How to Cooperate in Producing and Sharing Information—Two Examples of Mechanism Design with Capacity Constrained Agents

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Abstract: I study the mechanisms that render two agents with limited capacity cooperate in producing and sharing information. Two typical scenarios ("auction" and "public good") are analyzed. Necessary and sufficient condition is identified for the implementability of social efficiency. The informational efficiency of the mechanisms is evaluated under Shannon’s information theory. The agents’ limited capacity leads to the endogeneity of the information structure that distinguishes this problem from most models studied in the mechanism design literature. The main implications of the model are: i) in order to achieve social efficiency, information acquisition should be conducted in a sequential manner—one agent’s observation of the state of nature should be made conditional on the other; ii) social efficiency can be implemented by a budget-balanced Bayesian incentive compatible mechanism if and only if the extent of conflict of the two agents’ objectives is upper bounded by the smaller ratio of their sensitivities to shocks; 3) abilities of the two agents do not affect the implementability of social efficiency, but do affect their equilibrium payoffs.

I. Introduction

I study the problem that how individuals can be organized into most efficiently processing information and thus maximizing the social welfare. My model is rigorously built upon Shannon’s information theory, which provides an objective and natural criterion for evaluating the informational efficiency of different mechanisms. Specifically, I investigate how to design a mechanism to make two capacity constrained agents cooperate in producing and sharing information, and thus achieve social efficiency. Here "capacity" is a concept borrowed from the literature of "rational inattention" in macroeconomics. It captures the fact that "people are constrained in their ability to acquire and process information" (Sims (2005)). It postulates a deviation from the usual assumption of totally rational agents in economic models. Agents are no longer able to perfectly observe the outside shocks and thus can only take actions based on their imperfect observations through a communication channel with finite Shannon capacity.

Since the capacity is limited, the agents can not collect all the information of the environment. They must decide which kind of information to be collected and concentrate on the information most relevant to their objectives and rationally ignore others, i.e. rational inattention. For example, suppose an agent has 90% of his wealth in the stock market and 10% in the bond market, which makes him more sensitive to the stock market price fluctuations than the other. Thus he should pay more attention to the shocks to the stock market, say, using 80% of his capacity to trace the stock prices and only 20% to trace the bond prices. Roughly speaking, the ratios "80%" and "20%" here result from the specific conditions.
channel used by the agent to collect information, i.e. the channel determines what information is collected. Since collecting information is actually producing information about the environment, I don’t distinguish these two terms in the remaining part of the paper.

Since both agents are constrained by their capacity, they can benefit from cooperating in producing and sharing information. Consider the following scenario: two firms, 1 and 2, each has some resources which can be used to produce output. The firms’ productivities depend on the state of the environment, which is described by a set of random variables, called shocks thereafter. The shocks affect the two firms’ productivities in different ways, thus in some states productivity is higher for Firm 1 and for Firm 2 in other states. Define the social welfare as the sum of the two firms’ output. Then social efficiency requires all the resource to be allocated to the firm with higher productivity. However, the social planner can only know about the environment through the the firms’ reports and then make decisions (allocation of the resource and each firm’s payment) based on such reports. Since the two firms’ capacity is limited, the social planner must decide what information is the most useful and try to induce the firms to produce such information as much as possible. The shocks are public to both firms, but the firms can only observe signals of the shocks instead of the shocks themselves. The amount of information contained in the signals is upper bounded by each firm’s capacity. However, the channels through which the shocks are observed are not fixed. The social planner can choose any channels he’d like provided that the amount of information produced by each firm doesn’t exceed its capacity. In other words, the social planner can choose appropriate signals that convey the information most important to him and ignore others, given that the capacity constraints are satisfied.

Now the problem is, under what conditions social efficiency can be obtained. And, when such conditions are satisfied, how to design a mechanism to achieve social efficiency. The process can be roughly described as the following: firstly the social planner announces a mechanism, then each firm produces some information and reports it to the social planner, who then decides the payments and the allocation of the resource according to the mechanism announced at the beginning. At the first glance, this problem is no more than those we usually encounter in mechanism design theory, where an incentive compatible mechanism is designed to elicit private information from different agents and thus overcome the problem of information asymmetry. The problem here is similar in that it also requires an incentive compatible mechanism to make the agents tell their true information (i.e. the true values of the signals they received), but this is not

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2Whether there is a social planner is not essential here. He can be replaced by a contract that both firms agree on.

3In this paper, only the social planner can choose the channels, the agents are not able to do so. Thus the channels being used to collect information are themselves part of the mechanism. This is natural in many situations. For example, the agents lack the technique to change the channels assigned to them. However, when the agents are able to do so, no matter publicly or privately, the optimal mechanism (if it exists) in this paper is not sustainable. Another paper of the author addresses this problem in a repeated games framework and provides conditions for the existence of an optimal mechanism.
enough. The key difference is that the structure of the agents’ information (e.g. the distribution and the interdependence of the agents’ private signals) is not exogenously given as those in the usual mechanism design models. Actually, it is endogenously determined in the sense that a) which kind of information to be collected is itself a part of the mechanism rather than outside the model, and it does affects both agents’ benifits and the social welfare ; b) how the two agents cooperate affects the amount of information that can be produced; c) both a) and b) affect the incentives. This endogeneity makes the revelation principle difficult to use and gives rise to two key features of this problem that distinguish it from those dealt in usual mechanism design theory.

The first one is to choose appropriate channels for the two agents to produce the most relevant information. This is not trivial when the shocks are multi-dimensional. Recall the example of observing the financial markets in the first paragraph. The allocation of capacity on different markets determines what information is produced. It is not efficient, from the perspective of the agent I considered there, to spend most of his capacity in observing bond market while his wealth mainly consists of stocks. However, this may not be true for the other agent if his wealth has a different composition. This generates a tension between the two agents’ objectives in producing information. As analyzed in the model, this tension matters greatly when designing the mechanism to provide correct incentives for truth telling.

The second key feature of this problem is that the timing of producing information is important. There are many "timings" telling the two agents how to cooperate in producing information. At least two cases should be considered. The first one is that each agent independently makes his observations and then reports his result to the social planner. Thus the two agents’ signals are conditionally independent (independent when conditioning on the true state). This is the extreme case of no cooperation in producing information. The other extreme is the case of total cooperation, i.e. one agent makes his observations first and reports his result to the social planner. The social planner tells it to the second agent who then makes his observations accordingly and reports the result to the social planner. In a "middle" case, the two agents can partly cooperate, say, by using part of their capacity to produce some observations, sharing some of them, and based on which using their remaining capacity to produce new information independently. This seems to make the problem too complicated to analyze, however, I prove that the extreme case of total cooperation is the most efficient in producing information. Since this is the prerequisite for efficient resource allocation, I only need to consider this case when designing the optimal mechanism to maximize social welfare. This result comes from the properties of information processing, which is derived from Shannon’s information theory. Thus it is axiomatically correct regardless of our set up of the model. In Section 4 where the result is proved, we give an intuitive explanation. It is worth noticing that, besides the choice of channels, the choice of timing also affects

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4Here the existence of the social planner is not essential. When there is no such social planner, just let the first agent inform the second one his result, based on which the second agent makes his observations and then tells the first agent his result.
the incentives for truth telling.

The literature of mechanism design with interdependent valuations is the most relevant to our model. In that literature, the agents’ valuations may depend on others’ signals. As discussed in Borgers (2008), with this informational interdependence, the first best may not be achieved through the famous Vickrey-Clarke-Groves mechanism (Vickrey (1961), Clarke (1971), Groves (1973)). Jehiel and Moldovanu (2001) consider the case that each agent’s signals affect their own utility as well as others’, but potentially with different weights. They show that the first best decision rules are impossible to be implemented by a Bayesian incentive compatible mechanism in generic situations. My model relates to the above interdependent valuations models in that the agents’ productivities also depend on other’s signals since both their signals reveal some information about the state of the environment. However, the endogeneity of the information structure discussed in the last three paragraphs makes our problem significantly different. Actually, if the channels are fixed (i.e. exogenously given and not itself a part of the mechanism) and the agents follow the “timing” of no cooperation in producing information, my model reduces to the usual case of mechanism design with interdependent valuations. As I prove in the main part of this paper, the above endogeneity must be considered if we want to implement the first best allocation rules (and thus social efficiency). Therefore, just use the models of interdependent valuations cannot solve our problem.

Dasgupta and Maskin (2001) consider a mechanism design problem with interdependent valuations and show that the Vickrey-Clarke-Groves mechanism can be generalized to attain efficiency when each agent’s information can be represented as a one-dimensional signal. Moreover, they show that no efficient mechanism exists when an agent’s information is multidimensional. In that case, only the constrained efficiency can be achieved. In my model, I do allow multidimensional signals but the social efficiency is still obtained. The key point is that, from their own perspectives, the social planner and the agents will find that using the single-dimensional signals is enough to achieve their objectives. Thus when designing the optimal mechanism, the social planner can just consider the channels that generate single-dimensional signals and the impossibility of implementing the efficiency discussed in Dasgupta and Maskin (2001) doesn’t matter here.

There are many approaches in economics studying human’s irrational behaviors and their effects (e.g. Benabou and Tirole (2003), Gul and Pesendorfer (2001)) and the learning literature (e.g. Fudenberg and Kreps (1993), Gennotte (1986)). This paper models people’s bounded rationality by imposing a limit to their capacity of information processing. This is not controversial, since nothing in the world can process infinite information in a finite period. It accords with our ordinary experiences, as argued in Sims (2003), and is particularly appealing since it provides a unified and relatively simple framework to account

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5 In this paper the agents are not allowed to change their channels. They can only make their observations through the channels assigned by the social planner. However, even if they are allowed to do so, as the case studied in the author’s other paper, they would still find it optimal to have single-dimensional signals.
for a wide range of observations, and "by exploiting ideas from the engineering
theory of coding, it arrives at predictions that do not depend on the details
of how information is processed" (Sims (2003)). Besides these reasons, it has
another advantage that matches well with my purpose in this paper: Shannon’s
information theory provides an objective and rigorous criterion to measure and
analyze the efficiency of information processing, which makes my conclusions
stand on a solid foundation.

I briefly present our main results here. The dependence of each firm’s pro-
ductivity on the shocks is represented by a vector. The length of the vector
represents how sensitive the firm’s productivity is to the shocks. The cosine of
the angle between the two firms’ vectors represents the relation between their
objectives. A negative cosine value means their objectives are compatible to
each other, while a positive one indicates some conflict between their objec-
tives. The cosine value is in fact a measure of the extent of their objectives’
conflict. Our main results are a) the social efficiency is achieved only if the two
agents totally cooperate in producing information; b) the social efficiency can
be implemented by a budget-balanced Bayesian incentive compatible mechanism
if and only if the confliction between their objectives is upper bounded by the
smaller raitos of their sensitivities; c) the firms’ capacity of information process-
ing and benchmark productivities do not affect the implementability of social
efficiency, but they do affect the equilibrium payo¤s. I will give some intuitive
interpretation of the above results in the main part of the paper.

I proceed as follows. Section II provides some basic knowledge of informa-
tion theory and rational inattention. Section III sets up the model. Section
IV proves an important property of information processing and our first main
result. Section V studies the first best case. Section VI proves the necessary
and sufficient condition for the implementability of the "first best" and designs
a mechanism to implement it. Section VII studies the problem of "public good"
with capacity constrained agents and compares it to the previous problem. I
conclude in Section VIII with several directions for further research.

II. Basics of Informaiton Theory and Rational Inattention

In Shannon’s information theory, information is defined as the reduction of
uncertainty, while uncertainty is measured by entropy. For a discrete random
vector $\mathbf{X}$ with probability weights $p(\mathbf{x})$, $\mathbf{x} \in \mathbf{X}$, its entropy is

$$H(\mathbf{X}) = -E[\log p(\mathbf{x})] = -\sum_{\mathbf{x} \in \mathbf{X}} p(\mathbf{x}) \cdot \log p(\mathbf{x})$$

where we define $p(\mathbf{x}) \log p(\mathbf{x}) = 0$ when $p(\mathbf{x}) = 0$. Shannon proves that
any function measuring the uncertainty and satisfying three axioms must have
this form. Thus it is a natural and objective measurement of uncertainty. The
base of the logarithm is not essensial, it just changes the unit of entropy. For

\footnote{A firm’s benchmark productivity is the constant component of its productivity, i.e. the component doesn’t fluctuate with the shocks. It measures the basic profitability of the firm.}
example, when the base is 2, the entropy of a discrete random variable with equal probability on two values is 1 bit.

When \( X = (X_1, X_2) \), we also call \( H(X) = H(X_1, X_2) \) the joint entropy of \( X_1 \) and \( X_2 \). The conditional entropy of \( X_1 \) given \( X_2 \) is the expected conditional entropy over \( X_2 \), which is defined as

\[
H(X_1|X_2) = E_{X_2}[H(X_1|X_2)]
\]

\[
= - \sum_{x_2 \in X_2} p(x_2) \cdot \sum_{x_1 \in X_1} p(x_1|x_2) \cdot \log p(x_1|x_2)
\]

\[
= - \sum_{x \in X} p(x_1, x_2) \cdot \log \frac{p(x_1, x_2)}{p(x_2)}
\]

Note that \( H(X_1|X_2) = H(X_1, X_2) - H(X_2) \). \( H(X_1|X_2) \) measures the remaining uncertainty of \( (X_1, X_2) \) when \( X_2 \) is known.

The mutual information between two random vectors measures the amount of information that can be obtained about one random vector when the other one is known. Its definition is

\[
I(X_1; X_2) = \sum_{x \in X} p(x_1, x_2) \log \frac{p(x_1, x_2)}{p(x_1)p(x_2)}
\]

Note that

\[
I(X_1; X_2) = I(X_2; X_1)
\]

\[
= H(X_2) - H(X_2|X_1) = H(X_1) - H(X_1|X_2)
\]

\[
= H(X_1) + H(X_2) - H(X_1, X_2)
\]

Mutual information is always non-negative. If the above joint probability \( p(x_1, x_2) \) is replaced by the conditional joint probability \( p(x_1, x_2|\mathcal{Y}) \) on some random vector \( \mathcal{Y} \), we get the conditional mutual information, which measures the mutual information between two random vectors when the third one is known. Conditional mutual information is also always non-negative.

For a continuous random vector, its Shannon entropy is infinity, since it can take a continuum of possible values. In this case the differential entropy is defined as an extension of the Shannon entropy. Formally, the differential entropy of a continuous random vector \( X \) with probability density function \( p(x) \) is defined as

\[
h(X) = - \int_{\mathbb{R}^n} \log p(x) \cdot d\mathcal{L}_n
\]
where \( n \) is the dimension of \( \overrightarrow{X} \). All the above properties of Shannon entropy still hold for differential entropy. Since the base of the logarithm is not essential, we just use the natural logarithm in the rest of this paper.

An example of normal random vectors: let \( \overrightarrow{X_1}, \overrightarrow{X_2} \sim N((\mu_1, \mu_2), \Sigma) \), where \( \mu_i \in \mathbb{R}^{n_i} \), \( i = 1, 2 \) and

\[
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\]

Then the joint entropy of \( \overrightarrow{X_1}, \overrightarrow{X_2} \) is

\[
h(\overrightarrow{X_1}, \overrightarrow{X_2}) = \frac{n}{2} \cdot [\ln (2\pi) + 1] + \frac{1}{2} \cdot \ln |\Sigma|,
\]

where \( n = n_1 + n_2 \).

The conditional entropy is

\[
h(\overrightarrow{X_1} | \overrightarrow{X_2}) = \frac{n}{2} \cdot [\ln (2\pi) + 1] + \frac{1}{2} \cdot \ln |\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|,
\]

and the mutual information is

\[
I(\overrightarrow{X_1}; \overrightarrow{X_2}) = -\frac{1}{2} \cdot \ln |I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|,
\]

where \( I \) is the identical matrix with dimension \( n_1 \).

Channel refers to the medium to convey information. The channel between two random vectors \( \overrightarrow{X_1} \) and \( \overrightarrow{X_2} \) is described by the conditional distribution \( p(\overrightarrow{x}_1 | \overrightarrow{x}_2) \) or \( p(\overrightarrow{x}_2 | \overrightarrow{x}_1) \). Given the unconditional distributions of \( \overrightarrow{X_1} \) and \( \overrightarrow{X_2} \), different conditional distributions define different channels. Thus a specific channel determines which kind of information about one random vector is conveyed by the other. The mutual information \( I(\overrightarrow{X_1}; \overrightarrow{X_2}) \) measures the amount of information transmitted through the channel between \( \overrightarrow{X_1} \) and \( \overrightarrow{X_2} \).

The basic idea of rational inattention is that the agent’s actions can depend on observations of the state variables only through a channel with finite capacity. Thus he should appropriately allocate his capacity to collect the information most relevant to his objective and ignore others. Specifically, if \( \overrightarrow{\theta} \) denotes the state random variables, then the agent can choose an appropriate channel to generate the signals \( \overrightarrow{S} \) subject to the constraint that the mutual information \( I(\overrightarrow{\theta}; \overrightarrow{S}) \) is upper bounded by his capacity. For more about rational inattention, please see Sims (2003 and 2005).

III. The Model

Suppose there are two risk-neutral firms, 1 and 2, and an indivisible\(^7\) resource that can be used to make profit. The amount of the resource is normalized to unit. The firm’s productivity is \( v_i = \overrightarrow{a_i} \cdot \overrightarrow{\theta} + b_i \) (i.e. when the resource is allocated to Firm \( i \), it generates \( \overrightarrow{a_i} \cdot \overrightarrow{\theta} + b_i \) amount of profit), where \( i = 1, 2 \), \( b_i \in \mathbb{R} \) and \( \overrightarrow{\theta} \in \mathbb{R}^n \) describes the state of the environment. We call \( \overrightarrow{\theta} \) shocks thereafter. Assume \( \overrightarrow{a}_i \neq \overrightarrow{0} \), \( i = 1, 2 \) and \( \overrightarrow{a}_1 \neq \overrightarrow{a}_2 \). Otherwise the problem is triv-

\(^7\)The resource can be divisible. In this case I just interpret the allocation rule \( q_i \) as allocating \( q_i \) share of the resource to agent \( i \), while in the case of indivisible resource, \( q_i \) is agent \( i \)’s probability of winning the resource.
ial. Without any loss of generality, let \( \bar{\theta} \sim N \left( \bar{0}, I \right) \). \( \bar{\omega}_i \in \mathbb{R}^n \) describes how Firm \( i \)'s productivity depends on the shocks. The length \( || \bar{\omega}_i || \) represents Firm \( i \)'s sensitivity to the shocks and \( \cos \alpha \triangleq \frac{\bar{\omega}_i \cdot \bar{\omega}_2}{|| \bar{\omega}_1 || \cdot || \bar{\omega}_2 ||} \) measures the extent of conflict between the two firms' objectives. Note that when \( \text{corr}(\nu_1, \nu_2) = \cos \alpha \leq 0 \), the two firms' valuations of the resource are negatively correlated, thus the extent of conflict between their objectives is negative. In other words, their objectives are compatible. Conversely, a positive value of \( \cos \alpha \) indicates some conflict between their objectives. Our result shows that the sensitivities and the objectives' conflict are two key factors that determine whether there exists some budget balanced Bayesian incentive compatible mechanism implementing the "First Best".

The two firms are constrained in their capacity to collect information about the shocks. Specifically, Firm \( i \) can only observe his signal \( \bar{S}_i \) such that \( \beta(\bar{S}_i) \leq c_i \), where \( i = 1, 2 \) and \( c_i \) is his capacity of information processing.

To get a closed form solution, we assume \( \bar{\theta} \) and \( \bar{S}_i \) are jointly normal. This assumption facilitates my derivations and does not hurt my main results. Also assume that the true value of \( \bar{\theta} \) is revealed by the nature after the resource is allocated and consumed.

All the values of the parameters are assumed to be common knowledge.

In this situation, each channel of agent \( i \) is defined by a joint distribution of \( \bar{\theta} \) and \( \bar{S}_i \):

\[
\mathcal{N} \left( \frac{\bar{\theta}}{\mu_i}, \left( \begin{array}{cc} I & \Sigma_{\bar{\theta} \bar{S}_i} \\ \Sigma_{\bar{S}_i \bar{\theta}} & \Sigma_{\bar{S}_i \bar{S}_i} \end{array} \right) \right)
\]

and the agents can make inference of \( \bar{\theta} \) from their observation of \( \bar{S}_i \), i.e.

\[
\left( \bar{\theta} | \bar{S}_i \right) \sim \mathcal{N} \left( \Sigma_{\bar{\theta} \bar{S}_i} \Sigma_{\bar{S}_i \bar{S}_i}^{-1} \bar{S}_i - \mu_i, I - \Sigma_{\bar{\theta} \bar{S}_i} \Sigma_{\bar{S}_i \bar{S}_i}^{-1} \Sigma_{\bar{S}_i \bar{\theta}} \right).
\]

Now I can define a mechanism as the triplet of the following three elements:

a) timing, i.e. how should the agents cooperate in producing information;

b) channel, i.e. the joint distributions of \( \left( \frac{\bar{\theta}}{\bar{S}_i} \right) \) and \( \left( \frac{\bar{\theta}}{\bar{S}_j} \right) | \bar{S}_i \), where \( i, j \in \{1, 2\} \) and \( i \neq j \). This is equivalent to defining the joint distribution of \( \left( \frac{\bar{\theta}}{\bar{S}_1}, \bar{S}_2 \right) \) and payments \( t_i \left( \bar{S}_1, \bar{S}_2 \right) \), where \( i = 1 \)

c) the allocation rule \( q_i \left( \bar{S}_1, \bar{S}_2 \right) \) and payments \( t_i \left( \bar{S}_1, \bar{S}_2 \right) \), where \( i = 1 \)

\footnote{For the sake of calculation, I use the normal distribution here, which makes the productivity negative in some states. Just imagine the resource can sometimes be a burden and you are not allowed to discard it for some reason. For example, the resource is labor and you can not fire the workers whenever you want. You should always pay for the labor and thus your profit might be negative when the sale is really bad. All in all, this assumption is just for our modeling convenience.}

\footnote{Actually, this has already represented the consideration of "timing". To emphasize its importance, I write it down separately in a).}
or 2. Agent i wins the good with probability \( q_i \left( S_1, S_2 \right) \) and pays \( t_i \left( S_1, S_2 \right) \) when the reported signals are \( S_1, S_2 \). And then his expected payoff is
\[
E \left[ v_i \cdot q_i \left( S_1, S_2 \right) - t_i \left( S_1, S_2 \right) \right] 
\]
Comment: a) and b) show that the information structure is itself a part of the mechanism rather than being exogenously given, while c) is the usual part of a mechanism.

The social planner’s objective is to design a mechanism to achieve the "First Best", i.e. social efficiency. Before pursue this objective, let’s first introduce an important property of information processing in the next section.

IV. Timing and Information Acquisition

Suppose agent 1’s capacity is \( k \cdot c \) and agent 2’s capacity is \((1 - k) \cdot c \), where \( k \in [0, 1] \) and \( c \geq 0 \) is the total capacity. Now I compare the two situations below:

a) No cooperation: in this case, the two agents independently make their observations, which means \( I(\theta; S_1) = k \cdot c \), \( I(\theta; S_2) = (1 - k) \cdot c \) and \( I(S_1; S_2; \theta) = 0 \). Then the total amount of information being produced is
\[
I(\theta; S_1, S_2) = I(\theta; S_1) + I(\theta; S_2) - I(S_1; S_2; \theta) = c - I(S_1; S_2)
\]

i.e. there is a capacity loss of \( I(S_1; S_2) \). Note that \( I(S_1; S_2) \) is always non-negative\(^\text{10} \) and in most situations it is strictly positive. \( I(S_1; S_2) = 0 \) is the trivial case and we’ll see it not happen in this model.

b) Total cooperation: in this case, agent 1 makes observations \( S_1 \) first and agent 2 can make observations \( S_2 \) based on \( S_1 \), which means \( I(\theta; S_1) = k \cdot c \) and \( I(\theta; S_2|S_1) = (1 - k) \cdot c \). Then the total amount of information being produced is
\[
I(\theta; S_1, S_2) = I(\theta; S_1) + I(\theta; S_2|S_1) = c
\]

This simple calculation suggests that the "timing" of "total cooperation" is the most efficient in producing information and the "timing" of "no cooperation" suffers from an information loss of \( I(S_1; S_2) \). This is intuitive. Agent 1’s

\(^{10}\)Non-negativity is a basic property of mutual information.
observations provide some information that reduces the uncertainty of \( \overrightarrow{\theta} \), and thus agent 2 can allocate his capacity more efficiently based on such information. The example below illustrates this point.

Let \( \theta \sim N(0,1) \). \( S_1 \) and \( S_2 \) are two one-dimensional signals jointly normal with \( \theta \). The two observations are made independently, i.e. \( I \left( \overrightarrow{S}_1; \overrightarrow{S}_2 \mid \overrightarrow{\theta} \right) = 0 \). \( r_i = \text{corr}(\theta, S_i) \) \( i = 1, 2 \) and \( r_{12} = \text{corr}(S_1, S_2) \) are the correlations between the pairs of random variables. With loss of generality, we assume all the three correlations are non-negative. Thus the mutual information is 
\[ I \left( \overrightarrow{S}_1; \overrightarrow{S}_2 \right) = -\frac{1}{2} \cdot \ln (1 - r_1^2) = k \cdot c \quad \text{and} \quad I \left( \overrightarrow{S}_2; \overrightarrow{S}_1 \right) = -\frac{1}{2} \cdot \ln (1 - r_2^2) = (1 - k) \cdot c , \]
where \( k \in [0,1/2] \). Recall that \( I \left( \overrightarrow{S}_1; \overrightarrow{S}_2 \mid \overrightarrow{\theta} \right) = 0 \), which implies \( r_{12} = r_1 \cdot r_2 \).

Therefore, the information loss when there is no cooperation is 
\[ I \left( \overrightarrow{S}_1; \overrightarrow{S}_2 \right) = -\frac{1}{2} \cdot \ln (1 - r_1^2 \cdot r_2^2) = -\frac{1}{2} \cdot \ln \left( e^{-2k \cdot c} + e^{-2(1-k) \cdot c} - e^{-2c} \right) \leq k \cdot c , \]

"\( = \) holds if \( k \cdot c = 0 \). Note that
\[ \frac{\partial I \left( \overrightarrow{S}_1; \overrightarrow{S}_2 \right)}{\partial c} = \frac{(1 - k) \cdot e^{2k \cdot c} + k \cdot e^{2(1-k) \cdot c} - 1}{e^{2k \cdot c} + e^{2(1-k) \cdot c} - 1} = \begin{cases} 0 & \text{if } c = 0 \\ k & \text{as } c \to \infty \end{cases} \]

i.e. the speed of information loss becomes larger as total capacity increases, and approaches \( k \) when total capacity approaches \( \infty \). The figure below shows the relation between the information loss and total capacity when \( k = 1/2 \):

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\[ ^{11} \text{By the symmetry of the two observations, I can assume } k \in [0,1/2] \text{ without any loss of generality.} \]
Note that $\tan \beta$ equals the ratio of information loss to total capacity $c$. This "loss-ratio" increases with $c$ and approaches $k$ when $c \to \infty$. Also note that when $k = 0$, the first observation conveys no information. It reduces to the case of single observation with total capacity, and thus no information loss occurs.

I summarize the above results in the following proposition:

**Proposition 1:** Let total capacity $c$ be divided into $N$ ($\geq 1$) parts, $c_1, c_2, \cdots, c_N$, i.e. $c_n > 0$, $n = 1, 2, \cdots, N$, and $\sum_{n=1}^{N} c_n = c$. The agents make $N$ observations $\{S_n\}_{n=1}^{N}$ of a random vector $\vec{\theta}$. Each observation $S_n$ uses capacity $c_n$, i.e. $S_n$ conveys $c_n$ amount of information about $\vec{\theta}$. Then, the total information about $\vec{\theta}$ contained in $\{S_n\}_{n=1}^{N}$ equals $c$, i.e. $I(\vec{\theta};\{S_n\}_{n=1}^{N}) = c$, if and only if the $n$'th observation can be made based on all its previous observations, i.e. $I(\vec{\theta};S_{n}\mid\{S_{j}\}_{j=1}^{n-1}) = c_n$, $n = 2, 3, \cdots, N$. Otherwise, $I(\vec{\theta};\{S_n\}_{n=1}^{N}) < c$.

Proof: just recursively use the derivation for the case of two signals above.
I omit the details here.

Proposition 1 shows that serial cooperation in making observations can produce the largest amount of information, otherwise part of the capacity would be wasted. Since social efficiency requires the agents to collect information as much as possible, I have the following corollary:

Corollary: Total cooperation in producing information is a necessary condition for social efficiency\textsuperscript{12}.

V. The Social Planner’s First Best Case

To analyze the first best case, I just ignore the two agents and assume the social planner owns the total capacity $c = c_1 + c_2$. If the social planner knows the value of $\theta$, he should give the resource to agent 2 if $v_2 - v_1 = (\frac{a_2}{\mu_2} - \frac{a_1}{\mu_1}) + (b_2 - b_1) \geq 0$ and give it to agent 1 otherwise\textsuperscript{13}. However, with limited capacity, he can make decisions only based on finite amount of information about $\theta$. He must decide which kind of information is most relevant to his objective, i.e. choose a channel such that a) the capacity constraint is not violated; b) the probability of mis-allocation is minimized. That is

$$\min_{\bar{\mu}, \Sigma_{\bar{\theta} \bar{S}}, \Sigma_{SS}} \text{Var} \left[ (\frac{\bar{a}_2}{\mu_2} - \frac{a_1}{\mu_1}) \theta | \bar{S} \right]$$

s.t. $(\frac{\bar{\theta}}{\bar{S}}) \sim N \left( \frac{0}{\bar{\mu}}, \left( \begin{array}{cc} I & \Sigma_{\bar{\theta} \bar{S}} \\ \Sigma_{\bar{S} \bar{\theta}} & \Sigma_{SS} \end{array} \right) \right)$

and $I \left( \frac{\bar{\theta}}{\bar{S}} \right) \leq c$

Note that many channels are equivalent in the sense that they generate the same inference of $\bar{\theta}$\textsuperscript{14}. We can choose any one of them when designing the mechanism. Here, we choose the joint distribution(thus also the channel)

$$\left( \frac{\bar{\theta}}{\bar{S}} \right) \sim N \left( \frac{0}{\bar{\mu}}, \left( \begin{array}{cc} I & \Sigma_{\bar{\theta} \bar{S}} \\ \Sigma_{\bar{S} \bar{\theta}} & I \end{array} \right) \right)$$

Now the planner’s problem can be restated as

$$\min_{\Sigma_{\bar{\theta} \bar{S}}} \text{Var} \left[ (\frac{\bar{a}_2}{\mu_2} - \frac{a_1}{\mu_1}) \theta | \bar{S} \right]$$

\textsuperscript{12}Notice that in this paper, we don’t distinguish the two terms "social efficiency" and "first best". They have the same meaning here.

\textsuperscript{13}In the case of a tie, I assume that the resource is allocated to Agent 2. This assumption is not critical to our results, since the "tie" is a zero-probability event and the social planner is indifferent in the case of a tie.

\textsuperscript{14}i.e. many joint distributions of $\bar{\theta}$ and $\bar{S}$ lead to the same conditional distribution of $(\bar{\theta} | \bar{S})$.

\textsuperscript{15}We can always do this by using the new signal $\bar{\xi} = P \left( \bar{S} - \bar{\mu} \right)$ to replace $\bar{S}$, where $\Sigma_{\bar{S} \bar{\xi}} = P^T P$. 

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The solution of this problem gives the channel in the first best case.

Proposition 2: The social planner achieves "first best" if and only if

\[ \Sigma \bar{\theta} \bar{S} = \bar{\epsilon} \bar{\alpha}' = \bar{\epsilon}' \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{array} \right) = (\alpha_1 \cdot \bar{\epsilon}' \ \alpha_2 \cdot \bar{\epsilon}' \ \cdots \ \alpha_m \cdot \bar{\epsilon}') \],

where \( \bar{\epsilon} = (\bar{a}_2 - \bar{a}_1) / ||\bar{a}_2 - \bar{a}_1|| \), \( \sum_{i=1}^{m} \alpha_i^2 = 1 - e^{-2c} \) and \( m \geq 1 \) is the dimension of the signal \( \bar{S} \);

b) the allocation rule is

\[
q_2 = \begin{cases} 
1 & \text{if } E \left[ (\bar{a}_2 - \bar{a}_1)' \bar{\theta} + (b_2 - b_1) | \bar{S} \right] \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[ q_1 = 1 - q_2 \]

Moreover, the minimized conditional variance is \( \min_{\Sigma \bar{\theta} \bar{S}} \text{Var} \left[ (\bar{a}_2 - \bar{a}_1)' \bar{\theta} | \bar{S} \right] = ||\bar{a}_2 - \bar{a}_1||^2 \cdot e^{-2c} \), where \( c = c_1 + c_2 \).

Proof: see the appendix.

Comments:

1) The meaning of Condition b) is obvious. It states that the social planner should correctly use all the information being collected.

2) Condition a) states that all capacity should be allocated to the "right" channel to collect the information most relevant to social welfare. That's why all the columns of \( \Sigma \bar{\theta} \bar{S} \) should be proportional to \( \bar{\epsilon} = (\bar{a}_2 - \bar{a}_1) / ||\bar{a}_2 - \bar{a}_1|| \).

3) Condition a) also implies that the minimized conditional variance can be achieved regardless of the dimension of the signals. Intuitively, multi-dimensional signals is equivalent to making serial observations, each of which is made based on the previous ones. The coefficients \( \{\alpha_i\}_{i=1}^{m} \) determine the allocation of total capacity among different observations. Specifically, the capacity used in the \( j \)th \( (j = 1, 2, \cdots, m) \) observation is

\[
I \left( \bar{\theta} ; S_j | S_1, \cdots, S_{j-1} \right) = I \left( \bar{\theta} ; S_1, \cdots, S_j \right) - I \left( \bar{\theta} ; S_1, \cdots, S_{j-1} \right)
\]

\[ = -\frac{1}{2} \cdot \ln |I - \sum_{i=1}^{j} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}'| + \left( \frac{1}{2} \cdot \ln |I - \sum_{i=1}^{j-1} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}'| \right) \]

\[ \vdots \text{rank} \left( \sum_{i=1}^{j} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}' \right) = 1 
\]

\[ \vdots \text{sum} \sum_{i=1}^{j} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}' \] has only one non-zero eigenvalue \( \sum_{i=1}^{j} \alpha_i^2 \) (and the corresponding eigenvector is \( \bar{\epsilon}' \))

Note that the eigenvalues of \( I - \sum_{i=1}^{j} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}' \) can be expressed as \( 1 - \lambda_k \), \( k = 1, 2, \cdots, n \), where \( \lambda \)'s are the eigenvalues of \( \sum_{i=1}^{j} \alpha_i^2 \cdot \bar{\epsilon} \cdot \bar{\epsilon}' \). Thus \( I -
\[ \sum_{i=1}^{j} \alpha_i^2 \cdot \vec{e} \cdot \vec{e}' \] has \( n - 1 \) unit eigenvalues and a non-unit eigenvalue \( 1 - \sum_{i=1}^{j} \alpha_i^2 \).

This implies \( |I - \sum_{i=1}^{j} \alpha_i^2 \cdot \vec{e} \cdot \vec{e}'| = 1 - \sum_{i=1}^{j} \alpha_i^2 \).

Therefore I have

\[
I \left( \vec{\theta}; S_j \right) = 1 - \ln \left( 1 - \sum_{i=1}^{j} \alpha_i^2 \right) + \frac{1}{2} \cdot \ln \left( 1 - \sum_{i=1}^{j-2} \alpha_i^2 \right)
\]

where \( -\frac{1}{2} \cdot \ln \left( 1 - \sum_{i=1}^{j} \alpha_i^2 \right) \) is the capacity used in the first \( j \) observations, i.e. \( I \left( \vec{\theta}; S_j \right) \).

VI. The Necessary and Sufficient Condition for Implementing the First Best

In this section, I give a necessary and sufficient condition for the existence of a budget-balanced Bayesian incentive compatible mechanism that implements the social efficiency. I also design the mechanism under this condition.

As discussed in Section V, I can choose \( \text{dim} \left( S \right) = 1 \) without any loss of generality. This makes my model sidestep the multi-dimensional signal problem in Dasgupta and Maskin (2001)\(^{16}\).

Proposition 3: Under non-trivial conditions \( \vec{a}_1 \neq \vec{a}_2 \) and \( \| \vec{a}_i \| \neq 0, i = 1 \) or \( 2 \), there exists a budget balanced Bayesian incentive compatible mechanism implementing the social efficiency (i.e. the First Best) if and only if

\[
\cos \alpha \leq \min \left\{ \frac{\| \vec{a}_2 \|}{\| \vec{a}_1 \|}, \frac{\| \vec{a}_1 \|}{\| \vec{a}_2 \|} \right\} \quad (6.3.1)
\]

where

\[
\cos \alpha = \frac{\vec{a}_2 \cdot \vec{a}_1}{\| \vec{a}_2 \| \cdot \| \vec{a}_1 \|}
\]

i.e. \( \alpha \) is the angle between the two vectors \( \vec{a}_1 \) and \( \vec{a}_2 \).

Proof: see the appendix.

Comments:

1) As mentioned in Section III, \( \| \vec{a}_i \|, i = 1 \) or \( 2 \) measures the agents’ sensitivity to the shocks and \( \cos \alpha = \text{corr} (v_2, v_1) = \text{corr} \left( \vec{a}_2 \vec{\theta} + b_2, \vec{a}_1 \vec{\theta} + b_1 \right) \) measures the extent of conflict between their objectives. Proposition 3 states that the two agents can cooperate in producing and sharing information and achieve social efficiency if and only if their conflict is upper bounded by the smaller ratio of their sensitivities. Since the right hand side of (6.3.1) is non-negative, the inequality holds for all \( \cos \alpha \leq 0 \). \( \cos \alpha \leq 0 \) means that the agents’ valuations of the resource is negatively correlated, i.e. statistically, the two agents are not likely to desire the resource at the same time. In other words, the social efficiency is always implementable if the agents’ objectives are

\(^{16}\)Actually, the problem of multi-dimensional signal is not essential here. The social planner can just fix the channel "\( \vec{e}' \)" and ask agent \( i \) to report his estimation of \( \left( 1 - e^{-2 \epsilon_i} \right)^{1/2} \cdot \vec{e}' \vec{\theta} \).

Given the mechanism we design in this section, the agents will make their report as if they use a single-dimensional signal, although he is allowed to use multi-dimensional signals.
compatible. When $\cos \alpha > 0$, there exists some conflict between the agents’ objectives and social efficiency may not be implementable. Note that the right hand side of (6.3.1) achieves its maximum 1 if and only if $||\vec{a}_2|| = ||\vec{a}_1||$. In this case, (6.3.1) always holds and hence social efficiency can be obtained. Also note that the larger the difference between $||\vec{a}_2||$ and $||\vec{a}_1||$ is, the smaller is the right hand side of (6.3.1) and the harder can the inequality hold. In other words, when there exists some conflict between the agents’ objectives, they can cooperate to achieve social efficiency if and only if their sensitivities are not too different. As shown in the figure below, when $||\vec{a}_2||$ and $\cos \alpha > 0$ are fixed, social efficiency can be obtained if and only if $||\vec{a}_1||$ belongs to interval $\overline{AB}$.

The intuition is simple. Suppose Agent 1 is much more sensitive to the shocks than Agent 2 is. In order to achieve social efficiency, the allocation should be most often determined by Agent 1’s valuation. This gives rise to a tension between Agent 2’s incentive of cooperation and his objective, since his objective conflicts with Agent 1’s. Moreover, Proposition 3 states that this tension cannot be alleviated through adjusting the payment rule. When the two agents are firms and the sensitivities are interpreted as the volatility of their productivities, this result suggests that the firms similar in productivity volatility are more likely to cooperate in acquiring and sharing information.

2) As shown in the proof, the sufficient and necessary condition (6.3.1) is equivalent to

$$\overline{a}_2' (\overline{a}_2 - \overline{a}_1) (\overline{a}_2 - \overline{a}_1)' \overline{a}_1 \leq 0 \quad (A.3.17)$$
this formula gives the geometric meaning of the necessary and sufficient condition: both angles $\omega_1$ and $\omega_2$ shown in the figure below are less or equal to $\pi/2$:

Note that the two agents are symmetric in condition (6.3.1) and (A.3.17). In situations where inequalities hold strictly, either agent can be the second observer. With this symmetry, I can discuss the "fairness" constraint in the comments of Proposition 4.

3) Note that the agents' benchmark productivities $b_1$, $b_2$ and information processing capacities $c_1$, $c_2$ do not affect the implementability of the social efficiency. These are the parameters describing the "ability" of the agents while $\alpha_1$ and $\alpha_2$ describe the relation of the agents' objectives. Proposition 3 shows that whether the agents can cooperate to achieve social efficiency only depends on their objectives, regardless of their abilities. We will find in Proposition 4 that "abilities" affect the agents' equilibrium payoffs.

Since $\overrightarrow{a}_2' (\overrightarrow{a}_2 - \overrightarrow{a}_1) + \overrightarrow{a}_1' (\overrightarrow{a}_1 - \overrightarrow{a}_2) = ||\overrightarrow{a}_2 - \overrightarrow{a}_1||^2 > 0$, at least one of $\overrightarrow{a}_2' (\overrightarrow{a}_2 - \overrightarrow{a}_1)$ and $\overrightarrow{a}_1' (\overrightarrow{a}_1 - \overrightarrow{a}_2)$ is strictly positive and either Agent $i$ with $\overrightarrow{a}_1' (\overrightarrow{a}_i - \overrightarrow{a}_{-i}) > 0$ can be the second observer.

Proposition 4: Suppose the conditions in Proposition 3 are satisfied (and without any loss of generality, let $\overrightarrow{a}_2' (\overrightarrow{a}_2 - \overrightarrow{a}_1) > 0$), then the budget balanced Bayesian incentive compatible mechanism that implements the social efficiency has the form:

a) Agent 2 is the second observer;
b) the social planner chooses the channels:

\[ \begin{pmatrix} \overline{\theta} \\
S_1 \\
S_2 \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\
0 \\
0 \end{pmatrix}, \begin{pmatrix} I & r_1 \cdot \overline{e'} \\
r_1 \cdot \overline{e'} & 1 \\
r_2 \cdot \overline{e'} & 0 \end{pmatrix} \right) \]

where \( r_1 = (1 - e^{-2c_1})^{1/2} \) and \( r_2 = (1 - e^{-2c_2})^{1/2} \);

c) the allocation rule is

\[ q_2 (s_1, s_2) = \begin{cases} 1 & \text{if } s_2 \geq \frac{(b_1 - b_2) / ||\overline{\theta}_2 - \overline{\theta}_1|| - r_1 s_1}{r_2 \cdot (1 - r_2^2)^{1/2}} \\
0 & \text{otherwise} \end{cases} \]

and the payment rule is

\[ t_2 (s_1, s_2) = \begin{cases} K + \frac{(b_1 - b_2) / ||\overline{\theta}_2 - \overline{\theta}_1|| - r_1 s_1}{r_2 \cdot (1 - r_2^2)^{1/2}} & \text{if } s_2 \geq \frac{(b_1 - b_2) / ||\overline{\theta}_2 - \overline{\theta}_1|| - r_1 s_1}{r_2 \cdot (1 - r_2^2)^{1/2}} \\
K & \text{otherwise} \end{cases} \]

\[ t_1 (s_1, s_2) = -t_2 (s_1, s_2) \]

where \( K \) is a constant that determines the initial value of the payment function. \( K \) doesn’t affect incentives and budget balance. Its value can be determined by other constraints, such as individual rationality constraint and the "fairness" constraint discussed below.

Proof: see the Appendix.

Proposition 5: Under the mechanism given in Proposition 4, the two agents’ total expected payoff is

\[ EU = b_i + \sqrt{1 - e^{-2(c_i + c_j)}} \cdot ||\overline{\theta}_2 - \overline{\theta}_1|| \cdot \int_{-\infty}^{\frac{b_j - b_i}{||\overline{\theta}_2 - \overline{\theta}_1|| - r_1 s_1}} \sqrt{1 - e^{-2(c_i + c_j)}} \cdot ||\overline{\theta}_2 - \overline{\theta}_1|| \cdot \Phi (x) \cdot dx \]

where \( i, j \in \{1, 2\}, i \neq j \), \( \Phi (x) \) is the cumulative distribution function of a standard normal random variable. The first observer’s expected payoff is

\[ EU_1 = K + \frac{(b_1 \cdot \overline{\theta}_2 - b_2 \cdot \overline{\theta}_1)' \overline{e'} - \overline{\theta}_1' \overline{e'}}{||\overline{\theta}_2 - \overline{\theta}_1||} \cdot \int_{-\infty}^{\frac{b_j - b_i}{||\overline{\theta}_2 - \overline{\theta}_1|| - r_1 s_1}} \Phi \left( \frac{x}{r_1} \right) \cdot \Phi \left( \frac{(b_1 - b_2) / ||\overline{\theta}_2 - \overline{\theta}_1|| - x}{r_2 \cdot (1 - r_2^2)^{1/2}} \right) dx \]

The proof is just calculating \( EU \) and \( EU_1 \) by definition. I omit it here. Notice that the second observer’s expected payoff \( EU_2 \) can be expressed by \( EU - EU_1 \).

Comments:

1) In the above mechanism I only consider the incentive compatible and the budget balance conditions. Now let’s consider the individual rationality condition. Since \( \Pr \left( \left\{ \overline{\theta} \in \mathbb{R}^n \mid \overline{\theta}_1' \overline{\theta} + b_1 < 0 \text{ and } \overline{\theta}_2' \overline{\theta} + b_2 < 0 \right\} \right) > 0 \), ex post individual rationality condition is not always satisfied. Hence we consider ex
ante individual rationality here, i.e. the agents decide whether to join the mechanism before observing their signals and are not allowed to quit after signing the contract. When the necessary and sufficient condition in Proposition 3 holds, the first equation in Proposition 5 shows that the total expected payoff is larger than $\max \{b_1, b_2\}$. Thus if $\max \{b_1, b_2\}$ is large enough, we can make both agents' ex ante payoff strictly positive through adjusting the constant $K$ in the mechanism given in Proposition 4. Therefore, individual rationality condition is satisfied. Moreover, this leads to a conclusion different from the usual cases in mechanism design literature, where no budget-balanced, individual rational and Bayesian incentive compatible mechanism exists to implement social efficiency when the good is privately owned (Myerson and Satterthwaite (1983)). In this model, even if the good is privately owned by the agents, the individual rationality condition can also hold provided that the condition in Proposition 3 is satisfied. Suppose Agent 1 owns $k$ shares and Agent 2 owns $(1 - k)$ shares of the good, $k \in [0, 1]$. Without this mechanism, the agents’ expected payoff is $k \cdot b_1$ and $(1 - k) \cdot b_2$, respectively, while the total expected payoff under the proposed mechanism is larger than $\max \{b_1, b_2\} \geq k \cdot b_1 + (1 - k) \cdot b_2$. Hence we can always make both agents strictly better off through choosing a proper value for the constant $K$. The key reason for this to happen is that cooperation in producing information greatly improves the efficiency of allocation and thus generates enough surplus to improve both agents’ welfare.

2) It is worth noticing that the constant \( \frac{(b_1, a_2 - b_2, a_1)^2}{||a_2 - a_1||} \) can be interpreted as price of the good when $k = 1$ or 0. This price only depends on the agents’ valuations (or say, preferences described by $a_i$, $b_i$ $i = 1$ or 2), regardless of their capacity. In other words, when social efficiency is achieved, the agents’ "IQ" does not affect the price. However, as shown in Propostion 4, the agents’ capacity do affect the allocation rule. Maintain other parameters constant, the higher Agent $i$’s capacity, the higher the probability he wins the good. This is the only channel through which capacity affects the agents payoffs.

3) The total payoff consists of three parts. The first part is the benchmark valuation, which is at most $\max \{b_1, b_2\}$. The second part comes from the freedom of allocation, i.e. allocating the good to the agent who values it more improves total welfare. The third part results from the cooperation in producing information. The second part is what we usually encounter in economic analysis, but the third one is a new feature. It is worth noticing the relation between the second and third effects. Cooperation in information production enhances the efficiency of the allocation, while the possibility of improving social welfare through allocating provides the usefulness of capacity. If $a_2 = a_1$, the problem is trivial and capacity is of no use. On the other hand, if $c_1 = c_2 = 0$, the allocative effect can not be employed. The first equation in Proposition shows that even if one agent’s capacity is zero, it is still beneficial for the other one to cooperate with him. All the three factors discussed above contribute to this result.

4) As mentioned before Proposition 4, both agents can be the first observer in generic situations. We can choose a proper value for the constant $K$ such
that both agents are indifferent between being the first and the second observer. We call this the "fairness" constraint and the corresponding constant $K_{\text{fair}}$. Proposition 5 implies

$$K_{\text{fair}} = \frac{1}{2} \cdot \left[ b_1 + \sqrt{1 - e^{-2(c_1 + c_2)}} \cdot \| \overrightarrow{a}_2 - \overrightarrow{a}_1 \| \cdot \int_{-\infty}^{b_1} e^{f(x)} \cdot \Phi\left( \frac{x}{r_1} \right) \cdot \Phi\left( \frac{x}{r_2} \right) \cdot dx \right]$$

It is obvious that $K_{\text{fair}}$ renders both agents enjoy half of the total expected payoff.

### VII. The Public Good Scenario

Denote the cost of the public good by $g \in \mathbb{R}$. The decision rule is $q \in [0, 1]$, which specifies the probability of producing the good. All the other notations are the same as those in the previous auction-like scenario. The results here are very similar to the previous ones.

Proposition 6: The social planner achieves "first best" if and only if

a) $\sum_{\overrightarrow{S}} \alpha \cdot \overrightarrow{a} = \overrightarrow{e}$

where $\overrightarrow{e} = (\overrightarrow{a}_2 + \overrightarrow{a}_1) / \| \overrightarrow{a}_2 + \overrightarrow{a}_1 \|$, $\sum_{i=1}^{m} \alpha_i^2 = 1 - e^{-2c}$ and $m \geq 1$ is the dimension of the signal $\overrightarrow{S}$.

b) the decision rule is

$$q = \begin{cases} 
1 & \text{if } E \left[ (\overrightarrow{a}_2 + \overrightarrow{a}_1)' \overrightarrow{\theta} + (b_2 + b_1) \mid \overrightarrow{S} \right] \geq g \\
0 & \text{otherwise}
\end{cases}$$

Moreover, the minimized conditional variance is

$$\min_{\overrightarrow{S}} \text{Var} \left[ (\overrightarrow{a}_2 + \overrightarrow{a}_1)' \overrightarrow{\theta} \mid \overrightarrow{S} \right] = \| \overrightarrow{a}_2 + \overrightarrow{a}_1 \|^2 \cdot e^{-2c},$$

where $c = c_1 + c_2$.

Proof: omitted.

This proposition is the counterpart of Proposition 2 in the auction-like scenario.

Proposition 7: Under non-trivial conditions $\overrightarrow{a}_1 \neq -\overrightarrow{a}_2$ and $\| \overrightarrow{a}_i \| \neq 0$, $i = 1$ or $2$, there exists a budget balanced Bayesian incentive compatible mechanism implementing the social efficiency (i.e. the First Best) if and only if

$$-\cos \alpha \leq \min \left\{ \frac{\| \overrightarrow{a}_2 \|}{\| \overrightarrow{a}_1 \|}, \frac{\| \overrightarrow{a}_1 \|}{\| \overrightarrow{a}_2 \|} \right\}$$

where

$$\cos \alpha = \frac{\overrightarrow{a}_2 \cdot \overrightarrow{a}_1}{\| \overrightarrow{a}_2 \| \cdot \| \overrightarrow{a}_1 \|}$$

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i.e. $\alpha$ is the angle between the two vectors $\vec{a}_1$ and $\vec{a}_2$.

Proof: omitted.

This proposition is the counterpart of Proposition 3 in the auction-like scenario.

Comments:

1) Notice that except the minus sign before $\cos \alpha$, this result is almost the same as Proposition 3. Actually, since in the public good scenario $\cos \alpha$ measures the extent of compatibility between the two agents’ objectives and thus $-\cos \alpha$ measures the extent of conflict, Proposition 3 and 7 do convey the same intuition: maintaining cooperation requires the agents to have similar sensitivities to the shock.

2) This necessary and sufficient condition is equivalent to

$$\vec{a}_2' (\vec{a}_2 + \vec{a}_1) (\vec{a}_2 + \vec{a}_1)' \vec{a}_1 \geq 0$$

which gives rise to a geometric interpretation of the necessary and sufficient condition: both angles $\omega_1$ and $\omega_2$ shown in the figure below are less or equal to $\pi/2$:

Moreover, this result can be generalized to $n$-agents public good scenario.

Proposition 8: Under non-trivial conditions $\sum_{i=1}^{n} \vec{a}_i' \neq 0$ and $||\vec{a}_i|| \neq 0$, $i = 1, 2, \ldots, n$, there exists a budget balanced Bayesian incentive compatible mechanism implementing social efficiency (i.e. the First Best) if and only if

$$\vec{a}_i' \sum_{j=1}^{n} \vec{a}_j \geq 0 \quad i = 1, 2, \ldots, n$$
Proof: omitted.

This proposition does not have a simple counterpart in the auction-like scenario\(^{17}\). The reason is that, in order to achieve social efficiency in the public good scenario, we only need to compare \(\hat{\theta} S \sum_{j=1}^{n} \overline{a}_j + \sum_{j=1}^{n} b_j \) and \(g\), while in the auction-like scenario we have to compare 0 and each of the \(n-1\) terms 
\((\overline{a}_j - \overline{a}_{j-1}) \hat{\theta} + (b_j - b_{j-1})\), \(j = 2, 3, \ldots, n\).

Proposition 9: Suppose the conditions in Proposition 7 are satisfied, then the budget balanced Bayesian incentive compatible mechanism that implements the social efficiency has the form (denote the \(i\)th observer as Agent \(i\), \(i = 1\) or 2):

a) the social planner chooses the channels:

\[
\begin{pmatrix}
\overline{\theta} \\
S_1 \\
S_2
\end{pmatrix} \sim N \left( \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
I & r_1 \cdot \overrightarrow{e} & r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot \overrightarrow{e}

r_1 \cdot \overrightarrow{e} & 1 & 0

r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot \overrightarrow{e} & 0 & 1
\end{pmatrix} \right)
\]

where \(r_1 = \left(1 - e^{-2c_1}\right)^{1/2}\), \(r_2 = \left(1 - e^{-2c_2}\right)^{1/2}\) and \(\overrightarrow{e} = (\overrightarrow{a}_2 + \overrightarrow{a}_1) / ||\overrightarrow{a}_2 + \overrightarrow{a}_1||\);

c) the allocation rule is

\[
q(s_1, s_2) = \begin{cases}
1 & \text{if } s_2 \geq \frac{(g-b_1-b_2)/||\overrightarrow{a}_2 + \overrightarrow{a}_1|| - r_1 \cdot s_1}{r_2 \cdot \left(1 - r_1^2\right)^{1/2}} \\
0 & \text{otherwise}
\end{cases}
\]

and the payment rule is

\[
t_2(s_1, s_2) = \begin{cases}
K + \frac{(b_2 \cdot \overrightarrow{a}_1 - (b_1 - g) \cdot \overrightarrow{a}_2) \cdot \overrightarrow{e}}{||\overrightarrow{a}_2 + \overrightarrow{a}_1||} & \text{if } s_2 \geq \frac{(g-b_1-b_2)/||\overrightarrow{a}_2 + \overrightarrow{a}_1|| - r_1 \cdot s_1}{r_2 \cdot \left(1 - r_1^2\right)^{1/2}} \\
K & \text{otherwise}
\end{cases}
\]

\[
t_1(s_1, s_2) = q(s_1, s_2) \cdot g - t_2(s_1, s_2)
\]

\[
= \begin{cases}
g - K - \frac{(b_2 \cdot \overrightarrow{a}_1 - (b_1 - g) \cdot \overrightarrow{a}_2) \cdot \overrightarrow{e}}{||\overrightarrow{a}_2 + \overrightarrow{a}_1||} & \text{if } s_2 \geq \frac{(g-b_1-b_2)/||\overrightarrow{a}_2 + \overrightarrow{a}_1|| - r_1 \cdot s_1}{r_2 \cdot \left(1 - r_1^2\right)^{1/2}} \\
-K & \text{otherwise}
\end{cases}
\]

where \(K\) is a constant that determines the initial value of the payment function. \(K\) doesn’t affect incentives and budget balance. Its value can be determined by other constraints, such as individual rationality constraint and the "fairness" constraint discussed in Section VI.

Proof: omitted.

VIII. Conclusion

I have provided the necessary and sufficient condition for the existence of a budget-balanced Bayesian incentive compatible mechanism implementing social efficiency when the agents are constrained in their capacity of acquiring

\(^{17}\) The auction-like scenario with \(n \geq 3\) agents has its own generalization of Proposition 3, but the expression is complicated.
information. Individual rationality condition is automatically satisfied in these mechanisms. Under these mechanisms, the agents produce information in a serial manner, which generates the largest amount of information. The necessary and sufficient condition says that social efficiency is implementable if and only if the extent of conflict between the two agents' objectives is upper bounded by the smaller ratio of their sensitivities, i.e. the implementability is totally determined by the relation between the agents' objectives. Although the agents' abilities (benchmark valuations and capacity of information processing) do not affect the implementability of social efficiency, they affect the equilibrium payoffs. Cooperation in producing information makes the equilibrium payoffs large enough to satisfy the individual rationality condition.

Several topics worth further research. The first one is to study the senario where the agents are allowed to choose the channels by themselves. The second one is to study the case of allocating multiple goods among multiple agents. The third one is to directly study the general games with endogenous information acquisition. Actually, I have already done some research about these topics and got some interesting results. Due to the limited time before the deadline of the second-year paper, I can not incorporate them here. I plan to write several other papers to address these topics.

References:


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Appendix:
Proof of Proposition 2: b) is obvious, we just prove a). Note that
\[ \text{Var} \left( \frac{\theta - \bar{\theta}}{\sigma} \right) = \left| \frac{\theta - \bar{\theta}}{\sigma} \right|^2 \cdot e^{\theta^T (I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}) e} \]
and
\[ I \left( \bar{\theta}; \bar{S} \right) = -\frac{1}{2} \cdot \ln |I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}| = e^{18} \]
i.e.
\[ |I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}| = e^{-2c} \]
Thus the problem can be restated as
\[ \min_{\Sigma_{\bar{\theta}}} e^{\theta^T (I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}) e} \quad \text{s.t.} \quad |I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}| = e^{-2c} \]
\[ \cdot I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1} \]
is a real and symmetric matrix,
\[ \exists \text{ orthogonal matrix } P \text{ s.t. } I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1} = P^T \Lambda P , \text{ where } \Lambda \text{ is a } n \times n \text{ diagonal matrix with diagonal entries } \lambda_1, \lambda_2, \ldots, \lambda_n \, . \]
Actually, \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1} \) and \( \prod_{j=1}^{n} \lambda_j = |\Lambda| = |I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1}| = e^{-2c} \). Also note that \( \lambda_j > 0, j = 1, 2, \ldots, n \), since \( I - \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1} = \text{Var} \left( \frac{\theta}{\sigma} \right) \) is positive definite.
On the other hand, \( I - P^T \Lambda P = \Sigma_{\theta}^{-1} \Sigma_{\bar{\theta}}^{-1} \) is positive semi-definite, i.e. all of its eigenvalues \( \mu_j, j = 1, 2, \ldots, n \) are non-negative. Let \( \mu \) be an arbitrary eigenvalue of \( I - P^T \Lambda P \), then we have
\[ 0 = |\mu \cdot I - I + P^T \Lambda P| = |(\mu - 1) \cdot I + P^T \Lambda P| = |(\mu - 1) \cdot PIP^T + (PP^T) \Lambda (PP^T)| = |(\mu - 1) \cdot I + \Lambda| \]
i.e. the eigenvalues of \( I - P^T \Lambda P \) can be expressed as \( \mu_j = 1 - \lambda_j, j = 1, 2, \ldots, n \). Since \( \mu_j \geq 0 \), we have \( \lambda_j \leq 1, j = 1, 2, \ldots, n \).
Now we restate the problem as
\[ \min \left( P \bar{\theta} \right)^T \Lambda (P \bar{\theta}) \]
s.t. \( P^T P = I, \prod_{j=1}^{m} \lambda_j = e^{-2c} \) and \( \lambda_j \in (0,1], j = 1, 2, \ldots, n \).
Let \( P' = (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_n) \), then
\[ (P \bar{\theta})^T \Lambda (P \bar{\theta}) = \sum_{j=1}^{n} \lambda_j \cdot (\bar{\theta}_j \bar{\theta}_j)^2 \]
\[\^{18}\text{Here we can let the constraint } I \left( \bar{\theta}; \bar{S} \right) \leq c \text{ bind, since it is always beneficial to collect more information.}\]
Since $P$ is orthogonal, we have $\vec{p}_j \cdot \vec{p}_k = \delta (j, k)$, where $\delta (j, k) = \{\begin{array}{ll} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{array}$

Note $\sum_{j=1}^{n} \lambda_j \cdot (\overrightarrow{p}_j \cdot \overrightarrow{e})^2 > 0$ and $\lambda_j \in (0, 1]$, then by the concavity of the logarithmic function, we have

$$
\ln \left( (\overrightarrow{P} \overrightarrow{e})' A (\overrightarrow{P} \overrightarrow{e}) \right) = \ln \left( \sum_{j=1}^{n} \lambda_j \cdot (\overrightarrow{p}_j \cdot \overrightarrow{e})^2 \right) \\
\geq \sum_{j=1}^{n} (\overrightarrow{p}_j \cdot \overrightarrow{e})^2 \cdot \ln \lambda_j \geq \sum_{j=1}^{n} (\overrightarrow{p}_j \cdot \overrightarrow{e})^2 \cdot \ln \left( \min_{k=1,2,\ldots,n} \lambda_k \right) \\
= \ln \left( \min_{k=1,2,\ldots,n} \lambda_k \right)
$$

Thus, $\prod_{j=1}^{n} \lambda_j = e^{-2c}$ and $\lambda_j \in (0, 1]$, $j = 1, 2, \ldots, n$ , \text{ and } \ln (\min_{k=1,2,\ldots,n} \lambda_k) \geq \ln (e^{-2c})$. Therefore, $(\overrightarrow{P} \overrightarrow{e})' A (\overrightarrow{P} \overrightarrow{e}) \geq e^{-2c}$. Note that this minimum can be obtained if and only if $\exists j$ s.t. $\lambda_j = e^{-2c}$, $(\overrightarrow{p}_j \overrightarrow{e})^2 = 1$ and $\forall k \neq j$, $\lambda_k = 1$, $\overrightarrow{p}_k \overrightarrow{e} = 0$. Without loss of any generality, let $\lambda_1 = e^{-2c}$ and $\overrightarrow{p}_1 \overrightarrow{e} = 1$. Since $\forall j, k$, $\overrightarrow{p}_j \overrightarrow{p}_k = \delta (j, k)$ and $||\overrightarrow{e}|| = 1$, we have $\overrightarrow{p}_1 = \overrightarrow{e}$.

Let $\overrightarrow{p}_i = (P_{i1} \quad P_{i2} \quad \cdots \quad P_{in})'$, $i = 1, 2, \cdots, n$, then $P$ can be expressed by $(P_{ij})_{n \times n}$. Thus the $(i, k)$ entry of matrix $P'AP$ is

$$
(P'AP)_{ik} = \sum_{j=1}^{n} \lambda_j \cdot P_{ji} \cdot P_{jk} \\
= e^{-2c} \cdot P_{i1} \cdot P_{1k} + \sum_{j=2}^{n} \lambda_j \cdot P_{ji} \cdot P_{jk}
$$

Note that $\lambda_j = 1$, $j = 2, 3, \cdots, n$ and $\sum_{j=1}^{n} P_{ji} \cdot P_{jk} = \delta (i, k)$, we have

$$
(P'AP)_{ik} = e^{-2c} \cdot P_{i1} \cdot P_{1k} + \sum_{j=2}^{n} P_{ji} \cdot P_{jk} \\
= e^{-2c} \cdot P_{i1} \cdot P_{1k} + \delta (i, k) \cdot P_{i1} \cdot P_{1k} \\
= \delta (i, k) \cdot (1 - e^{-2c}) \cdot P_{i1} \cdot P_{1k}
$$

$$
(I - P'AP)_{ik} = \delta (i, k) - \delta (i, k) + (1 - e^{-2c}) \cdot P_{i1} \cdot P_{1k} \\
= (1 - e^{-2c}) \cdot P_{i1} \cdot P_{1k}
$$

Because the columns of an orthogonal matrix also form an orthonormal basis of $\mathbb{R}^n$. 

24
implies that
t of argument we have concludes our proof.

second observer. Denote !
tion 2, the mechanism can implement the First Best only if the joint distribution

As discussed in Section V, there are many other forms of joint distribution with non-zero

Proof of Proposition 3: Without any loss of generality, let Agent 2 be the
second observer. Denote \( \bar{e} = (\bar{a}_2 - \bar{a}_1) / \|\bar{a}_2 - \bar{a}_1\| \). As proved in Proposition 2, the mechanism can implement the First Best only if the joint distribution of \( \bar{\theta} \), \( S_1 \) and \( S_2 \) has the form\(^\text{20}\)

\[
\begin{pmatrix}
\bar{\theta} \\
S_1 \\
S_2
\end{pmatrix}
\sim N
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
, \begin{pmatrix}
I \\
\alpha_1 \cdot \bar{e} \\
\alpha_2 \cdot \bar{e}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \cdot \bar{e}' \\
1 \\
0
\end{pmatrix}
\]

and \( \alpha_1^2 + \alpha_2^2 = 1 - e^{-2c_1 + c_2} \). Since \( c_1 = I \left( \bar{\theta} ; S_1 \right) = -\frac{1}{2} \ln \left( 1 - \alpha_1^2 \right) \), we have \( \alpha_1 = (1 - e^{-2c_1})^{1/2} \) and \( \alpha_2 = [e^{-2c_1} - e^{-2(c_1 + c_2)}]^{1/2} \). Note that the signs of \( \alpha_1 \) and \( \alpha_2 \) don’t affect the agents’ inference of \( \bar{\theta} \), just let \( \alpha_1 \) and

\(^\text{20}\)As discussed in Section V, there are many other forms of joint distribution with non-zero expection and non-identity covariance of \( (S_1, S_2) \). All these forms are equivalent in that they generate the same inference of \( \bar{\theta} \).
$\alpha_2$ be positive. Let $r_1 = \alpha_1 = (1 - e^{-2c_1})^{1/2}$ and $r_2 = (1 - e^{-2c_2})^{1/2}$, then $\alpha_2 = \left[ e^{-2c_1} - e^{-2(c_1+c_2)} \right]^{1/2} = r_2 \cdot (1 - r_1^2)^{1/2}$. Actually, $r_1$ can be interpreted as the "correlation" between $\bar{\theta}$ and $S_1$, and $r_2$ is the "correlation" between $\bar{\theta}$ and $S_2$ conditional on $S_1$. Thus the joint distribution of $\bar{\theta}$, $S_1$ and $S_2$ can be restated as

$$
\begin{pmatrix}
\bar{\theta} \\
S_1 \\
S_2
\end{pmatrix}
\sim N
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
I & r_1 \cdot \bar{c}' & r_2 \cdot (1 - r_1^2)^{1/2} \cdot \bar{c}' \\
r_1 \cdot \bar{c}' & 1 & 0 \\
r_2 \cdot (1 - r_1^2)^{1/2} \cdot \bar{c}' & 0 & 1
\end{pmatrix}
$$

(A.3.1)

Note that $S_1$ and $S_2$ are unconditionally independent and conditionally dependent. The unconditional independence is not essential here. We can always make $S_2$ unconditionally correlate to $S_1$ through some linear transform of $(S_1, S_2)$, but this makes no sense since it won’t affect the agents’ inference of $\bar{\theta}$ . However, the conditional dependence between $S_1$ and $S_2$ is essential (recall that conditional independence is possible only when the two agents make their observations independently, i.e. non-cooperation in information production). It results from the fact that $S_2$ is made based on $S_1$, i.e. it comes from the "timing" of total cooperation in producing information.

Now let’s focus on the remaining part of the mechanism: the allocation rule $q_i(s_1, s_2)$ and payments $t_i(s_1, s_2)$ \(^{21}\), where $i = 1$ or $2$. Agent $i$ wins the good with probability $q_i(s_1, s_2)$ and pays $t_i(s_1, s_2)$ when the reported signals are $(s_1, s_2)$. Then Agent $i$’s expected payoff is $E[v_i \cdot q_i(s_1, s_2) - t_i(s_1, s_2) | Agent i’ s information]$. To achieve the first best, the agents’ reports should one-to-one correspond to their true values. Since the information structure has been fixed at this stage, the revelation principle is applicable and we only need to consider incentive compatible mechanisms. When truth telling is obtained in the equilibrium, social efficiency requires

$$
q_2(s_1, s_2) = \begin{cases} 
1 \text{ if } E[(\bar{a}_2 - \bar{a}_1) \cdot \bar{\theta} + (b_2 - b_1) | s_1, s_2] \geq 0 \\
0 \text{ otherwise}
\end{cases}
$$

$$
q_1(s_1, s_2) = 1 - q_2(s_1, s_2)
$$

i.e.

$$
q_2(s_1, s_2) = \begin{cases} 
1 \text{ if } s_2 \geq \frac{(b_2 - b_1) \cdot |\bar{a}_2 - \bar{a}_1| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \\
0 \text{ otherwise}
\end{cases}
$$

(A.3.2)

$$
q_1(s_1, s_2) = 1 - q_2(s_1, s_2)
$$

\(^{22}\). Since the allocation rule is totally determined by the requirement of social efficiency, we have to provide incentives for truth telling through the payments $t_i(s_1, s_2)$, $i = 1$ or $2$.

\(^{21}\)We use capital letters to refer to the random variables and the corresponding lower-case letters to refer to the specific values such random variables take.

\(^{22}\)In the case of a tie, we assume that the resource is allocated to Agent 2. This assumption is not critical to our results, since the "tie" is a zero-probability event and the social planner is indifferent in the case of a tie.
First we study the incentive compatible condition for Agent 2.

Suppose the mechanism is incentive compatible with respect to Agent 2.

Given that Agent 1 tells the true value of his signal $s_1$, Agent 2’s expected payoff when he observes $s_2$ but reports $s'_2$ is

$$U_2(s'_2|s_1,s_2) = \left[\frac{\partial u'_2}{\partial \theta} \cdot q_2(s_1,s'_2) + b_2\right] \cdot q_2(s_1,s'_2) - t_2(s_1,s'_2)$$

$$= \left[\frac{\partial u'_2}{\partial \theta} \cdot \left(r_1 \cdot s_1 + r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot s_2\right) + b_2\right] \cdot q_2(s_1,s'_2) - t_2(s_1,s'_2)$$

the incentive compatible condition for Agent 2 is: $\forall s_1, s_2$ and $s'_2$, $U_2(s'_2|s_1,s_2) \geq U_2(s'_2|s_1,s_2)$ and $U_2(s_2|s_1,s'_2) \leq U_2(s'_2|s_1,s_2)$, which implies

$$\frac{\partial u'_2}{\partial \theta} \cdot r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot (s_2 - s'_2) \cdot [q_2(s_1,s_2) - q_2(s_1,s'_2)] \geq 0$$

$$\therefore r_2 \cdot \left(1 - r_1^2\right)^{1/2} > 0 \text{ and } q_2(s_1,s_2) \text{ is increasing with respect to } s_2 \text{ as required by the social efficiency.}$$

we need $\frac{\partial u'_2}{\partial \theta} \cdot r_2 > 0$. Actually, this is not really a restriction, since we can always make it satisfied. Note that since $\frac{\partial u'_2}{\partial \theta} \cdot (\theta_2 - \theta_1) + \frac{\partial u'_1}{\partial \theta} \cdot (\theta_1 - \theta_2) = \left[\frac{\partial u'_2}{\partial \theta} - \frac{\partial u'_1}{\partial \theta}\right] > 0$, at least one of $\frac{\partial u'_2}{\partial \theta} - \frac{\partial u'_1}{\partial \theta}$ is strictly positive. Without any loss of generality, let $\frac{\partial u'_2}{\partial \theta} - \frac{\partial u'_1}{\partial \theta}$ > 0 and Agent 2 be the second observer. Hence, we always have

$$\frac{\partial u'_2}{\partial \theta} > 0 \quad (A.3.3)$$

i.e.

$$\frac{\partial u'_2}{\partial \theta} (\theta_2 - \theta_1) > 0 \quad (A.3.3')$$

Define

$$U_2(s_1,s_2) = \max_{s'_2} U_2(s'_2|s_1,s_2) = U_2(s_2|s_1,s_2)$$

Note that $U_2(s_1, \cdot)$ is the maximum of increasing and affine, hence convex functions. The maximum of increasing functions is increasing, and the maximum of convex functions is convex. Therefore $U_2(s_1, \cdot)$ is increasing and convex. Convex functions are indifferentiable in at most countably many points, thus by the envelope theorem,

$$\frac{\partial U_2(s_1,s_2)}{\partial s_2} = \frac{\partial u'_2}{\partial \theta} \cdot r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot q_2(s_1,s_2)$$

whenever $U_2(s_1,s_2)$ is differentiable with respect to $s_2$. Moreover,

$$U_2(s_1,s_2) = U_2(s_1,-\infty)+\frac{\partial u'_2}{\partial \theta} \cdot r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot \int_{-\infty}^{s_2} q_2(s_1,s_2) \cdot d\bar{s_2} \quad (A.3.4)$$

23 We assume $\theta_2 \neq \theta_1$, since $\theta_2 = \theta_1$ is the trivial case.

24 This means Agent 2’s signal is positively correlated with his valuation, since

$$\text{cov}(\theta'_2, S_2) = \frac{\partial u'_2}{\partial \theta} \text{cov}(\theta, S_2)$$

$$= r_2 \cdot \left(1 - r_1^2\right)^{1/2} \cdot \frac{\partial u'_2}{\partial \theta} > 0$$

We can also make $S_2$ negatively correlated with Agent 2’s valuation by letting $r_2 < 0$ and this won’t affect the inference of $\theta'$. 

27
\[ U_2 (s_1, s_2) = \left[ \overline{a}_2' \overline{e} \cdot (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2) + b_2 \right] \cdot q_2 (s_1, s_2) - t_2 (s_1, s_2) \]

and \( q_2 (s_1, -\infty) = 0 \) implies \( U_2 (s_1, -\infty) = - t_2 (s_1, -\infty) \),

\[
t_2 (s_1, s_2) = t_2 (s_1, -\infty) + \left[ \overline{a}_2' \overline{e} \cdot (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2) + b_2 \right] \cdot q_2 (s_1, s_2) - \overline{a}_2' \overline{e} \cdot r_2 \cdot (1 - r_1^2)^{1/2} \cdot \int_{-\infty}^{s_2} q_2 (s_1, \hat{s}_2) \cdot d\hat{s}_2 \quad (A.3.5)\]

Note that \((A.3.3)\) and \((A.3.4)\)(or \((A.3.5)\)) is also sufficient to make Agent 2 tell the truth, since

\[
U_2 (s_1, s_2) \geq U_2 (s'_2 | s_1, s_2) \]

iff

\[
U_2 (s_1, s_2) \geq \left[ \overline{a}_2' \overline{e} \cdot (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2) + b_2 \right] \cdot q_2 (s_1, s_2') - t_2 (s_1, s_2') \]

iff

\[
U_2 (s_1, s_2) \geq \left[ \overline{a}_2' \overline{e} \cdot (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2) + b_2 \right] \cdot q_2 (s_1, s_2') - \overline{a}_2' \overline{e} \cdot r_2 \cdot (1 - r_1^2)^{1/2} \cdot \int_{s_2}^{s_2'} q_2 (s_1, \hat{s}_2) \cdot d\hat{s}_2 \]

iff

\[
U_2 (s_1, s_2) - U_2 (s_1, s_2') \geq \overline{a}_2' \overline{e} \cdot r_2 \cdot (1 - r_1^2)^{1/2} \cdot (s_2 - s_2') \cdot q_2 (s_1, s_2') \]

iff

\[
\overline{a}_2' \overline{e} \cdot r_2 \cdot (1 - r_1^2)^{1/2} \cdot \int_{s_2}^{s_2'} q_2 (s_1, \hat{s}_2) \cdot d\hat{s}_2 \geq \overline{a}_2' \overline{e} \cdot r_2 \cdot (1 - r_1^2)^{1/2} \cdot \int_{s_2'}^{s_2} q_2 (s_1, s_2') \cdot d\hat{s}_2 \]

iff

\[
\int_{s_2'}^{s_2} q_2 (s_1, \hat{s}_2) \cdot d\hat{s}_2 \geq \int_{s_2'}^{s_2} q_2 (s_1, s_2') \cdot d\hat{s}_2 \quad (A.3.6)\]

Since \((A.3.6)\) holds for all \( s_2 \) and \( s_2' \), this proves the sufficiency.

We continue to derive \( t_2 (s_1, s_2) \).

When

\[
s_2 \geq \frac{(b_1 - b_2) / \left\| \overline{a}' - \overline{a}_1 \right\| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}}
\]

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substitute (A.3.2) into (A.3.5) we get

\[ t_2 (s_1, s_2) = t_2 (s_1, -\infty) + \left[ \frac{\overrightarrow{a_2 \bar{e} \cdot (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2)} + b_2}{|| \overrightarrow{a_2 - a_1} ||} \right] \cdot q_2 (s_1, s_2) \]

\[-\overrightarrow{a_2 \bar{e} \cdot \left[ r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2 + \frac{(b_2 - b_1)}{|| \overrightarrow{a_2 - a_1} ||} \right] \]

\[ = t_2 (s_1, -\infty) + b_2 - \overrightarrow{a_2 \bar{e}} \cdot \frac{(b_2 - b_1)}{|| \overrightarrow{a_2 - a_1} ||} \]

\[ = t_2 (s_1, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2 - b_2 \cdot a_1} \cdot \bar{e})}{|| \overrightarrow{a_2 - a_1} ||} \]

when

\[ s_2 < \frac{(b_1 - b_2) / || \overrightarrow{a_2 - a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \]

(A.3.2) and (A.3.5) imply that \( t_2 (s_1, s_2) = t_2 (s_1, -\infty) \). Therefore, we have

\[ t_2 (s_1, s_2) = \begin{cases} 
  t_2 (s_1, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2 - b_2 \cdot a_1} \cdot \bar{e})}{|| \overrightarrow{a_2 - a_1} ||} & \text{if } s_2 \geq \frac{(b_1 - b_2) / || \overrightarrow{a_2 - a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \\
  t_2 (s_1, -\infty) & \text{otherwise}
\end{cases} \] (A.3.7)

and "budget balance" implies that

\[ t_1 (s_1, s_2) \]

\[ = -t_2 (s_1, s_2) = \begin{cases} 
  -t_2 (s_1, -\infty) - \frac{(b_1 \cdot \overrightarrow{a_2 - b_2 \cdot a_1} \cdot \bar{e})}{|| \overrightarrow{a_2 - a_1} ||} & \text{if } s_2 \geq \frac{(b_1 - b_2) / || \overrightarrow{a_2 - a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \\
  -t_2 (s_1, -\infty) & \text{otherwise}
\end{cases} \] (A.3.8)

Now we can turn to Agent 1’s incentive compatible condition.

Note that the joint distribution of \( \overrightarrow{\theta} \) and \( S_2 \) conditional on \( S_1 \) is

\[
\begin{pmatrix}
\overrightarrow{\theta} \\
S_2
\end{pmatrix} | S_1 \sim N \left( \begin{pmatrix} r_1 \cdot s_1 \cdot \bar{e} \\ 0 \end{pmatrix}, \begin{pmatrix} I - r_1^2 \cdot \bar{e} \cdot \bar{e}' & r_2 \cdot (1 - r_1^2)^{1/2} \cdot \bar{e}' \\ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \bar{e}' & 1 \end{pmatrix} \right)
\]

Agent 1’s expected payoff when he receives \( s_1 \) but reports \( s_1' \) is

\[ U_1 (s_1'|s_1) = E \left[ \left( \overrightarrow{a_1} \overrightarrow{\bar{\theta}} + b_1 \right) \cdot q_1 (s_1', s_2) - t_1 (s_1', s_2) | s_1 \right] \]

\[ = E \left[ \left( \overrightarrow{a_1} \overrightarrow{\bar{\theta}} + b_1 \right) \cdot q_1 (s_1', s_2) | s_1 \right] + E \left[ t_2 (s_1', s_2) | s_1 \right] \]
by the law of iterated expectation

\[
E \left[ \left( \mathbf{a}'_1 \mathbf{\hat{\theta}} + b_1 \right) \cdot q_1 (s'_1, s_2) | s_1 \right] \\
= E \left[ \mathbf{a}'_1 E \left( \mathbf{\hat{\theta}} | s_1, s_2 \right) + b_1 \right) \cdot q_1 (s'_1, s_2) | s_1 \right] \\
= E \left[ \mathbf{a}'_1 e^t (r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2) + b_1 \right) \cdot q_1 (s'_1, s_2) | s_1 \right] \\
= \int_{-\infty}^{(b_1 - b_2)/||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1} \mathbf{a}'_1 e^t \cdot \left( r_1 \cdot s_1 + r_2 \cdot (1 - r_1^2)^{1/2} \cdot s_2 \right) + b_1 \right) \cdot \phi (s_2) \cdot ds_2 \\
= \mathbf{a}'_1 e^t \cdot r_1 \cdot s_1 + b_1 \right) \cdot \phi \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \\
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \mathbf{a}'_1 e^t \cdot \int_{-\infty}^{(b_1 - b_2)/||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1} \phi (s_2) \cdot ds_2 \\
\]

where \( \phi (\cdot) \) and \( \phi (\cdot) \) are the cumulative distribution and density functions of the standard normal distribution, respectively.

By (A.3.7),

\[
E \left[ t_2 (s'_1, s_2) | s_1 \right] \\
= t_2 (s'_1, -\infty) + \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot \left[ 1 - \Phi \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \right] \\
\]

\[
U_1 (s'_1 | s_1) \\
= E \left[ \mathbf{a}'_1 \mathbf{\hat{\theta}} + b_1 \right) \cdot q_1 (s'_1, s_2) | s_1 \right] + E \left[ t_2 (s'_1, s_2) | s_1 \right] \\
= \left( \mathbf{a}'_1 e^t \cdot r_1 \cdot s_1 + b_1 \right) \cdot \phi \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \\
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \mathbf{a}'_1 e^t \cdot \int_{-\infty}^{(b_1 - b_2)/||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1} \phi (s_2) \cdot ds_2 \\
+ t_2 (s'_1, -\infty) + \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{||\mathbf{a}_2 - \mathbf{a}_1||} \right) \cdot \left[ 1 - \Phi \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \right] \\
= t_2 (s'_1, -\infty) + \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{||\mathbf{a}_2 - \mathbf{a}_1||} \right) \\
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \mathbf{a}'_1 e^t \cdot \int_{-\infty}^{(b_1 - b_2)/||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1} \phi (s_2) \cdot ds_2 \\
+ \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{||\mathbf{a}_2 - \mathbf{a}_1||} \right) \cdot \phi \left( \frac{(b_1 - b_2) / ||\mathbf{a}_2 - \mathbf{a}_1|| - r_1 s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \\
\]

The third equality comes from (A.3.2).
\begin{align*}
&= t_2(s_1', -\infty) + \frac{(b_1 \cdot \overrightarrow{a}_2 - b_2 \cdot \overrightarrow{a}_1)' \overrightarrow{e}}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||} \\
&\quad + r_2 \cdot (1 - r^2_1)^{1/2} \cdot \overrightarrow{a}_1' \overrightarrow{e} \cdot \int_{-\infty}^{\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1'}{r_2 \cdot (1 - r^2_1)^{1/2}}} \varphi(s_2) \cdot ds_2 \\
&\quad + \overrightarrow{a}_1' \overrightarrow{e} \cdot (r_1 \cdot s_1 + \frac{(b_2 - b_1)}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||}) \cdot \Phi(\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1'}{r_2 \cdot (1 - r^2_1)^{1/2}}) \tag{A.3.8*}
\end{align*}

The incentive compatible condition for Agent 1 is: \( \forall s_1 \) and \( s_1' \), \( U_1(s_1'|s_1) \geq U_1(s_1'|s_1) \) and \( U_1(s_1'|s_1) \leq U_1(s_1'|s_1) \), which implies

\begin{align*}
\overrightarrow{a}_1' \overrightarrow{e} \cdot r_1 \cdot (s_1 - s_1') \cdot \left[ \Phi(\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1'}{r_2 \cdot (1 - r^2_1)^{1/2}}) - \Phi(\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1'}{r_2 \cdot (1 - r^2_1)^{1/2}}) \right] \geq 0
\end{align*}

Since \( \Phi(\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1'}{r_2 \cdot (1 - r^2_1)^{1/2}}) \) is decreasing with respect to \( s_1 \), we have

\begin{align*}
\overrightarrow{a}_1' \overrightarrow{e} \leq 0 \tag{A.3.9}^{26}
\end{align*}

i.e.

\begin{align*}
\overrightarrow{a}_1' (\overrightarrow{a}_2 - \overrightarrow{a}_1) \leq 0 \tag{A.3.9'}
\end{align*}

Define

\begin{align*}
U_1(s_1) = \max_{s_1'} U_1(s_1'|s_1) = U_1(s_1|s_1)
\end{align*}

i.e.

\begin{align*}
U_1(s_1) &= t_2(s_1, -\infty) + \frac{(b_1 \cdot \overrightarrow{a}_2 - b_2 \cdot \overrightarrow{a}_1)' \overrightarrow{e}}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||} \\
&\quad + r_2 \cdot (1 - r^2_1)^{1/2} \cdot \overrightarrow{a}_1' \overrightarrow{e} \cdot \int_{-\infty}^{\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1}{r_2 \cdot (1 - r^2_1)^{1/2}}} \varphi(s_2) \cdot ds_2 \\
&\quad + \overrightarrow{a}_1' \overrightarrow{e} \cdot (r_1 \cdot s_1 + \frac{(b_2 - b_1)}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||}) \cdot \Phi(\frac{(b_1 - b_2)/||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - r_1 \cdot s_1}{r_2 \cdot (1 - r^2_1)^{1/2}}) \tag{A.3.10}
\end{align*}

Note that \( U_1(\cdot) \) is the maximum of increasing and affine, hence convex functions. The maximum of increasing functions is increasing, and the maximum of convex functions is convex. Therefore \( U_1(\cdot) \) is increasing and convex. Convex

\footnote{This means Agent 1’s signal is negatively correlated with his valuation, since

\begin{align*}
cov(\overrightarrow{a}_1', \overrightarrow{e}, s_1) &= \overrightarrow{a}_1' \cdot \text{cov}(\overrightarrow{e}, s_1) \\
&= r_1 \cdot \overrightarrow{a}_1' \overrightarrow{e} \leq 0
\end{align*}

We can also make \( S_1 \) positively correlated with Agent 1’s valuation by letting \( r_1 < 0 \) and this won’t affect the inference of \( \overrightarrow{e} \).}

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functions are indiﬀerentiable in at most countably many points, thus by the
envelope theorem,

\[ \frac{dU_1(s_1)}{ds_1} = -a_1' \varepsilon \cdot r_1 \cdot \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \leq 0 \]

whenever \( U_1(s_1) \) is diﬀerentiable with respect to \( s_1 \). Moreover,

\[ U_1(s_1) = U_1(+\infty) - a_1' \varepsilon \cdot r_1 \cdot \int_{s_1}^{+\infty} \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1 \quad (A.3.11) \]

Substitute (A.3.10) into (A.3.11), we have

\[ t_2(s_1, -\infty) = U_1(+\infty) - \frac{(b_1 \cdot \overline{a}_2 - b_2 \cdot \overline{a}_1)'}{\| \overline{a}_2 - \overline{a}_1 \|} \cdot \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \]

\[ -a_1' \varepsilon \cdot \int_{-\infty}^{s_1} \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1 \quad (A.3.11^*) \]

Note that (A.3.9) and (A.3.11*) are also suﬃcient to make Agent 1 tell the
truth, since

\[ U_1(s_1) \geq U_1(s_1'|s_1) \]

iff (note that \( U_1(s_1'|s_1) \) is deﬁned by (A.3.8*)

\[ U_1(s_1) \geq t_2(s_1', -\infty) + \frac{(b_1 \cdot \overline{a}_2 - b_2 \cdot \overline{a}_1)'}{\| \overline{a}_2 - \overline{a}_1 \|} \cdot \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1'}{r_2 \cdot (1 - r_1^2)^{1/2}} \]

\[ + \overline{a}_1' \varepsilon \cdot \int_{-\infty}^{s_1'} \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1'}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1' \]

iff

\[ U_1(s_1) \geq U_1(s_1') + a_1' \varepsilon \cdot r_1 \cdot (s_1 - s_1') \cdot \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1'}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \]

iff

\[ U_1(s_1) - U_1(s_1') \geq a_1' \varepsilon \cdot r_1 \cdot (s_1 - s_1') \Phi \left( \frac{(b_1 - b_2) / \| \overline{a}_2 - \overline{a}_1 \| - r_1 \cdot s_1'}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \]

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iff
\[ \overline{\alpha}' \overline{e} \cdot r_1 \int_{s'_1}^{s_1} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot x}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) dx \geq \overline{\alpha}' \overline{e} \cdot r_1 \int_{s'_1}^{s_1} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) dx \]

iff (by (A.3.9))
\[ \int_{s'_1}^{s_1} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot x}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) dx \leq \int_{s'_1}^{s_1} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s'_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) dx \] (A.3.11 **)

Since the integrand is positive and decreasing with respect to \( x \), (A.3.11 **) holds for all \( (s_1, s'_1) \). This proves the sufficiency.

Let’s continue to derive \( t_2 (s_1, -\infty) \). By (A.3.11*)

\[
\begin{aligned}
t_2 (s_1, -\infty) &= U_1 (+\infty) - \frac{(b_1 \cdot \overline{a_2} - b_2 \cdot \overline{a_1}) \cdot \overline{e}}{|| \overline{a_2} - \overline{a_1} ||} \left( -r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overline{a_1}' \overline{e} \cdot \left[ s_2 \cdot \Phi (s_2) \right]_{-\infty}^{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1} - \int_{-\infty}^{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1} \Phi (s_2) \cdot ds_2 \right) - \overline{a_1}' \overline{e} \cdot r_1 \cdot \left( \frac{(b_1 - b_1) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \int_{s_1}^{+\infty} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot \overline{s}_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1 \\
&= U_1 (+\infty) - \frac{(b_1 \cdot \overline{a_2} - b_2 \cdot \overline{a_1}) \cdot \overline{e}}{|| \overline{a_2} - \overline{a_1} ||} \left( -\overline{a_1}' \overline{e} \cdot \left[ \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1}{|| \overline{a_2} - \overline{a_1} ||} \right] \cdot \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) + r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overline{a_1}' \overline{e} \cdot \int_{-\infty}^{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot s_1} \Phi (s_2) \cdot ds_2 \right) - \overline{a_1}' \overline{e} \cdot r_1 \cdot \left( \frac{(b_1 - b_1) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot \overline{s}_1}{|| \overline{a_2} - \overline{a_1} ||} \right) \int_{s_1}^{+\infty} \Phi \left( \frac{(b_1 - b_2) / || \overline{a_2} - \overline{a_1} || - r_1 \cdot \overline{s}_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1 \\
&= \ldots
\end{aligned}
\]

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\[= \ U_1(\infty) - \frac{(b_1 \cdot \overrightarrow{a}_2 - b_2 \cdot \overrightarrow{a}_1) \cdot \overrightarrow{e}}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||}
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overrightarrow{a}_1 \cdot \overrightarrow{e} \cdot \int_{-\infty}^{+\infty} \Phi \left( \frac{\sqrt{(b_1 - b_2) \cdot (||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - \overrightarrow{s}_1 \cdot \overrightarrow{s}_1)}}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1
- \frac{r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overrightarrow{a}_1 \cdot \overrightarrow{e}}{r_1} \cdot r_1 \cdot \int_{-\infty}^{+\infty} \Phi(x) \cdot \frac{r_2 \cdot (1 - r_1^2)^{1/2}}{r_1} \cdot dx
\]

\[= \ U_1(\infty) - \frac{(b_1 \cdot \overrightarrow{a}_2 - b_2 \cdot \overrightarrow{a}_1) \cdot \overrightarrow{e}}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||}
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overrightarrow{a}_1 \cdot \overrightarrow{e} \cdot \int_{-\infty}^{+\infty} \Phi \left( \frac{\sqrt{(b_1 - b_2) \cdot (||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - \overrightarrow{s}_1 \cdot \overrightarrow{s}_1)}}{r_2 \cdot (1 - r_1^2)^{1/2}} \right) \cdot ds_1
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overrightarrow{a}_1 \cdot \overrightarrow{e} \cdot r_1 \cdot \int_{-\infty}^{+\infty} \Phi(x) \cdot dx
\]

\[= \ U_1(\infty) - \frac{(b_1 \cdot \overrightarrow{a}_2 - b_2 \cdot \overrightarrow{a}_1) \cdot \overrightarrow{e}}{||\overrightarrow{a}_2 - \overrightarrow{a}_1||} \tag{A.3.12} \]

\[= \ U_1(\infty) - \frac{(b_1 - b_2) \cdot (||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - \overrightarrow{s}_1 \cdot \overrightarrow{s}_1)}{r_2 \cdot (1 - r_1^2)^{1/2}} \]

\[x = \frac{(b_1 - b_2) \cdot (||\overrightarrow{a}_2 - \overrightarrow{a}_1|| - \overrightarrow{s}_1 \cdot \overrightarrow{s}_1)}{r_2 \cdot (1 - r_1^2)^{1/2}} \]
By (A.3.10),

\[ U_1 (+\infty) = \lim_{s_1 \to +\infty} U_1 (s_1) \]

\[ = t_2 (+\infty, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} 
+ r_2 \cdot (1 - r_1^2)^{1/2} \cdot \overrightarrow{a_1}' \overrightarrow{e} \cdot \lim_{s_1 \to +\infty} \int_{-\infty}^{(b_1 - b_2)/|| \overrightarrow{a_2} - \overrightarrow{a_1} || - r_1 \cdot s_1} \frac{r_2}{r_2 \cdot (1 - r_1^2)^{1/2}} s_2 \cdot \varphi (s_2) \cdot ds_2 \]

\[ + \overrightarrow{a_1}' \overrightarrow{e} \cdot \lim_{s_1 \to +\infty} \left[ \left( r_1 \cdot s_1 + \frac{(b_2 - b_1)}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} \right) \cdot \Phi \left( \frac{(b_1 - b_2)}{|| \overrightarrow{a_2} - \overrightarrow{a_1} || - r_1 \cdot s_1} \right) \right] \]

\[ = t_2 (+\infty, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} \] (A.4.13)

Substitute (A.3.13) into (A.3.12), we have

\[ t_2 (s_1, -\infty) = U_1 (+\infty) - \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} \]

\[ = t_2 (+\infty, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} - \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} \]

\[ = t_2 (+\infty, -\infty) \] (A.3.14)

Combine (A.3.7), (A.3.8) and (A.3.14), we have

\[ t_2 (s_1, s_2) = \begin{cases} 
    t_2 (+\infty, -\infty) + \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} & \text{if } s_2 \geq \frac{(b_1 - b_2)/|| \overrightarrow{a_2} - \overrightarrow{a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \\
    t_2 (+\infty, -\infty) & \text{otherwise}
\end{cases} \] (A.3.15)

and

\[ t_1 (s_1, s_2) = \begin{cases} 
    -t_2 (+\infty, -\infty) - \frac{(b_1 \cdot \overrightarrow{a_2} - b_2 \cdot \overrightarrow{a_1})' \overrightarrow{e}}{|| \overrightarrow{a_2} - \overrightarrow{a_1} ||} & \text{if } s_2 \geq \frac{(b_1 - b_2)/|| \overrightarrow{a_2} - \overrightarrow{a_1} || - r_1 \cdot s_1}{r_2 \cdot (1 - r_1^2)^{1/2}} \\
    -t_2 (+\infty, -\infty) & \text{otherwise}
\end{cases} \] (A.3.16)

Now we can summarize our results: When Agent 2 is the second observer,

\[ \overrightarrow{a_1}' \cdot (\overrightarrow{a_2} - \overrightarrow{a_1}) > 0 \] (A.3.3')

\[ \text{b) The last equality holds since } x \cdot \varphi (x) \text{ is integrable on } \mathbb{R}^- \text{, which implies that} \]

\[ \lim_{s_1 \to +\infty} \int_{-\infty}^{(b_1 - b_2)/|| \overrightarrow{a_2} - \overrightarrow{a_1} || - r_1 \cdot s_1} \frac{r_2}{r_2 \cdot (1 - r_1^2)^{1/2}} s_2 \cdot \varphi (s_2) \cdot ds_2 = 0 \]
and
\[ \overline{a}'_1 (\overline{a}_2 - \overline{a}_1) \leq 0 \quad \text{(A.3.9')} \]
are the sufficient and necessary condition for the existence of a budget balanced Bayesian incentive compatible mechanism that implements social efficiency.

The corresponding mechanism is:

a) Agent 2 is the second observer;
b) the social planner choose the channels:

\[
\begin{pmatrix}
\theta \\
S_1 \\
S_2
\end{pmatrix} \sim N
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
, \begin{pmatrix}
I & r_1 \cdot \overline{e}' \\
0 & r_2 \cdot (1-r_1^2)^{1/2} \cdot \overline{e}'
\end{pmatrix}
\]

where \( r_1 = (1 - e^{-2c_1})^{1/2} \) and \( r_2 = (1 - e^{-2c_2})^{1/2} \);

c) the allocation rule is

\[ q_2(s_1, s_2) = \begin{cases} 1 \text{ if } s_2 \geq \frac{(b_1-b_2)/||\overline{v}_2 - \overline{v}_1|| - r_1 \cdot s_1}{r_2 \cdot (1-r_1^2)^{1/2}} \\ 0 \text{ otherwise} \end{cases} \quad \text{(A.3.2)} \]

\[ q_1(s_1, s_2) = 1 - q_2(s_1, s_2) \]

and the payment rule is

\[ t_2(s_1, s_2) = \begin{cases} t_2(\infty, -\infty) + \frac{(b_1 - b_2 - b_2 \cdot \overline{a}_1)' \overline{e}}{||\overline{v}_2 - \overline{v}_1||} \text{ if } s_2 \geq \frac{(b_1-b_2)/||\overline{v}_2 - \overline{v}_1|| - r_1 \cdot s_1}{r_2 \cdot (1-r_1^2)^{1/2}} \\ t_2(\infty, -\infty) \text{ otherwise} \end{cases} \quad \text{(A.3.15)} \]

\[ t_1(s_1, s_2) = -t_2(s_1, s_2) \]

where \( t_2(\infty, -\infty) \) is a constant that determines the initial value of the payment function. \( t_2(\infty, -\infty) \) doesn’t affect incentives and budget balance. Its value can be determined by other constraints, such as individual rationality condition and the "fairness" condition, which is discussed in the comments of Proposition 4.

Now we prove that

\[ \overline{a}'_2 (\overline{a}_2 - \overline{a}_1) (\overline{a}_2 - \overline{a}_1)' \overline{a}_1 \leq 0 \quad \text{(A.3.17)} \]

is the sufficient and necessary condition for the existence of a budget balanced Bayesian incentive compatible mechanism that implements social efficiency.

Note that \( \overline{a}'_2 (\overline{a}_2 - \overline{a}_1) + \overline{a}'_1 (\overline{a}_1 - \overline{a}_2) = ||\overline{v}_2 - \overline{v}_1||^2 > 0 \) implies at least one of \( \overline{a}'_2 (\overline{a}_2 - \overline{a}_1) \) and \( \overline{a}'_1 (\overline{a}_1 - \overline{a}_2) \) is strictly positive. Hence (A.3.17) is equivalent to

\[ \overline{a}'_2 (\overline{a}_2 - \overline{a}_1) > 0 \text{ and } \overline{a}'_1 (\overline{a}_1 - \overline{a}_1) \leq 0 \]

or

\[ \overline{a}'_1 (\overline{a}_1 - \overline{a}_2) > 0 \text{ and } \overline{a}'_2 (\overline{a}_2 - \overline{a}_2) \leq 0 \]

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The former case is what we have proved in the above derivation. In the latter case, let Agent 1 be the second observer and redefine \( \vec{e}' = (\vec{a}_1 - \vec{a}_2) / \|\vec{a}_1 - \vec{a}_2\| \), then all the previous derivations still apply (or just exchange the labels of the two agents and it becomes the former case).

Let

\[
\cos \alpha = \frac{\vec{a}'_2 \cdot \vec{a}_1}{\|\vec{a}_2\| \cdot \|\vec{a}_1\|}
\]

i.e. \( \alpha \) is the angle between the two vectors \( \vec{a}_1 \) and \( \vec{a}_2 \). Then we can restate (A.3.17) as

\[
0 \geq \vec{a}'_2 (\vec{a}_2 - \vec{a}_1)' - \vec{a}_1 
= \vec{a}'_2 \vec{a}_2 \cdot \vec{a}_1 - \vec{a}'_2 \vec{a}_2 \vec{a}_1 - \vec{a}'_2 \vec{a}_1 \vec{a}_1 + \vec{a}'_2 \vec{a}_1 \vec{a}_1 
= -\|\vec{a}_2\|^2 \cdot \|\vec{a}_1\|^2 \cdot \left[ \frac{\|\vec{a}_2\|}{\|\vec{a}_1\|} \cdot \frac{\vec{a}'_2 \cdot \vec{a}_1}{\|\vec{a}_2\| \cdot \|\vec{a}_1\|} - 1 - \left( \frac{\vec{a}'_2 \cdot \vec{a}_1}{\|\vec{a}_2\| \cdot \|\vec{a}_1\|} \right)^2 + \frac{\|\vec{a}_1\|}{\|\vec{a}_2\| \cdot \|\vec{a}_1\|} \right] 
= -\|\vec{a}_2\|^2 \cdot \|\vec{a}_1\|^2 \cdot \left[ \cos^2 \alpha - \frac{\|\vec{a}_2\|}{\|\vec{a}_1\|} \cdot \cos \alpha - \frac{\|\vec{a}_1\|}{\|\vec{a}_2\|} \cdot \cos \alpha + 1 \right] 
= -\|\vec{a}_2\|^2 \cdot \|\vec{a}_1\|^2 \cdot \left( \cos^2 \alpha - \frac{\|\vec{a}_2\|}{\|\vec{a}_1\|} \right) \cdot \left( \cos \alpha - \frac{\|\vec{a}_1\|}{\|\vec{a}_2\|} \right)
\]

therefore (A.3.17) is equivalent to

\[
\left( \cos \alpha - \frac{\|\vec{a}_2\|}{\|\vec{a}_1\|} \right) \cdot \left( \cos \alpha - \frac{\|\vec{a}_1\|}{\|\vec{a}_2\|} \right) \geq 0
\]

i.e.

\[
\cos \alpha \leq \min \left\{ \frac{\|\vec{a}_2\|}{\|\vec{a}_1\|}, \frac{\|\vec{a}_1\|}{\|\vec{a}_2\|} \right\}
\]

This concludes the proof of Proposition 3.

Proof of Proposition 4: Note that the proof of Proposition 3 has already proved Proposition 4. The results are stated in (A.3.1), (A.3.2) and (A.3.15), where \( K \) is replaced by \( t_2 (\pm \infty, -\infty) \).