longer static; an intertemporal (intergenerational) margin appears. Since shocks hit different generations asymmetrically, the government has to pay attention to providing some amount of intergenerational insurance. To achieve this, the government may opt to trade off intratemporal for intertemporal efficiency and this may cause optimal monetary policy to deviate from the zero inflation policy.

When shocks are introduced, we are able to make a fair bit of analytical progress under the assumption of logarithmic (henceforth “log”) utility. When the economy is hit with i.i.d shocks to the endowment drawn from a general distribution, we can show that the stationary welfare maximizing (henceforth “optimal”) net money growth rate is zero (fixed money supply). The corresponding optimal inflation targeting policy calls for positive inflation. We are also able to show that an optimally chosen fixed money growth rate is (stationary) welfare superior to an optimally chosen fixed inflation rate. For the class of CRRA utility functions with coefficient of relative risk aversion $\phi$, computations reveal that the optimal net money growth rate is negative for $\phi < 1$, zero for $\phi = 1$, and negative again for $\phi > 1$. For the entire range of $\phi$ studied, the corresponding optimal inflation targeting policy calls for positive inflation and optimal monetary targeting is (stationary) welfare superior to optimal inflation rate targeting.

For the most part, the situation is exactly reversed when the shocks are monetary in nature, that is they affect the fraction of agents relocating. For log utility, we can prove
that optimal monetary targeting involves a negative net money growth rate while the corresponding optimal inflation targeting policy calls for positive inflation. For “small enough” liquidity shocks, we can also prove that inflation targeting does a better job than monetary targeting; this result is however reversed for “large” shocks. Numerical experiments confirm that the flavor of these results carries over to the CRRA case for a wide range of $\phi$.

Overall, two strong themes emerge. First, our results indicate that for the most part (i.e., for small enough liquidity shocks), Poole’s original IS-LM based insight remains valid in our more microfounded modern setup. Secondly, while under inflation rate targeting it is always optimal to pursue an expansionary policy, it is never optimal to do so under money growth targeting.

In a fairly narrow sense, this paper has few antecedents. Almost all the work done in this area employs models with sticky or staggered prices and very few, as Collard and Dellas (2005) point out, use welfare criteria to answer Poole’s original question. Prominent examples of work in the rigid prices tradition surveyed in Walsh (2003) and Woodford (2003) include Carlstrom and Fuerst (1996), Ireland (2000), Khan, King, and Wolman (2003), among others. In a recent paper, Collard and Dellas (2005) answer Poole’s question “in an economy that represents a faithful, general equilibrium rendition of Poole.”: the model is Neo-Keynesian with capital accumulation, staggered prices (monopolistic competition), money-in-the-utility-function, and supply, fiscal, and money demand shocks. Their main findings are: a) contrary to Poole (1970), monetary targeting generates higher welfare for money demand shocks irrespective of the degree of risk aversion, b) for real shocks, interest rate targeting produces higher welfare only when risk aversion is high.

Finally, using a deterministic OG model with legal restrictions, Smith (1988, 1994) compares the two targeting procedures in terms of their efficiency properties and goes on to isolate a “tension between efficiency and determinacy” of monetary equilibria reminiscent of the nineteenth century quantity theory versus real bills doctrine controversy. This tension is not a focus of our analysis. Liquidity shocks in the random relocation environment (and their relation to banking crises) have also been studied in Champ, Smith, and Williamson (1997), Paal and Smith (2000), Smith (2002) and Antinolfi, Huybens, and Keister (2001), and Antinolfi and Keister (forthcoming). Our treatment of liquidity shocks is different from these papers (see footnote 11 on this).
The plan for the rest of the paper is as follows. In the next section, we outline the baseline model without uncertainty and compute optimal monetary policies. In Section 3, we study the role of endowment uncertainty in shaping the optimal choice of monetary instruments. In Section 4, we do the same with money demand shocks. Section 5 presents the results from the computational experiments under CRRA utility while Section 6 concludes. Proofs of all major results are in the appendices.

2. The Environment

2.1. Primitives. We consider an economy consisting of an infinite sequence of two period lived overlapping generations. Time $t$ is discrete and runs from $\{t\}_{-\infty}^{\infty}$. At each date $t$, young agents are symmetrically assigned to one of two locations. Each location contains a continuum of young agents with unit mass, and our assumptions will imply that locations are always symmetric. There is a single good that may be consumed or stored. Each two-period-lived agent is endowed with $w_t > 0$ units of this good at date $t$ when young and nothing when old; in Section 3 below we study a setting in which $w$ is stochastic where we assume $w$ is revealed at the start of any date.

Let $c_{2t+1}$ denote the consumption of the final good by a representative old agent born at $t$. All such agents have preferences representable by the utility function $U(c_{2,t+1}) = E_t u(c_{2,t+1})$ where $u$ is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments. At points below, we will specialize to $u(c) = \frac{c^{1-\phi} - 1}{1-\phi}$, with $\phi > 0$, and $u(c) = \ln c$ when $\phi = 1$.

The assets available to the agents are goods, which they may store, and fiat currency (money). If $\zeta > 0$ units of the good are placed in storage at any date $t \geq 1$, then $x \zeta$ units are recovered from storage at date $t + 1$, where $x > 1$. The quantity of money in circulation at the end of period $t \geq 1$, per young agent, is denoted $M_t$. Let $0 < p_t < \infty$ denote the price level at date $t$. Let $\frac{p_{t+1}}{p_t} = \pi_t$ denote the inflation rate between period $t$ and $t + 1$. Then the gross real rate of return on money ($R_{m,t}$) between period $t$ and $t + 1$ is given by $R_{m,t} \equiv \frac{p_t}{p_{t+1}} = \frac{1}{\pi_t}$. Also, let $m_t \equiv M_t/p_t$ denote real money balances at date $t$, and $I_t \equiv \frac{x}{R_{m,t}} = x \pi_t$ denote the gross nominal interest rate between $t$ and $t + 1$. Note that $I_t$ represents the opportunity cost of cash relative to storage. We assume that money is a “bad” asset; when $R_{m,t}$ is non-stochastic, this translates into the assumption that

$$x > R_{m,t} \text{ for all } t > 1.$$ (2.1)
In addition to the store-of-value function of money, spatial separation and limited communication generate a transactions role of money as in Townsend (1987). As such, money can be valued even if it is dominated in return by storage. The details are outlined below and follow standard conventions setup in Schreft and Smith (1997) or Smith (2002).

2.2. **Random relocation.** Each period, a fraction $\alpha_t$ of the young agents is relocated to the other location. An agent that is relocated cannot collect the return on any goods she has stored, or that have been stored on her behalf, since goods cannot be transported across locations. However, if an agent is carrying fiat currency when she is relocated, then the currency is relocated with it.

Under the circumstances, there are two strategies an agent can use to transfer income over time. First, she can save on her own, storing some quantity of goods and acquiring some quantity of fiat currency. The problem is that if she is relocated then she must abandon her stored goods, and if not, then she is stuck holding fiat currency, a “bad” asset (more below on this). Alternatively, she can deposit her entire endowment in a perfectly competitive bank. The bank pools the goods deposited by all the young agents and uses them to acquire a portfolio of stored goods and fiat currency. Banks can transport fiat currency across locations. It issues claims to the agents whose nature, timing and size are contingent on their relocation status. If an agent does not get relocated, then she gets a return on her deposit next period that is funded by the goods the bank has stored. If she gets relocated, then she gets a return on her deposit in the same period that takes the form of fiat currency payment (whose real value will depend on the following period’s price level) funded by the bank’s holdings of fiat currency. Since banks can pool individual risks, it can be checked that the latter strategy always dominates the former and we will analyze the economy on this basis.⁶

In Section 4 below, we will study a setting where $\alpha$ is stochastic; there we will assume $\alpha$ is revealed to the bank before it makes its portfolio decisions even though individual agents do not learn their relocation status until the very end of the period.

2.3. **Conduct of monetary policy.** We allow the government to conduct monetary policy in one of two possible ways. The first, called “monetary targeting”, is one where the government changes the nominal stock of fiat currency at a fixed non-stochastic gross rate

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⁶Relocation status is public information; concerns regarding bank runs do not appear here.
\( \mu > 0 \) per period, so that \( M_t = \mu M_{t-1} \) for all \( t \). The second, called “inflation rate targeting”, is one where the government changes the nominal stock of fiat currency in such a way as to keep the long-run gross inflation rate fixed at \( \pi \).

As an aside, also note that in this setting, since \( I_t = \frac{x}{R_{m,t}} = x\pi_t \) holds, nominal interest rate targeting and inflation rate targeting are exactly identical instruments. Recall in steady states, money growth targeting and inflation rate targeting are identical goals. Below we will show that in steady states and in the presence of shocks, money growth targeting and inflation rate targeting are very dissimilar.

If the net money growth rate is positive then the government uses the additional currency it issues to purchase goods, which it gives to current young agents (at the start of a period) in the form of lump-sum transfers. If the net money growth rate is negative, then the government collects lump-sum taxes from the current young agents, which it uses to retire some of the currency. The tax (+) or transfer (−) is denoted \( \tau_t \). The budget constraint of the government is

\[
\tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t - m_{t-1}R_{m,t-1} \tag{2.2}
\]

for all \( t \geq 1 \).

### 2.4. The bank’s problem.

As a benchmark, it is useful to start by studying the bank’s problem for the purely deterministic environment. As discussed earlier, the asset holdings of young agents are assumed to be costlessly intermediated by perfectly competitive banks. These banks hold portfolios of fiat currency and physical assets, which consist of stored goods. Every young agent deposits her after-tax/transfer income in the bank. The banks divide their deposits between stored goods \( s_t \) and real balances of fiat currency \( m_t \), so that

\[
w_t + \tau_t = m_t + s_t. \tag{2.3}
\]

Define \( \gamma_t \equiv \frac{m_t}{w_t + \tau_t} \) as the ratio of cash reserves to deposits. Banks announce a return of \( d_t^m \) to each mover (one who gets relocated) and \( d_t^n \) to each non-mover (one who stays on in the location she was born). These returns satisfy some constraints. First, relocated agents, of whom there are \( \alpha_t \), have to be given money and so the bank has to use its holdings of cash reserves to pay them. Under assumption (2.1), it can be checked that banks will not hold

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\(^7\)Taxes and transfers to old agents are discussed in Bhattacharya, Haslag, and Russell (2005).
cash reserves to pay non-movers. Consequently,

$$\alpha_t d^m_t \leq \gamma_t R_{m,t}$$

must hold, since money earns a return of $R_{m,t} = \frac{p_t}{p_{t+1}}$ between $t$ and $t+1$ (which the bank takes as given). Similarly, the promised return to the non-movers must satisfy

$$(1 - \alpha_t) d^n_t \leq (1 - \gamma_t) x.$$  

Additionally, $\gamma_t \in [0, 1]$ must hold. Competition among banks for depositors will, in equilibrium, force banks to choose return schedules and portfolio allocations so as to maximize the expected utility of a representative depositor, subject to the equality versions of the constraints we have described. For future reference, define $c_{m,t} \equiv d^m_t (w_t + \tau_t)$ as the old-age consumption of each mover and $c_{n,t} \equiv d^n_t (w_t + \tau_t)$ as the old-age consumption of each non-mover.

If $w_t = w$ and $\alpha_t = \alpha \forall t$, i.e., the endowment and relocation probability are known and fixed (as in the standard deterministic random relocation model analyzed by Schreft and Smith (1997)), the bank’s problem can be rewritten as

$$\max_{\gamma_t \in [0,1]} \left\{ \alpha u \left( \frac{\gamma_t}{\alpha} R_{m,t} (w_t + \tau_t) \right) + (1 - \alpha) u \left( \frac{(1 - \gamma_t)x}{1 - \alpha} (w_t + \tau_t) \right) \right\}.$$ 

(2.6)

The first order conditions for this problem are given by

$$R_m \cdot u'(c_{m,t}) = x \cdot u'(c_{n,t}) \text{ for } \gamma > 0.$$ 

(2.7)

The bank equates the value of marginal utility across the two types. Since $x > R_m$, it follows that $u'(c_{m,t}) > u'(c_{n,t})$; it follows from the assumed concavity of $u$ that the bank’s promised per unit return to each mover will be less than that of a non-mover. Since money is a poor asset and movers can use only money, their marginal utility of consumption is higher than that of the non-movers; indeed it is higher by a margin $x/R_m$. As we show below, a planner (constrained by limited communication) would set this margin to $x$. For future reference, bear in mind that the higher marginal utility of the movers implies that transfers of resources to them have a presumption of being desirable.

It is then easily checked that the first order conditions to the problem in (2.6) for $u(c) = \left[ e^{1-\phi} - 1 \right]/(1-\phi)$ is given by

$$R_m \left( \frac{\gamma_t}{\alpha} R_{m,t} \right)^{-\phi} = x \left( \frac{(1 - \gamma_t)x}{(1 - \alpha)} \right)^{-\phi},$$ 

(2.8)
the solution to which is given by
\[ \gamma_t (I_t) = \frac{\alpha}{\alpha + (1 - \alpha) (I_t)^{1 - \phi}}. \] (2.9)

Notice that when \( \phi = 1 \), the case of logarithmic utility studied extensively below, (2.8) reduces to
\[ \frac{\gamma_t}{\alpha} = \frac{1 - \gamma_t}{1 - \alpha} \] (2.10)
the solution to which is given by
\[ \gamma_t = \alpha; \] (2.11)
in this case, the bank’s choice of the reserve-deposit ratio is independent of the nominal interest rate. Then \( d_m^m = R_{m,t} \) and \( d_n^n = x \) and the consumption of each mover is given by \( R_{m,t} (w_t + \tau_t) \) while that of a non-mover is \( x (w_t + \tau_t) \).

Several points, some discussed in Schreft and Smith (1997), deserve mention here. First, for CRRA utility, notice that the optimal \( \gamma \) does not depend on \( w \). When shocks are introduced (see below), this changes; then the nominal interest rate depends on future money demand which in turn depends on future realizations of the shocks. Second, monetary policy influences the optimal \( \gamma \) in the case of CRRA utility only insofar as it determines the relative return on money, \( I_t \). Thirdly, as is clear from (2.9), for all \( I > 1 \), \( \gamma \gtrless \alpha \) iff \( \phi \gtrless 1 \). Intuitively, think of the bank allocating its deposit base among two “goods”, the consumption of movers and the consumption of non-movers. When the two are complements (substitutes) a low return on money relative to storage (i.e., \( I > 1 \)) requires that the share of current “income” allocated to consumption of movers be relatively high (low).

Finally, \( \phi ^{(I)} \gtrless 0 \), \( \forall I \) if \( \phi \gtrless 1 \). An increase in \( I \) has both income and substitution effects. First, it decreases the combined income available for consumption next period. However, for any fixed share \( \gamma \), it affects movers relatively more. Thus, when the consumptions of movers and non-movers are complements, movers’ share \( \gamma \) must be increased. On the other hand, when the two consumptions are substitutes, it is better to shift consumption from movers to non-movers; hence, \( \gamma \) should be lowered.

2.5. Welfare. Finally, steady state welfare (indirect utility) for CRRA utility can be defined as
\[ W (R_m) = \frac{w + \tau (R_m)}{1 - \phi} \left\{ \alpha^\phi [\gamma (R_m) R_m]^{1 - \phi} + (1 - \alpha)^\phi [(1 - \gamma (R_m)) x]^{1 - \phi} \right\} - \frac{1}{1 - \phi} \]
For future reference, note that for logarithmic utility, using (2.10), we can write
\[
W(R_m) = \ln(w + \tau(R_m)) + \alpha \ln R_m + (1 - \alpha) \ln x.
\]

(2.13)

Under monetary targeting, the gross money growth rate is set to \(\mu\). Then in a steady state, \(R_m = 1/\mu\) and so
\[
\tau = \frac{M_t - M_{t-1}}{p_t} = \left(1 - \frac{1}{\mu}\right) m(\mu)
\]
holds. For future reference, note that \(\tau\) maybe thought of as the amount of seigniorage, which itself is the product of the inflation tax rate (the term in parenthesis in the above, \(\left(1 - \frac{1}{\mu}\right)\)) and the inflation tax base \((m(\mu))\).

Since \(\gamma(w + \tau) = m\), we have
\[
\tau(\mu) = \left(1 - \frac{1}{\mu}\right) \left[\frac{\gamma(\mu) w}{1 - \left(1 - \frac{1}{\mu}\right) \gamma(\mu)}\right].
\]

(2.14)

The volume of equilibrium real balances \((m)\) in the economy is given by
\[
m = \frac{\gamma(\mu)}{1 - \left(1 - \frac{1}{\mu}\right) \gamma(\mu)} w,
\]
and the size of the inflation tax base is given by the term in square parenthesis on the r.h.s of (2.14) above.

Since \(I = x\mu\), we can rewrite (2.9) in steady states as
\[
\gamma(\mu) = \frac{\alpha}{\alpha + (1 - \alpha) (x\mu)^{-\phi}}.
\]

(2.15)

Then the problem of choosing the optimal ("steady state welfare maximizing") money growth rate under monetary targeting reduces to
\[
\max_{\mu} W(\mu) = \max_{\mu} \left\{w + \tau(\mu)\right\}^{1-\phi} x \left(\frac{(1 - \gamma(\mu))}{(1 - \alpha)} x\right)^{-\phi} - \frac{1}{1 - \phi}
\]

Under inflation rate targeting, the return to money is set to \(1/\pi\), i.e., \(R_{m,t} = 1/\pi \ \forall t\). As such, in steady states, monetary targeting and inflation targeting are exactly identical goals; see (2.12). We close this section with a fairly well-known result about optimal monetary policy in this environment, the proof of which may be found in Bhattacharya, Haslag, and Russell (2005).
Proposition 1. Under inflation rate targeting or equivalently under monetary targeting, the optimal policy is to hold the money stock fixed (zero inflation) if there are no shocks to endowments or liquidity preference.

Notice that this result holds irrespective of the degree of risk-aversion. The intuition is as follows. Focus attention solely on steady states. A planner constrained by limited communication faces a rate of return of $x$ on stored goods since a unit of the good invested in the storage technology any period yields $x$ next period.\footnote{We assume that the planner is constrained by limited communication to the extent that she has to allocate some portion of the current endowment to the movers, i.e., she cannot store the entire endowment and equalize consumption across movers and non-movers by giving them $xw$ next period.} At any date, such a planner who allocates $w$ between the movers and the non-movers would choose an allocation $(c_m, c_n)$ so as to maximize $\alpha u(c_m) + (1 - \alpha) u(c_n)$ subject to $\alpha c_m + (1 - \alpha) c_n / x = w$; the marginal conditions would reduce to

$$\frac{u'(c_m)}{u'(c_n)} = x.$$  

This is the usual within-period $MRS = MRT$ condition which, at points below, will be referred to as the “intratemporal efficiency” or “intragenerational efficiency” condition.

The government’s objective, of course, is to choose a $\mu$ that maximizes stationary welfare in a decentralized equilibrium. In such an equilibrium involving money, using $R_m = \frac{1}{\mu}$, (2.4), (2.5), and the assumed CRRA form of $u$, it is easily checked that

$$\frac{u'(c_m)}{u'(c_n)} = \left[ \frac{\gamma (\mu)}{1 - \gamma (\mu)} \frac{1 - \alpha}{\alpha} \frac{1}{\mu x} \right]^{-\phi},$$

which using (2.15) reduces to

$$\frac{u'(c_m)}{u'(c_n)} = \mu x.$$  

Thus the government can select an efficient allocation only by setting $\mu = 1$. As Bhattacharya, Haslag, and Russell (2005) argue, in a OG model, in steady states, every unit of goods devoted to holding money is an unit that is not devoted to acquiring storage; as such, the social opportunity cost of money is the return on storage. Optimality requires that the private opportunity cost of holding money be the same as the social opportunity cost of money. Of course, the private opportunity cost of money is the nominal interest rate,
$I = x\mu$. Hence, $\mu = 1$ is the best choice.$^9$ Indeed, as Wallace (1980) and Bhattacharya, Haslag, and Russell (2005) point out, zero inflation often has the presumption of being the optimal monetary policy in monetary overlapping generation models.

In steady states, intratemporal efficiency is achieved with a fixed money stock (zero inflation). Since every period is exactly the same, the government faces a static problem and hence cares only about this intratemporal margin. With shocks, however, the government’s problem is generically no longer static; an intertemporal (intergenerational) margin appears. Since shocks hit different generations asymmetrically, the government pays attention to providing some amount of intergenerational insurance. To achieve this, the government may opt to trade off intratemporal for intertemporal efficiency and this causes optimal monetary policy to deviate from the zero inflation policy. This is the subject matter of the next two sections.

3. Endowment Uncertainty

We now analyze an economy that is identical to the one studied above, except that the endowment $w$ is assumed to be stochastic. In particular, we assume that $w$ is drawn each period from a i.i.d distribution $f(w)$ with support $[w, \bar{w}]$, $w > 0, \bar{w} < \infty$. We denote by $w^e$ the expected value of $w$, and by $\sigma_w^2$, its variance. Shocks to the endowment represent real shocks. Our goal is to investigate how monetary policy should respond to such intrinsic real uncertainty.

Recall that at the point at which the bank solves its problem, the current endowment is known. But the realization of next period’s endowment has not happened yet. The bank indirectly cares about next period’s endowment because the latter will potentially influence next period’s money demand, hence the next period’s price level and thus the return on money between this period and the next. In this sense, the bank cannot promise a fixed real return to the movers anymore. All it can do is let movers know how much nominal balances are being kept aside for them. The bank knows the distribution for $w$ and forms expectations on the return on money conditional on $f(w)$; in a rational expectations

$^9$ As Bhattacharya, Haslag, and Russell (2005) argue, in standard infinitely-lived agent models, the gross social opportunity cost of providing money is one (not $x$ as in OG models) so it is optimal for the gross private opportunity cost of holding money to be one – the “Friedman rule”. This explains why the Friedman rule, $\mu = 1/x$, is not the best monetary policy choice for the government in the OG model we study.
equilibrium, these expectations are correct. We will focus on stationary versions of such equilibria below.

The bank’s problem (assuming it never finds it optimal to carry cash across periods; see below) in general is described by

$$\max_{\gamma_t \in [0, 1]} \left\{ \alpha E \left( u \left( \frac{\gamma_t}{\alpha} R_{m,t} (w_t + \tau_t) \right) \right) + (1 - \alpha) E \left( u \left( \frac{(1 - \gamma_t)x}{1 - \alpha} (w_t + \tau_t) \right) \right) \right\} .$$

(3.1)

Analogous to (2.7), the first order conditions for this problem are now given by

$$E \left( u'(c_{m,t}) \cdot R_{m,t} \right) = E \left( x \cdot u'(c_{n,t}) \right) \Rightarrow E \left( u'(c_{m,t}) \cdot R_{m,t} \right) = xE \left( u'(c_{n,t}) \right)$$

(3.2)

As in Smith (2002), we start by exploring in detail the analytically manageable case of logarithmic utility. We assume that the bank never finds it optimal to use a part of its cash reserves to pay non-movers. Below we will write down a sufficient condition, a stochastic analog to (2.1), under which this assumption will be validated. The bank’s problem is now described by

$$\max_{\gamma_t} \int_{w_t}^{\bar{w}} \left\{ \alpha \ln \left[ \frac{\gamma_t}{\alpha} R_{m,t} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t)x}{1 - \alpha} \right] + \ln (w_t + \tau_t) \right\} f(w_{t+1}) dw_{t+1},$$

which simplifies to

$$\max_{\gamma_t} \int_{w_t}^{\bar{w}} \alpha \ln (R_{m,t}) f(w_{t+1}) dw_{t+1} + \alpha \ln \left[ \frac{\gamma_t}{\alpha} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t)x}{1 - \alpha} \right] + \ln (w_t + \tau_t).$$

Observe that since it knows the current period endowment and takes the return on money and the size of the transfer as given, the bank’s choice of $\gamma$ will respond only to the second and the third terms of the previous expression. Then, the choice of $\gamma_t$ is given by

$$\gamma_t = \alpha \text{ for all } t,$$

making the decision rule of the bank identical to that in the non-stochastic endowment case [see (2.11)].

Even though endowment uncertainty has the potential to affect the real return on money, under log utility, the choice of $\gamma$ is separate and is not influenced by this return. Indeed, the choice of $\gamma$ is not state-contingent. Uncertainty on the rate of return is then irrelevant for the choice of $\gamma$. Intuitively, an increase in the uncertainty about the rate of return on money effectively increases its opportunity cost, making it less desirable. This has income
and substitution effects that cancel each other out for logarithmic utility. This will not be the case in the more general CRRA formulation discussed below.

3.1. **Monetary targeting.** Under monetary targeting, the government fixes the money growth rate at $\mu$. Since $m_t = \gamma_t (w_t + \tau_t)$, it follows that the real return to money is given by

$$R_{m,t} = \frac{m_{t+1}}{\mu m_t} = \frac{\gamma_{t+1} (w_{t+1} + \tau_{t+1})}{\mu \gamma_t (w_t + \tau_t)}. \quad (3.3)$$

Also, since $\tau_t = \frac{M_t - M_{t-1}}{\rho_t} = m_t (1 - \frac{1}{\mu})$, $m_t = \gamma_t (w_t + \tau_t)$, and $\gamma_t = \alpha$, we have

$$\tau_t = \left(1 - \frac{1}{\mu}\right) \left[\frac{\alpha w_t}{1 - \alpha \left(1 - \frac{1}{\mu}\right)}\right], \quad (3.4)$$

and

$$m_t = \left[\frac{\alpha}{1 - \alpha \left(1 - \frac{1}{\mu}\right)}\right] w_t.$$

Notice that equilibrium money demand is a fixed non-stochastic fraction of the stochastic endowment; as such, the volatility of money demand is induced *entirely* by the volatility in the endowment and not by the share of the endowment going towards money.

From (3.4), it follows that $w_t + \tau_t = w_t/[1 - \alpha \left(1 - \frac{1}{\mu}\right)]$, and hence the equilibrium return on money [*using (3.3) and (3.4)] is given by

$$R_{m,t} = \frac{w_{t+1} + \tau_{t+1}}{\mu (w_t + \tau_t)} = \frac{w_{t+1}}{\mu w_t}. \quad (3.5)$$

and the gross nominal interest rate by

$$I_t = \frac{x}{R_{m,t}} = \mu x \frac{\gamma_t w_t}{\gamma_{t+1} w_{t+1}} \frac{1 - \gamma_t (1 - \frac{1}{\mu})}{1 - \gamma_{t+1} (1 - \frac{1}{\mu})}. \quad (3.6)$$

Notice since $w_{t+1}$ is not known at $t$, the bank cannot promise a fixed real return to the movers anymore.

When $\mu = 1$, $\tau_t = 0$; in this case, from the ex-ante standpoint of the government, monetary policy does not contribute to post-tax/transfer income, $(w_t + \tau_t)$, volatility. In that case, the only uncertainty that the bank faces would come from the return on money given by (3.5). When $\mu < 1$ (a contractionary monetary policy is implemented), $\tau_t < 0$ obtains, and so $w_t + \tau_t$ falls; i.e., the government imposes a lump-sum tax on all agents
and uses the proceeds to retire some of the currency. Since less money “chases” the same amount of goods, the price level falls raising the return on money making storage less attractive. The fall in \( w_t + \tau_t \) also contributes to less investment in storage.

Monetary policy has different effects on the two groups, movers and non-movers. The latter’s consumption is given by \( x(w_t + \tau_t) \) while the formers’ by \( R_{m,t}(w_t + \tau_t) \). For future reference, note that since \( w_t + \tau_t = w_t/[1 - \alpha \left(1 - \frac{1}{\mu}\right)] \), the consumption of movers is given by \( c_{m,t+1} = \frac{1}{\mu} \frac{w_{t+1}}{1 - \alpha + \mu} \), and that of the non-movers by \( c_{n,t+1} = \frac{xw_t}{1 - \alpha + \mu} \).

We are now in a position to write down a sufficient condition under which the bank would not carry cash balances across periods (to pay non-movers).

**Condition 1.** If agents have logarithmic utility, then in the economy with i.i.d endowment shocks, the bank will use currency only to pay the movers if and only if \( x > \frac{w_e}{\mu w} \) holds.

We proceed to evaluate welfare. Welfare at \( t \) is given by

\[
W_t = \int_{w}^{w_t} \alpha \ln(R_{m,t}) f(w_{t+1}) dw_{t+1} + \alpha \ln \left[ \frac{\gamma_t}{\alpha} \right] + (1 - \alpha) \ln \left[ \frac{(1 - \gamma_t) x}{1 - \alpha} \right] + \ln [w_t + \tau_t]
\]

(3.7)

Using (3.3) and (3.5) and \( \gamma_t = \alpha \), \( W_t \) in (3.7) reduces to

\[
W_t = \int_{w}^{w_t} \alpha \ln \left[ \frac{w_{t+1}}{\mu w_t} \right] f(w_{t+1}) dw_{t+1} + (1 - \alpha) \ln x + \ln w_t - \ln \left[ 1 - \alpha \left(1 - \frac{1}{\mu}\right) \right]
\]

(3.8)

and finally to

\[
W_t = \alpha \int_{w}^{w_t} \{ \ln w_{t+1} \} f(w_{t+1}) dw_{t+1} + (1 - \alpha) \ln (xw_t) - \ln \left[ 1 - \alpha \left(1 - \frac{1}{\mu}\right) \right] - \alpha \ln \mu.
\]

(3.9)

Notice from (3.9) that monetary policy \( (\mu) \) has no effect on the intertemporal margin (the first two terms on the r.h.s. of (3.9)). As such, as argued before, the presumption is still in favor of zero inflation being optimal since that would be intra and inter – temporally efficient.
Since $w$ is assumed to be drawn from a time-invariant i.i.d. distribution, ex-ante stationary welfare is given by $\int W f (w) \, dw$. Thus we have

\[
\int W f (w) \, dw = \alpha \int_{\ln w}^{\ln w} \{ \ln w \} f (w) \, dw + (1 - \alpha) \ln x + (1 - \alpha) \int_{\ln w}^{\ln w} \{ \ln w \} f (w) \, dw
\]

\[
- \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] - \alpha \ln \mu
\]

\[
= (1 - \alpha) \ln x + \int_{\ln w}^{\ln w} \{ \ln w \} f (w) \, dw - \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] - \alpha \ln \mu \quad (3.10)
\]

What $\mu$ is the best from the standpoint of stationary welfare? The one that solves $\frac{d}{d\mu} \int W f (w) \, dw = 0$; call it $\tilde{\mu}$ [notice $\tilde{\mu}$ maximizes the last two terms in (3.10)].

**Proposition 2.** Under logarithmic utility and i.i.d shocks to the endowment, the optimal monetary policy is to keep the money supply fixed, i.e., $\tilde{\mu} = 1$.

It is interesting to note that the optimal prescription for the money growth rate coincides with that in the economy with no real shocks studied in Section 2.5.

To see the intuition behind this result, it is useful to revisit the planner’s problem in this environment. Analogous to the deterministic case discussed earlier, a planner who allocates $w$ between the movers and the non-movers would choose an allocation $(c_m, c_n)$ so as to maximize $\alpha E \left[ u(c_m) \right] + (1 - \alpha) E \left[ u(c_n) \right]$ subject to $\alpha c_m + (1 - \alpha) c_n / x = w$; the marginal condition is

\[
\frac{E \left[ u'(c_m) \right]}{E \left[ u'(c_n) \right]} = x. \quad (3.11)
\]

which is the “intratemporal efficiency” or the “intragenerational efficiency” condition. In the decentralized problem, recall the consumption of movers is given by $c_m = \frac{w}{1 - \alpha + \mu}$, and that of the non-movers by $c_n = \frac{xw}{1 - \alpha + \mu}$. Then, for log utility, it is easy to check that $\frac{E[u'(c_m)]}{E[u'(c_n)]} = x$ holds only when $\mu = 1$. In other words, intratemporal efficiency is ensured at $\mu = 1$. The question then arises: does cutting or raising $\mu$ from unity have any intertemporal benefits that may overwhelm the loss of deviating from intratemporal efficiency? As discussed earlier and clear from (3.9), monetary policy $(\mu)$ has no effect on the intertemporal margin and hence such an action has no benefits at all.

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Footnote 10: By stationary welfare, we mean the indirect utility accruing to two-period-lived agents in a steady state.
As we demonstrate below, unlike in the deterministic case, the prescription for optimal monetary policy will be different from zero inflation under inflation rate targeting.

3.2. Inflation rate targeting. Under inflation rate targeting, the government fixes the inflation rate at $\pi$. It follows that the real return to money is given by

$$R_{m,t} = \frac{1}{\pi}.$$  \hfill (3.12)

Clearly in this case, any uncertainty about the rate of return on money is removed. Since $\tau_t = \frac{M_t - M_{t-1}}{\pi t} = m_t - \frac{m_{t-1}}{\pi}$, we get (using $\gamma_t = \alpha$),

$$m_t = \alpha (w_t + \tau_t) = \alpha \left( w_t + m_t - \frac{m_{t-1}}{\pi} \right)$$

implying

$$m_t = -\frac{\alpha}{(1 - \alpha) \pi} m_{t-1} + \frac{\alpha}{1 - \alpha} w_t$$  \hfill (3.13)

Notice that (3.13) represents an AR(1) process for real balances where $w_t$ is a stochastic “forcing function”.

The invariant (long run) distribution will depend on $w$ and $\pi$. Denote the stationary distribution by $\Omega (m; f (w), \pi)$. A necessary condition for its existence is

$$\left| \frac{\alpha}{(1 - \alpha) \pi} \right| \leq 1;$$  \hfill (3.14)

then $m_\infty$ will not depend on $m_0$. Denote the mean and variance of the stochastic process for $m$ defined in (3.13) by $m^e$ and $\sigma_m^2$ respectively. Then it is easily checked that

$$E (m) \equiv m^e = \frac{w^e}{1 - \alpha + \frac{1}{\pi}}; \quad \sigma_m^2 \equiv E \left\{ (m - m^e)^2 \right\} = \frac{\left( \frac{\alpha}{1 - \alpha} \right)^2 \sigma_w^2}{1 - \left( \frac{\alpha}{(1 - \alpha) \pi} \right)^2}.$$  \hfill (3.15)

Notice that $m^e$ rises with $\pi$ but $\sigma_m^2$ falls with $\pi$.

Using $\gamma_t = \alpha$ and (3.12), in (3.8), we get

$$W_t (\pi) = -\int_{w^e}^{\infty} \alpha \ln \pi f (w_{t+1}) \, dw_{t+1} + (1 - \alpha) \ln x + \ln (w_t + \tau_t)$$

$$= -\alpha \ln \pi + (1 - \alpha) \ln x + \ln (w_t + \tau_t) = -\alpha \ln \pi + (1 - \alpha) \ln x + \ln \left( \frac{m_t}{\alpha} \right)$$ \hfill (3.16)

From (3.16), it is apparent that $\pi$ has two effects on welfare, one through its effect on the return on money (captured by the $\int_{w^e}^{\infty} (\ln \pi) f (w_{t+1}) \, dw_{t+1}$ term above) and the other via its effect on post-tax/transfer income (captured by the $\ln (w_t + \tau_t)$ term above). As
such, the government will have to pay attention not just to intratemporal but also to intertemporal efficiency when choosing \( \pi \). Using (3.16), stationary welfare is written as

\[
E(W) \equiv \int W f(w) dw = -\alpha \ln \pi - \ln \alpha + (1 - \alpha) \ln x + \int_{\Omega(m; f(w), \pi)} \ln m \ dm
\]

where the support of \( \Omega(m; f(w), \pi) \) given by \([\bar{m}, f(w), m]\) corresponds to the support of \( w \) given by \([w, \bar{w}]\). Notice that the last term on the r.h.s of (3.17) corresponds to \( E(\ln(w_t + \tau_t)) \) or the mean value of log post tax income. Henceforth, we assume that \( E(W) \) is strictly concave in \( \pi \). What is the best \( \pi \)? The one that solves \( \frac{d}{d\pi} \int W f(w) dw = 0 \); call it \( \pi^- \).

**Proposition 3.** Under logarithmic utility, optimal inflation targeting involves setting a positive inflation rate, or \( \pi > 1 \) obtains.

A rough intuition for this result is as follows. The first thing to note is that inflation rate targeting completely eliminates uncertainty concerning the rate of return on money. The only uncertainty that remains as such is the one in post-tax/transfer income. As is evident from (3.16) and (3.15), a risk-averse agent would prefer a high expected value for \( \ln m \) and as little variability in it as possible. Since raising the inflation rate achieves both, it follows that choosing a positive inflation rate \( (\pi > 1) \) may be desirable.

Notice that zero inflation \( (\pi = 1) \) would be intratemporally efficient but not so intertemporally. The idea is as follows. Consider a starting date at which the current endowment is realized to be relatively high and so the current money demand is high. Since the previous period endowment was lower, the amount of money balances being brought over by the old movers is relatively small. For the government to preserve the return on money at \( 1/\pi \), it will have to inject fresh money to supply this higher money demand. Relative to \( \pi = 1 \), i.e., the return on money is 1, the size of the government’s current injection will have to be higher if \( \pi > 1 \) (return on money is less than one). If next period’s endowment is closer to the mean, then relative to \( \pi = 1 \), the size of the government’s injection in that period will be lower if \( \pi > 1 \). Thus the transmission of the volatility across time is lessened when \( \pi \) rises; as (3.13) makes clear,

\[
(w_t + \tau_t) = -\frac{1}{(1 - \alpha)\pi} (w_{t-1} + \tau_{t-1}) + \frac{1}{1 - \alpha} w_t
\]
the autocorrelation between total income at two adjacent periods is negative and the strength of this correlation becomes smaller as \( \pi \) rises. Thus setting \( \pi > 1 \) maybe be a good idea from the standpoint of intertemporal efficiency; this way shocks to income get transmitted over time in a more muted fashion.

We now proceed to answer the question: given the real shocks to endowment, what should a benevolent government do, target the money growth rate (and set the net money growth rate to zero) or target a positive inflation rate? which action would generate the higher aggregate welfare?

**Proposition 4.** Under logarithmic utility, optimal targeting of the money growth rate is stationary-welfare superior to optimal targeting of the inflation rate.

Recall that optimal monetary targeting involves fixing the money supply which eliminates any post tax/transfer income uncertainty; any remaining uncertainty is with respect to the return on money. On the other hand, optimal inflation rate targeting involves fixing the inflation rate thereby eliminating any uncertainty with respect to the return on money; the residual uncertainty, in this case, is with regard to the post-tax/transfer income.

Why is monetary targeting superior? A fixed money supply rule achieves ex-ante intratemporal efficiency; it does/can not affect the intrinsic endowment uncertainty, \( \sigma_w^2 \). Compare this to a zero net inflation rate policy. Of course, \( \pi = 1 \) can achieve ex-ante intratemporal efficiency, but is associated with a higher volatility of post-tax income, since the variance of post-tax income at \( \pi = 1 \) (denoted \( \sigma_{w+\tau}|_{\pi=1} \)) is \( \sigma_{w+\tau}|_{\pi=1} = \frac{\sigma_w^2}{1-2\alpha} > \sigma_w^2 \). Any \( \pi > 1 \) would distort intratemporal efficiency but would be associated with reduced post tax income volatility. However, the volatility of post-tax income still exceeds that under monetary targeting since

\[
\sigma_{w+\tau}|_{\pi>1} = \frac{\sigma_w^2}{1-2\alpha + \alpha^2(1-\frac{1}{\pi^2})} > \sigma_{w+\tau}|_{\pi=1} = \frac{\sigma_w^2}{1-2\alpha} > \sigma_w^2
\]

holds for all \( \pi \). Overall, relative to monetary targeting, optimal inflation targeting distorts intratemporal efficiency margin and leaves post-tax/transfer income more volatile.

4. **Liquidity shocks**

We now pursue another variation on the standard random relocation model by introducing liquidity shocks. Specifically, we assume that \( \alpha_t \), the fraction of young agents relocating to the other location, is drawn each period from an i.i.d distribution \( g(\alpha) \) with support
Denote by $\alpha^e$ the expected value of $\alpha$ and by $\sigma^2_\alpha$ its variance. As described earlier, shocks to $\alpha$ represent money demand or liquidity shocks. We assume that these shocks are realized each period before the bank makes its portfolio decisions. Also, to motivate the continued need for banks, we assume that individual agents do not learn their relocation status until the end of the period.

Our goal as before is to investigate how monetary policy should respond to shocks to liquidity preference. Analogous to the setting with endowment uncertainty, the bank cares about next period’s liquidity demand because it will potentially influence next period’s price level and thus the return on money between this period and the next. As before, we assume the bank knows the distribution for $\alpha$ and forms expectations on the return on money conditional on $g(\alpha)$; in a rational expectations equilibrium, these expectations are correct. We will focus on stationary versions of such equilibria below. As we demonstrate below, the impact of such liquidity shocks is entirely different from the endowment shocks studied earlier, even though at first blush it may seem that they ought to have similar effects (after all, both shocks work through liquidity demand and the return on money).

As before, for analytical tractability and intuition building, we start by working through the case with logarithmic utility. Since our current focus is on shocks to liquidity, we hold $w$ fixed for all $t$. Just as before, we assume that the bank never finds it optimal to use a part of its cash reserves to pay non-movers. Below we will write down a sufficient condition, analogous to that in Condition 1, under which this assumption will be validated.

The bank’s problem is now described by

$$\max_{\gamma_t} \int_\alpha^{\tilde{\alpha}} \left\{ \alpha_t \ln \left( \frac{\gamma_t p_t}{\alpha_t p_{t+1}} \right) + (1 - \alpha_t) \ln \left( \frac{(1 - \gamma_t)x}{1 - \alpha_t} \right) + \ln (w + \tau_t) \right\} g(\alpha_{t+1}) d\alpha_{t+1},$$

which straightforwardly reduces to

$$\max_{\gamma_t} \int_\alpha^{\tilde{\alpha}} \ln \left( \frac{p_t}{p_{t+1}} \right) g(\alpha_{t+1}) d\alpha_{t+1} + \alpha_t \ln \left( \frac{\gamma_t}{\alpha_t} \right) + (1 - \alpha_t) \ln \left( \frac{(1 - \gamma_t)x}{1 - \alpha_t} \right) + \ln (w + \tau_t).$$

$$\text{(4.1)}$$

---

11Smith (2002) and Antinolfi, Huybens, and Keister (2001), and Antinolfi and Keister (forthcoming) consider settings where such shocks are realized after the bank has made its portfolio decisions. In such situations, “banking crises” may arise, i.e., if the realized value of the liquidity shock is “too high”, the bank may run out of all its cash reserves and even be forced to prematurely liquidate storage. These issues are the subject matter of a companion paper.
Note that bank’s choice of $\gamma$ will only consider the second and the third term. It is easy to verify that the optimal choice of $\gamma_t$ is given by

$$\gamma_t = \alpha_t \text{ for all } t. \quad (4.2)$$

Now the choice rule for $\gamma$ is time-dependent irrespective of the monetary policy regime. This is an important difference with the corresponding problem with endowment shocks.

4.1. **Monetary targeting.** Under monetary targeting, the government fixes the money growth rate at $\mu$. It follows that the real return to money is given by (3.3). Also, since $\tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t(1 - \frac{1}{\mu})$, $m_t = \gamma_t(w + \tau_t)$, and $\gamma_t = \alpha_t$, we have

$$\tau_t = \frac{\alpha_t (1 - \frac{1}{\mu})}{1 - \alpha_t (1 - \frac{1}{\mu})} w, \quad (4.3)$$

and

$$m_t = \left[ \frac{\alpha_t}{1 - \alpha_t (1 - \frac{1}{\mu})} \right] w.$$  

In contrast to (3.4), equilibrium money demand is a stochastic fraction of the non-stochastic endowment; as such, the volatility of money demand is induced entirely by the volatility in the share of the endowment going towards money and not by the endowment itself. Herein lies the crucial difference between the way the two types of shocks influence the economy.

From (4.3), it follows that $w + \tau_t = w/[1 - \alpha_t(1 - \frac{1}{\mu})]$, and hence

$$R_{m,t} = \frac{w + \tau_{t+1}}{\mu (w + \tau_t)} = \frac{\alpha_{t+1}}{\alpha_t} \frac{1 - \alpha_t (1 - \frac{1}{\mu})}{\mu [1 - \alpha_{t+1}(1 - \frac{1}{\mu})]]. \quad (4.4)$$

As before, $\mu = 1$ would render $\tau_t = 0$ (set the inflation tax rate to zero) thereby removing any income uncertainty.

**Condition 2.** Under logarithmic utility, the bank never finds it optimal to use a part of its cash reserves to pay non-movers if

$$x > \frac{1}{\mu} E \left\{ \frac{\alpha \mu - \alpha (\mu - 1)}{\alpha \mu - \alpha (\mu - 1)} \right\}$$

holds.
Using (4.4), we can write indirect utility as
\[
W_t = \int_\alpha^\bar{\alpha} \alpha_t \ln \left[ \frac{\alpha_{t+1}}{\alpha_t} \frac{1 - \alpha_t \left( 1 - \frac{1}{\mu} \right)}{1 - \alpha_{t+1} \left( 1 - \frac{1}{\mu} \right)} \right] g(\alpha_{t+1}) \, d\alpha_{t+1} \\
+ (1 - \alpha_t) \ln x + \ln w - \ln \left[ 1 - \alpha_t \left( 1 - \frac{1}{\mu} \right) \right],
\]
which reduces to
\[
W_t = \alpha_t \int_\alpha^\bar{\alpha} \ln \left[ \frac{\alpha_{t+1}}{1 - \alpha_{t+1} \left( 1 - \frac{1}{\mu} \right)} \right] g(\alpha_{t+1}) \, d\alpha_{t+1} + (1 - \alpha_t) \ln x \\
+ \ln w - \alpha_t \ln \alpha_t - (1 - \alpha_t) \ln \left[ 1 - \alpha_t \left( 1 - \frac{1}{\mu} \right) \right] - \alpha_t \ln \mu. \tag{4.5}
\]

Notice that monetary policy, unlike in the corresponding economy with endowment shocks, can affect the intertemporal margin here via its effect on the first term on the r.h.s of (4.5). Hence, there would no longer exist a presumption in favor of a zero money growth rate being optimal. It can be checked that stationary welfare is given by
\[
\int W g(\alpha) \, d\alpha = (1 - \alpha^e) \ln x + \ln w - \alpha^e \ln \mu \\
- \int_\alpha^\bar{\alpha} \left\{ (\alpha - \alpha^e) \ln \alpha + (1 - (\alpha - \alpha^e)) \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] \right\} g(\alpha) \, d\alpha. \tag{4.6}
\]

What \( \mu \) maximizes stationary welfare? The one (call it \( \bar{\mu} \)) that solves \( \frac{d}{d\bar{\mu}} \int W f(\alpha) \, d\alpha = 0 \). It follows that \( \bar{\mu} \) is implicitly defined by
\[
\bar{\mu} = \frac{1}{\alpha^e} \int_\alpha^\bar{\alpha} \frac{1 - (\alpha - \alpha^e)}{1 - \alpha \left( 1 - \frac{1}{\mu} \right)} \alpha g(\alpha) \, d\alpha. \tag{4.7}
\]

**Proposition 5.** In the presence of liquidity shocks, the optimal monetary target involves a negative net money growth rate, i.e., \( \bar{\mu} < 1 \).

In this setup, monetary policy has different effects on the two groups, movers and non-movers. The latter’s consumption is given by \( x(w_t + \tau_t) \) while the formers’ is given by \( R_{m,t}(w_t + \tau_t) \). Clearly, constant money supply, unlike in the case of real endowment shocks, is no longer optimal. Why? At \( \mu = 1 \), the consumption of non-movers is simply \( xw \) while that of the movers (using (4.4) is \( \frac{\alpha_{t+1}}{\alpha_t} w \). As before, intragenerational efficiency requires that \( E \{ u'(c_m) \} = xE \{ u'(c_n) \} \). However, in this setting \( E \{ u'(c_m) \} = \)
\[ \frac{1}{w} E \left\{ \frac{\alpha_t}{\alpha_{t+1}} \right\} > \frac{1}{w}, \text{ whereas } x E \{ u'(c_m) \} = \frac{1}{w}. \]  

Thus, in the decentralized equilibrium \[ E \{ u'(c_m) \} > x E \{ u'(c_n) \} \]  

obtains. Thus, even though \( \mu = 1 \) ensures a constant post-tax/transfer income of \( w \), the uncertain return on money makes the movers’ expected utility of consumption higher than what intergenerational efficiency requires. This makes a case for transferring some income to movers by cutting \( \mu \) below \( 1 \). Doing so, however, comes with a cost; since \( \mu < 1 \) implies \( \tau < 0 \), such an action lowers overall income \( w + \tau \), and reduces storage investment. Since movers have a higher expected marginal utility of consumption, the social benefit obtained by such a transfer may overwhelm the cost of lowered overall income and storage investment. In addition, an overall lowering of income level caused by the cut in \( \mu \) also lowers the overall volatility of income (though this presumably has only a second order beneficial effect).

4.2. **Inflation rate targeting.** Under inflation rate targeting, the government fixes the inflation rate at \( \pi \). It follows that the real return to money is given by (3.12). In this case, using (4.1), indirect utility reduces to

\[ W_t = -\alpha_t \ln \pi + (1 - \alpha_t) \ln x + \ln \left( \frac{m_t}{\alpha_t} \right). \]  

(4.8)

Since \( \tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t - \frac{m_{t-1}}{\pi}, \) we get

\[ m_t = \alpha_t (w + \tau_t) = \alpha_t \left( w + m_t - \frac{m_{t-1}}{\pi} \right) \]

implying, analogous to (3.13),

\[ m_t = -\frac{\alpha_t}{(1 - \alpha_t) \pi} m_{t-1} + \frac{\alpha_t}{1 - \alpha_t} w \]  

(4.9)

Note that \( m \) and \( \alpha \) are jointly distributed, and from (4.8), it is clear that we seek the invariant distribution of \( \frac{m_t}{\alpha_t} \) which will depend on the distribution of \( \alpha \) and the value of \( \pi \). Denote it as \( \Omega \left( g(\alpha), \pi \right) \). Then, stationary welfare is given by

\[ \int W g(\alpha) d\alpha = -\alpha^e \ln \pi - (1 - \alpha^e) \ln x + \int \left( \frac{m}{\alpha} \right) \ln \left( \frac{m}{\alpha} \right) \Omega \left( g(\alpha), \pi \right) d \left( \frac{m}{\alpha} \right) \]

and optimal choice of \( \pi \) (denoted by \( \tilde{\pi} \)) is given by the solution to \( \frac{d}{d\tilde{\pi}} \int W f(\alpha) d\alpha = 0 \).

**Proposition 6.** In the presence of liquidity shocks, the optimal inflation target involves a positive net inflation rate, i.e., \( \tilde{\pi} > 1 \).
The intuition for this result closely follows the argument laid out in the endowment uncertainty case. Notice that zero inflation ($\pi = 1$) would be intratemporally efficient but not so intertemporally. Consider a starting date at which the current fraction of movers is realized to be relatively high and so the current money demand is high. Since the previous period’s $\alpha$ was lower, the amount of money balances being brought over by the old movers (born last period) is relatively small. For the government to preserve the return on money at $1/\pi$, it will have to inject fresh money to supply this higher money demand. Relative to $\pi = 1$, i.e., the return on money is 1, the size of the government’s current injection will have to be higher if $\pi > 1$ (return on money is less than one). If next period’s endowment is closer to the mean, then relative to $\pi = 1$, the size of the government’s injection in that period will be lower if $\pi > 1$. Relative to $\pi = 1$, the transmission of the volatility of income across time is lessened when $\pi$ rises; as (4.9) makes clear,

$$ (w + \tau_t) = -\frac{\alpha_{t-1}}{1 - \alpha_t} \pi (w + \tau_{t-1}) + \frac{1}{1 - \alpha_t} w $$

the autocorrelation between post tax income at two adjacent periods is negative and the strength of this correlation becomes smaller as $\pi$ rises. Thus setting $\pi > 1$ maybe be a good idea from the standpoint of intertemporal efficiency. Doing so ensures that shocks to income get less transmitted over time.

Finally, we ask: in the presence of liquidity shocks, can we rank the two policy instruments in terms of welfare?

**Proposition 7.** For any given $\frac{\alpha}{\alpha^e}$, there exists an $\hat{\alpha} < 1/2$ such that, $\bar{\bar{W}}^{\pi} \succeq \bar{\bar{W}}^\mu$ for all $\alpha^e \leq \hat{\alpha}$, where $\bar{\bar{W}}^{\pi}$ ($\bar{\bar{W}}^\mu$) is the maximized value of stationary welfare under inflation (money growth) targeting.

In other words, if the liquidity shock is “small enough”, inflation targeting does a better job. A rough intuition for this is as follows. As we have seen, optimal monetary targeting involves cutting $\mu$ below unity since such an action improves intragenerational efficiency. This comes at a cost; after all lowering $\mu$ diverts resources from being invested in storage for two reasons: a) cutting $\mu$ raises taxes cuts post-tax income available for investment, and b) reducing $\mu$ raises the return on money making storage less attractive. As a result of this reduced investment in storage, both movers and non-movers receive a relatively low level of consumption under monetary targeting. Inflation targeting, on the other hand, achieves intragenerational efficiency without shifting resources from non-movers to movers.
Of course, because of the aforediscussed intertemporal transmission of shocks, inflation targeting also imposes a cost. It turns out if $\alpha$ is small enough (the “persistence” of shocks is not too high), i.e., its value does not generate near-unit root process for money balances, inflation targeting does a better job.

5. Numerical results: extending to CRRA utility

We now extend our analysis to incorporate the more general CRRA utility form: $u(c) = \left[ c^{1-\phi} - 1 \right] / (1 - \phi)$ where $\phi$ is the coefficient of relative risk aversion. Our objective here is to verify whether the flavor of the results from Section 3 and 4 continue to obtain for away from unity. Since it is not possible pursue this analytically, we will resort to numerical analysis below. First we briefly sketch the formulation of the bank and the government’s problem for a general specification of shocks.

Analogous to (2.6), we first rewrite the bank’s problem in period $t$ as

$$\max_{\gamma_t \in [0,1]} \left\{ \frac{(w_t + \tau_t)^{1-\phi}}{1-\phi} \left( \alpha_t \gamma_t^{1-\phi} E_t \left\{ R_{m,t}^{1-\phi} \right\} + (1 - \alpha_t)^\phi \left[ (1 - \gamma_t)x \right]^{1-\phi} \right) - \frac{1}{1-\phi} \right\} \right.$$

where $E_t \{ \cdot \}$ denotes the expectation operator conditional on the information available at $t$. Notice $E_t \left\{ R_{m,t}^{1-\phi} \right\} \equiv \int_{\omega}^{\pi} \left( R_{m,t} \right)^{1-\phi} f (w_{t+1}) \, dw_{t+1}$ in the case of the endowment shocks and $E_t \left\{ R_{m,t}^{1-\phi} \right\} \equiv \int_{\omega}^{\pi} \left( R_{m,t} \right)^{1-\phi} g (\alpha_{t+1}) \, d\alpha_{t+1}$ for the case of liquidity shocks. The first order condition to the bank’s problem is given by

$$\left( \frac{\gamma_t}{\alpha_t} \right)^{-\phi} E_t \left\{ R_{m,t}^{1-\phi} \right\} = x \left( \frac{1 - \gamma_t}{1 - \alpha_t} x \right)^{-\phi},$$

which readily yields

$$\gamma_t = \frac{\alpha_t}{\alpha_t + (1 - \alpha_t) E_t \left\{ I_t^{\frac{1}{\mu} - 1} \right\}}. \tag{5.2}$$

As described earlier, under monetary targeting, $I_t$ is given by (3.6), i.e.,

$$I_t = \mu x \frac{1 - \gamma_{t+1}}{\gamma_{t+1} w_{t+1}} \frac{1 - \gamma_t}{1 - \gamma_t \left( 1 - \frac{1}{\mu} \right)},$$
Thus, for given probability distributions for $\alpha$ and $w$, the equilibrium function $\gamma$ is obtained as a fixed point of (5.2). Evidently, under monetary targeting, the equilibrium $\gamma_t [\text{denoted } \gamma_t (\mu)]$ is a function of $\mu$, and the period $t$ realization of $\alpha$ and/or $w$. Under inflation targeting, however, $I_t = \pi x$ and the equilibrium $\gamma_t [\text{denoted } \gamma_t (\pi)]$ is readily obtained from (5.2).

Let $i \in \{\pi, \mu\}$ denote the index for inflation rate and monetary growth targeting respectively. We can then write the indirect utility at $t$ as

$$W^i_t = \frac{1}{1 - \phi} \left[ x^{1-\phi} (1 - \alpha^i)^{\phi} (w_t + \tau_t (i))^{1-\phi} (1 - \gamma_t (i))^{-\phi} - 1 \right], \quad i \in \{\pi, \mu\}. \tag{5.3}$$

As discussed earlier, under monetary targeting,

$$w_t + \tau_t (\mu) = \frac{\gamma_t (\mu) w_t}{1 - \gamma_t (\mu) \left( 1 - \frac{t}{\mu} \right)},$$

whereas under inflation targeting, the post-tax income $w + \tau_t (\pi)$ is obtained as $\frac{m_t}{\pi}$, where $m$ follows a stochastic process given by

$$m_t = -\frac{\gamma_t (\pi)}{1 - \gamma_t (\pi) \frac{1}{\pi}} m_{t-1} + \frac{\gamma_t (\pi)}{1 - \gamma_t (\pi)} w_t. \tag{5.4}$$

We define ex-ante stationary welfare under the two policies as

$$W^i \equiv \frac{1}{1 - \phi} \left[ x^{1-\phi} (1 - \alpha^i)^{\phi} (w^i + \tau^i (i))^{1-\phi} (1 - \gamma^i (i))^{-\phi} - 1 \right], \quad i \in \{\pi, \mu\}, \tag{5.5}$$

where the expectations are obtained under the stationary distribution of all variables. Next, we denote optimal policies and optimal welfare levels as

$$\bar{W}^i \equiv \max_i \{ W^i \}, \quad \bar{i} \equiv \arg \max \{ W^i \}, \quad i \in \{\pi, \mu\}$$

Finally, we represent $\bar{W}^i$ in terms of its consumption equivalent $\bar{c}^i$ by using

$$\bar{W}^i = \left[ (\bar{c}^i)^{1-\phi} - 1 \right] / (1 - \phi).$$

Our choice of parametric specification is as follows. In the case of endowments shocks, we fix $\alpha = 0.2$ and assume that $w$ is i.i.d. and uniformly distributed over support $[0.9, 1.1]$. Thus, $w^e = 1$, $\sigma^2_w = 0.00333$, and $\frac{\sigma_w}{\pi} = 0.0577$. In the case of liquidity shocks, we fix $w = 1$ and assume that $\alpha$ is i.i.d. and uniformly distributed over $[0.18, 0.22]$. This implies
that $\alpha = 0.2$ and $\frac{e}{\alpha} = 0.0577$. Below, we compare optimal money growth and inflation rates under the two policies, along with their respective welfare levels, for $\phi \in [0.5, 2.1]$.\textsuperscript{12}

A few words about the computational algorithm is in order. Under monetary targeting, the main step entails computing the fixed point of $\gamma$ as a function of $w$ or $\alpha$, depending on the nature of the shock. To do so, we guess an initial function\textsuperscript{13}, and numerically iterate on (5.2) to convergence. This is done for a fixed $\mu$. Once $\gamma$ function is obtained, evaluating (5.5) is straightforward. By repeating this exercise for different values of $\mu$, we easily obtain $\tilde{\mu}$ and $\tilde{W}^\mu$. Under inflation targeting, however, the money supply (and hence the post-tax income) process (5.4), even under the assumption of a uniform distribution for $w$ and $\alpha$, does not yield any analytical closed form stationary distribution. Hence, we resort to simulations. Specifically, we first fix $\pi$. This yields $\gamma$ explicitly as a function of $w$ or $\alpha$. After assuming an initial value of $m_0$, we then simulate (5.4), \textit{more details? what does simulate mean? where is $m_0$ chosen from?} and for each observation \textit{of $m_0$?} compute (5.3). An average obtained from the previous step yields (5.5) simply by the law of large numbers. By repeating this exercise for different values of $\pi$, we obtain $\tilde{\pi}$ and $\tilde{W}^\pi$.

5.1. Endowment shocks. We now report on the optimal choices of the inflation rate and the money growth rate as they vary with risk aversion. Our earlier results have established that when $\phi = 1$ (log utility), we have $\tilde{\mu} = 1$ and $\tilde{\pi} > 1$. Figure 1 shows that $\tilde{\mu} \leq 1$ and $\tilde{\pi} > 1$ for $\phi \geq 1$. Both optimal money growth and inflation rates vary monotonically with

\textsuperscript{12}We find that $\phi \in [0.5, 2.1]$ is fairly representative, and our results continue to hold when this range is enlarged.

\textsuperscript{13}Our initial guess is $\gamma (w_t) = \alpha$, or $\gamma (\alpha_t) = \alpha_t$. The convergence to the equilibrium function at any desired accuracy is reasonably fast.
A rough intuition for why $\hat{\mu} \leq 1$ obtains is as follows. To begin with, notice that $\gamma$ and $w$ are positively correlated when $\phi > 1$. Why? When $\phi > 1$, and the current realized endowment is high (low), the current price level is lower (higher) than the average and banks expect a lower (higher) return on money, leading to a higher (lower) choice of $\gamma$. As a result, $\gamma$ and $w$ are positively correlated; it follows that $(1 - \gamma)$ and $w$ are negatively correlated. At $\mu = 1$, the consumption of each mover is given by $\gamma w/\alpha$ while that of each of the non-movers is always $(1 - \gamma)wx/(1 - \alpha)$. Ex-ante, therefore, the consumption of the movers is relatively more volatile than that of the nonmovers. As a result, the formers’ marginal utility of consumption is higher, and transferring income to movers by cutting $\mu$ turns out to be efficient.

To see why $\tilde{\pi} > 1$, recall from our earlier discussion for log utility that a higher inflation target reduces intertemporal volatility of income. It seems that as $\phi$ gets higher, the relative importance of reducing income volatility rises relative to the need of ensuring intragenerational efficiency.

Figure 2 presents the percentage gain in steady state welfare, expressed in terms of equivalent consumption, that is obtained under monetary targeting relative to inflation.
targeting.

![Figure 2: % change in $c_i$ from following $\tilde{\mu}$ over $\tilde{\pi}$](image)

Monetary targeting is more desirable relative to inflation targeting as $\phi$ increases. Our earlier analysis with log utility showed that inflation targeting generates a higher intertemporal income volatility. As $\phi$ increases, income volatility hurts more and more, thus making monetary targeting even more desirable. The opposite is the case when $\phi < 1$.

5.2. **Liquidity shocks.** Figure 3 below combines optimal money growth and inflation rates in the case of liquidity shocks for a range of $\phi$. The main upshot of the figure is that $\tilde{\mu} \leq 1$ and $\tilde{\pi} > 1$ as $\phi \leq 1$; optimal monetary targeting always involves implementing a non-expansionary monetary policy while optimal inflation targeting always involves a
expansionary policy.

\[
\pi_t, \mu
\]

$0.99$ $0.9925$ $0.995$ $0.9975$ $1.0025$ $1.005$ $1.0075$

Optimal Inflation Rate

$0.5$ $0.75$ $1.25$ $1.5$ $1.75$ $2$

Optimal Money Growth Rate

Figure 3: Optimal $\pi$ and $\mu$ against $\phi$ (liquidity shocks)

A rough intuition is as follows. At $\mu = 1$, the consumption of each mover is given by $\gamma_{t+1} w$, while that of each non-mover is $\frac{1-\gamma_t}{\alpha_t} x w$. It is evident that movers’ consumption is more volatile than that of non-movers.\(^\text{14}\) Recall from our discussion under log utility that transferring income to movers by cutting $\mu$ ensures intragenerational efficiency. The need for intertemporal insurance becomes stronger as $\phi$ increases. Hence, a higher reduction in $\mu$ is required for larger values of $\phi$. The intuition behind an increasing relationship between $\pi$ and $\phi$ is similar to that under endowment shocks.

\(^\text{14}\) When $\phi = 1$ (log utility), the consumption of each mover is given by $\frac{\alpha_{t+1}}{\alpha_t} w$, while that of each non-mover is $x w$ which is non-stochastic. For general $\phi$, the consumption of nonmovers will be stochastic but not as volatile as that of the movers.
Finally, Figure 4 shows the percentage loss in steady state welfare, expressed in terms of equivalent consumption, under monetary targeting relative to inflation targeting.

A rough intuition for this follows the discussion after Proposition 7. Here, under monetary targeting a higher $\phi$ will imply that $\mu$ will have to be lowered “a lot” (as evident from Figure 3), which, as discussed earlier can be quite costly in terms of lost income from storage. Thus, the desirability of inflation targeting increases with $\phi$.

6. Conclusion

This paper revisits the classic issue of the optimal choice of monetary instruments faced by central bankers around the world. The issue received its first formal treatment in Poole (1970). Using a stochastic IS-LM model, with reduction in variability of aggregate output as the criterion, Poole showed that when the shocks to the economy are real in nature, the central bank should target the money growth rate; if the shocks are monetary in origin, it is the interest rate that should be targeted. We update the analysis of Poole’s “instrument problem” within the context of a modern optimizing agents framework with flexible prices and an explicit welfare criterion.

To that end, we produce a two-period lived pure-exchange overlapping generations model in the tradition of Townsend (1987) and Champ, Smith, and Williamson (1997) where
limited communication and stochastic relocation create an endogenous transactions role for fiat money. We study two kinds of shocks, real shocks (shocks to the endowment) and liquidity shocks (shocks to the fraction of agents relocating). Overall, two important results transpire. Our results indicate that for the most part (i.e., for small enough liquidity shocks), Poole’s original IS-LM based insight remains valid in our more microfounded modern setup. Additionally, under inflation rate targeting it is always optimal to pursue an expansionary policy; it is never optimal to do so under money growth targeting.

A simple model like ours, needless to say, misses many aspects of reality. Introducing government debt, neoclassical production, and state-contingent monetary policies seem like fruitful directions for further work. It would also be useful to allow for persistent shocks.
References


Appendix

Appendix A. Proof of Condition 1

Suppose after reserving a fraction $\gamma$ of deposits for movers, banks may further split the rest $(1 - \gamma)$ into money (a fraction $1 - \theta$) and storage ($\theta$) for non-movers. Then, the appropriate problem problem for the bank is

$$\max_{\{\gamma_t \in [0,1], \theta_t \in [0,1]\}} E_t \left\{ \alpha u \left( \frac{\gamma_t}{\alpha} R_{m,t} (w_t + \tau_t) \right) + (1 - \alpha) u \left( \frac{(1 - \gamma_t) (\theta_t x + (1 - \theta_t) R_{m,t})}{(1 - \alpha)} (w_t + \tau_t) \right) \right\}.$$ 

Define $c_n \equiv \frac{(1 - \gamma_t) (\theta_t x + (1 - \theta_t) R_{m,t})}{(1 - \alpha)} (w_t + \tau_t)$. The first order condition with respect to $\theta_t$ can be compactly written as

$$E_t \left[ u'(c_n) x - u'(c_n) R_{m,t} \right] \left\{ \begin{array}{l} \geq 0, \quad \text{"= " if } \theta_t < 1 \\
\leq 0, \quad \text{"= " if } \theta_t > 0. \end{array} \right.$$ 

Note that when $\theta = 1$, $c_n$ is non-stochastic, and the above condition reduces to

$$x > E_t \{ R_{m,t} \}.$$ 

With logarithmic utility, we know from (3.5) that $\frac{p_t}{p_{t+1}} = \frac{w_{t+1}}{\mu w_t}$. Then $x > E_t \{ R_{m,t} \}$ will hold for all possible realizations of $w_t$ if and only if the condition in the statement of Condition 1 holds. 

Appendix B. Proof of Proposition 3

Using (3.16),

$$E(W) = -\alpha \ln \pi - \ln \alpha + (1 - \alpha) \ln x + E(\ln m(\pi)), \quad \text{(B.1)}$$

where the distribution of $m$ is governed by (3.13). We first take a second-order Taylor approximation of the last term around $E(m) \equiv m^e$:

$$E \{ \ln m \} \approx E \left\{ \ln m^e + \frac{1}{m^e} (m - m^e) - \frac{1}{2} \frac{1}{(m^e)^2} (m - m^e)^2 \right\} = \ln m^e - \frac{1}{2} \frac{\sigma_m^2}{(m^e)^2}. \quad \text{(B.2)}$$

Using (3.15) with (B.2) in (B.1), and differentiating with respect to $\pi$ yields

$$\frac{dE(W)}{d\pi} = -\frac{\alpha}{\pi} + \frac{\pi^2}{1 - \alpha + \frac{\alpha}{\pi}} + \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2} \frac{\alpha (1 - \alpha)}{[(1 - \alpha) \pi - \alpha]^2}.$$ 

Thus,

$$\left. \frac{dE \{ W \}}{d\pi} \right|_{\pi=1} = \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2} \frac{\alpha (1 - \alpha)}{(1 - 2\alpha)^2} > 0.$$ 

Assuming $W(\pi)$ is concave in $\pi$, the above implies $\pi > 1$. 

Appendix C. Proof of Condition 2

The proof follows that of Condition 1 closely. To begin with, note that $\alpha_t$ is known at the start of $t$. Then, as in the proof of Condition 1, the required condition reduces to $x > E_t \{R_{m,t}\}$. Using (??), it follows that

$$x > E_t \left\{ \frac{\alpha_{t+1}}{\alpha} \frac{1 - \alpha}{\mu} \left( 1 - \frac{1}{\tilde{m}} \right) \right\} \Leftrightarrow$$

$$x > E_t \left\{ \frac{\alpha}{\mu} \left[ 1 - \alpha_{t+1} \left( 1 - \frac{1}{\tilde{m}} \right) \right] \right\} \Leftrightarrow$$

$$x > \frac{1}{\mu} E \left\{ \frac{\alpha - \alpha (\mu - 1)}{\alpha - \alpha (\mu - 1)} \right\}.$$

Appendix D. Proof of Proposition 4

We start by computing the maximized value of stationary welfare under monetary targeting, denoted by $W^\mu$. From (3.10), it can be checked that at $\mu = 1$,

$$W^\mu = (1 - \alpha) \ln x + \int_\omega \{ \ln w \} f(w) dw.$$

Analogous to (B.2), we can write $\int_\omega \{ \ln w \} f(w) dw \equiv E \{ \ln w \} \simeq \ln w^e - \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2}$ and so

$$W^\mu = (1 - \alpha) \ln x + \ln w^e - \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2}.$$

Denote by $W^\pi$, the maximized value of stationary welfare under inflation targeting. From (3.17), it follows that

$$W^\pi = -\alpha \ln \tilde{\pi} - \ln \alpha + (1 - \alpha) \ln x + E \{ \ln m (\tilde{\pi}) \},$$

where it is known from Proposition 3 that $\tilde{\pi} > 1$. Using (B.2) and (3.15), note that

$$E \{ \ln m (\tilde{\pi}) \} = \ln m_E - \frac{1}{2} \frac{\sigma_m^2}{(m^e)^2} = \ln \frac{\alpha w^e}{1 - \alpha + \frac{\alpha}{\tilde{\pi}}} - \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2} \left( 1 - \alpha \right) \tilde{\pi} + \alpha$$

then

$$W^\pi = -\alpha \ln \tilde{\pi} - \ln \alpha + (1 - \alpha) \ln x + \ln \left( \frac{\alpha w^e}{1 - \alpha + \frac{\alpha}{\tilde{\pi}}} - \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2} \left( 1 - \alpha \right) \tilde{\pi} + \alpha \right)$$

$$= -\alpha \ln \tilde{\pi} - \ln \alpha + (1 - \alpha) \ln x + \ln w^e - \ln \left( \frac{1}{\alpha} + \frac{1}{\tilde{\pi}} \right) - \frac{1}{2} \frac{\sigma_w^2}{(w^e)^2} \left( 1 - \alpha \right) \tilde{\pi} + \alpha$$
Then it is easily checked that
\[
W^\mu - W^\pi = \ln w^\epsilon - \frac{1}{2} \sigma_w^2 (w^\epsilon)^2 + \alpha (1 - \alpha) \frac{1}{\pi} \ln \varpi + \ln (1 - \alpha) \frac{1}{\pi} \ln w^\epsilon + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right) + \frac{1}{2} \sigma_w^2 (1 - \alpha) \frac{1}{\pi} - \alpha
\]
which reduces to
\[
W^\mu - W^\pi = \alpha \ln \varpi + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right) + \frac{\sigma_w^2}{2} \left(1 - \alpha + \frac{\alpha}{\pi}\right) \frac{1}{\pi} \quad (D.1)
\]
The third term on the r.h.s of (D.1) is positive by virtue of assumption (3.14). How about the first two terms? The first one is positive as and the second one is negative as it is known that \(\varpi > 1\). Notice that the first term increases with \(\varpi\) while the second one decreases. What is the minimum of the sum of the first two terms in terms of \(\varpi\)? If we can show that the sum reaches a minimum at \(\varpi = 1\) and is non-positive, then we would know that the sum is positive for \(\varpi > 1\): Di\(\grave{e}\)ferentiating \(\ln \varpi + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right)\) with respect to \(\varpi\) yields a turning point of \(\varpi = 1\) and \(\varpi = 1\): Checking the second derivative reveals that the sign is positive at \(\varpi = 1\) implying that \(\ln \varpi + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right)\) is a global minimum at \(\varpi = 1\) and so the minimum value of \(\ln \varpi + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right) = 0\); clearly since \(\varpi > 1\), \(\alpha \ln \varpi + \ln \left(1 - \alpha + \frac{\alpha}{\pi}\right) > 0\). \(\Box\)

**Appendix E. Proof of Proposition 5**

Differentiating the ex-ante welfare expression in (4.6) with respect to \(\mu\) yields
\[
\frac{d}{d\mu} \int W g(\alpha) d\alpha = -\frac{\alpha^e}{\mu} + \frac{1}{\mu^2} \int_{\alpha}^{\bar{\alpha}} \frac{1 - (\alpha - \alpha^e)}{1 - \alpha \left(1 - \frac{1}{\pi}\right)} \alpha g(\alpha) d\alpha.
\]
Evaluating the above expression at \(\mu = 1\) obtains
\[
\left. \frac{d}{d\mu} \int W g(\alpha) d\alpha \right|_{\mu = 1} = \alpha^e \left(-1 + \frac{1}{\alpha^e} \int_{\alpha}^{\bar{\alpha}} \alpha - \alpha \left(\alpha - \alpha^e\right) g(\alpha) d\alpha\right)
\]
\[
= - \int_{\alpha}^{\bar{\alpha}} \alpha - \alpha^e g(\alpha) d\alpha = - \left[ \int_{\alpha}^{\bar{\alpha}} \alpha^2 g(\alpha) d\alpha - \alpha^e \int_{\alpha}^{\bar{\alpha}} \alpha g(\alpha) d\alpha \right]
\]
\[
= - \left[ E\left(\alpha^2\right) - (E(\alpha))^2\right] = -\sigma_\alpha^2 < 0
\]
\(\Box\)

**Appendix F. Proof of Proposition 6**

Define \(\frac{\alpha_t}{1 - \alpha_t} \equiv \chi_t\). Then \(\chi_E \equiv E\{\chi\}\) and \(\sigma_\chi^2 \equiv E\{(\chi - \chi_E)^2\}\). Then
\[
m_t = -\frac{\alpha_t}{(1 - \alpha_t) \pi} m_{t-1} + \frac{\alpha_t}{1 - \alpha_t} w_t \Leftrightarrow m_t = \chi_t \left(w - \frac{1}{\pi} m_{t-1}\right) \quad (F.1)
\]
Taking expectations on both sides, we get
\[ m_E = \chi_E \left( w - \frac{1}{\pi} m_E \right), \text{ or } m_E = \frac{\chi_E}{1 + \frac{\chi_E}{\pi}} w \]

Squaring both sides of (F.1) and taking expectations yields
\[ \sigma_m^2 + m_E^2 = E \left( \chi^2 \right) \left[ \frac{1}{\pi^2} \sigma_m^2 + \left( w - \frac{1}{\pi} m_E \right)^2 \right] \]

which using the preceding equation can be rewritten as
\[ \frac{\sigma_m^2}{m_E^2} = \frac{\sigma^2}{1 - E(\chi^2)} \]

Using (3.15) with (B.2) in (B.1), differentiating (B.1) with respect to \( \pi \) yields
\[ \frac{dE \{ W \}}{d\pi} = -\frac{\alpha_E}{\pi} + \frac{\chi_E}{1 + \chi_E} + \frac{1}{2} \frac{\sigma^2 E(\chi^2)}{\left( 1 - E(\chi^2) \right)^2} \]

It follows that
\[ \left| \frac{dE \{ W \}}{d\pi} \right|_{\pi=1} = -\alpha_E + \frac{\chi_E}{1 + \chi_E} + \frac{1}{2} \frac{\sigma^2 E(\chi^2)}{\left( 1 - E(\chi^2) \right)^2} \]

Note that \( \chi_E = E \left\{ \frac{\alpha}{1-\alpha} \right\} = E \left\{ \frac{1}{1-\alpha} - 1 \right\} > \frac{1}{E(1-\alpha)} - 1 = \frac{\alpha_E}{1-\alpha_E} \). Hence, \( \frac{\chi_E}{1+\chi_E} > \alpha_E \). Thus,
\[ \left| \frac{dE \{ W \}}{d\pi} \right|_{\pi=1} > \frac{1}{2} \frac{\sigma^2 E(\chi^2)}{\left( 1 - E(\chi^2) \right)^2} > 0 \]

Assuming \( W(\pi) \) is concave in \( \pi \), the above implies that \( \tilde{\pi} > 1 \).\( \blacksquare \)

**Appendix G. Proof of Proposition 7**

Recall
\[
W^\pi = -\alpha_E \ln \pi + (1 - \alpha_E) \ln x + E \{ \ln (w + \tau(\pi)) \} \\
W^\mu = (1 - \alpha_E) \ln x + \ln w - \alpha_E \ln \mu - E \{(\alpha - \alpha_E) \ln \alpha\} - E \left\{ (1 - (\alpha - \alpha_E)) \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] \right\}
\]

Then
\[
W^\pi - W^\mu = -\alpha_E \ln \frac{\pi}{\mu} + E \{ \ln (w + \tau(\pi)) \} - \ln w + E \{(\alpha - \alpha_E) \ln \alpha\} \\
+ E \left\{ (1 - (\alpha - \alpha_E)) \ln \left[ 1 - \alpha \left( 1 - \frac{1}{\mu} \right) \right] \right\} \quad \text{(G.1)}
\]
Note that for small amount of uncertainty, using Taylor expansion, we can get

$$E \{ (\alpha - \alpha_E) \ln \alpha \} \simeq E \left( \frac{(\alpha - \alpha_E)^2}{\alpha_E} \right) = \frac{\sigma^2}{\alpha_E} \tag{G.2}$$

where the inequality follows from the assumption of a symmetric distribution for $\alpha$. Similarly,

$$E \{ \ln (w + \tau (\pi)) \} \simeq \ln (E \{ w + \tau (\pi) \}) - \frac{1}{2} \frac{\sigma^2_{w+\tau}}{(E \{ w + \tau \})^2} \tag{G.3}$$

For future use, let’s first compute the mean and the variance of $w + \tau (\pi) = \frac{m(\pi)}{\alpha}$ under inflation targeting. Recall that $m$’s distribution is governed by:

$$m_t = \frac{-\rho_t}{\pi} m_{t-1} + \rho_t w \tag{G.4}$$

where $\rho \equiv \frac{\alpha}{1-\alpha}$. It has been shown that

$$m_E = \frac{\rho_E}{1 + \frac{\rho_E}{\pi}} w$$

$$\sigma^2_m = \frac{\sigma^2}{1 - \frac{E(\rho^2)}{\pi^2}} \left( \frac{1}{1 + \frac{\rho_E}{\pi}} \right)^2 w^2$$

Note that (G.4) can be written as

$$m_t = w + \tau_t = w + m_t - \frac{m_{t-1}}{\pi}$$

Then,

$$E \{ w + \tau \} = E \left\{ \frac{m}{\alpha} \right\} = w + \left( 1 - \frac{1}{\pi} \right) m_E$$

$$= \frac{1 + \rho_E}{1 + \frac{\rho_E}{\pi}} w \tag{G.5}$$
We want to show that introducing an infinitesimal amount of uncertainty makes one regime superior to another. We know that when both \( w \) and \( \alpha \) are deterministic, \( \tilde{W} = \tilde{W} = 1 \). Our idea is to use Envelope theorem to see if introducing an infinitesimal uncertainty (around certainty) makes \( \tilde{W} - \tilde{W} \neq 0 \). Can we sign the welfare difference at the margin then? As an example, we want to conduct the following experiment. Suppose \( \alpha \) is stochastic, uniform and i.i.d over support \([\alpha_E - \epsilon, \alpha_E + \epsilon]\). Thus, \( E\{\alpha\} = \alpha_E \). (To ensure that the above process is stationary, we assume that and \( \frac{\alpha_E + \epsilon}{1-(\alpha_E + \epsilon)} < 1 \). If \( \epsilon = 0 \), then

\[ \tilde{W} = \tilde{W} = 0 \]

We want to evaluate

\[ \frac{d}{d\epsilon} (\tilde{W} - \tilde{W}) \bigg|_{\epsilon=0} \]

Invoking Envelope theorem implies that we only need to evaluate

\[ \frac{d}{d\epsilon} (\tilde{W} - \tilde{W}) \bigg|_{\epsilon=0} \]

to sign (G.7). Thus, substituting \( \pi = \mu = 1 \) in (G.1) yields

\[ W - W = E \{ \ln (w + \tau(\pi)) \} - \ln w + E \{ (\alpha - \alpha_E) \ln \alpha \} \]
For any distribution, fix $\frac{\alpha}{\alpha_E} = \delta$, where $\delta$ can be made arbitrarily small. Then, we can use Taylor series approximation (G.2) and (G.3) in (G.8) as higher order terms can be readily neglected. Substituting $\pi = 1$ in (G.5) and (G.6), and using (G.2) and (G.3) in (G.8), we get

$$W^\pi=1 - W^\mu=1 \Big|_{\frac{\alpha}{\alpha_E} = \delta} = \frac{\sigma^2_\alpha}{\alpha_E} - \frac{\sigma^2_\rho}{1 - E(\rho^2) 1 + \rho_E} \frac{1}{1 - E(\rho^2) 1 + \rho_E}$$

$$= \frac{\sigma^2_\alpha}{\alpha_E} - \frac{\sigma^2_\rho}{(1 + \rho_E)^2} \frac{1}{1 - \rho_E - \frac{\sigma^2_\rho}{(1 + \rho_E)^2}}$$

$$= \alpha_E \frac{\sigma^2_\alpha}{(\alpha_E)^2} - \frac{\sigma^2_\rho}{1 - \rho_E - \frac{\sigma^2_\rho}{\chi^2}}$$

where $\chi \equiv \frac{1}{1 - \alpha}$. It can be readily shown that there exists an $\hat{\alpha}$ for each $\delta$ such that $W^\pi=1 - W^\mu=1 \Big|_{\frac{\alpha}{\alpha_E} = \delta} \geq 0$ for all $\alpha_E \leq \hat{\alpha}$. Clearly, for any arbitrarily small $\delta$ as $\alpha_E \to 0.5$ (such that $\rho_{\text{max}} < 1$ is always satisfied), $W^\pi > W^\mu$.

\[
E_t \left\{ \left( \frac{w_t \gamma_t}{1 - \gamma_t (1 - \frac{1}{\mu})} \right)^{-\phi} \left( \frac{\gamma_{t+1}}{\mu \gamma_t} \right)^{-\phi} \left( \frac{1 - \gamma_t (1 - \frac{1}{\mu})}{1 - \gamma_{t+1} (1 - \frac{1}{\mu})} \right)^{-\phi} \left( \frac{w_{t+1}}{w_t} \right)^{-\phi} \right\}
\]

\[
= E_t \left\{ \left( \frac{1}{\mu} \right)^{-\phi} \left( \frac{w_{t+1} \gamma_{t+1}}{1 - \gamma_{t+1} (1 - \frac{1}{\mu})} \right)^{-\phi} \right\}
\]

\[
E_t \left\{ \left( \frac{1}{w_{t+1} \gamma_{t+1}} \right)^{\phi} \right\}, \text{ if } \mu = 1 \quad E_t \left\{ \left( \frac{1}{w_t (1 - \gamma_t)} \right)^{\phi} \right\}, \phi > 1 \quad \gamma_t > \alpha_t
\]