Equilibrium Open Interest

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Abstract

Open interest in a financial contract describes the total number that are held long at the close of the exchange; it is a stylized fact that open interest peaks at-the-money. In a two-period setup this paper investigates agents’ equilibrium holdings in a portfolio of call options and the resulting curve of open interest across strikes. We explain that agents’ preferences over skewness induce equilibrium option demand and that the shape of the open interest curve is preference-independent (in a first approximation). Our analysis indicates that (1) in incomplete markets the shape of the open interest curve is very sensitive to characteristics of the strike grid and the type of the distribution, (2) in incomplete markets a judiciously chosen strike grid and a suitable distribution could match the stylized fact, (3) in complete markets open interest curves are flat and cannot match the stylized fact.

Keywords

option demand, open interest, co-skewness, skewness preference
1 Introduction

Our paper starts pointing to a stylized fact of option markets: for a fixed maturity a plot of call option open interest across strikes peaks for the at-the-money contract\textsuperscript{1}. This paper studies the conditions under which the stylized fact is consistent with the equilibrium holdings of agents in a portfolio of options. We also provide intuition about equilibrium trade in options; little is known so far about the driving forces behind derivatives demand since the literature focuses almost exclusively on the pricing of options.

We set up a two-period general equilibrium exchange economy with two agents that can trade a stock and a portfolio of options on that stock and interpret the absolute value of either agent’s demand as open interest. We derive individual demand schedules and the market clearing allocation using the “small-noise” expansion, a technique that was introduced by Samuelson (1970) and has recently been extended by Judd and Guu (2001). An approximation technique is necessary since neither individual demand schedules nor the equilibrium allocation can be calculated in closed-form in general. The major advantage of this technique over other expansion techniques is twofold: first, the leading terms in the small-noise expansion can be calculated easily in closed-form and second these terms can be interpreted intuitively as a generalized mean-variance-skewness analysis.

Our view is that agents typically have elaborate preferences over risk, e.g. they care about asymmetric events like liquidity shocks and market crashes\textsuperscript{2}; we parametrize these as preferences over skewness. The presence of skewness preference justifies equilibrium trade in options in our economy: call options are contingent claims that allow agents to trade events that are in the upper tail of the stock distribution; thereby they can alter the skewness of their portfolio payoff.

We find that in our economy the relative size of option demand is determined (in a first approximation) by distributional characteristics of the underlying stock distribution and the

\textsuperscript{1}Open interest in a financial assets denotes the total number of that contract that are held long at the daily close of the exchange; it is quoted at the end of each day for all financial contracts traded. This shape appears for both calls and puts.

\textsuperscript{2}Others have taken similar views: For example Bates (2001) considers an economy with investors who differ in their risk aversion and where crashes can occur in the market. In this economy, the less crash-averse investors insure the more risk-averse investors.
strike grid, only; in particular agent’s risk-preferences do not play a role. We then look at the curve of open interest across strikes when stock prices are either uniformly, (truncated) normally or (truncated lognormally) distributed. We interpret the cases where only a small number of options can be traded as incomplete markets; we document that the curve is then sensitive to assumptions about the type of distribution and that for the uniform distribution the stylized fact is an outcome of our model with a judiciously chosen strike grid. We then extend our analysis to complete markets\(^3\); we find that the open interest is flat across strikes and that sizes of open interest are (in a first approximation) independent of the type of distribution.

Our paper makes a case for the presence of skewness risk and its importance on the trading and pricing of derivatives. Our analysis confirms theoretically previous empirical studies that point out the importance of skewness risk for pricing: Kraus and Litzenberger (1976) considered moments higher than variance for agent’s expected utility\(^4\) and found empirically that systematic skewness risk is priced. Recently Harvey and Siddique (2000), Dittmar (2002) and Chang, Johnson, and Schill (2002) studied extensions of the basic CAPM setup and concluded that co-skewness risk is priced in the market in addition to variance risk. We make three contributions to this strand of literature about skewness: first our approach tells specifically how the price for skewness risk depends on agents’ risk-/skew-tolerances and co-skewness risk. Second, we link skewness to the demand in options and argue that the presence of such risks could explain equilibrium option trade. Third, we provide a framework by which volume, open interest and prices can be analyzed jointly; so far the literature only looks at prices and volume; adding open interest could potentially enrich current empirical studies.

Ross (1976), Green and Jarrow (1987) and Nachman (1988) explained that options complete the market. This suggests that options that are introduced into a market are always

\(^3\)From a theoretical perspective any economy with a finite number of traded securities is incomplete; we adopt here a practical view and call a market complete when the strike grid is “fine.” This will be discussed in detail in subsections 4.5 and 5.2.

\(^4\)Kraus and Litzenberger (1976) perform a Taylor series expansion of agent’s utility function that is truncated at the third moment. The small-noise expansion provides a theoretical basis for this ad-hoc procedure. In particular Samuelson (1970) pointed out the importance of skewness risk in his small-noise expansion; yet only the extension by Judd and Guu (2001) allows to analyze this rigorously.
traded. However this is questionable for at least two reasons: First, Elul (1995) shows that the introduction of options can decrease social welfare in a two agent setup. If one agent is worse off when an option is introduced, why would he then be willing to trade options in the first place?\(^5\) Second, it is known that mean-variance preferences will not induce derivatives demand\(^6\). We overcome this criticism through a discussion of skewness risk and skewness preference; we explain option demand by the presence of skewness risk. Furthermore we point out that in equilibrium the common belief that markets are “basically” complete does not support the stylized fact about the open interest curve.

Besides the market completeness argument the literature has largely ignored the reasons to trade options\(^7\): Leland (1980), and Brennan and Solanki (1981) compared the risk-tolerance of the price-setting “representive” agent to that of an individual agent and used this to infer who is taking long and short positions. Recently Franke, Stapelton, and Subrahmanyan (1998) looked at the effect of background risk on optimal sharing rules in a complete market; they find that agents take either convex or concave sharing rules and interpret this as long/short positions in options. The contribution of our paper lies in pointing out the links between skewness risk and agent’s skewness preferences for trade in derivatives.

The remainder of the paper is organized as follows: Section 3 discusses the stylized fact about the shape across strikes. Section 4 describes our setup for the expansion; the following section derives the equilibrium allocation, relates co-skewness to demand and open interest, and explains that the shape of the open interest curve in our setup is mainly preference independent. Section 5 discuss the open interest curves that result in incomplete markets and section 6 looks at complete markets. Section 7 concludes the paper.

\(^5\)Detemple and Selden (1991) first pointed out that introducing a new contract might not lead to trade.
\(^6\)The argument is as follows: Under such preference structure two fund-separation rules hold; since options are in zero net-supply they are never part of the market portfolio and under such preferences options would therefore not be traded.

\(^7\)Market participants use changes in overall open interest across calls and puts to predict future price changes in the underlying, see Shaleen (1997). Cassano (2000), and Carr and Madan (2001) provide a theoretical rationale when agents “trade” volatility. In a market microstructure setup Easley, O’Hara, and Srinivas (1998) link option volume, prices and “information.” Buraschi and Jiltsov (2002) argue that overall open interest proxies for differences in beliefs. Our analysis does not look at incremental changes in demand over time and informational asymmetries; we are interested here in market participants option holdings at the end of a trading day that is due to risk-sharing and comparing the particular the shape on the markets with complete/incomplete market setups, various distributions and strike grid.
2 The Shape of the Open Interest Curve Across Strikes

Open interest in a call option denotes the total number of that contract that are held long at the close of the exchange; at the end of each trading day it is quoted not only for call options but for all financial contracts traded. Several plain vanilla options with various exercise prices and for various expiration months can be traded at any date. In this paper we look at the dependence of open interest on the strike price for fixed maturity and fixed underlying security; It is a stylized fact that a plot of the open interest curve for calls and puts has a peak for the at-the-money contract. In this paper we are particularly interested in the series with the shortest maturity for call options on a fixed underlying security. (This series is commonly thought to be the most actively traded one.)

An econometric analysis is beyond the scope of this paper. Only as an example we will present that feature for Microsoft and the options on it with July 2002 maturity. Expiration for that series was on Saturday July 20, 2002; we look at the last three trading days before that date\(^8\).

The structure of the market in the third week of July is the following: options on Microsoft can be traded with strike prices in $5 unit intervals between $25 and $105 with July 2002 maturity. There is also a contract traded at $47.5 for that maturity and and options with maturity at the “Saturday immediately following the third Friday” in August 2002, October 2002, January 2003 and January 2004. The closing price for Microsoft on Wednesday July 17 was $51.25, on Thursday it was $52 and on Friday it was $51.11.

Figure 1 looks in its left-hand part at the open interest and in its right-hand part at the ratio of volume to open interest on Wednesday July 17, Thursday July 18 and Friday July 19 for the July maturity\(^9\). The ratio of volume to open interest is calculated for each contract

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\(^8\) The day of July 18, 2002 was a particular one for Microsoft: after the stock market closed on that day the company released earnings figures for the fourth quarter 2002 and announced results for its fiscal year 2002. We therefore expect that on/before this date there is a particularly large disagreement about the future evolution of the stock and that this is then resolved on Friday July 19, 2002. We ignore this fact here.

\(^9\) The grid is identical for calls and puts, i.e. for every call with a maturity \(T\) and a strike \(K\) also a put with same maturity and strike can be traded. Put-call parity allows replicating a long put position in a strike \(K\) through joint long positions in the stock, the call with strike \(K\) and an investment of $\(K\) in the bond. In
as $\frac{\text{volume}}{\text{open interest}}$, if the open interest is larger than 10 and as 0 if it is smaller than 10.

On all three days figure 1 shows the stylized fact about the open interest curve; we also find that over these three days the open interest curve is very stable; even the actual numbers do not change significantly\(^\text{10}\).

It could be conjectured that the stylized fact is due to insufficient liquidity, i.e. that daily volume is not large enough to change considerably the shape of the curve. To address this issue we calculated the ratio of volume to open interest and plotted it in the right-hand part of figure 1. We find that on Thursday July 18 this ratio is close to 0.4 twice; in these two contracts we could therefore expect a significant change in the open interest curve from Wednesday to Thursday; yet this does not materialize. On Friday this ratio is even larger but even then it does not lead to a change in the shape of open interest. This is even more surprising given that the ratio for the contract with strike $50$ is 0.8, i.e. the volume on that day is 80% of the open interest in that contract: yet the end-of-day open positions (open interest) do not change significantly.

\section{The Setup}

We consider a two-period financial economy populated by two agents; they trade only today (date 0) and at date $T$. Agents can invest into a riskless bond with constant price over time\(^\text{11}\) and $N + 1$ risky securities: a stock (security 0) and $N$ call options with maturity $T$ written on that stock with strike prices $K_1, \ldots, K_N$.

We introduce a parameter $\varepsilon \geq 0$ that parametrizes a series of economies and corresponding portfolio problems. At maturity the payoffs $\Pi_0(\varepsilon)$ from the stock and $\Pi_j(\varepsilon)$ ($j = 1, \ldots, N$) for the options are given in the $\varepsilon$-economy as

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\(^{10}\)For example, open interest (volume) in the contracts with strikes $50$, $55$, $60$ was 33038, 59350, 77676 (3932, 10316, 1639) on Wednesday; it was 31233, 60140, 76486 (8246, 21345, 82) on Thursday and it was 33852, 61227, 76435 (24184, 608, 204) on Friday.

\(^{11}\)We do not analyze the consumption-savings decision here and therefore set the riskless rate equal to 0 for simplicity.
\( \Pi_0(\varepsilon) = 1 + \xi_0 \varepsilon \) and \( \Pi_j(\varepsilon) = \varepsilon (1 + \xi_0 - K_j)^+ \).

Here \( \xi_0 \) is a zero-mean random variable that is of bounded support\(^{12}\). The distribution \( \xi_0 \) is known to both agents. (Note that \( E[\Pi_0(\varepsilon)] = 1 \) for all \( \varepsilon \) since \( E[\xi_0] = 0 \).) A characteristic feature of the expansion is that \( \varepsilon = 0 \) corresponds to an economy without risk.

Our expansion for the option may initially seem odd, but it has three important features: First, we get a standard option for \( \varepsilon = 1 \). Second, note that the variance of options is \( \text{var}(\Pi_j(\varepsilon)) = \text{var}((1 + \xi_0 - K_j)^+) \varepsilon^2 \); therefore our expansion keeps the relation between stock and option variance \( \frac{\text{var}(\Pi_j(\varepsilon))}{\text{var}(\Pi_0(\varepsilon))} = \frac{\text{var}((1 + \xi_0 - K_j)^+)}{\text{var}(\xi_0)} \) constant in \( \varepsilon \). Third, the cumulative probability of exercise is unaffected by \( \varepsilon \): as we expand the distribution of stock prices we also expand strikes and therefore the relationship between distribution and strikes remains constant.

We denote the date 0 price of securities by

\[
P_j(\varepsilon) = E[\Pi_j(\varepsilon)] - \pi_j(\varepsilon) \cdot \varepsilon^2,
\]

and interpret \( \pi_j(\varepsilon) \varepsilon^2 \) as the risk-premium in the \( \varepsilon \)-economy. The variance of the stock and the option is \( \text{var}(\Pi_0(\varepsilon)) = \text{var}(\xi_0) \varepsilon^2 \) and \( \text{var}(\Pi_j(\varepsilon)) = \text{var}((1 + \xi_0 - K_j)^+) \varepsilon^2 \), respectively. When \( \pi_j(\varepsilon) \varepsilon^2 \) would be constant in \( \varepsilon \) this would model risk premia that are linear in variance; here we allow \( \pi_j(\varepsilon) \) to depend on \( \varepsilon \) to capture nonlinear dependence on the variance. Note that in the no-risk economy (\( \varepsilon = 0 \)) the returns from the bond, stock and the call options coincide\(^{13}\), i.e., \( \Pi_j(0) = P_j(0) \) for all \( j = 0, \ldots, N \).

Each agent is endowed with 1/2 units of the stock, i.e. the total supply consists of one unit of stock, which is infinitely divisible. All call options are in zero net–supply. In each \( \varepsilon \)-economy agents pursue trading strategies over time: between 0 and \( T \) agent \( i \) invests \$\( b_i(\varepsilon) \) of his initial wealth \( W_{0i}(\varepsilon) = \frac{P_0(\varepsilon)}{2} \) into the riskless security and holds \( d_{ij}(\varepsilon) \) units of risky security \( j \) (\( 0 \leq j \leq N \)), i.e. \( W_{0i}(\varepsilon) = b_i(\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon) P_j(\varepsilon) \). We re-express this as \( b_i(\varepsilon) = W_{0i}(\varepsilon) - \sum_{j=0}^{N} d_{ij}(\varepsilon) P_j(\varepsilon) \) and derive his total payoff (wealth) at date \( T \) as

\(^{12}\)This condition is necessary for the purposes of the solution technique we will use below. It also ensures that stock payoffs are non-negative for a sufficiently small \( \varepsilon \).

\(^{13}\)In particular here we force the payoff and the price of the derivative to be zero when \( \varepsilon = 0 \).
\[ W_{Ti}(\varepsilon) = W_{0i}(\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot (\Pi_j(\varepsilon) - P_j(\varepsilon)). \]

We assume he does not consume today and his date \( T \) preferences over wealth can be represented by von Neumann-Morgenstern utility functions \( E[u_i(W_{Ti}(\varepsilon))] \), where \( u_i \) is an increasing and strictly concave function. The agent then chooses the strategy that maximizes his expected utility

\[
E[u_i(W_{Ti}(\varepsilon))] = E \left[ u_i \left( W_{0i}(\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot (\Pi_j(\varepsilon) - P_j(\varepsilon)) \right) \right],
\]

over all holdings \( d_{ij}(\varepsilon) \) in the risky securities. Both agents trade competitively; we derive both agents’ demand and prices in each security by the following equilibrium concept:

**Definition 1** A financial equilibrium in the \( \varepsilon \)-economy consists of prices \( P_j(\varepsilon) \) for the available assets, and portfolio demand vectors \( (b_i(\varepsilon), d_i(\varepsilon)) \) for both agents, such that \( (b_i(\varepsilon), d_i(\varepsilon)) \) maximizes agent \( i \)'s utility, and stock and option markets clear, i.e. \( d_{10}(\varepsilon) + d_{20}(\varepsilon) = 1 \) and \( d_{1j}(\varepsilon) + d_{2j}(\varepsilon) = 0 \) \( (j = 1, \ldots, N) \).

In each \( \varepsilon \)-economy open interest in an option \( (j = 1, \ldots, N) \) corresponds to the absolute value of either agent’s demand in that option. Our setup will therefore provide us with a risk-sharing prediction of the open interest in options.

For each agent \( i = 1, 2 \) we define the \((N + 1)\)-dimensional function \( H_i(d_i(\varepsilon), \varepsilon) \) by

\[
H_{ij}(d_i(\varepsilon), \varepsilon) = \frac{1}{\varepsilon} \cdot \frac{\partial E[u_i(W_{Ti}(\varepsilon))]}{\partial d_{ij}} = E \left[ \frac{\partial u_i}{\partial W} (W_{Ti}(\varepsilon)) \cdot (\xi_j + \pi_j(\varepsilon)\varepsilon) \right].
\] (3)

The first order conditions\(^{14} \) of agent \( i \) in the \( \varepsilon \) economy are then \( H_i(d_i(\varepsilon), \varepsilon) = 0 \). To solve for individual demand and the equilibrium allocation we expand the demand vector, \( d_i(\varepsilon) \), and the vector of premia, \( \pi_j(\varepsilon)\varepsilon^2 \), into series with respect to \( \varepsilon \) around \( \varepsilon = 0 \):

\[
d_1(\varepsilon) = d_1(0) + d_1'(0)\varepsilon + \ldots, d_2(\varepsilon) = d_2(0) + d_2'(0)\varepsilon + \ldots
\]

and

\[
\pi(\varepsilon) = \pi(0) + \pi'(0)\varepsilon + \ldots,
\]

where \( \pi'(\varepsilon) = \frac{\partial \pi}{\partial \varepsilon}(\varepsilon) \), and \( d_i'(\varepsilon) = \frac{\partial d_i}{\partial \varepsilon}(\varepsilon) \). Our idea is to derive agent’s holdings through an application of the Implicit Function Theorem: taking derivatives \( H_i \) with respect to \( \varepsilon \),

\(^{14}\)

To simplify the exposition we divided by \( \varepsilon \) when defining \( H_i \) in equation (3). A repeated application of our procedure would yield the same result, see Judd and Guu (2001).
(formally) we get $0 = \frac{\partial H_i}{\partial d_i} \cdot \frac{\partial d_i}{\partial \epsilon} + \frac{\partial H_i}{\partial \epsilon}$ and we would like to derive $\frac{\partial d_i}{\partial \epsilon}$ by dividing $\frac{\partial H_i}{\partial \epsilon}$ by $\frac{\partial H_i}{\partial d_i}$. Yet, the way our expansion is set up, we have $\frac{\partial H_i}{\partial d_i}(d_i(0), \epsilon = 0) = 0$: in the $\epsilon = 0$ economy all assets coincide and therefore their demand is indeterminate. (A proof is provided in the appendix.) In the spirit of Hospital’s rule we “require” $\frac{\partial H_i}{\partial \epsilon}(d_i(0), \epsilon = 0)$ to be also equal to zero and use second order derivatives of $H_i$ to express $\frac{\partial d_i}{\partial \epsilon}$.

This approach has been introduced by Judd and Guu (2001) (see their Theorem 7); here we will apply their procedure and for completeness we quote their generalized Implicit Function Theorem as Theorem 4 in the appendix. We refer to this as the small-noise expansion technique; it is a perturbation technique that was first introduced by Samuelson (1970) (see also Merton and Samuelson (1974)). The technique used here is an extension of Samuelson’s: While Samuelson’s analysis is asymptotically valid only for the zero-order term, however, Judd and Guu (2001) developed this to a technique that is asymptotically valid for all terms in the polynomial expansion.

An alternative to the perturbation technique used here would be to expand agent’s utility function around the logarithmic one as in Kogan and Uppal (2001) or to log-linearize the budget constraint as in Campbell (1993), and Campbell and Viceira (2002). The technique we are using has the advantage of providing results that can be interpreted as a generalized mean-variance-skewness analysis, i.e. it can be interpreted in terms that are common to financial economics.

The first three derivatives of agent $i$’s utility function are denoted $u'_i = \frac{\partial u_i}{\partial W}$, $u''_i = \frac{\partial^2 u_i}{\partial W^2}$, and $u'''_i = \frac{\partial^3 u_i}{\partial W^3}$, which are all evaluated at the agent’s “safe” wealth $W_0(0) = \frac{P_0(0)}{2} = \frac{1}{2}$. Moreover, for agent $i$ we define the risk-tolerance $\tau_i$ and the skew-tolerance $\rho_i$ by

$$
\tau_i = -\frac{u'_i}{u''_i}, \quad \rho_i = \frac{\tau_i^2 u'''_i}{2 u'_i^2} = \frac{1}{2} \frac{u''_i u'''_i}{u'_i^2}.
$$

It is important to note that the risk-tolerance $\tau_i$ consists of first and second order derivatives of agents’ utility functions; it describes the marginal rate of substitution between mean and variance and is a usual term in the financial economics literature. The term skew-tolerance$^{15}$ appeared first in Judd and Guu (2001); additionally it contains a third order

$^{15}$Kimball (1990) refers to $\frac{\partial^2 \tau_i}{\partial \epsilon^2} = -\frac{u'''_i}{2u''_i}$ as prudence. We will not pursue this relation here further.
derivative of agent’s utility function. We explain below that this term describes the marginal rate of substitution between skewness and variance risk.

We also denote by $V$ the $(N+1) \times (N+1)$ dimensional covariance matrix of pure random components (of the stock and all options), by $\chi$ the $N + 1$ dimensional co-skewness vector, and by $\zeta_i$ the $N + 1$ dimensional (third order) co-moment vector, for agent $i = 1, 2,$ and securities $j, k = 0, \ldots, N$,

$$V_{jk} = E[\xi_j \xi_k], \chi_j = E[\xi_0^2 \xi_j], \zeta_{ij} = \sum_{k,l=0}^{N} d_{ik}(0)d_{il}(0)E[\xi_j \xi_k \xi_l]. \quad (5)$$

Here we denote $\xi_j = (1 + \xi_0 - K_j)^+ - E[(1 + \xi_0 - K_j)^+]$ for the options $j = 1, \ldots, N$; note that $\xi_j$ is a zero-mean random variable.

Co-skewness is a known term in the financial economics literature, see Huang and Litzenberger (1988). It has recently found renewed interest in empirical studies by Harvey and Siddique (2000), Dittmar (2002), and Chang, Johnson, and Schill (2002) who look at extensions of the CAPM framework to determine if the squared return on the market portfolio is priced\(^\dagger\). We argue that a third moment leads to option demand and is priced.

For technical reasons we assume throughout that the variance-covariance matrix $V$ of the $N + 1$ securities is not singular, i.e. that $V$ is invertible. (This assumption is weaker than the one that a security could not be redundant. Redundant securities need to be excluded, since its demand would be indeterminate.) It is important to note that throughout we do not require completeness of markets.

4 The Small-Noise Expansion

4.1 How to Invest into a Stock and a Portfolio of Calls

The appendix proves:

**Theorem 1** Agent $i$’s demand vector $d_i(\epsilon) = d_i(0) + d_i'(0)\epsilon + \ldots$ is described through

\(^\dagger\)Co-skewness is defined in the literature for a security as the covariance of the squared market risk with the idiosyncratic risk of that security. In our setup this definition of co-skewness and ours coincide. To focus on the question of option demand we will not pursue the relationship between coskewness as defined here and the one defined typically in the literature.
\[ d_i(0) = \tau_i \cdot V^{-1} \cdot \pi(0) \quad \text{and} \quad d_i'(0) = \tau_i \cdot V^{-1} \cdot \left( \pi'(0) + \frac{\rho_{ij}}{\tau_i} \cdot \zeta_i \right). \quad (6) \]

We interpret the demand terms in the asymptotic expansion \( d_i(\varepsilon) = d_i(0) + d_i'(0) \varepsilon + \mathcal{O}(\varepsilon^2) \) as follows: the \( d_i(0) \) term consists of the premium, \( \pi(0) \), standardized by the variance, \( V \), of the stock risk; the agent’s risk-tolerance \( \tau_i \) describes the marginal rate of substitution between variance risk and the premium gained for taking that risk. This result is common to economies in which an agent’s preferences can be summarized in terms of means and variances, see, Huang and Litzenberger (1988).

The \( d_i'(0) \) term takes into account all third-order cross moments via \( \zeta_i \). We rewrite demand using equation (6) as

\[ d_i(\varepsilon) = \tau_i \cdot V^{-1} \cdot (\pi(0) + \pi'(0) \varepsilon) + \frac{\rho_{ij}}{\tau_i} V^{-1} \zeta_i \varepsilon + \mathcal{O}(\varepsilon^2) \]

and note that \( \zeta_i = E[\eta_i^2 \xi_i] = \text{Cov}(\eta_i^2, \xi_i) \) corresponds to a linear regression of a security’s risk \( \xi_i \) on agent \( i \)'s (zero-order) wealth risk \( \eta_i = \sum_{j=0}^{N} d_{ij}(0) \xi_j \). We interpret this as a correction to the agents’ risk coming from his zero-order position in assets (the zero order wealth risk) \( \eta_i = \sum_{j=0}^{N} d_{ij}(0) \xi_j \). (Note that in the zero-order term the agent cares how his squared wealth risk covaries with security \( i \), i.e. the \( \text{Cov}(\eta, \xi_i) \) term determines the position in security \( j \).) Equation (6) explains that the term \( \frac{\rho_{ij}}{\tau_i} = \frac{\sigma_{ij}}{2\sigma_i} \) characterizes the marginal rate of substitution between (third moment) risk \( \zeta_i \) and the compensation (part of the premium) received for taking that kind or risk.

4.2 The Equilibrium Allocation

In equilibrium agents have to agree on a market clearing price in all \( \varepsilon \)-economies, i.e. \( d_{10}(\varepsilon) + d_{20}(\varepsilon) = 1 \) for the stock and \( d_{1j}(\varepsilon) + d_{2j}(\varepsilon) = 0 \) for the options \( (j = 1, \ldots, N) \). For the zero and first order expansion terms of \( d_i(\varepsilon) \) in our series expansion this translates into

\[ d_{10}(0) + d_{20}(0) = 1, \quad d_{1j}(0) + d_{2j}(0) = 0 \]

for options \( j = 1, \ldots, N \). The appendix proves:

**Theorem 2** The premium of asset \( j = 0, \ldots, N \) is:

\[ \pi_j(\varepsilon) \varepsilon^2 = \frac{1}{\tau_1 + \tau_2} V_0 \varepsilon^2 - \frac{\rho_{1j} \tau_1 + \rho_{2j} \tau_2}{(\tau_1 + \tau_2)^3} \chi_j \varepsilon^3 + \mathcal{O}(\varepsilon^4) \quad (7) \]

and the first agent’s demand is:
\[d_{10}(\varepsilon) = \frac{\tau_1}{\tau_1 + \tau_2} - \tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} \cdot (V^{-1} \chi)_0 \varepsilon + \mathcal{O}(\varepsilon^2) \] for the stock, and

\[d_{1j}(\varepsilon) = -\tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} \cdot (V^{-1} \chi)_j \varepsilon + \mathcal{O}(\varepsilon^2) \] for options \((j = 1, \ldots, N)\).

Note that \(d_{1j}(0) = 0\) \((j = 1, \ldots, N)\) and that agent’s zero order wealth risk therefore simplifies to \(\eta_i = d_{i0}(0) \xi_0\). This implies that \(\zeta_i = Cov(\eta_i^2; \xi_i) = d_{i0}^2(0) \chi_i\). (Here the stock plays a distinct role since it is the only non-financial asset, i.e. it is the only asset in positive aggregate supply. This distinct role is responsible for the simplification of \(\zeta_i\).) In our asymptotic analysis we focus on the first term that is not equal to zero; since \(d_{1j}(0) = 0\) for options \(j = 1, \ldots, N\), zero and first order terms need to be analyzed throughout.

It is important to note that according to theorem 2 we have separation of tastes (risk-preferences) and distributional characteristics in \(d_i(0)\) and \(d_i'(0)\). In particular, in a first approximation, tastes (risk-preferences) enter only as a multiplicative factor for the demand in all contracts. This will play a crucial role in our analysis of the shape of the open interest curve across strikes.

Based on theorem 2 we find that prices are equal to

\[P_j(\varepsilon) = E[\Pi_j(\varepsilon)] - \pi_j(0) \varepsilon^2 - \pi_j'(0) \varepsilon^3 + \mathcal{O}(\varepsilon^4)\]

\[= E[\Pi_j(\varepsilon)] - \frac{1}{\tau_1 + \tau_2} V_{0j} \varepsilon^2 + \frac{\rho_1 \tau_1 + \rho_2 \tau_2}{(\tau_1 + \tau_2)^3} \chi_j \varepsilon^3 + \mathcal{O}(\varepsilon^4).\]

Here the \(\frac{1}{\tau_1 + \tau_2} V_{0j} \varepsilon^2\) term is exactly the pricing term that would result in a CAPM world. In addition we have the \(\pi'(0) \varepsilon^2 = \frac{\rho_1 \tau_1 + \rho_2 \tau_2}{(\tau_1 + \tau_2)^3} \chi_j \varepsilon^2\) term which depends on preferences \(\tau_1, \tau_2, \rho_1, \rho_2\) and co-skewness \(\chi_j\). Therefore, in our models co-skewness risk is priced. This confirms from a theoretical perspective the empirical evidence in Harvey and Siddique (2000), Dittmar (2002) and Chang, Johnson, and Schill (2002)\(^{17}\).

### 4.3 Why Agents Trade Options

Theorem 2 states that for options the zero order equilibrium demand term is zero. The zero-order term corresponds to a mean-variance framework and that result is in line with common

---

\(^{17}\)We will not pursue this relation further, since our focus is on option demand and open interest; we elaborate in the next subsection that co-skewness induces option demand.
knowledge about mean-variance frameworks: it is well known that in such frameworks two-
fund separation holds, i.e. agents hold the bond and the market portfolio. Options are
not contained in the “market” portfolio, since they are in zero net-supply. Therefore, in
our economy the “market” portfolio consists of one unit of the stock only and this part of
demand does not induce agents to trade options.

We can write \( V^{-1} \cdot \chi = V^{-1} \cdot E[\xi_0^2 \xi_1] = E[\xi_0^2 \cdot (V^{-1} \xi)] = \text{cov}(\xi_0^2, V^{-1} \xi) \) and then option
demand as

\[
d_1(\varepsilon) = -\tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} E[\xi_0^2 \cdot (V^{-1} \xi)] \varepsilon + O(\varepsilon^2)
\] (8)

In this equation the term \( V^{-1} \xi \) describes the orthogonal decomposition (in variance-
covariance terms) of risks of all contracts; this decomposition characterizes the contributions
each contract makes for hedging purposes. Agents care about the covariance with wealth
risk. We explained above that with respect to the first order demand term the agent cares
about hedging his zero-order wealth risk: since zero order wealth risk is given through the
stock risk he looks for that purpose at the covariance of these components with squared
stock risk \( \xi_0 \) to determine the demand in each contract. Therefore these terms are driven by
co-skewness risk \( \chi \), i.e. the presence of third-order moment risk.

Demand is scaled by taste parameters \(-\tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3}\). Our view is that agents typically
have elaborate preferences over risk, e.g. they care about asymmetric events like liquidity
shocks and market crashes; we parametrize these as preferences over skewness and assume
that neither \( \rho_1 \) nor \( \rho_2 \) are equal to zero.

Note that when both agents have identical skew-tolerance (\( \rho_1 = \rho_2 \)), then no option
demand arises in a first approximation in \( \varepsilon \), i.e. \( d_j'(0) = 0 \) for \( j = 1, \ldots, N \). (This captures,
e.g. the case of identical agents in the economy.) However equilibrium demand in options
will arise from skew-tolerance and co-skewness as long as \( \rho_1 \neq \rho_2 \). In general we expect
\( \rho_1 \neq \rho_2 \) and find that this justifies equilibrium trade in options in our setup. The intuitive
basis of our argument is that call options are contracts that allow trading of events that are
in the upper tail of the stock distribution; trading options allows agents to alter the skewness
of their portfolio payoff.
4.4 What Determines The Shape of The Open Interest Curve

Theorem 2 implies that the open interest is

\[ \tau_1 \tau_2 \frac{|\rho_1 - \rho_2|}{(\tau_1 + \tau_2)^3} \cdot |V^{-1}\chi| \varepsilon + \mathcal{O}(\varepsilon^2) = \text{tastes} \cdot |V^{-1}\chi| \varepsilon + \mathcal{O}(\varepsilon^2), \]

where “tastes” are described as \( \tau_1 \tau_2 \frac{|\rho_1 - \rho_2|}{(\tau_1 + \tau_2)^3} \). Throughout we assume that “taste” parameter being different than 0 and that exists at least one option \( k \) with non-vanishing demand, i.e. with \((V^{-1}\chi)_k \neq 0\). Since open interest separates into tastes and preferences the relative size (the shape of the open interest curve) is determined through co-variance and skewness terms by \( V^{-1}\chi \), i.e. for all economies with \( \varepsilon > 0 \) we have\(^{18}\):

\[ \frac{d_{1j}(\varepsilon)}{d_{1k}(\varepsilon)} = \frac{(V^{-1}\chi)_j}{(V^{-1}\chi)_k} + \mathcal{O}(\varepsilon), \quad \frac{|d_{1j}(\varepsilon)|}{|d_{1k}(\varepsilon)|} = \frac{|(V^{-1}\chi)_j|}{|(V^{-1}\chi)_k|} + \mathcal{O}(\varepsilon). \]

Therefore, in a first approximation, the relative size of demand and open interest across strikes neither depends on the particular \( \varepsilon \) nor on agent’s tastes; it can be described entirely through \( V^{-1}\chi \). In the remainder of this paper we study further what shapes of the open interest curve result for various distributions of the underlying security where the support is continuous and \( \xi_0 \) is distributed over the interval \((-1, 1)\), i.e. for distributions where \( 1 + \xi_0 \varepsilon \) is distributed over the interval \((1 - \varepsilon, 1 + \varepsilon)\).

4.5 The Strike Grid and Complete Markets

Agents trade options to maximize their expected utility derived from the portfolio at maturity. Ross (1976), Green and Jarrow (1987) and Nachman (1988) point out that the introduction of additional options into a market makes the market “more complete.” The benchmark is here a “complete” set of Arrow-Debreu contingencies, a so-called “complete” market.

Most securities have a very large number of call options outstanding and it seems natural to conjecture that the market is complete; it is commonly believed that most markets are “complete.” However, throughout we only look at distributions that have continuous support

\(^{18}\text{In our two agent setup open interest is the absolute value of either agent’s demand, } |d_{1j}(\varepsilon)| = |d_{2j}(\varepsilon)|.\)
and according to the (theoretical) definition used in the literature a complete market can only be attained by trading an infinitely countable number of options. In practice, however, there are always only a finite number of options traded and it might be more interesting to ask how close the optimal payoff comes to the actual one. We note the following: It is known that the butterfly spread based on options with strikes $K < K' < K''$ converges to Arrow-Debreu contingencies when the difference $K'' - K$ shrinks to zero. The distance between traded strikes is therefore a “natural” measure of completeness. (We apply the interpretation that the stock is an option with strike 0.)

Based on this we call a market complete when the (maximal) difference between adjacent strikes is “small;” otherwise we call the market incomplete. (We include here the difference to the option with strike 0 and interpret that one as the stock.) In the following section we compare the predictions for agents’ demand in complete and incomplete markets with the stylized fact about the shape of the open interest curve.

Throughout our analysis we only look at strike grids that are equidistant and symmetric around the center. We vary the number $N$ of traded options and the minimal strike $K_1$, thereby controlling how dense the strike grid is and how far it spreads out: Given $K_1$ and $N$ we take the difference between adjacent traded strikes as $\delta = \frac{1-K_1}{N-1}$. For each of the distributions we allow only strikes $K_i$ between 0 and 2, since strikes outside the support of the distributions would violate the non-redundancy condition; this restricts the interval within which $K_1$ varies to $[0,1]$.

The strike grid we are taking here is symmetric around the center $\frac{1}{2}$. Whenever the distribution $\xi_0$ is symmetric around 0 and the strike grid is symmetric around 1 the parameter $\chi$ as a function of the strike will be symmetric around 1 and therefore the (first order) demand for options is symmetric around the strike 1, i.e. $d'_{i,j}(0) = d'_{i,N-j}(0)$ for $j = 1, \ldots, N$.

However, when the distribution is not symmetric we expect the open interest curve to be

\[ d'_{i,j}(0) = d'_{i,N-j}(0) \quad \text{for} \quad j = 1, \ldots, N. \]

19 This butterfly spread consists of holding $\frac{1}{\pi_{-K}}$ units of the option with strike $K$, $-\left(\frac{1}{\pi_{-K}} - \frac{1}{\pi_{-K'}}\right)$ units of the option with strike $K'$ and $\frac{1}{\pi_{-K''}}$ units of the option with strike $K''$. It pays nothing outside the interval $[K,K'']$ and is linear continuous on the intervals $[K,K']$ and $[K',K'']$.

20 When $N$ is even, this defines the strike grid $K_{N/2-j} = 1 - \frac{2N}{2} - 1 - \frac{1}{2}\delta$ for $j = 1, \ldots, N/2$ and $K_{N/2+j} = 1 - \frac{2N}{2} - 1 - \frac{1}{2}\delta$; when $N$ is odd this defines the strike grid $K_{(N-1)/2-j} = 1 - j\delta$ and $K_{(N-1)/2+j} = 1 + j\delta$ for $j = 0, \ldots, (N-1)/2$. 

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skewed. In this paper we will not pursue how the structure of the distribution influences the fine structure of the open interest curve.

When the number of traded options is “small,” then the difference between adjacent strikes of traded options can not made “small” and for this reason (as explained in subsection 4.4) this will be studied in the following section as an example of an incomplete market. In section 6, however, we take a large number of traded options, $K_1$ small, and interpret this as examples of complete markets. We define the $N$-dimensional vector $\theta_j$ for $j = 1, \ldots, N$ by

$$
\theta_j = \frac{d'_{ij}(0)}{\delta \cdot d'_{i0}(0)}, \quad \text{so that} \quad \frac{|d_{ij}(\varepsilon)|}{|d'_{i0}(0)|} = \frac{|(V^{-1} \chi)_j|}{\delta \cdot |(V^{-1} \chi)_0|} + O(\varepsilon) = |\theta_j| + O(\varepsilon). \quad (10)
$$

Throughout plots will look at $\theta$. This is a standardization with respect to the first order demand term in the stock $d'_{i0}(0)$ and the distance $\delta = 2^{\frac{1-K_1}{N-1}}$ between traded strikes.

5 Equilibrium Open Interest Across Strikes In Incomplete Markets

Our purpose in this section is to explain that the shape of the open interest curve is sensitive to the number of traded options and the underlying type of distribution and that a peak could result under suitable conditions. In the first subsection we make the assumption that the underlying distribution is uniform and calculate all terms in closed-form; this allows us to look in more detail at the driving forces behind the shape of the open interest curves. In general, however, variance/skewness terms cannot be calculated in closed form and therefore the second subsection compares graphically the resulting open interest curves for various distributions.

5.1 Theoretical Analysis of The Open Interest Curve For The Uniform Distribution

In this subsection we assume that $\xi_0$ is uniformly distributed over the interval $(-1, 1)$, i.e. $1 + \xi_0 \varepsilon$ is uniformly distributed over the interval $(1 - \varepsilon, 1 + \varepsilon)$. The density is

$$
1_{(-1,1)}(x) = \begin{cases} 
1 & \text{if } -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}.
$$
Straightforward calculations reveal that for this distribution the mean is $E[1 + \xi_0] = 1$, and that the variance $Var(1+\xi_0) = \frac{1}{3}$, $E[(1+\xi_0-K_j)^+] = \frac{(K_j-2)^2}{4}$, $E[(1+\xi_0)\cdot(1+\xi_0-K_j)] = \frac{(K_j-2)^2(K_j-4)}{12}$, and $E[((1+\xi_0-K_j)^+)^2] = -\frac{(K_j-2)^3}{6}$ and for $K_j < K_k$ that $E[(1+\xi_0-K_j)^+\cdot(1+\xi_0-K_k)^+] = -\frac{(K_j-2)^2(3K_j-K_k-4)}{12}$. Also we calculate $E[(1+\xi_0)^2\cdot(1+\xi_0-K_j)^+] = \frac{(K_j^2+2)(-2+K_j)^2}{24}$.

This implies:

**Lemma 3** Co-variance terms are $V_{00} = \frac{1}{3}$ for the stock, and for options $j, k = 1, \ldots, N$: $V_{0j} = \frac{(K_j+1)(-2+K_j)^2}{12}$, $V_{jj} = \frac{3K_j+2(-2+K_j)^3}{48}$, and $V_{jk} = \frac{-3K_j+2(-4K_k-4+3K_j^2)}{48}$. Skewness is $\chi_0 = 0$, $\chi_j = \frac{K_j^2(-2+K_j)^2}{24}$.

We will now discuss two market setups: one in which 3 options are traded with one in which 4 options are traded. When 3 options are traded we get:

$$\theta = \left(\begin{array}{c}
\frac{1}{(K_1-1)^2(2K_1^2-5K_1-1)} \\
2 \frac{(2K_1^2-7K_1+4)K_1}{(K_1-1)^2(2K_1^2-5K_1-1)} \\
\frac{(K_1-1)^2(2K_1^2-5K_1-1)}{(K_1-1)^2(2K_1^2-5K_1-1)}
\end{array}\right), \quad (11)$$

where $\theta$ is as defined in equation (10). (Note that this standardized demand is symmetric, since the uniform distribution is symmetric.) A peak in the open interest curve is characterized (in a first order approximation in $\varepsilon$) by $|\theta_1| < |\theta_2|$. A straightforward analysis of maxima based on equation (11) proves that up to higher order terms in $\varepsilon$, the open interest peaks at the center strike, i.e. $|\theta_1| < |\theta_2|$ or equivalently $|d_1'(0)| < d_2'(0)|$, if and only if either $0.1771 < K_1 < 0.5$ or $0.8696 < \delta < 1$. Therefore we need to distinguish between four subintervals of the interval $[0,1]$ with cutoff points $0 < 0.1771 < 0.5 < 0.8696 < 1$; when $K_1$ varies within two out of these four subintervals our setup supports the stylized fact that there is a peak for the at-the-money option.

Note that for $K_1 \to 1$ all strikes “collapse” to a single one; then all options become more and more “similar” and agents try to “leverage” their position in these “close” securities by holding a long position in one of them and a short in the other, i.e. $|d_{11}'(0)|, |d_{12}'(0)|, |d_{13}'(0)| \to \infty$ and also $|\theta_1|, |\theta_2|, |\theta_3| \to \infty$. However, we are interested in checking for the presence of a peak and this is the case if $|\theta_1| < |\theta_2|$; we find for $K_1 \to 1$ that $|\frac{\theta_2}{\theta_1}| \to 2$, i.e. a peak.

Similarly to the case $K_1 \to 1$ we get for $K_1 \to 0$ that the first option becomes more and more similar to the stock and the agent tries to leverage his position in these “close”
securities by holding a long position in one of them and a short in the other; therefore \(|d_{11}'(0)|, |d_{12}'(0)|, |d_{13}'(0)| \to \infty\) and also \(|\theta_1|, |\theta_2|, |\theta_3| \to \infty\). For the presence of a peak we need to compare whether \(|\theta_1| < |\theta_2|\); we find for \(K_1 \to 0\) that \(\frac{|\theta_2|}{|\theta_1|} \to 0\), i.e. there is no a peak, but a dip.

We are mostly interested in the behavior where \(K_1\) is neither close to 0 nor close to 1. When we look at the ratio \(\frac{|\theta_2|}{|\theta_1|}\) as a function of \(K_1\) then we find that on the interval \([0, 0.5]\) this ratio is maximal at \(K_1 = 1/3\). Then the ratio is equal to 1.2593, i.e. we find a slightly pronounced peak.

In the second setup we assume that 4 options are traded and get:

\[
\theta = \begin{pmatrix}
-3 \frac{(K_1^2 - 2K_1 - 2)(2 + K_1)^2}{4(1+K_1)^2(-4 - 60K_1 + 15K_1^2 + 4K_1^3)} \\
9 \frac{(3K_1^2 + 8K_1^2 - 52K_1 + 32)K_1}{(3K_1^2 + 8K_1^2 - 52K_1 + 32)K_1} \\
9 \frac{4(-1 + K_1)^2(-4 - 60K_1 + 15K_1^2 + 4K_1^3)}{(K_1^2 - 2K_1 - 2)(2 + K_1)^2} \\
-3 \frac{4(-1 + K_1)^2(-4 - 60K_1 + 15K_1^2 + 4K_1^3)}{(K_1^2 - 2K_1 - 2)(2 + K_1)^2}
\end{pmatrix} \tag{12}
\]

A straightforward analysis of maxima based on equation (12) yields that up to higher order terms in \(\varepsilon\), the open interest curve peaks at the center strike, i.e. \(|d_1'(0)| < |d_2'(0)|\) if and only if \(0.1372 < K_1 < 0.4\). Therefore we need to distinguish between three subintervals with cutoff points \(0 < 0.1372 < 0.4 < K_1\); when \(K_1\) varies within one out of these three subintervals our setup supports the stylized fact that there is a peak for the at-the-money option. As in the previous case where 3 options are traded problems arise when \(K_1 \to 0\) or \(K_1 \to 1\).

5.2 Graphical Analysis of The Open Interest Curve For The Uniform, Normal and Lognormal Distribution

In the previous section we analyzed the open interest curve for the uniform distribution, only. Here we analyze plots of the open interest curve for the uniform, the truncated normal and the truncated lognormal distribution. The latter two distributions for \(\xi_0\) are defined as follows:

1. Normal truncated to the interval \((-1, 1)\): We take \(\xi_0 \sim \mathcal{N}(0, 1/3)\) conditional on \(\xi_0\) in \((-1, 1)\); the density is
\[ f(x) = 1_{(-1,1)}(x) \frac{1}{\sqrt{2 \cdot \pi \cdot 1/3}} \exp \left( -\frac{x^2}{2 \cdot 1/3} \right). \]

We find that the mean is 0 and the variance is 0.2215. (Note that the variance is not equal to 1/3 since we truncate the distribution.)

2. (Transformed) Lognormal truncated to the interval \([-1, 1] \): \( \xi_0 \sim \exp(X) - 1 \) where \( X \sim \mathcal{N}(0.036, 1/3) \), conditional on \( \xi_0 \) in \([-1, 1] \); the density is

\[ f(x) = 1_{(-1,1)}(x) \frac{1}{(1 + x)\sqrt{2 \cdot \pi \cdot 1/3}} \exp \left( -\frac{(\ln(1 + x))^2}{2 \cdot 1/3} \right). \]

We find that the mean is 0 and the variance is 0.1840. (The variance here is not equal to 1/3 since we truncate the distribution.)

[Figure 2 about here.]

We illustrate in figure 2 the open interest curve when \( K_1 = 1/3 \): they differ in the type of distribution (columns one to three) and in the number of traded options (rows one to two). For both the uniform and the normal distribution we find that the open interest curve has a peak for the at-the-money option under the parameters chosen. This peak is more pronounced for the uniform distribution than for the normal distribution.

The truncated lognormal is the only distribution of the three that we look at that is not symmetric. This leads to a skewness vector that is not symmetric around the strike 1 and therefore demand/open interest is not symmetric around the center option (with strike 1). We see this clearly in figure 2. Besides, in figure 2 it seems that with the lognormal distribution the overall tendency is that open interest in incomplete markets decreases the higher the strike price of the option.

Overall it seems that in incomplete markets the open interest curve is very sensitive to both the number of options traded and to the type of distribution. We would like to stress that the uniform distribution case matches partly the stylized fact. Further analysis of the link between distributions and the open interest curve could lead that future empirical studies might be able to extract information from open interest.
6 The Open Interest Curve In Complete Markets

In this section we look at complete markets as we defined them in subsection 4.5, i.e. we take the minimal strike $K_1$ to be "small" so that the strike grid spreads out far and we take the number $N$ of options that are traded to be large, so that the distance between traded strikes is "small."

As in the previous section we will first perform an analysis for the uniform distribution: in subsection 6.1 we look at 10 traded options and vary the size of the minimal strike $K_1$, thereby controlling how far the strike grid spreads out. We will then compare graphically in subsection 6.2 the curves for uniform, normal and lognormal by varying $N$.

6.1 For The Uniform Distribution

We will look in this subsection at the uniform distribution and discuss a case where ten options are traded. Since demand is symmetric around the center strike we have $d'_{i6}(0) = d'_{i5}(0)$, $d'_{i7}(0) = d'_{i4}(0)$, $d'_{i8}(0) = d'_{i3}(0)$, $d'_{i9}(0) = d'_{i2}(0)$, $d'_{i10}(0) = d'_{i1}(0)$. We calculate (using the symbolic algebra feature of MATLAB) that $d'_{i0}(0) = \frac{14648K_1^3 - 25233K_1^2 - 10812K_1 - 68}{81(197K_1 + 68)K_1}$ and

$$\theta = \begin{pmatrix} \frac{(116K_1^2 + 119K_1 + 17)(2 + 7K_1)}{2(14648K_1^3 - 25233K_1^2 - 10812K_1 - 68)(K_1 - 1)^2} \\ \frac{3891K_1^3 + 3176K_1^2 - 772K_1 - 544}{81} \\ \frac{14648K_1^3 - 25233K_1^2 - 10812K_1 - 68}{81} \\ \frac{3039K_1^3 - 2456K_1^2 + 412K_1 + 544}{K_1} \\ \frac{1191K_1^3 - 2648K_1^2 + 508K_1 + 544}{K_1} \\ \frac{1653K_1^3 - 2600K_1^2 + 484K_1 + 544}{K_1} \\ \frac{1191K_1^3 - 2648K_1^2 + 508K_1 + 544}{K_1} \\ \frac{1653K_1^3 - 2600K_1^2 + 484K_1 + 544}{K_1} \\ \frac{1191K_1^3 - 2648K_1^2 + 508K_1 + 544}{K_1} \\ \frac{1653K_1^3 - 2600K_1^2 + 484K_1 + 544}{K_1} \end{pmatrix},$$

where $\theta$ is as defined in equation (10). On the interval $[0,1]$ we define the function

$$\eta(K_1) = \frac{K_1^2(7K_1^2 + 2)(11K_1^2 - 2)}{(K_1 - 1)^2 \cdot (14648K_1^3 - 25233K_1^2 - 10812K_1 - 68)}.$$

It is a straightforward linear manipulation to check that for the inner options the (first order) differences in demand are described through the function $\eta$ as

$$\theta_2 - \theta_5 = -1458 \cdot \eta(K_1), \theta_3 - \theta_5 = \frac{729}{2} \cdot \eta(K_1), \theta_4 - \theta_5 = -\frac{243}{2} \eta(K_1).$$
For the options with minimal and maximal strike \((K_1 \text{ and } K_{10})\) the difference in demand to the center options is \(\theta_1 - \theta_5 = -\frac{9}{4} \cdot \frac{(11K_1-2)(319K_1^2-3720K_1^2-1122K_1+68)}{(K_1-1)^2(14648K_1^2-25733K_1^2-10812K_1-68)}\) and this can not be re-expressed in terms of \(\eta\); we will not investigate these further here and focus instead on the center behavior, i.e. for options \(i = 2, \ldots, 9\).

The function \(\eta\) has the following characteristics: \(\eta(0) = 0\); it is monotonically increasing on the interval 0 to 0.0111, attains a maximum at \(\delta = 0.1111\) (with \(\eta(0.1111) = -2.1629 \cdot 10^{-5}\)), is decreasing on the interval 0.1111 to 1 and has an asymptotic at 1, i.e. \(\eta(1) = +\infty\).

We also find that the function \(\eta\) has a single zero at \(K_1 = 2/11\); we also calculate for that \(K_1 = 2/11\) that \(\theta_1 - \theta_5 = 0\) and that \(\theta_5 = -1.1\). Herefore (up to terms in \(\varepsilon\) of order higher than 2) all \(d''_j(0)\) (for \(j = 1, \ldots, 10\)) are equal for that value of \(K_1\), i.e. the open interest curve is flat.

We refer to the case where \(K_1\) varies between 0 and \(2/11 \approx 0.1818\) as the one where the market is “complete.” We find that the standardized open interest is then close to 1.1 and does not change (up to 2 digits). Therefore in markets the open interest curve would appear as flat. We will illustrate this in the following subsection, see the upper left-hand plot (first column, first row) in figure 3 below. (In that figure rows one to four correspond to the cases where 10, 15, 20, 25 options can be traded.)

6.2 Differences and Similarities of The Open Interest Curve For Various Distributions

In this subsection we analyze the same distributions we looked at in subsection 5.2 and plot the standardized open interest curves for large numbers of traded options by varying the minimal strike \(K_1\). We will calculate approximations of the co-variance and skewness terms through a numerical integration scheme with 20000 nodes on the interval \((-1, 1)\), i.e. the step size is 0.0001. Therefore we expect our results to be accurate at four digits.

[Figure 3 about here.]

Figure 3 takes \(K_1 = 0.2\) and plots the resulting standardized open interest curves (the vector \(\theta\)) when \(N = 10, 15, 20, 25\) options can be traded (rows 1 to 4) and for the uniform, truncated normal and truncated lognormal distributions (columns 1 to 3).
The plot in the first row, first column matches the case we analyzed in the previous subsection when \( K_1 = 0.2 \). We see here graphically what we pointed out there: the options with outer strikes behave differently than the ones with center strikes and in the center the standardized open interest is flat.

Figure 3 suggests that the shape of the open interest curve is fairly independent of the type of distributions when \( N \) is “large” (greater than 10 here). When we ignore the open interest for the four outer strikes (the two with the smallest strikes and the two with the largest strikes) then the open interest curve seems to be flat; the more options are traded the smaller the differences become.

Figure where \( K_1 \) is made smaller behave the same way than those of figure 3; they would correspond even more to “complete” markets. Therefore we conclude that in complete markets the standardized open interest curve almost does not change, that it appears in plots as being flat and that actual size seem to be distribution independent.

7 Conclusion

This paper derived the leading terms in an asymptotic expansion that describes agents’ individual demand schedules and the equilibrium allocation. We related them to preferences over mean, variance and co-skewness terms, highlighted that option demand is driven by co-skewness of the option with the underlying security. We discussed that, in a first approximation, the shape of the open interest curve across strikes in our model is independent of tastes and the expansion parameter; it only depends on distributional characteristics (variance and co-skewness). We explained that the uniform distribution could produce the stylized fact with a judiciously chosen strike grid in incomplete markets. Simulations indicate that in a first approximation the open interest curve is “flat” and sizes are distribution independent.
Appendix

A  Perturbation Analysis

We prove in this appendix theorem 1 for any agent $i = 1, 2$. We denote $\eta_i = \sum_{j=0}^{N} d_{ij}(0)\xi_j$, and find that $\eta_i = \frac{\partial W_{Ti}(0)}{\partial \varepsilon}$ and $\sum_{j=0}^{N} \frac{\partial^2 W_{Ti}(0)}{\partial \varepsilon^2} = \pi_0(0) + \sum_{j=0}^{N} d_{ij}(0)\xi_j$, since $W_0(\varepsilon) = \frac{1}{2}P_0(\varepsilon) = \frac{1}{2}(1 - \pi_0(0)\varepsilon^2)$ and therefore the wealth dynamics is

$$W_{Ti}(\varepsilon) = \frac{1}{2}(1 - \pi_0(0)\varepsilon^2) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot (\xi_j \varepsilon + \pi_j(\varepsilon) \cdot \varepsilon^2)$$

$$\frac{\partial W_{Ti}}{\partial \varepsilon} = \frac{1}{2}(\pi_0'(0)\varepsilon^2 + \pi_0(0)2\varepsilon) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot (\xi_j + \pi_j'(\varepsilon) \cdot \varepsilon^2 + \pi_j(\varepsilon)2\varepsilon)$$

$$\frac{\partial^2 W_{Ti}}{\partial \varepsilon^2} = \frac{1}{2}(\pi_0''(0)\varepsilon^2 + \pi_0'(0)4\varepsilon + \pi_0(0)2) + \sum_{j=0}^{N} d_{ij}(\varepsilon) \cdot (\pi_j''(\varepsilon)\varepsilon^2 + \pi_j'(\varepsilon)4\varepsilon + \pi_j(\varepsilon)2).$$

Note that $E[\eta_i] = 0$ and $\zeta_{ij} = E\left[(\sum_{k=0}^{N} d_{ik}(0)\xi_k)^2 \cdot \xi_j\right] = E[\eta_i^2 \xi_j]$. 

**Theorem 4** (Theorem 7 in Judd and Guu (2001)) Suppose $H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is analytic, and $H(x, 0) = 0$ for all $x \in \mathbb{R}^n$. Furthermore, suppose that for some $(x_0, 0)$

$$\frac{\partial H}{\partial x}(x_0, 0) = 0_{n \times n}, \quad \frac{\partial H}{\partial \varepsilon}(x_0, 0) = 0_n, \quad \text{and det} \left( \frac{\partial^2 H}{\partial x \partial \varepsilon}(x_0, 0) \right) \neq 0$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$, and a function $h(\varepsilon) : \mathbb{R} \to \mathbb{R}^n$, $h(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, such that

$$H(h(\varepsilon), \varepsilon) = 0 \text{ for } (h(\varepsilon), \varepsilon) \in \mathcal{N}$$

Furthermore, $h$ is analytic and can be approximated by a Taylor series. In particular, the first order derivatives equal

$$h'(0) = -\frac{1}{2} \cdot \left( \frac{\partial^2 H}{\partial x \partial \varepsilon} \right)^{-1} \cdot \frac{\partial^2 H}{\partial \varepsilon^2}.$$

A.1  Calculating Derivatives

To apply theorem 4 we will now calculate the (first order) derivatives of $H_i$ with respect to $d$ and $\varepsilon$ and then the (second order) derivatives with respect to $(d, \varepsilon)$ and $(\varepsilon, \varepsilon)$ and evaluate
them at $\varepsilon = 0$:

\[
\begin{align*}
\frac{\partial H_{ij}}{\partial d_{ik}} &= E \left[ \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot (\xi_j + \pi_j(\varepsilon)\varepsilon) \cdot (\xi_k\varepsilon + \pi_k(\varepsilon)^2) \right] \\
\frac{\partial H_{ij}}{\partial \varepsilon} &= E \left[ \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot \frac{\partial W_{Ti}}{\partial \varepsilon} \cdot (\xi_j + \pi_j(\varepsilon)\cdot \varepsilon) + \frac{\partial u_i}{\partial W} (W_{Ti}(\varepsilon)) \cdot (\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon) \right].
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \varepsilon} &= E \left[ \frac{\partial^3 u_i}{\partial W^3} (W_{Ti}(\varepsilon)) \cdot \frac{\partial W_{Ti}}{\partial \varepsilon} \cdot (\xi_j + \pi_j(\varepsilon)\varepsilon) \cdot (\xi_k\varepsilon + \pi_k(\varepsilon)^2) \\
&\quad + \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot \frac{\partial W_{Ti}}{\partial \varepsilon} \cdot (\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon) \cdot (\xi_k\varepsilon + \pi_k(\varepsilon)^2) \\
&\quad + \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot (\xi_j + \pi_j(\varepsilon)\varepsilon^2) \cdot (\xi_k + \pi_k'(\varepsilon)\varepsilon^2 + \pi_k 2\varepsilon) \right].
\end{align*}
\]

We also calculate

\[
\begin{align*}
\frac{\partial^2 H_{ij}}{\partial \varepsilon^2} &= E \left[ \frac{\partial^3 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot \left( \frac{\partial W_{Ti}}{\partial \varepsilon} \right)^2 \cdot (\xi_j + \pi_j(\varepsilon)\varepsilon) \\
&\quad + \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot \frac{\partial W_{Ti}}{\partial \varepsilon} \cdot (\pi_j(\varepsilon) + \pi_j'(\varepsilon)\varepsilon) \\
&\quad + \frac{\partial^2 u_i}{\partial W^2} (W_{Ti}(\varepsilon)) \cdot \frac{\partial W_{Ti}}{\partial \varepsilon} \cdot (\xi_j + \pi_j(\varepsilon)) + \frac{\partial u_i}{\partial W} (W_{Ti}(\varepsilon)) \cdot \left( 2\frac{\partial \pi_j}{\partial \varepsilon} + \pi_j''(\varepsilon)\varepsilon \right) \right].
\end{align*}
\]

### A.2 Deriving The Zero and First Order Terms

We see that $\frac{\partial H_{ij}}{\partial d_{ik}}$ is equal to zero at $\varepsilon = 0$ for any $j, k$ so that $\frac{\partial H_{ij}}{\partial d}(\varepsilon = 0) = 0$. In the next steps we will check that $det \left( \frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \varepsilon} \right)_{\varepsilon = 0} \neq 0$ and require $\frac{\partial H_{ij}}{\partial \varepsilon}(\varepsilon = 0) = 0$ to apply theorem 4:

We can deduce from the above equations that $\frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \varepsilon}(\varepsilon = 0) = u''_i E[\xi_j, \xi_k]$, i.e.

\[
\frac{\partial^2 H_{ij}}{\partial d \partial \varepsilon}(\varepsilon = 0) = u''_i \cdot V.
\]

(All other terms are equal to zero at $\varepsilon = 0$.) Therefore $det \left( \frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \varepsilon} \right)_{\varepsilon = 0} = u''_i \cdot det(V)$ and $\left( \frac{\partial H^2}{\partial d \partial \varepsilon}(0) \right)^{-1} = \frac{1}{u''_i} V^{-1}$. Since $u''_i > 0$ ($u_i$ is strictly concave), and $det(V) \neq 0$ we have that $det \left( \frac{\partial^2 H_{ij}}{\partial d_{ik} \partial \varepsilon} \right)_{\varepsilon = 0} \neq 0$.

The condition $H_{\varepsilon}(\varepsilon = 0) = 0$ becomes the following system of equations:

\[
0 = \frac{\partial H_{ij}}{\partial \varepsilon}(0) = u''_i \cdot \sum_{k=0}^{N} d_{ik}(0) Cov(\xi_j, \xi_k) + u'_i \cdot \pi_j(0),
\]

i.e. $0 = u''_i \cdot V \cdot d_i + u'_i \cdot \pi(0)$ or equivalently $d_i(0) = \tau_i \cdot (V^{-1}\pi(0))$. 

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This gives us the zero order demand term. Now that the conditions have been checked to apply theorem 4 we calculate

\[
\frac{\partial^2 H_{ij}}{\partial \varepsilon^2} (0) = u_i'' E[\eta_i^2 \xi_j] + 2u_i'' E[\eta_i] \pi_j(0) + 2u_i' \pi_j'(0) = u_i'' \zeta_j + 2u_i' \pi_j'(0).
\]

This follows since \( E[\eta_i] = 0 \). Theorem 4 tells us

\[
\frac{\partial d_i}{\partial \varepsilon} (0) = -\frac{1}{2} \left( \left( \frac{\partial^2 H_i}{\partial d_i \partial \varepsilon} (\varepsilon = 0) \right)^{-1} \frac{\partial^2 H_i}{\partial \varepsilon^2} \right) = -\frac{u_i''}{2u_i''} (V^{-1} \xi_i) - \frac{u_i'}{u_i''} (V^{-1} \pi'(0)).
\]

### A.3 Calculating The Equilibrium

Using \( d_i(0) = \tau_i \cdot (V^{-1} \pi) \) (equation 6) we get that the zero order aggregate demand is \( (\tau_1 + \tau_2) \cdot (V^{-1} \pi) \). This translates into (for assets \( j = 1, \ldots, N \))

\[
\pi_0(0) = \frac{1}{\tau_1 + \tau_2} V_{00}, \pi_j(0) = \frac{1}{\tau_1 + \tau_2} V_{0j}, d_{i0}(0) = \frac{\tau_i}{\tau_1 + \tau_2}, d_{ij}(0) = 0.
\]

This implies that \( \zeta_{ij} = E \left[ \left( \sum_{k=0}^{N} d_{ik}(0) \xi_k \right)^2 \xi_j \right] = d_{i0}^2(0) E[\xi_0^2 \xi_j] = d_{i0}^2(0) \cdot \chi_j \). Therefore, the \( \zeta_i \) vector reduces to the vector \( d_{i0}^2(0) \cdot \chi \). We calculate the first order equilibrium demand and price terms using equation (6) as

\[
d_i'(0) = V^{-1} \left( \tau_i \pi'(0) + \frac{\rho_i}{\tau_i} \left( \frac{\tau_i}{\tau_1 + \tau_2} \right)^2 \chi \right).
\]

The market clearing condition for the first order demand is then:

\[
0 = d_1'(0) + d_2'(0) = V^{-1} \left( (\tau_1 + \tau_2) \pi'(0) + \left( \frac{\tau_1 \rho_1}{(\tau_1 + \tau_2)^2} + \frac{\tau_2 \rho_2}{(\tau_1 + \tau_2)^2} \right) \chi \right)
\]

which implies

\[
\pi'(0) = -\frac{\rho_1 \tau_1 + \rho_2 \tau_2}{(\tau_1 + \tau_2)^3} \chi,
\]

and so

\[
d_1'(0) = \left( \frac{\rho_1 \tau_1}{(\tau_1 + \tau_2)^2} - \frac{\tau_1}{(\tau_1 + \tau_2)^3} (\rho \tau_1 + \rho_2 \tau_2) \right) V^{-1} \chi = -\tau_1 \tau_2 \frac{\rho_1 - \rho_2}{(\tau_1 + \tau_2)^3} V^{-1} \chi.
\]
References


Figure 1: Empirically observed open interest and the ratio of volume to open interest for Microsoft on July 17, 18, 19 for July 2002 maturity.
Figure 2: With a minimal strike $K_1 = 1/3$: Comparing the resulting open interest curve when
3 options (first row) or 4 options (second row) can be traded and the underlying security
has a uniform (first column), truncated normal (second column), truncated lognormal (third
column) distribution.
Figure 3: With a minimal strike at 0.2 and a maximal strike at 1.8: Comparing the resulting open interest curve when 10 options (first row), 15 options (second row), 20 options (third row), 25 options (fourth row) can be traded and the underlying security has a uniform (first column), truncated normal (second column), truncated lognormal (third column) distribution.