Dynamic Education Signaling with Dropout∗

–Preliminary–

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Abstract

We present a dynamic signaling model where wasteful education takes place over several periods of time. Workers pay an education cost per unit of time and can not commit to a prefixed education length. By introducing exogenous dropout, low-productivity workers endogenously choose to drop out over time to avoid a high education cost. This allows us to provide a neat characterization of all equilibria of our model. In contrast to Swinkels (1999), we restore the presence of wasteful education signals even when job offers are privately made and the length of the period is small. Furthermore, we show that the maximum education length is decreasing in the prior about a worker being highly productive. The joint dynamics of returns to education and the dropout rate are characterized, which is consistent with previous empirical evidence.

Keywords: Dynamic Education Signaling, Dropout

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1 Introduction

In his seminal paper, Spence (1973) argues that people may rationally engage in useless education to reveal their ability. Cho and Kreps (1987) provide a game-theoretic analysis of the Spence model. In their model, a worker, whose productivity is either high or low, makes a one-shot education choice, which is a positive real number, to which he commits. Education is assumed to be more costly for a worker with low productivity. When a worker finishes his education, he goes on the job market and firms simultaneously make him offers. Many perfect Bayesian equilibria exist in their model, but only the least costly separation equilibrium, found by Riley (1979), satisfies the intuitive criterion. In this equilibrium, workers fully separate: the low type worker chooses zero education and obtains a wage equal to his productivity, and the high type worker obtains a wage equal to his productivity by choosing the least costly separation education duration such that a low type worker has no incentive to mimic him.

One interesting feature of this equilibrium was pointed out by Weiss (1983) and Admati and Perry (1987) (henceforth WAP): when a student arrives on the first day of school, the separation has already happened. For example, Weiss (1983) asks, "Why don’t firms hire individuals immediately after they have sorted themselves?". Cho and Kreps (1987) avoid this challenge by assuming that a worker can commit to his decision about education duration. In practice, it is hard to see where the commitment power comes from. Noldeke and Van Damme (1990) formulate an explicitly dynamic game-theoretic version of the Spence model and try to answer WAP’s question. In their model, long-lived firms simultaneously make public offers to the worker in each period, and the worker decides to accept an offer or continue receiving education. They focus on equilibria that satisfy never a weak best response (NWBR) requirement provided by Kohlberg and Mertens (1986), and they find that equilibria outcome converges to the Riley outcome when the time interval between two education decision points goes to zero. Nonetheless, Swinkels (1999) argues that Noldeke and Van Damme’s result crucially depends on the fact that job offers are publicly made. Hence, he considers a model where two short-lived firms enter and simultaneously make private offers to the worker in each period before the worker’s decision on whether to receive further education and provides a second answer to WAP’s question. He finds that, when the interval between offers goes to zero, the unique sequential equilibrium in this game is a pooling one at no education.

We think it is important to understand the degree to which Swinkels’ rather surprising result depends on the presence of preemptive offers. In Cho and Kreps (1987), the Riley outcome can always be supported since, off the path of play, a belief threat can be imposed to punish early dropouts with low wages. However, in Swinkels (1999), firms can directly interfere on workers’
education by making private preemptive offers. Consequently, the ”small game” between firms and the worker in each period is a screening game instead of a signaling one, and, as a by-product, the belief threat in Cho and Kreps (1987) can not be used. It is not clear whether the result in Swinkels (1999) results from the non-existence of a belief threat in Cho and Kreps (1987) or other factors induced by his screening-style timing. To eliminate the effect of a belief threat in Cho and Kreps (1987), we assume that a worker in school faces an exogenous dropout risk, even though he does not choose to do so, and firms can make private offers to a worker only when he has dropped out of school and is available on the market. By doing so, we can provide the third answer to WAP’s question.

Particularly, in every period, a worker may have to drop out and go on the job market with some exogenous probability. We interpret this exogenous dropout process as random shocks faced by students, and the shocks are driven by exogenous reasons such as financial constraints, family reasons and the arrival of utility shocks. Since whether and when the student will be forced to drop out is not known with certainty by the student at the beginning of his education, one can expect, under these features, that students who drop out do not have any offer when they leave college. This establishes a timing that distinguishes our work from the literature on dynamic signal with preemptive offers: in our model, the informed party, the student, moves first (going to the job market or not) and, consequently, conditional on being in the job market, the uninformed agents, firms, make him offers.

Since a high-productivity worker leaves education with positive probability, a low-productivity worker may have incentives to mimic him by voluntarily quitting school in order to save future education costs. Nevertheless, if in some period a low-productivity worker dropped out with probability one while the high-productivity worker stayed with positive probability, the next period’s beliefs about the worker being a high productivity worker would jump to one. If the corresponding jump in wage was large enough, the low-productivity worker would have incentives not to drop out at the current period, leading to a contradiction. On the other hand, if low-productivity workers did not endogenously drop out at some period, then learning would be slow, which makes education less attractive for them, incentivizing drop out at the current period. We show that, in our benchmark model, in equilibrium, low-productivity workers will mix between dropping out and staying in almost all periods, in order to balance these two forces. Hence, as the first main result of our model, the joint dynamics of education signaling and the dropout rate are charac-

\textsuperscript{1}Empirical evidence suggests that the dropout rate in US’ colleges is not small: only about half of those who begin first-level degree programs actually obtain their degrees, rates of completion exceed 70% in many other countries. See Bound and Turner (2011) for a detailed survey of related empirical evidence. A study from the Bill & Melinda Gates Foundation, With Their Whole Lives Ahead of Them, shows that the primary reason students drop out of college is that they need money for survival.
terized. We show, for most reasonable parameters, that wasteful education signaling appear as an equilibrium phenomenon. Furthermore, to ensure the low-type worker’s randomization, wage increment in each period must equal the marginal cost of education for the low-type worker. As a result, in the two-type model, the return to education is linear.

Note, our model should be interpreted as a complement rather than a substitute for Swinkels (1999). In practice, both in-school offers and in-market offers take place in many industries. Our goal is to understand how the interaction between the decision sequence and the privacy of offers affects the equilibrium dynamics and the existence of wasteful signaling.

Another interesting feature of the Riley outcome, as pointed out by Mailath, Okuno-Fujiwara and Postlewaite (1993), is that education length is strictly positive (significantly different from that in the symmetric information case, which is zero) and it does not depend on the initial prior of a worker’s type. They ask: “But is it reasonable to believe that the outcome in a game with a 1 in 1,000,000 chance of a worker of low ability will differ significantly from that in a game with no chance of such a worker? If not, this discontinuity with respect to the probability distribution over the two types of (workers) is disturbing.” They provide a belief-based refinement concept, undefeated equilibria, to eliminate unreasonable equilibria. When the prior is close to 1, the Riley outcome is defeated. Our paper provides a dynamic foundation for their static argument. In our model, the maximum education duration is decreasing in the prior. In particular, when the prior goes to one, no wasteful education appears in any perfect Bayesian equilibrium. The intuition is as follows. Since the worker exogenously drops out with a positive rate, in any equilibrium, early dropouts are on the path of play, and the posterior about workers’ type conditional on the dropout is specified by Bayes’ rule. When the prior about a worker being high type is close to one, the lower bound of the posterior about workers’ type conditional on dropping out is close to one; thus, the maximum marginal benefit by waiting one more period is very small, but the cost is non-trivial. Hence, neither the low-type nor the high-type worker have incentives to take one more period of education, and the game ends immediately at no education.

We extend our two-type model to a finite-many-type one and show that equilibria properties are similar. In particular, at each period, there is a marginal type who is indifferent between dropping out and receiving more education. Workers whose cost is higher than this marginal type’s strictly prefer to drop out, while those whose cost is lower strictly prefer to receive more education. Over time, the cutoff type’s cost decreases, and therefore, the return to education is concave, which is consistent with the previous empirical findings.

We also relax the assumption of a homogeneous exogenous dropout rate. When the exogenous dropout rate of the high-type worker is greater than that of the low-type worker, the set of equilibria is identical to that in the benchmark model. When the exogenous dropout rate of the
high-type worker is smaller than that of the low-type worker, equilibria may be different from those in the benchmark model. We show that the difference vanishes when the difference between the exogenous dropout rates is small.

**Other Related Literature**

Apart from papers we have already mentioned, our paper is related to the literature in three ways. First, there is a growing literature considers the impact of extra signaling in Swinkels’ mode. Kremer and Skrzypacz (2007) introduce an extra (noisy) signal, which is observed at a given time, into Swinkels’ model. They interpret the extra signal as students’ grade. By doing this, they restore the wasteful signaling and show the presence of a degree premium. Daley and Green (2011) follow Kremer and Skrzypacz (2007) and consider a dynamic lemons market model where the seller’s (worker) type is gradually revealed to buyers (firms). In particular, they assume that the news follows a type-contingent diffusion process: the high type has a greater drift rate than the low type. They show that the game ends in one of two ways: either enough good news arrives, restoring confidence and transaction happens with a high wage, or enough bad news arrives, making the worker more pessimistic so that he accepts a low wage offer. Our paper is different from those in the following aspects. (1) Kremer and Skrzypacz (2007) follow Swinkels’ timing and introduce an extra signal at the deadline of the education. Daley and Green (2011) consider a continuous time model where a type-contingent diffusion process, as a sequence of extra signals, is realized over time. In contrast, we do not introduce any extra signal in addition to education length, and we adopt a different decision sequence. (2) In Daley and Green (2011), the extra signal for informed agents with different types is generated by different processes. In our benchmark model, both high-type and low-type workers are forced to drop out at the same exogenous rate, and the difference in the total dropout rate are determined by the equilibrium instead of by the exogenous assumption.

Second, our model is also related to the dynamic adverse selection literature. Janssen and Roy (2002) study a dynamic lemons market problem and show that each equilibrium involves a sequence of increasing prices and qualities traded over time. Trade is delayed and therefore inefficient, but all goods are sold out in finitely many periods. In their model, the time-on-the-market of a good is used to signal the quality of the good. Horner and Vieille (2009) study a dynamic bargaining game in which a single seller faces a sequence of buyers and show that the observability of previously rejected-prices can cause a bargaining impasse. Kim (2011) examines the roles of different pieces of information about sellers’ past behavior in a dynamic decentralized lemons market. He suggests that market efficiency is not monotone in the amount of information available to buyers but depends crucially on what information is available under what market
conditions. Camargo and Lester (2011) investigate a dynamic decentralized lemons market with one-time entry. They demonstrate how prices and the composition of assets evolve over time given an initial fraction of lemons. They find that the patterns of trade depend systematically on the initial fraction of lemons, which is similar to the structure of our result. However, they focus on the dynamics of trade and price.

Last, in our model, since an exogenous dropout rate exists, education length is a noisy signal of the worker’s type. Some papers also consider signaling models with noisy signals. Matthews and Mirman (1983) consider a noisy signaling model and show that the equilibrium can be unique, which depends on prior beliefs. Bar-Isaac (2003) investigates learning and reputation in a dynamic signaling model where a privately informed monopolist faces a type-contingent but random demand, which can be treated as a noisy signal, and decides whether to sell in each period.

The rest of this paper is organized as follows. In the next section we present the model with a type-independent dropout rate and characterize the set of equilibria. We consider a multiple-type version of our model in Section 3. In Section 4 we turn to a model with a type-dependent dropout rate. Section 5 concludes. All omitted proofs are in Appendix.

2 Benchmark Model

Time is discrete, \( t = 0, 1, 2, ... \). There is one uneducated worker who has a type \( \theta \in \{ H, L \} \), which is his private information with a common prior \( p_0 = \Pr(\theta = H) \in (0, 1) \). The productivity of type \( \theta \) worker is \( Y_\theta \). We normalize \( Y_H = 1 \) and \( Y_L = 0 \). The worker can receive education by a paying type-contingent cost per unit of time, \( c_\theta \), where \( 0 < c_H < c_L \) and \( c_H < 1 \). We assume that there is some probability of exogenous dropout in each period, that is, nature may force the worker to leave education. We understand this process as attributable to some financial shocks, utility shocks and so on. In the benchmark model, we assume each worker, regardless of his type, is forced to drop out with a probability \( \lambda \in (0, 1) \).

The timing is summarized as follows. Before education begins, nature decides the type of the worker, choosing \( H \) with probability \( p_0 \). In period \( t \): (1) nature decides that the worker exogenously drops out with probability \( \lambda \). (2) If nature decides that the worker does not drop out, the worker decides whether to endogenously drop out. (3) If the worker decides not to drop out, he pays the education cost and goes to the next period. If the worker drops out, he goes on the job market. (4) Two short-lived firms enter the job market and simultaneously make private job offers to the worker who has dropped out. (5) The worker can choose to take either offer or a

\(^2\)Note, in contrast to Swinkels (1999), firms cannot make offers to a worker in school. This will play an important role in the existence of equilibria with education, which will be discussed below.
zero value outside option.

The utility of a worker of type $\theta$ who received $t$ periods of education and accepts a wage of $w$ is $U(w, t) = w - c_\theta t$. The profit of a firm that employs a worker of type $\theta$ at a wage $w$ is given by $Y_\theta - w$. When a firm hires no worker, its profit is zero.

A dropout (behavior) strategy for $\theta$-workers is $\alpha^\theta : \{0, 1, \ldots\} \rightarrow [0, 1]$, that is the probability that type $\theta$ worker chooses to drop out at $t$ conditional on reaching its decision point. We use $s_t^\theta \equiv \lambda + (1 - \lambda)\alpha_t^\theta$ to denote the total probability of dropping out at period $t$. Finally, $S_t^\theta$ denotes the probability of reaching $t$, that is

$$S_t^\theta \equiv \prod_{\tau=0}^{t-1} (1 - s_\tau^\theta).$$

For each strategy profile, let $T^\theta \equiv \min\{t|S_{t+1}^\theta = 0\} \in \{0\} \cup \mathbb{N} \cup \infty$, which is the maximum number of education periods the type $\theta$ worker may receive under the given strategy profile.

Define $p_t$ to be the beliefs about a worker who reached period $t$, and $\hat{p}_t$ be the beliefs about a worker who dropped out at $t$. When a worker goes on the job market, two firms Bertrand-compete given their updated belief $\hat{p}_t$, so they both will offer $w_t = \hat{p}_t$. On the path of play, firms can correctly predict $\hat{p}_t$; thus, they obtain zero expected profit. The worker will take the offer with the higher wage if it is positive. The solution concept we employ is a perfect Bayesian equilibrium, which is defined as follows.

**Definition 1.** A perfect Bayesian equilibrium (PBE) in this game is a strategy profile $\{(\alpha^\theta)_{\theta=L,H}, w\}$ and two belief sequences $p$ and $\hat{p}$ such that:

1. The $\theta$-worker chooses $\alpha^\theta$ to maximize their expected payoff given $w$.
2. If a worker drops out with education $t$, firms offer $w_t = \hat{p}_t$.
3. When it is well defined, $\hat{p}_t$ satisfies the Bayes’ rule

$$\hat{p}_t = \frac{p_t s_t^H}{p_t s_t^H + (1 - p_t) s_t^L}. \quad (1)$$

4. When it is well defined, $p_t$ is updated following the Bayes’ rule

$$p_{t+1} = \frac{p_t(1 - s_t^H)}{p_t(1 - s_t^H) + (1 - p_t)(1 - s_t^L)}. \quad (2)$$

The value function of the $\theta$-workers at period $t$ is

$$V_t^\theta = \lambda \hat{p}_t + (1 - \lambda)W_t^\theta,$$
where $\hat{p}_t$ is his payoff when he exogenously drops out, and $W_t^\theta \equiv \max\{\hat{p}_t, V_{t+1}^\theta - c_\theta\}$ is his continuation value in the complementary event. The worker will decide to endogenously drop out when $\hat{p}_t > V_{t+1}^\theta - c_\theta$, stay in school when $\hat{p}_t < V_{t+1}^\theta - c_\theta$, and potentially randomize when $\hat{p}_t = V_{t+1}^\theta - c_\theta$.

**Lemma 1.** For any equilibrium, in all periods $t < T^L$,

1. There is positive voluntary dropout by the $L$-worker.
2. There is no voluntary dropout by the $H$-worker.

**Proof.** The proof is in the appendix on page 16.

**Remark 1.** When $p = 1$, there is no reason to receive any extra education for the high-type worker. Combining this fact and Lemma 1 yields $T^H \in \{T^L, T^L + 1\}$.

Lemma 1 implies that, in any PBE, for all periods $t < T^L$,

$$\hat{p}_{t+1} - \hat{p}_t = c_L,$$

so the low-type worker is always indifferent between dropping out and staying in school except (possibly) at his last possible period $T^L$. This fact implies that the wage must linearly increase before $T^L$. In other words, except (possibly) in the last period, education has constant returns over time. As we will show, this implication depends on the two-type assumption. When it comes to the multiple-type model, the returns to education become concave.

**Theorem 1.** If $p_0 > p_0^1$ the only equilibrium is pooling at no education, where

$$p_0^1 \equiv \frac{1 - c_H}{1 - (1 - \lambda)c_H} < 1.$$  

**Proof.** The proof is in the appendix on page 16.

The intuition behind the previous theorem is as follows. Given a $p_0$, the lowest possible $\hat{p}_0$ is obtained when $s^L_0 = 1$ and $s^H_0 = \lambda$, given by $\frac{\lambda p_0}{\lambda p_0+1-\lambda}$. Since the maximum wage next period is 1, an upper bound on the gain from not dropping out at period 0 is $1 - \frac{\lambda p_0}{\lambda p_0+1-\lambda}$. If $p_0$ is close to 1, the maximum gain gets close to 0. Nevertheless, the marginal cost for type $\theta \in \{L, H\}$ is $c_\theta > 0$. Hence, when $p_0$ is close to 1, receiving education is not attractive for both types, and the game ends immediately with a pooling equilibrium at no education.

In the standard signaling model by Cho and Kreps (1987), wasteful signaling can be supported even when the prior about the type being high ($p_0$) is very close to 1. The reason is that, off the path of play, a belief threat may be imposed by the firms, so early dropouts are punished with
low wages. In our model, since $\lambda > 0$, there is positive dropout in any period before $T^H$. Hence, there cannot be belief threats off the path of play for dropouts in all periods before $T^H$.

When $p_0 = 1$, i.e. when there is common knowledge that the worker is high type, the unique equilibrium both in our model and in the standard model exhibits no wasteful education signaling. However, in the standard model, if information is asymmetric between firms and the worker ($p_0 < 1$), the set of equilibria can support nontrivial wasteful signaling, which is very different from the equilibrium education choice in the symmetric information game. What is more, for any $p_0 \in (0, 1)$, by imposing some refinement concept, for example, an intuitive criterion or a D1, the equilibrium prediction of the standard signaling model is the Riley outcome, in which a nontrivial wasteful signal is sent. In other words, a discontinuity appears as the information asymmetry, measured by $1 - p_0$, vanishes. Motivated by this unsatisfactory property, Mailath, Okuno-Fujiwara and Postlewaite (1993) provide a belief-based solution concept, undefeated equilibria, to fix this problem. In our model, without imposing any refinement concept, there is no signaling waste when $p_0 \to 1$ in any equilibrium. Consequently, the equilibrium education length converges to that in the symmetric information model as $p_0$ goes to 1.

The following theorem characterizes possible education lengths in the set of all equilibria:

**Theorem 2.** Let $T^* \equiv \lceil \frac{1-c_H}{c_L} \rceil$. There exists a partition of $[0, 1]$ characterized by $\{p_0^k\}_{k=0}^{T^*+1}$, with $p_0^0 = 1$, $p_0^k > p_0^{k+1}$ for all $k$ and $p_0^{T^*+1} = 0$, such that for all $0 \leq k \leq T^*$ and $0 \leq T \leq k$, if $p_0 \in (p_0^{k+1}, p_0^k)$ then there exists an equilibrium with number of education periods $T$ and there is no equilibrium more than $k$ periods of education.

**Proof.** The proof is in the appendix on page 17.

In Theorem 1, we already discussed the case where $p_0$ is close to 1. Now, consider the case in which $p_0$ is not close to 1. As we shown in Lemma 1, the low type endogenously drops out with positive probability and the high type does not voluntarily drop out; thus $s_t^L > s_t^H$, which means that $p_t$ is pushed up over time. The low type indifference condition (3) implies that $\hat{p}_t$ is linear before $T^L$. These two observations imply that $p_t$ and $\hat{p}_t$ will be high enough (close to 1) after finitely many periods. The smaller the prior $p_0$, the more periods of education can be supported in an equilibrium. This suggests that the upper bound of the education duration supported by an equilibrium is non-increasing in $p_0$.

As the main result of this paper, Theorem 2 characterizes the set of PBE. It shows that the maximum equilibrium education duration is finite, and it is non-increasing in the prior $p_0$ and shrinks to zero as $p_0$ goes to 1. Since $\lambda > 0$, both the high-type and low-type workers drop out

\[3\lceil x \rceil \text{ denotes the smallest integer greater than } x.\]
before \( T^H \) with positive probability, and therefore, there is no fully separating equilibrium. When \( p_0 \leq p_0^1 \), for any positive \( c_H, c_L \), and \( \lambda \) such that \( c_H < c_L \), there exists an equilibrium with wasteful signaling. This is in sharp contrast to the result in Swinkels (1999), who studies a dynamic education signaling model with preemptive offers. In Swinkels (1999), the only equilibrium is pooling at no education when the worker can adjust his education choice very frequently. In other words, wasteful education cannot appear in any equilibrium. In our model, we assume the time length between two consecutive periods is 1. Our main result does not qualitatively change when the time length is \( \Delta \in (0, 1) \); the education cost and dropout probability in each period are given by \( c_\theta \Delta \) and \( \lambda \Delta \), respectively. Specifically, \( T^*_\Delta \equiv \lceil \frac{1 - \Delta c_H}{\Delta c_L} \rceil \) goes up as \( \Delta \) decreases, but the real time \( \Delta T^*_\Delta \) is finite and bounded away from zero. The contrast in these results illustrates the critical role of timing in the two models. In both models, offers are privately made. However, when firms can make preemptive offers, they can attract the worker in school and end the game immediately. When the time interval between two consecutive periods is small, firms can post an appropriate wage \( w \) to (1) attract both types, and (2) obtain a non negative profit. In our model, firms cannot directly disturb the worker’s signaling process by making an in-school offer, and therefore, semi-separating equilibria can survive.

Without imposing any refinement, multiple equilibria do exist for most \( p_0 \); thus, wasteful signaling does appear as an equilibrium phenomenon when \( p_0 \) is low. The reason we do not have uniqueness is the arbitrariness of belief after \( T^H \) off the path of play, which is the same as that in Cho and Kreps (1987). By imposing an appropriate criterion on the belief off the path of play, for example D1 defined by Banks and Sobel (1990) or NWBR defined by Kohlberg and Mertens (1986), one can shrink the equilibrium set. Specifically, off-the-path-of-play, these concepts require
that for each type, firms put positive probability only on that type is most likely to deviate. In our model, since the marginal cost of education of the high type worker is strictly smaller than that of the low type worker, for any sequence of wage off-the-path-of-play, \( w_t, t = T^H + 1, T^H + 2, \ldots \), induces a deviation of the low type worker deviate must induce a deviation of the high type worker. As a result, off-the-path-of-play, firms put positive belief only on the high type worker, i.e. \( p_t = 1 \) for any \( t = T^H + 1, T^H + 2, \ldots \). Given this belief sequence off the path of play, a PBE is eliminated by NWBR if \( \hat{p}_{T^H} < 1 - c_H \). However, by applying these concepts, one cannot select a unique equilibrium, which is similar to Noldeke and Van Damme (1990). The key reason of the multiplicity is that, in our model, the education choice is an integer instead of a real number.

Consider the following case as an example.

**Example 1.** Suppose \( p_0 \in (1 - c_H, \frac{1 - c_H}{1 - (1 - \lambda)c_H}) \). There is a PBE in which \( s^H_0 = \lambda, s^L_0 = 1, \hat{p}_0 = \frac{\lambda p_0}{\lambda p_0 + (1 - p_0)} < 1 - c_H \). Since \( p_1 = \hat{p}_1 = 1 \), it survives the elimination of NWBR. However, there is another PBE in which \( s^H_0 > \lambda, s^L_0 = 1 \), and \( \hat{p}_0 = \frac{s^H_0 p_0}{s^H_0 p_0 + 1 - p_0} = 1 - c_H \). Since \( p_1 = 1 \), this PBE also survives the elimination of NWBR.

A natural question is how the set of equilibria depends on \( \lambda \). Figure 2 plots \( \{p^T_0\}_{k=1}^{T^*} \) for different values of \( \lambda \). As we see, when \( \lambda \to 0 \), \( p^T_0 \) for all \( T \) collapses to 1. This implies that, when \( \lambda \) is low, for almost all priors the maximum length of an equilibrium is \( T^* \). This is consistent with the \( \lambda = 0 \) case, explained below, so the limit is continuous. In the other limit, when \( \lambda \to 1 \), \( p^k_0 - p^{k+1}_0 = c_L \) for all \( T > 1 \). This is a consequence of the fact that when \( \lambda \) is close to 1, so are \( s^L \) and \( s^H \). Therefore, as we see in (1), \( \hat{p}_t \) is close to \( p_t \) for all \( t \), which imposes a nearly linear evolution for \( p_t \).
3 Multiple Types

Now we consider the $N > 2$ types case in which $\theta \in \{1, 2, 3, ..., N\}$ with a prior $p_0^\theta$, where $\sum_{\theta=1}^{N} p_0^\theta = 1$. Type $\theta$ worker has a cost of waiting $c^\theta$, $c^\theta > c^{\theta+1}$. The productivity of $\theta$ is $Y^\theta$, $Y^\theta < Y^{\theta+1}$. All types exogenously drop out with probability $\lambda$.

The equilibrium concept is the same as in Definition 1 but adapted to the fact that now we have many types. Note that firms’ offers depend only on the expected productivity and not on other moments of the productivity distribution. This fact helps us to keep our definition simple:

**Definition 2.** A perfect Bayesian equilibrium (PBE) is a strategy profile $\{(\alpha^\theta)_{\theta=1,...,N}, w\}$, $N$ beliefs sequences $p^\theta$ and a process $\hat{Y}$ such that:

1. $\theta$-worker chooses $\alpha^\theta$ to maximize her expected payoff given $w$.
2. If a worker drops out with education $t$, firms offer $w_t = \hat{Y}_t$.
3. When it is well defined, $\hat{Y}_t$ satisfies
   \[ \hat{Y}_t = \sum Y^\theta \frac{p_t^\theta s_t^\theta}{\sum_{\theta'=1}^{N} p_t^{\theta'} s_t^{\theta'}}. \]  
   \[ (5) \]
4. When it is well defined, $p_{t+1}^\theta$ is updated according to the Bayes’ rule
   \[ p_{t+1}^\theta = \frac{p_t^\theta (1 - s_t^\theta)}{\sum_{\theta'=1}^{N} p_t^{\theta'} (1 - s_t^{\theta'})}. \]  
   \[ (6) \]

Let $T^\theta$ be the last time the $\theta$-worker is in school. The following theorem shows that our insight into the binary-type model can be easily extended to a multiple-types model.

**Theorem 3.** Under the previous assumptions, in any equilibrium:

1. In each period $t$, there is at most one type, indifferent to dropping out.
2. More productive types stay longer in education, $T^\theta \leq T^{\theta+1}$.
3. There is a positive voluntary dropout in all periods.
4. $\hat{Y}_t$ is concave in $t$.

**Proof.** The proof can be found in the appendix at page 22. \[ \square \]
Most features in the two-type model are preserved. However, note that under many types we have decreasing returns to education instead of linear ones, since lower types are skimmed out before higher types in equilibria. This pattern of decreasing returns to education is consistent with many empirical studies, for example, Frazis (2002), Habermalz (2003) and Manoli (2008). The equilibrium construction in multiple-type models is almost identical to that in the two-type model, and thus is omitted.

4 Type-Dependent Exogenous Dropout Rate

In our benchmark model, the dropout rate is independent of the worker’s type. In most equilibria, the total dropout rate of low-type workers is higher than that of high-type workers. Since both the posterior about a worker being high-type and the endogenous dropout rate of low-type workers change over time, the observable dropout rate changes over time. Hence, assuming a homogeneous exogenous dropout rate is enough to generate an endogenous partial separation and interesting dropout rate dynamics. In this section, we consider a model in which a worker’s dropout rate is correlated with his productivity. It turns out that our predictions in section 2 are robust. There are three relevant cases: (1) $\lambda_H > \lambda_L \geq 0$, (2) $\lambda_L > \lambda_H > 0$, and (3) $\lambda_L \geq \lambda_H = 0$.

4.1 $\lambda_H > \lambda_L \geq 0$ Case

The first case we consider is $\lambda_H > \lambda_L \geq 0$, that is, the high-type worker exogenously drops out at a higher rate than the low-type worker. The following lemma implies that the equilibrium set in this case coincides with the base model when $\lambda = \lambda_H$:

**Lemma 2.** Assume $\lambda_H > \lambda_L \geq 0$. Then, $(\alpha^L, \alpha^H, w, p, \tilde{p})$ is a PBE if and only if it is also a PBE in the benchmark model with $\lambda = \lambda_H$.

**Proof.** The proof can be found in the appendix at page 23.

The intuition behind this lemma is that, in our original model, by Lemma 1, the endogenous dropout rate of the low-type worker is positive in all periods before (maybe) the last. So, the constraint $s_t^L \geq \lambda$ was never binding in equilibrium. Therefore, all equilibria from the base model for $\lambda = \lambda_H$ are also equilibria for the case $\lambda_H > \lambda_L \geq 0$. On the other hand, for any equilibrium in the case where $\lambda_H > \lambda_L$, let $\tilde{\alpha}_t^L$ denote the low type’s strategy. It must be true that $\tilde{\alpha}_t^L \geq \lambda_H - \lambda_L$. Define $\bar{\alpha}_t^L = \tilde{\alpha}_t^L - (\lambda_H - \lambda_L) \geq 0$. One can easily verify that $\bar{\alpha}_t^L$ can be supported in a PBE of the game with a symmetric exogenous dropout rate, $\lambda = \lambda_H$. 

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4.2 $\lambda_L > \lambda_H > 0$ Case

As we can see in Figure 3, $s^L$ may be non-monotone. In particular, there are some equilibria where it is initially decreasing and then increasing and finally it goes down again. Now, $s^L$ is restricted to be no lower than $\lambda_L > \lambda_H$. We may guess that this constraint will be potentially binding in two connected regions, one for large $\hat{p}$ and the other for intermediate values. In any equilibrium, when this constraint is binding, both types strictly prefer to wait. Different from the benchmark model, the equilibrium belief $p_t$ still goes up since $\lambda_L > \lambda_H$. After some periods, the constraint may become not binding anymore, and the low-type worker starts to play a mixed strategy again. However, the neat equilibrium characterization in the benchmark model can not survive for some parameters. Fortunately, the following theorem shows that the equilibrium characterization in the benchmark model still works when $\lambda_L$ is not significantly larger than $\lambda_H$.

**Theorem 4.** For any given set of parameters $(\lambda, c_L, c_H, p_0)$ there exist $\varepsilon > 0$ such that if $\lambda_H = \lambda$ and $\lambda_L = (\lambda, \lambda + \varepsilon)$ then the set of PBE is the same.

**Proof.** The proof can be found in the appendix at page 24.

4.3 $\lambda_L \geq \lambda_H = 0$ Case

In this case, there is no exogenous drop out by the $H$-worker. Consider first $\lambda_L = 0$. In this case our model is equivalent to Cho and Kreps (1987), only corrected by the fact that the education choice is restricted to be discrete. The reason is that the worker decides his education without interacting with the firms. Once the decision to drop out has been done, the worker...
cannot change the market’s belief about his type. Furthermore, early dropping out may be off the path of play, so beliefs can be arbitrarily assigned in those events. Therefore, the equilibrium predictions of both models share the same characteristics.

Intuitively, when $\lambda_L > 0$, nothing essential changes. The reason is that the belief threats off the path of play when $\lambda_L = 0$ are replaced by potentially exogenous dropping out by the $L$-worker, so now deviations to early drop-out are still punished.

Note that our main mechanism in the benchmark model does not present here. Indeed, in our benchmark model, as it is proven in Lemma 1, the $L$-worker uses the fact that the $H$-worker exogenously drop out to mimic him in order to save a high cost of education. Since the $H$-worker exogenously drops out, early drop-out cannot be punished too much, constraining the beliefs threats by the firms. This is no longer true when $\lambda_H = 0$, so the set of equilibria is qualitatively different from the $\lambda_H > 0$ case.

5 Concluding Remarks

This paper presents a new dynamic signaling model where wasteful education takes place over several periods of time. Workers pay an education cost per unit of time and can not commit to a pre-fixed education length. We adopt a timing that is different from that in the traditional literature and introduce an exogenous dropout rate. By doing this, we make three contributions to the literature. First, we highlight the importance of timing in the dynamic signaling model without commitment power. If a job offer is private but not preemptive, the education signaling will not disappear in a model where education is not productive. Second, in our equilibrium, the maximum length of education is decreasing in the prior about the worker being productive, and therefore, the equilibrium correspondence is lower hemicontinuous with respect to the information asymmetry perturbation. Third, our model provides rich empirical implications: the joint dynamics of education returns and the dropout rate can be derived.

Even though we present our model in an education signaling environment, our insight is also useful to understand some other environments where sending signals is not only costly but also time-consuming. For example consider a firm owner trying to sell his firm. In order to signal the type of the firm, the owner may wait some time. The opportunity cost of waiting is likely to be low if the quality of the firm is good. Drop out may be reinterpreted as liquidity shocks that force the owner to sell the firm early. Another example is given by central banks defending themselves from currency attacks. In this case, the cost of defending may be depending on the fundamentals of the economy, only known by the central bank. As time passes, the posterior about the economy being healthy increases, so the size of the attacks decreases and eventually vanishes. The exogenous
shocks may be given by random events in the international markets, such as a devaluation of the foreign currency used to defend attacks.

In our model, there is no exogenous constraint on the education length. In practice, there is an upper bound on it. A possible extension of our paper is to consider the deadline effect on the set of equilibria. Also, as we show, the empirical implementation of our model is interesting and it may help to distinguish a dynamic signaling model from a human capital model empirically. In particular, one can consider a dynamic education choice model where a worker’s productivity depends on both his privately observed ability and on his accumulation of human capital in school. Fang (2006) estimates a static education choice model with both human capital accumulation and a signaling mechanism and claims that the signaling effect is at most about one-third of the actual college wage premium. By using data of workers’ education and earnings, one can estimate our model to fit the dynamic process of education returns and the dropout rate, instead of a premium, and decompose the time-varying education returns into human capital effects and signaling effects. We leave these questions for further research.
A Appendix

A.1 The Proof of Lemma 1

Let’s first prove a preliminary result:

Lemma A.1. (The L-worker does not beat the market) For all PBE and $t$, $V^L_t \leq p_t$.

Proof of Lemma A.1. Fix a PBE. Let $\tau$ be the time at which the game ends. Then,

$$p_t V^H_t + (1 - p_t) V^L_t \leq \mathbb{E}_t[w_\tau | \tau \geq t],$$

and

$$\mathbb{E}_t[w_\tau | \tau \geq t] = \sum_{\tau = t}^{\infty} \Pr(\tau, t) \hat{p}_\tau = \sum_{\tau = t}^{\infty} \Pr(\tau, t) \frac{s^H_\tau p_t \Pr^H(\tau, t)}{\Pr(\tau, t)} = p_t \sum_{\tau = t}^{\infty} s^H_\tau \Pr^H(\tau, t) = p_t.$$

where $\Pr(\tau, t)$ denotes the conditional probability in period $t$ that the game ends in period $\tau$, and $\Pr^H(\tau, t) = s^H_\tau \prod_{t' = t}^{\tau-1} (1 - s^H_{t'})$ is further conditioning on the dropout being type $H$. The last equality holds because the high type has strictly positive dropout rate and therefore he drops out in finite time with probability one. Since $V^H_t \geq V^L_t$ (the $H$-worker can mimic the $L$-worker at a cheaper price) the result holds.

Suppose there is no endogenous dropout by the $L$-worker in period $t$, then $p_{t+1} \leq p_t \leq \hat{p}_t$. But, $\hat{p}_t \leq W^L_t = V^L_{t+1} - c_L$ due to the fact that the $L$-worker does not voluntarily drop out. By Lemma A.1, $V^L_t \leq p_{t+1} \leq \hat{p}_t$; thus $\hat{p}_t \leq \hat{p}_t - c_L$, which is a clear contradiction. So (1) is true. Therefore (2) is also true, since $W^H_t \geq V^H_{t+1} - c_H \geq \hat{p}_{t+1} - c_H$ by definition of $W^H_t$ and $V^H_t$, and $\hat{p}_{t+1} - c_H = \hat{p}_t + c_L - c_H > \hat{p}_t$ by the indifferent condition of the $L$-worker. Q.E.D.

A.2 The Proof of Theorem 1

The wage in period $t = 1$ is bounded above by 1. This implies that for the $H$-worker to be (weakly) willing to receive one period education, it must be the case that $w_0 \leq 1 - c_H$. This implies that

$$1 - c_H \geq \hat{p}_0 = \frac{p_0 s^H_0}{p_0 s^H_0 + (1 - p_0) s^L_0} \geq \frac{p_0 \lambda}{p_0 \lambda + 1 - p_0}.$$

Solving for $p_0$ under the equality, we get that the threshold for the existence of an equilibrium with non zero education satisfies equation (4). Q.E.D.
A.3 The Proof of Theorem 2

The proof of Theorem 2 is divided in several steps. To make the proof clear to the reader, we clarify that we will be following this road map:

1. We begin defining and proving some properties of the “pull-back functions”, which will be used to construct equilibria in the rest of the proof (lemmas A.2 and A.3).

2. In subsection A.3.1 we define some putative values for \(p^k_0\), denoted \(\tilde{p}^k_0\), and we prove by induction that, if \(p_0 \in (\tilde{p}^{k+1}_0, \tilde{p}^k_0]\), then there is no equilibrium with more than \(k\) periods of education.

3. Then, in subsection A.3.2 we show that, if \(p_0 \in (\tilde{p}^{k+1}_0, \tilde{p}^k_0]\), there exists an equilibrium where the \(L\)-worker is indifferent on dropping out or not for all periods except (maybe) the last for all \(T \in \{0, \ldots, k-1\}\).

4. Finally, in subsection A.3.3 we show that, if \(p_0 \in (\tilde{p}^{k+1}_0, \tilde{p}^k_0]\), there exists an equilibrium with length \(k\). Therefore, \(p^k_0 = \tilde{p}^k_0\).

We begin this proof by stating and proving two results that will simplify the rest of the proof and the proofs of other results in our paper. The first one defined and states two properties of the “pull-back functions” \(\mu_\tau(\cdot, \cdot)\) and \(\hat{\mu}_\tau(\cdot)\):

**Lemma A.2.** For any \(\tau \in \mathbb{N}\), let \(\mu_\tau : [0, 1]^2 \to [0, 1]\) and \(\hat{\mu}_\tau : [0, 1] \to \mathbb{R}\) be the functions defined by

\[
\mu_\tau(p, \hat{p}) \equiv \frac{\mu_{\tau-1}(p, \hat{p})\mu_\tau(\hat{p})}{\hat{\mu}_\tau(\hat{p})(1 - \lambda) + \mu_{\tau-1}(p, \hat{p})\lambda}, \quad (A.1)
\]

\[
\hat{\mu}_\tau(\hat{p}) \equiv \hat{p} - \tau c_L, \quad (A.2)
\]

and \(\mu_0(p, \hat{p}) \equiv p\) and \(\hat{\mu}_0(\hat{p}) = \hat{p}\). Then, for any \(\tau > 0\), \(\mu_\tau(\cdot, \cdot)\) is continuous and strictly increasing in both arguments.

**Proof of Lemma A.2.** It is obvious when \(\tau = 1\), and it holds when \(\tau > 1\) by induction argument. \(\square\)

The meaning of the pull-back functions is the following. Assume that, for a given equilibrium, the \(L\)-worker is indifferent on dropping out or not in period \(t > 0\). Then, for all \(\tau \in \{0, \ldots, t\}\), we have \(p_{t-\tau} = \mu_\tau(p_t, \hat{p}_t)\) and \(\hat{p}_{t-\tau} = \hat{\mu}_\tau(\hat{p}_t)\). So, since by Lemma 1 the \(L\)-worker is indifferent on dropping out or not in all periods except their last period, the pull-back functions give us the values of the beliefs sequences \(p\) and \(\hat{p}\) for all periods prior to a given period. They are obtained using equations (1) and (2). The following lemma formalizes this intuition:
Lemma A.3. For any equilibrium with $T > 1$ periods of education and any $T > T' \geq 0$ we have

$$p_{T'} = \mu_{T-T'}(p_T, \hat{p}_T) \quad \text{and} \quad \hat{p}_{T'} = \mu_{T-T'}(\hat{p}_T) .$$

Proof of Lemma A.3. Note that, by Lemma 1, in all periods $t < T - 1$, the $L$-worker is indifferent on dropping out or not and $s_t^H = \lambda$. This implies that if $t < T - 1$, $\hat{p}_{t-1} = \hat{p}_t - c_L$. Combining equations (1) and (2) with $s_t^H = \lambda$, we have

$$p_t \equiv \frac{p_{t+1}\hat{p}_t}{\hat{p}_t(1-\lambda) + p_{t+1}\lambda} = \frac{p_{t+1}(\hat{p}_{t+1} - c_L)}{(\hat{p}_{t+1} - c_L)(1-\lambda) + p_{t+1}\lambda} = \mu_1(p_{t+1}, \hat{p}_{t+1}) .$$

Using this formula recursively and the fact that $\mu_T(p, \hat{p}) = \mu_{T-1}(\mu_1(p, \hat{p}), \mu_1(\hat{p}))$ we obtain the desired result. \hfill \Box

A.3.1 Constructing the Upper Bound on the Length

Define the sequence $\hat{p}_0^k \equiv \mu_{k-1}(p_0^1, 1 - c_H)$, where $p_0^1$ is defined in (4). Our goal is to show that $\hat{p}_0^k$ has the same properties that $p_0^k$ (stated in the statement of the theorem), so $p_0^k = \hat{p}_0^k$. We are going to prove first, by induction, that if $p_0 \in (\hat{p}_0^{k+1}, \hat{p}_0^k]$ then there is no equilibrium with more than $k$ periods of education:

Step 1 (induction hypothesis): If $p_0 \in (\hat{p}_0^{k+1}, \hat{p}_0^k]$ there is no equilibrium with more than $k$ periods of education. If an equilibrium has $k$ periods of education, then $\hat{p}_0 \leq \hat{p}_0^k \equiv \mu_{k-1}(1 - c_H).$

Step 2 (proof for $k = 0$ periods of education): By Theorem 1 there is no equilibrium with education for $p_0 > p_0^1$. Also, in the same proof, it is shown that all equilibria in this region, $\hat{p}_0 = p_0 \geq p_0^1 > 1 - c_H = \mu_0(1 - c_H)$.

Step 3 (proof for $k = 1$ period of education): Assume that $p_0$ is such that there is an equilibrium with 1 period of education. Then, $\hat{p}_0 \leq \hat{p}_0^1 \equiv 1 - c_H$ (at least the $H$-worker has to be willing to wait). Using Bayes’ update (equations (1) and (2)) we can express $\hat{p}_0 \equiv \hat{p}_0(p_0, s_0^L, s_0^H)$ and $p_1 = p_0(p_0, s_0^L, s_0^H)$. Therefore, using these equations, we can write $p_0$ in terms of $\hat{p}_0$, $p_1$ and $s_0^H$ in the following way:

$$p_0 = p_{-1}(p_1, \hat{p}_0, s_0^H) \equiv \frac{p_1\hat{p}_0}{\hat{p}_0(1 - s_0^H) + p_1s_0^H} .$$

The RHS of the previous expression is maximized when $s_0^H = \lambda$. Therefore, if an equilibrium ends with length of two periods, the initial prior is at most $p_0^1 \equiv \frac{1-c_H}{1-c_H(1-\lambda)}$.

---

\footnote{The second induction hypothesis is included in order to make the argument simple in the induction argument (step 4).}
Step 4 (induction argument for $k > 1$): Assume the induction hypothesis is true for $k - 1$, for $k > 1$. We need to verify whether it is true for $k$.

Assume that $p_0$ is such that there exists some equilibrium with $k$ periods of education. Denote the beliefs sequences for this equilibrium $p$ and $\hat{p}$. Note that, by the induction hypothesis, $p_1 \leq \hat{p}_0^{k-1}$ and $\hat{p}_1 \leq \hat{p}_0^{k-1}$, since the continuation play after 1 is itself an equilibrium with initial prior $p_1$. Since $k > 2$, by Lemma 1, the $H$-worker is strictly willing to wait in period 0, so $s_0^H = \lambda$, and the $L$-worker randomizes in period 0. Then, $\hat{p}_0 = \hat{p}_1 - c_L \leq \hat{p}_0^{k-1} - c_L = \hat{p}_0^k$. Therefore, by Lemma A.3, $p_0 = \mu_1(p_1, \hat{p}_1)$, and that this is increasing in both arguments. So, the maximum value it can take is $\tilde{p}_0^k \equiv \mu_1(\tilde{p}_0^{k-1}, \hat{p}_0^{k-1})$.

Step 5 ($T^*$ is the limit): Note that $T^*$ is such that

\[
\tilde{p}_0^{T^*+1} \leq 0 < \tilde{p}_0^{T^*}.
\]

Then, since $\tilde{p}_0^{T^*+1} \leq 0$, there is no equilibrium longer than $T^*$ periods of education.

A graphical intuition of the proof can be found in Figure 4. It graphically represents both $\tilde{p}_0^T$ and $\hat{p}_0^T$ used in the proof.

A.3.2 Constructing $L$-equilibria

Now, we prove a result related to the set of equilibria where the $L$-worker is indifferent in all periods, which is similar to Theorem 2 itself. For each $p_0 \in (0, 1)$, we use $\bar{T}_L(p_0)$ to denote the maximum number of education periods of an equilibrium where the $L$-worker is indifferent to drop out in all periods except (maybe) the last. We name these equilibria $L$-equilibria. The following lemma shows that, for any $p_0 \in (0, 1)$, there is a finite integer $k$ such that, for each $T = 0, 1, \ldots, k$,
there is an $L$-equilibrium that lasts for $T$ periods of education, and no $L$-equilibrium with length more than $k$.

**Lemma A.4.** Let’s define $T^{**} = \left[\frac{1-c_L}{c_L}\right]$, $p_0^{L,k} \equiv \mu_k(1,1)$ for $k = 0,\ldots,T^{**}$ and $p_0^{L,T^{**}+1} \equiv 0$. Then, if $p_0 \in (p_0^{L,k+1},p_0^{L,k})$ for some $k = 0,\ldots,T^{**}$, we have $\tilde{T}_L(p_0) = k$. Furthermore, for each $T \leq \tilde{T}_L(p_0)$, there is a unique $L$-equilibrium with $T$ periods of education.

**Proof of Lemma A.4.** Fix some $p_0 \in (0,1)$. If $p_0 > \mu_k(1,1)$ for some $k \leq T^{**}$ there is no $L$-equilibrium with $k$ periods of education. Indeed, if there was one (ending at $p_k = \hat{p}_k$), then $p_0 = \mu_k(p_k,p_k)$. But since $\mu_k(p_k,p_k)$ is strictly increasing in $p_k$ and $p_0 > \mu_T(1,1)$, then $p_0 > \mu_k(p,p)$ for all $p \in [0,1]$. This is clearly a contradiction. Note also that, in an $L$-equilibrium with $T$ periods of education, $\hat{p}_T - \hat{p}_0 = Tc_L \leq 1$. Since $(T^{**} + 1)c_L > 1$, we have $\tilde{T}(p_0) < T^{**} + 1$.

Fix $k < T^{**}$, $p_0 \in (p_0^{L,k+1},p_0^{L,k})$ and $T \leq k$. Note that $\mu_T(p,p)$ is continuous and strictly increasing when $p > Tc_L$ for any $T \leq T^{**}$ and $\lim_{p \searrow Tc_L} \mu_T(p,p) = 0$. So, since $p_0 \leq \mu_k(1,1) \leq \mu_T(1,1)$, there exists a unique $p_T \in (Tc_L,1)$ such that $p_0 = \mu_T(p_T,p_T)$. Furthermore, there is an equilibrium with length $T$ with $p_T = \mu_{T-t}(p_T,p_T)$ and $\hat{p}_t = \mu_{T-t}(p_T)$. The argument for $k = T^{**}$ is analogous.

**Lemma A.5.** For any $k \leq T^{**}$, we have $p_0^{L,k} \in (\bar{p}_0^{k+1},\bar{p}_0^{k})$.

**Proof of Lemma A.5.** Note first that

\[
\frac{p_0^L}{1 - (1 - \lambda)c_L} > \frac{1 - c_L}{1 - (1 - \lambda)c_H} = \mu_1(1,1) > \mu_1(p_0^L,1 - c_H)
\]

By definition, for $k > 1$, $p_0^k = \mu_{k-1}(p_0^k,1 - c_H) = \mu_1(p_0^{k-1},1 - c_H) = \mu_1(p_0^{k-1},1 - c_L)$ and $p_0^{k,L} = \mu_{k-1}(p_0^L,1 - c_L) = \mu_1(p_0^{k-1},1 - c_L))$. Also, note that $\hat{\mu}_k(1 - c_H) > \hat{\mu}_k(1 - c_L) > \hat{\mu}_{k+1}(1 - c_H)$. Therefore, since $\mu_1(\cdot,\cdot)$ is strictly increasing in both arguments, we have $p_0^{L,k} \in (\bar{p}_0^{k+1},\bar{p}_0^k)$.

**A.3.3 Constructing $H$-equilibria**

Lemma A.4 implies that for any $p_0 \in (0,1)$, an $L$-equilibrium lasting for at most $k$ periods can be constructed, where $k$ satisfies that $p_0 \in (p_0^{L,k+1},p_0^{L,k})$. However, Lemma A.5 shows that $p_0^{L,k} < \bar{p}_0^k$. For $p_0 \in (p_0^{L,k},\bar{p}_0^k]$, there is no $L$-equilibrium lasting for $k$ periods. The question is now whether any other equilibrium which lasts for $k$ periods in this last region. Lemma A.6 shows that the answer to this question is yes.

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5Note that, if $T \leq T^{**}$ then $Tc_L < 1$, and, by definition, $\hat{\mu}_T(Tc_L) = 0$. Using the definition of $\mu_1(\cdot,\cdot)$, we have that $\mu_T(c_LT,c_LT) = 0$. 

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An equilibrium which lasts for \( T > 0 \) periods of education is an \( H \)-equilibrium if and only if, in equilibrium, the \( L \)-worker strictly prefers dropping out in period \( T - 1 \). In other words, in an \( H \)-equilibrium \( p_T = 1 \). Note each equilibrium is either \( L \)-equilibrium or \( H \)-equilibrium, and never both.

**Lemma A.6.** If \( p_0 \in (p_{00}^{Lk}, \hat{p}_0^k) \), there exists an \( H \)-equilibrium of length \( k \), for \( k \in \{1, \ldots, T^{**} \} \). If \( p_0 \in (\hat{p}_0^{k+1}, p_{00}^{Lk}) \), there exists an \( L \)-equilibrium of length \( k \), for \( k \in \{1, \ldots, T^* - 1 \} \).

**Proof of Lemma A.6.** For \( p_0 \in (\hat{p}_0^{k+1}, p_{00}^{Lk}) \) the proof of the previous lemma tells us that there exists an \( L \)-equilibrium of length \( k \). To prove the case \( p_0 \in (p_{00}^{Lk}, \hat{p}_0^k) \), we define the function \( \hat{p} : (p_{00}^{1L}, p_{00}^0) \to (1 - c_L, 1 - c_H) \) as follows

\[
\hat{p}(p) \equiv \frac{\lambda p}{\lambda p + 1 - p}.
\]

Then for all \( p_0 \in (p_{00}^{Lk}, \hat{p}_0^k) \) there exists a unique \( f(p_0) \in (p_{00}^{1L}, p_{00}^0) \) such that \( p_0 \equiv \mu_{k-1}(f(p_0), \hat{p}(f(p_0))) \). Indeed, we have that \( \lim_{p \to p_{00}^{1L}} \hat{p}(p) = 1 - c_L \) and \( \hat{p}(p_{00}^0) = 1 - c_H \). So, we have

\[
\lim_{p \to p_{00}^{1L}} \mu_{k-1}(p, \hat{p}(p)) = p_{00}^{Lk} \quad \text{and} \quad \mu_{k-1}(p_{00}^0, \hat{p}(p_{00}^0)) = \hat{p}_0^k.
\]

Since \( \hat{p}(\cdot) \) is continuous and strictly increasing, \( \mu_{k-1}(|\cdot|, \cdot) \) is continuous in both arguments and strictly increasing, then there exists such \( f(p_0) \), and is unique.

Let’s construct one equilibrium with \( k \) education periods when \( p_0 \in (p_{00}^{Lk}, \hat{p}_0^k) \), for \( k \leq T^* - 1 \). Our claim is that it can be defined by \( p_k = \hat{p}_k = 1, p_t = \mu_{t-1}(f(p_0), \hat{p}(f(p_0))) \) and \( \hat{p}_t = \hat{p}(f(p_0)) - c_L(k - t - 1), \) for \( t \in \{0, \ldots, k - 1 \} \). To prove that we show that the corresponding strategies are well defined. Note that, if the \( L \)-worker is indifferent in period 0, we have

\[
s_{t}^{L} = \frac{1}{1 + \frac{1 - (1 - \lambda)(1 - p_i)\hat{p}}{\lambda \hat{p}(1 - \hat{p})}} = \frac{\lambda}{1 - \frac{(1 - \lambda)(1 - p_i)\hat{p}}{\lambda \hat{p}(1 - \hat{p})}}.
\]

The first inequality shows that \( s_{t}^{L} < 1 \). The second equality shows that, if \( p_1^l > \hat{p}_t \), then \( s_{t}^{L} > \lambda \), what is equivalent to \( p_t^2 < p_t^l \), which is true as long as \( \hat{p}_t > 0 \). Since, when \( k < T^* \), \( \hat{p}_0 = \hat{p}(f(p_0)) - c_L(k - 1) > 0 \), the result holds in this case.

Finally, there are two possible cases. If \( T^{**} = T^* \), we know from the previous lemma that there exists an \( L \)-equilibrium in with length \( T^{**} \) in \((0, p_{00}^{L,T^*})\). If \( T^{**} = T^* - 1 \) then there exists some \( p \in (p_{00}^{1L}, p_{00}^0) \) such that \( \hat{p}(p) = T^{**}c_L \). Indeed, in this case \( 1 \leq T^*c_L < 1 - c_H + c_L \), so \( T^*c_L \in (1 - c_H, 1 - c_L) \). Therefore, we can use the same argument as for \( p_0 \in (p_{00}^{Lk}, \hat{p}_0^k) \) for \( k \leq T^* - 1 \). The idea of the partition construction can be summarized in Figure 5.
Finally, note that the set \( \{ \tilde{p}_k^k \}_{k=0}^{T^*+1} \) is such that \( \tilde{p}_k^k > \tilde{p}_k^{k+1} \) for all \( k \). Furthermore, for all \( 0 \leq k \leq T^* \) and \( 0 \leq T \leq k \), if \( p_0 \in (\tilde{p}_k^{k+1}, \tilde{p}_k^k] \) then there exists an equilibrium with \( T \) periods of education and no equilibrium with length larger than \( k \). So, \( \tilde{p}_k^k \equiv \tilde{p}_k^k \), for \( k = 0, \ldots, T^*+1 \), satisfy the statement of Theorem 2, and therefore its proof is complete. \( Q.E.D. \)

A.4 The Proof of Theorem 3

1. Assume that, in period \( t \), there are two types \( \theta_1, \theta_2 \in \Theta \), with \( c^{\theta_1} < c^{\theta_2} \), both indifferent on dropping out or not. Let \( \tau_1 \) and \( \tau_2 \) denote, respectively, the stopping times of the continuation strategies that make players indifferent on dropping out or not.\(^6\) Then, we have

\[
\hat{Y}_t = \mathbb{E}[w_{\tau_{2}} - c^{\theta_2} \tau_2] \geq \mathbb{E}[w_{\tau_{1}} - c^{\theta_2} \tau_1] > \mathbb{E}[w_{\tau_{1}} - c^{\theta_1} \tau_1] = \hat{Y}_t .
\]

The first (weak) inequality is from the optimality of the \( \theta_2 \)-worker. The strong inequality is because \( \mathbb{E}[\tau_{\theta_1}] > 0 \) and \( c^{\theta_1} < c^{\theta_2} \). The equalities come from the fact that \( i \)-workers, with \( i \in \{1, 2\} \) are indifferent on taking on dropping out (and getting \( \hat{Y}_t \)) or staying and following \( \tau_i \). Therefore, we have a contradiction.

2. Assume otherwise, so there exist \( \theta_1, \theta_2 \in \Theta \) such that \( \theta_1 < \theta_2 \) and \( T^{\theta_1} > T^{\theta_2} \). Let \( \tau_{\theta_1} \) be the stopping time of the continuation strategy after \( T^{\theta_2} \), given by the strategy of \( \theta_1 \). Then, note that

\[
\hat{Y}_{T^{\theta_2}} \geq \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_2} \tau_{\theta_1}] > \mathbb{E}[w_{\tau_{\theta_1}} - c^{\theta_1} \tau_{\theta_1}] \geq \hat{Y}_{T^{\theta_2}} .
\]

This is clearly a contradiction. The first inequality comes from the optimality of the \( \theta_2 \)-worker choosing to drop out at \( T^{\theta_2} \) (since they could deviate to mimic the \( \theta_1 \)-worker). The second inequality is given by the fact that since \( \theta_1 < \theta_2 \), \( c^{\theta_2} < c^{\theta_1} \) and since \( T^{\theta_1} > T^{\theta_2} \), \( \mathbb{E}[\tau_{\theta_1}] > 0 \). The last inequality comes from the optimality of the \( \theta_1 \)-worker choosing to drop out at \( T^{\theta_1} > T^{\theta_2} \) (since they could deviate to mimic the \( \theta_2 \)-worker).

\(^6\)For this proof, for a given strategy, it is convenient to use the random variable \( \tau \), which gives the duration of the game.
3. Define $\Theta_t = \{\theta | T^\theta \geq t\}$ and $\theta_t = \min\{\Theta_t\}$. We proceed as in the proof of Lemma A.1. Now we have

$$
\mathbb{E}_t[w_t | \tau \geq t] = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \hat{Y}_\tau = \sum_{\tau=t}^{\infty} \Pr(\tau, t) \frac{\sum_{\theta} Y^\theta s^\theta_t p^\theta_t \Pr^\theta(\tau, t)}{\Pr(\tau, t)}
$$

where $\Pr(\tau, t)$ and $\Pr^\theta(\tau, t) = s^\theta_t \prod_{t'=t}^{\tau-1} (1 - s^\theta_{t'})$ are defined as in the proof of Lemma A.1.

Note that, by the previous result,

$$
\sum_{\theta = \theta_t}^{N} p_t^\theta V_t^\theta = \mathbb{E}_t[w_t | \tau \geq t] - \sum_{\theta = \theta_t}^{N} p_t^\theta c^\theta \tau^\theta(t) < \mathbb{E}_t[w_t | \tau \geq t],
$$

where $\tau^\theta(t)$ is the stopping time for the $\theta$-worker conditional on reaching $t$. Since $V_t^\theta \leq V_t^{\theta+1}$ (since the $(\theta + 1)$-worker can mimic the $\theta$-worker at a lower cost), and $\sum_{\theta = \theta_t}^{N} p_t^\theta = 1$ we have that $V_t^\theta < Y_t$.

Assume that in period $t$ there is no voluntary dropout. In this case, $\hat{Y}_t = Y_t$. Since we just showed $V_{\theta_t} < Y_t$, the $\theta_t$-worker is willing to drop out, which is a contradiction.

4. Note that, by part 3 of this theorem, we have that $\hat{Y}_{t+1} - c^\theta_t \leq \hat{Y}_t$. Furthermore, $\hat{Y}_{t+1} - c^{\theta_t+1} \geq \hat{Y}_t$. This implies that $\hat{Y}_{t+1} - \hat{Y}_{t+1} \in [c^{\theta_t+1}, c^\theta_t]$. Since $c^\theta$ is decreasing in $\theta$ and, by part 2 of this theorem, the $\theta_t$-worker is (weakly) increasing in $t$, $\hat{Y}_t$ is concave in $t$.

Q.E.D.

A.5 The Proof of Lemma 2

We first prove that Lemma 1 (that holds when $\lambda_H = \lambda_L$) is still valid when $\lambda_H \geq \lambda_L$. Consider $T$ as the maximum periods lower than $T^L$ where $s_t^L \leq s_t^H$. In this case

$$
p_{T+1} \leq p_T \leq \hat{p}_T.
$$

Furthermore, since the $L$-worker is voluntarily dropping out at time $T + 1$, this implies $\hat{p}_T \leq \hat{p}_{T+1} - c_L$. Nevertheless, since $s_{T+1}^L \geq s_{T+1}^H$, we have $\hat{p}_{T+1} \geq p_{T+1}$, which is a contradiction, since

$$
\hat{p}_{T+1} \leq p_{T+1} \leq p_T \leq \hat{p}_T \leq \hat{p}_{T+1} - c_L.
$$

So, when $\lambda_H \geq \lambda_L$, it is still true that $s_t^L > s_t^H$ in all periods of all equilibria before $T^L$. Therefore, the relaxation of the constraint $\lambda_L = \lambda_H = \lambda$ to $\lambda_L \leq \lambda_H = \lambda$ does not introduce new equilibria. Trivially, it does not destroy any equilibria, since in the model $\lambda_L = \lambda_H = \lambda$, in all equilibria, $s_t^L > \lambda$ for all equilibria and period $t \leq T^L$.

Q.E.D.
A.6 The Proof of Theorem 4

Note that Lemma A.1 still holds (the $H$-worker can imitate the strategy of the $L$-worker). Now we try to prove a result analogous to Lemma 1. Assume that the $L$-worker is not voluntarily dropping out at period $t$, so his dropout rate is $\lambda + \varepsilon$. First assume that the dropout rate of the $H$-worker is larger than $\lambda + \varepsilon$. In this case, we can apply the exact same argument as in the proof of Lemma 1, so we obtain again a contradiction. Assume now that $\lambda_H \in [\lambda, \lambda + \varepsilon)$. In this case 

$$p_{t+1} = p_t + O(\varepsilon) \quad \text{and} \quad \hat{p}_t = \hat{p}_t + O(\varepsilon),$$

so $\hat{p}_t - p_{t+1} = O(\varepsilon)$. Then, using the same logic as in the proof of Lemma 1 we have

$$\hat{p}_t \leq W^L_t \leq V^L_{t+1} - c_L \leq p_{t+1} - c_L.$$

Therefore, $\hat{p}_t - p_{t+1} \leq -c_L$. But this is inconsistent with $\hat{p}_t - p_{t+1} = O(\varepsilon)$. That proves that, if $\varepsilon > 0$ is small enough, the model with $\lambda_H = \lambda$ and $\lambda_L = \lambda + \varepsilon$ does not have more equilibria than for the case $\varepsilon = 0$.

Let’s prove the reverse. Assume that there exists a sequence $\{\varepsilon_n > 0\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \varepsilon_n = 0$ and, for each $n$, there exists an equilibrium in our original model and $t_n$ reached with positive probability on the path of play under this equilibrium such that $s^L_{t_n} \in [\lambda, \lambda + \varepsilon_n)$. This implies $p_{t_n+1} = p_{t_n} + O(\varepsilon_n)$ and $\hat{p}_{t_n} = p_{t_n} + O(\varepsilon_n)$, so $\hat{p}_{t_n} - p_{t_n+1} = O(\varepsilon_n)$. So,

$$\hat{p}_{t_n} = W^L_{t_n} = V^L_{t_n+1} - c_L \leq p_{t_n+1} - c_L.$$

This, again, is a contradiction. 

\[ Q.E.D. \]

\[ ^7 \text{Using some abuse of notation, } p_{t_n} \text{ and } \hat{p}_{t_n} \text{ denote the corresponding posteriors in the } n\text{-th equilibrium of the sequence.} \]
References


