Abstract

Raiffa (1961) criticizes ambiguity-averse preferences by claiming that hedging is possible with randomization of choices. We argue that the timing of randomization is crucial for hedging: If the decision maker thinks that the randomization is ex ante (i.e., before a realization of a state) then it provides no hedging; if it is ex post (i.e., after a realization of a state) then it provides hedging. We investigate the decision maker’s preference over sets of acts among which he can randomize. To capture a preference for randomization, we propose two new axioms. The dominance axiom claims that the decision maker should prefer a set \( A \) of acts to another set \( B \) if, for any randomization among \( B \), there exists a preferred randomization among \( A \), no matter whether he can randomize ex ante or ex post. The certainty strategic rationality axiom requires that if the decision maker prefers \( A \) to \( B \) and \( A \) consists of a constant act, then he should be indifferent between \( A \) and \( A \cup B \) because the randomization with the constant act does not provide hedging. By using these
axioms as well as standard axioms, we provide an extension of Gilboa and Schmeidler’s (1989) Maxmin preference that identifies the decision maker’s subjective probability that he uses to randomize ex post. In the representation, a single unique parameter captures the subjective probability; moreover, the decision maker’s optimal randomizations over sets is obtained endogenously.

**Keywords:** Ambiguity; randomization; Ellsberg’s paradox; Raiffa’s critique; maxmin utility; preferences over sets.

JEL Classification Numbers: D81, D03.

1 Introduction

Ellsberg (1961) proposed the following thought experiment: Consider an urn containing balls, each of which is either red or black. There is no further information about the contents of the urn. You bet on the color of the ball that an experimenter draws. If your bet turns out to be correct, then you get a payoff of 1. Otherwise, you get a payoff of 0. Typically, subjects are indifferent between betting on either color. However, they strictly prefer a fifty-fifty objective lottery between 1 and 0 to the bets. This behavior is called ambiguity aversion.

![Figure 1: Raiffa’s (1961) critique](image)

(The first and second coordinates in payoff profiles show the payoffs when the color of the drawn ball is red and black respectively.)

Raiffa (1961) criticizes ambiguity-averse preferences with this argument: by flipping a fair coin to choose which color to bet on, you can hedge and obtain a constant expected payoff (i.e., the fifty-fifty lottery between 1 and 0) for each color of the ball you will draw. (See Figure
1.) Since this argument has such strong intuitive and normative appeal, this preference for randomization has been assumed in the literature without through investigation.

However, whether the randomization provides hedging or not depends on the decision maker’s subjective belief about the timing of randomization, namely, before or after the realization of a state.\(^1\) For instance, in Ellsberg (1961)’s example, if the randomization by flipping a coin is \textit{ex ante} (i.e., before the realization of the state) as the right tree in Figure 2 shows, then one faces ambiguity again after either side of the coin appears. In contrast, if the randomization is \textit{ex post} (i.e., after a realization of a state) as the left tree in Figure 2 shows, then one removes all the ambiguity by flipping the coin. Indeed, recent experiments by Dominiak and Schnedler (2010) have found strong evidence for a preference for ex-post randomization but no evidence for a preference for ex-ante randomization.

![Figure 2: Ex-post Randomization (left) and Ex-ante Randomization (right)](image)

(The solid lines correspond to the risk introduced by flipping a coin, while the dotted lines correspond to the ambiguity of the color of the drawn ball.)

Understanding a preference for randomization is important to (stochastically) predict choices of a decision maker. The purpose of the present paper is to provide an axiomatic model of a preference for randomization. An axiomatic approach is useful to identify the decision maker’s

\(^1\)A careful reader might say that a state is supposed to describe all the payoff relevant information including beliefs about the timing of randomization. However, theory with such richer state space would lead to a model that is inherently not related to observed choices.
subjective belief about the timing of randomization behaviorally.

However, there is one essential difficulty to investigate a preference for randomization. It is often impossible for us to observe the decision maker’s randomization: If he randomizes implicitly without using observable randomization devices then all we can observe is an ex-post realization of the randomization, not the randomization itself. For example, in Ellsberg’s (1961) example, we cannot distinguish between the degenerate choice of red and the choice of red as an ex-post realization of the fifty-fifty implicit randomization between red and black. To overcome this difficulty, we investigate the decision maker’s preference over sets of acts (i.e., payoff profiles) among which he could randomize, instead of his preference over the randomizations themselves.

A preference for randomization may violate two standard rationality axioms in the literature on preferences over sets. A strict preference for randomization could violate the *indifference to (ex-post) randomization* proposed by Dekel, Lipman, and Rustichini (2001).\(^2\) Instead of that axiom, we propose the *dominance* axiom, which states that the decision maker should prefer a set \(A\) of acts to another set \(B\) if for any randomization among \(B\) there exists a preferred randomization among \(A\), no matter whether he can randomize ex ante or ex post. The dominance axiom allows a strict preference for randomization.

In addition, a preference for randomization could imply a *preference for flexibility*. For example, in Ellsberg’s (1961) experiment, a decision maker who prefer the fifty-fifty randomization would prefer to have the flexibility of colors to bet. Hence, a preference for randomization could

\(^2\)Since the indifference to randomization axiom is originally proposed for a preference over lotteries, not over acts, Dekel, Lipman, and Rustichini (2001) do not distinguish between ex-post and ex-ante randomizations.
violate the *strategic rationality* axiom proposed by Kreps (1979). Our weaker axiom, the *certainty strategic rationality* axiom, requires no preference for flexibility that is useless for hedging. In particular, the axiom states that if the decision maker prefers $A$ to $B$ and $A$ consists of a constant act then he should be indifferent between $A$ and $A \cup B$. This is because randomization with the constant act does not provide hedging.

By using these two new axioms with the standard axioms in Gilboa and Schmeidler (1989), we characterize the following representation: for all sets $A$ of acts

$$V(A) = \max_{P \in \Delta(A)} \delta U(\int f dP(f)) + (1 - \delta) \int U(f) dP(f),$$

(1)

where $\delta \in [0, 1]$ captures the decision maker’s *subjective probability that he can randomize ex post* and $U$ is the maxmin utility proposed by Gilboa and Schmeidler (1989). Given $\delta$ and $U$, the optimal randomization (i.e., probability distribution) $P$ over $A$ is determined endogenously as a maximizer of the right hand side of (1). Remember that the decision maker’s preference is *not* defined on randomizations because of the aforementioned difficulty of observing (implicit) randomizations. Nevertheless, the representation (1) allows us to predict how the decision maker randomizes given a set of acts.

The optimal randomization $P$ maximizes the weighted sum of the two terms in the right hand side of (1). The first term is the utility when the decision maker can randomize ex post because $\int f dP(f)$ denotes an ex-post (i.e., state-wise) mixture of acts with respect to $P$. For example, in this term, an equal randomization between red and black in Ellsberg’s (1961) experiment is perceived as the ex-post randomization in the left tree in Figure 2. Hence, the equal randomization yields the completely hedged utility $U(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0)$, where $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0$ denotes a lottery that puts equal weights on 1 and 0.

On the other hand, the second term in the right hand side of (1) is the utility when the decision maker can randomize only ex ante. Hence, $P$ provides a mere expectation over utilities. For example, the equal randomization in Ellsberg’s (1961) experiment is perceived as the ex-ante randomization in the right tree in Figure 2. Hence, the equal randomization yields an expected utility of $\frac{1}{2}U(1, 0) + \frac{1}{2}U(0, 1)$.
The representation (1) satisfies the strategic rationality axiom if and only if \( \delta = 0 \); the indifference to randomization axiom if and only if \( \delta = 1 \). In the generic case where \( \delta \in (0,1) \), both axioms are violated. For example, in Ellsberg’s (1961) experiment the decision maker with \( \delta \in (0,1) \) prefers to have the flexibility to bet on both colors so as to randomize. However, the decision maker thinks randomizing over the bets is not enough to hedge completely and still prefers to obtain the fifty-fifty objective lottery, as formally shown in Section 3.

The remainder of the paper is organized as follows: Section 2 introduces the setup. Section 3 introduces an example. Section 4 provides the axioms. Section 5 provides the representation theorem, uniqueness result, and the characterization of the special cases corresponding to \( \delta = 1 \) or \( \delta = 0 \). Section 6 discusses the related literature. All proofs are in the appendix.

## 2 Setup

For any set \( X \), let \( \Delta(X) \) be the set of distributions over \( X \) with finite supports. An element in \( \Delta(X) \) is called a lottery on \( X \). Let \( \delta_x \in \Delta(X) \) denote a point mass on \( x \). Let \( S \) be a finite set of states. Let \( Z \) denote a finite set of outcomes. A payoff profile \( f \) is called an act and defined to be a function from \( S \) into \( \Delta(Z) \). Let \( \mathcal{F} \) be the set of all acts. Let \( \mathcal{A} \) be the set of closed subsets of \( \mathcal{F} \). A preference relation \( \succeq \) is defined on \( \mathcal{A} \). As usual, \( \succ \) and \( \sim \) denote the asymmetric and symmetric parts of \( \succeq \) respectively. A constant act is an act \( f \) such that \( f(s) = f(s') \) for all \( s, s' \in S \). Elements in \( \Delta(Z) \) are identified as constant acts. For \( f \in \mathcal{F} \), an element \( l_f \in \Delta(Z) \) is a certainty equivalent of \( f \) if \( f \sim l_f \).

We define mixtures of acts and sets as follows:

**Definition:** For all \( \alpha \in [0,1] \) and \( f, g \in \mathcal{F} \), \( \alpha f + (1 - \alpha)g \in \mathcal{F} \) is an act such that \( (\alpha f + (1 - \alpha)g)(s)(z) = \alpha f(s)(z) + (1 - \alpha)g(s)(z) \in [0,1] \) for each \( s \in S \) and \( z \in Z \).

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\(^{3}\text{Since } S \text{ and } Z \text{ are finite, } \mathcal{F} \text{ is compact subset of } (|Z| - 1)|S| \text{ dimensional Euclidean space. Therefore } A \in \mathcal{A} \text{ is compact as well.} \)
**Definition:** For all $\alpha \in [0,1]$ and $A, B \in \mathcal{A}$, $\alpha A + (1-\alpha)B = \{ \alpha f + (1-\alpha)g | f \in A \text{ and } g \in B \}$.

Elements in $\mathcal{A}$ are denoted by $A$ and $B$. Elements in $\mathcal{F}$ are denoted by $f$ and $g$. Elements in $\Delta(Z)$ are denoted by $l$ and $r$. For simplicity, we denote singleton sets $\{f\}$ and $\{l\}$ by $f$ and $l$, respectively.

Finally, we introduce a notation for randomizations: Elements in $\Delta(\mathcal{F})$ are called randomization (over acts) and denoted by $P$ and $Q$. If $P$ is a randomization over acts $\{f^i\}_{i=1}^n$ with probabilities $\{\alpha_i\}_{i=1}^n$ then we denote $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$. Note that the domain of our primitive preference does not include $\Delta(\mathcal{F})$.

### 3 Example

We will investigate the implication of our model in Ellsberg’s (1961) example discussed in the introduction. Let $S = \{\text{red, black}\}$. Consider two acts $f = (1,0)$ and $g = (0,1)$ corresponding to betting on red and black respectively.

For simplicity, assume $u$ is linear and $C = \Delta(S)$. Then $U(f) = 0 = U(g)$ and for all $\alpha \in [0,1]$ \[
U(\frac{1}{2}f + \frac{1}{2}g) = \frac{1}{2} \geq \min\{\alpha, 1-\alpha\} = U(\alpha f + (1-\alpha)g). \tag{2}
\]

It follows that

\[
P(\{f, g\}) = \begin{cases} 
\frac{1}{2}f \oplus \frac{1}{2}g & \text{if } \delta > 0, \\
\Delta(\{f, g\}) & \text{if } \delta = 0.
\end{cases} \tag{3}
\]

Since $U(f) = 0 = U(g)$, (3) implies

\[
V(\{f, g\}) = \begin{cases} 
\delta U(\frac{1}{2}f + \frac{1}{2}g) & \text{if } \delta > 0, \\
0 & \text{if } \delta = 0.
\end{cases} \tag{4}
\]

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Moreover, (2) shows $V(\co\{f, g\}) = U(\frac{1}{2}f + \frac{1}{2}g)$, where $\co\{f, g\} = \{\alpha f + (1 - \alpha)g | \alpha \in [0, 1]\}$. This together with (4) shows

\[
\begin{align*}
\co\{f, g\} &\sim \{f, g\} \succ \{f\} \sim \{g\} & \text{if } \delta = 1, \\
\co\{f, g\} &\succ \{f, g\} \succ \{f\} \sim \{g\} & \text{if } \delta \in (0, 1), \\
\co\{f, g\} &\succ \{f, g\} \sim \{f\} \sim \{g\} & \text{if } \delta = 0.
\end{align*}
\]

The above preferences show that the indifference to randomization axiom (i.e., $\co(A) \sim A$) is not violated only when $\delta = 1$. The strategic rationality axiom (i.e., $A \succeq B \Rightarrow A \sim A \cup B$) is not violated only when $\delta = 0$. For $\delta \in (0, 1)$, both axioms are violated.

To see how the decision maker’s choice and his optimal randomization change as $\delta$ changes, let us introduce a constant act $h = (\varepsilon, \varepsilon)$, where $\varepsilon < \frac{1}{2}$. Then $U(h) = \varepsilon > 0 = U(f) = U(g)$. This together with (4) shows $\{f, g\} \succ \{h\}$ if and only if $\delta > 2\varepsilon$. This result means that if the decision maker thinks the probability that he can randomize ex post is high (i.e., $\delta > 2\varepsilon$) then he chooses $\{f, g\}$ to randomize; if he thinks the probability is low (i.e., $\delta < 2\varepsilon$) then he does not randomize and prefers the constant act $h$.

Finally, we will investigate how the optimal randomization among $\{f, g, h\}$ changes as $\delta$ changes. It is easy to see that the optimal randomization among $\{f, g, h\}$ is of the form $\alpha(\frac{1}{2}f + \frac{1}{2}g) \oplus (1 - \alpha)h$ for some $\alpha \in [0, 1]$.\textsuperscript{4} Since $h$ is constant, the utility of the randomization is $\alpha\delta + (1 - \alpha)\varepsilon$. Therefore, the optimal randomization is

\[
P(\{f, g, h\}) = \begin{cases} \frac{1}{2}f + \frac{1}{2}g & \text{if } \delta > 2\varepsilon, \\ \alpha(\frac{1}{2}f + \frac{1}{2}g) \oplus (1 - \alpha)h | \alpha \in [0, 1] & \text{if } \delta = 2\varepsilon, \\ h & \text{if } \delta < 2\varepsilon. \end{cases}
\]

4 Axioms

The first six axioms are due to Gilboa and Schmeidler (1989). However, since our domain is larger than that of Gilboa and Schmeidler (1989), the weak order axiom and the certainty

\textsuperscript{4}Since $C = \Delta(S)$, if probabilities on $f$ and $g$ are different then the decision maker can obtain a higher utility by making both probabilities the same.
independence axiom need to be assumed in our larger domain.

**Axiom (Weak Order):** $\succeq$ is complete and transitive on $\mathscr{A}$.

**Axiom (Continuity):** If $f \succ g$ and $g \succ h$ then there exist $\alpha$ and $\beta$ in $(0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g$ and $g \succ \beta f + (1 - \beta)h$.

**Axiom (Nondegeneracy):** There exist $z_+, z_- \in Z$ such that $z_+ \succ z_-$. 

**Axiom (Monotonicity):** If $f(s) \succeq g(s)$ for all $s \in S$ then $f \succeq g$.

If a preference relation $\succeq$ satisfies the axioms above, then each act $f \in \mathcal{F}$ admits a certainty equivalent $l_f \in \Delta(Z)$.

**Axiom (Ambiguity Aversion):** If $f \sim g$ then $\frac{1}{2}f + \frac{1}{2}g \succeq f$.

The next axiom is a direct extension of the certainty independence axiom proposed by Gilboa and Schmeidler (1989). The motivation is that *mixing constant acts does not provide hedging*.

**Axiom (Certainty Set Independence):**

$$A \succeq B \iff \alpha A + (1 - \alpha)l \succeq \alpha B + (1 - \alpha)l.$$ 

The decision maker who has a preference for randomization can have a preference for flexibility, so as to violate the strategic rationality axiom (i.e., $A \succeq B \Rightarrow A \sim A \cup B$). However, since mixing between a constant act and any other acts does not provide hedging, a weaker version of the axiom should hold as follows:
**Axiom** (Certainty Strategic Rationality):

\[ l \succeq B \Rightarrow l \sim l \cup B. \]

To explain the last axiom, dominance, remember that as we saw in Figure 2, whether a randomization provides hedging or not depends on the decision maker’s subjective belief about the timing of randomization, namely, before or after a realization of a state. For example, if a randomization \( \frac{1}{2}(1, 0) \oplus \frac{1}{2}(0, 1) \) is ex-post then the randomization is equivalent to a state-wise mixture \( \frac{1}{2}(1, 0) + \frac{1}{2}(0, 1) \equiv (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0) \), as the left tree in Figure 4 shows. In contrast, if the randomization is ex-ante then the randomization does not provide hedging. Therefore, the decision maker should be indifferent between the randomization and the mixture \( \frac{1}{2}l_{(1,0)} + \frac{1}{2}l_{(0,1)} \) over the certainty equivalents, as the right tree in Figure 4 shows.

![Figure 4: Ex-post (left) and Ex-ante (right) Perception of Randomization \( \frac{1}{2}(1, 0) \oplus \frac{1}{2}(0, 1) \)](https://example.com/figure4.png)

(The solid lines correspond to risk, while the dotted lines corresponds to ambiguity.)

This observation suggests two ways to evaluate randomizations depending on the timing as follows:

**Definition:** For all \( P, Q \in \Delta(\mathcal{F}) \) such that \( P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n, Q = \beta_1 g^1 \oplus \cdots \oplus \beta_m g^m \),

(i) \( P \) ex-post dominates \( Q \) if \( \alpha_1 f^1 + \cdots + \alpha_n f^n \succeq \beta_1 g^1 + \cdots + \beta_m g^m \),

(ii) \( P \) ex-ante dominates \( Q \) if \( \alpha_1 l_{f_1} + \cdots + \alpha_n l_{f^n} \succeq \beta_1 l_{g_1} + \cdots + \beta_m l_{g^m} \).
The dominance axiom claims that the decision maker should prefer $A$ to $B$ if for any randomization among $B$ there exists a preferred randomization among $A$ no matter which he thinks he can randomize, ex ante or ex post.\(^5\)

**Axiom (Dominance):** If for any $Q \in \Delta(B)$ there exists $P \in \Delta(A)$ such that $P$ ex-ante and ex-post dominates $Q$ then $A \succsim B$.

Note that in the dominance axiom, choice of $P \in \Delta(A)$ can depend on $Q \in \Delta(B)$. This would make sense because the decision maker chooses one randomization from a set after all. Hence, if for any randomization among $B$ there exists a preferred randomization among $A$, then he should prefer $A$ to $B$, even though there exists no uniformly dominant randomization among $A$.

## 5 Theorem

Before stating the main result, we introduce a notation: for all $P \in \Delta(\mathcal{F})$ such that $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$, we denote $\alpha_1 f^1 + \cdots + \alpha_n f^n \in \mathcal{F}$ by $\int_{\mathcal{F}} f P(f)$.

**Theorem:** $\succsim$ satisfies Weak Order, Continuity, Nondegeneracy, Monotonicity, Ambiguity Aversion, Certainty Set Independence, Certainty Strategic Rationality, and Dominance if and only if there exists a real number $\delta \in [0, 1]$ such that $\succsim$ is represented by

$$V(A) = \max_{P \in \Delta(A)} \delta U(\int_{\mathcal{F}} f dP(f)) + (1 - \delta) \int_{\mathcal{F}} U(f) dP(f),$$

where $U(f) = \min_{\mu \in C} u(f(s)) d\mu(s)$ for some nonempty convex set $C \subset \Delta(S)$ and a nonconstant mixture-linear utility function $u : \Delta(Z) \to \mathbb{R}$.

\(^5\)There are no logical implications between ex post and ex ante dominances. For example, suppose that there exist $f, g \in \mathcal{F}$ such that $\frac{1}{2} f + \frac{1}{2} g \succ f \sim g$. Suppose also that there exists a lottery $l$ such that $\frac{1}{2} f + \frac{1}{2} g \succ l \succ f \sim g$. Then, $\frac{1}{2} f \oplus \frac{1}{2} g$ ex-post dominates $l$ but $l$ ex-ante dominates $\frac{1}{2} f \oplus \frac{1}{2} g$. 

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5.1 Uniqueness

The representation has a strong uniqueness property.

**Proposition 1:** The following two statements are equivalent:

(i) Two triples \((\delta, C, u)\) and \((\delta', C', u')\) represent the same \(\preceq\).

(ii) (a) \(C = C'\), and there exist real numbers \(\alpha\) and \(\beta\) such that \(\alpha > 0\) and \(u = \alpha u' + \beta\); and

(b) If \(C'\) is nondegenerate, then \(\delta = \delta'\).

5.2 Characterization of Special Cases with \(\delta = 1\) and \(\delta = 0\)

The following two stronger versions of dominance axiom respectively characterize two special cases with \(\delta = 1\) and \(\delta = 0\). The first axiom assumes ex-post randomization; the second axiom assumes ex-ante randomization.

**Axiom (Ex-post Dominance):** If for any \(Q \in \Delta(B)\) there exists \(P \in \Delta(A)\) such that \(P\) ex-post dominates \(Q\) then \(A \succeq B\).

**Axiom (Ex-ante Dominance):** If for any \(Q \in \Delta(B)\) there exists \(P \in \Delta(A)\) such that \(P\) ex-ante dominates \(Q\) then \(A \succeq B\).

**Proposition 2:** Suppose that \(\preceq\) is represented by \((\delta, C, u)\) as in the theorem.

(i) \(\preceq\) satisfies Ex-post Dominance if and only if \(\delta = 1\).

(ii) \(\preceq\) satisfies Ex-ante Dominance if and only if \(\delta = 0\).

Since each axiom implies Dominance, each axiom together with the axioms in the theorem (except Dominance) characterizes each special cases with \(\delta = 1\) or \(0\), respectively.
5.3 Comparative Attitudes toward Randomization

Definition: $\succeq$ is said to exhibit a stronger preference for randomization than $\succeq'$ if, for every $A \in \mathcal{A}$ and $l \in \Delta(Z)$,

$$A \succ l \Rightarrow A \succeq l.$$ 

The next proposition shows that $\delta$ captures the attitude toward randomizations.\footnote{Our notion of comparative attitude toward ex-ante randomizations is similar in spirit to the literature on comparative ambiguity aversion such as Ghirardato and Marinacci (2002).}

**Proposition 3:** Suppose $\succeq$ and $\succeq'$ are respectively represented by $(\delta, C, u)$ and $(\delta', C', u')$, where $C_1$ and $C_2$ are nondegenerate. Then the following statements are equivalent:

(i) $\succeq$ exhibits a stronger preference for randomization than $\succeq'$.

(ii) $\delta \geq \delta'$, $C = C'$, and there exist real numbers $\alpha$ and $\beta$ such that $\alpha > 0$ and $u = \alpha u' + \beta$.

Note that in (ii), both of the preferences coincide in $C$ as well as in $u$ under normalization. Therefore, Proposition 3 says that a stronger preference for randomization is characterized only by a larger value of $\delta$, or higher subjective probability of ex-post randomization.

6 Related Literature

In this section we discuss the relationship between our paper and the literature on subjective state space. Dekel, Lipman, and Rustichini (2001) have proposed the indifference to randomization axiom to derive a unique subjective state space as a part of the representation theorem for a preference over sets of lotteries.

Epstein, Marinacci, and Seo (2007) argued that the appropriateness of the indifference to randomization axiom depends on the decision maker's subjective belief on the timing between his choice from a set and a realization of a state. Epstein et al. (2007) have proposed two models...
that are similar to the two special cases of our model, although they obtained the state space endogenously as a set of expected utility functions in both models.

In the first model, on the one hand, Epstein et al. (2007) assumed that a decision maker believes that his choice from a set is after a realization of a state; then they imposed the indifference to randomization axiom to obtain a representation that is similar to the special case with $\delta = 1$.

In the second model, on the other hand, Epstein et al. (2007) assumed that the decision maker believes that his choice from a set is before a realization of a state. To obtain a representation that is similar to the special case with $\delta = 0$, Epstein et al. (2007) expanded the domain of the decision maker’s preference to (two-stage) lotteries over sets of (one-stage) lotteries. Then, Epstein et al. (2007) assumed the (first-stage) independence axiom but not the indifference to (second-stage) randomization axiom.

In contrast we did not make any assumption on the decision maker’s subjective belief on the timing of realization of a state. Hence we did not assume both the indifference to randomization axiom and the independence axiom. By imposing new axioms, we obtain the subjective belief on the timing as a part of our representation, while assuming the state space as a primitive.

A Proof of Propositions

A.1 Proof of Proposition 1

It is easy to see that (ii) implies (i). In the following, we will show that (i) implies (ii). Fix $\zeta$ on $\Delta(\mathcal{F})$. Let $(\delta, C, u)$ and $(\delta', C', u')$ represent $\zeta$, then $u$ and $u'$ are affine representations of $\zeta$ restricted on $\Delta(Z)$. Hence by the standard uniqueness results, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u = \alpha u' + \beta$. The uniqueness of $C$ follows from Gilboa and Schmeidler (1989), so that $C = C'$.

Suppose that $C$ is nondegenerate to show $\delta = \delta'$. Let $V$ and $V'$ be the representations defined by $(\delta, C, u)$ and $(\delta', C', u')$, respectively. Let $U$ and $U'$ be the restrictions of $V$ and $V'$ on $\mathcal{F}$,
respectively. Then $U = \alpha U' + \beta$.

Since $C$ is nondegenerate, there exist $f^*, g^* \in \mathcal{F}$ such that $U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*) = U(g^*)$. Define $\alpha^* = \arg\max_{\alpha \in [0, 1]} U(\alpha f^* + (1 - \alpha)g^*)$. (Such $\alpha^*$ exists because $U$ is continuous with respect to $\alpha$ and $[0, 1]$ is compact.) Then $U(\alpha^* f^* + (1 - \alpha^*)g^*) > U(f^*)$ and

$$V(\{f^*, g^*\}) = \delta U(\alpha^* f^* + (1 - \alpha^*)g^*) + (1 - \delta)(\alpha^* U(f^*) + (1 - \alpha^*)U(g^*)) .$$

Let $l = \delta l_{\alpha^* f^* + (1 - \alpha^*)g^*} + (1 - \delta)(\alpha^* l_{f^*} + (1 - \alpha^*)l_{g^*})$. Then $l \sim \{f^*, g^*\}$. It follows that

$$u(l) = \delta (U(\alpha^* f^* + (1 - \alpha^*)g^*) - U(f^*)) + U(f^*) .$$

The same equation holds for $u', \delta'$, and $U'$. Therefore

$$\delta = \frac{u(l) - U(f^*)}{U(\alpha^* f^* + (1 - \alpha^*)g^*) - U(f^*)} = \frac{u'(l) - U'(f^*)}{U'(\alpha^* f^* + (1 - \alpha^*)g^*) - U'(f^*)} = \delta' ,$$

where the second equality holds because $U = \alpha U' + \beta$ and $u = \alpha u' + \beta$.

### A.2 Proof of Proposition 2

Fix $\succsim$ on $\mathcal{A}$. Suppose that $\succsim$ is represented by $(\delta, C, u)$ with nondegenerate $C$. Since $C$ is nondegenerate, there exist $f^*, g^* \in \mathcal{F}$ such that $U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*) = U(g^*)$. Since $U(\alpha f^* + (1 - \alpha)g^*)$ is continuous with respect to $\alpha$, there exists $\alpha^* \in [0, 1]$ such that $U(\alpha^* f^* + (1 - \alpha^*)g^*) \geq U(\alpha f^* + (1 - \alpha)g^*)$ for all $\alpha \in [0, 1]$. Let $A = \{f^*, g^*\}$.

**Step 1: (i)**

**Proof of Step 1:** It is easy to see that if $\delta = 1$ then Ex-post Dominance is satisfied. To show the other way, let $B = \{l_{\alpha^* f^* + (1 - \alpha^*)g^*}\}$. Then Ex-post Dominance shows that $A \sim B$. Therefore

$$\delta U(\alpha^* f^* + (1 - \alpha^*)g^*) + (1 - \delta)U(f^*) = V(A) = V(B) = u(l_{\alpha^* f^* + (1 - \alpha^*)g^*}) = U(\alpha^* f^* + (1 - \alpha^*)g^*) ,$$

so that $\delta = 1$. \qed
Step 2: (ii)

Proof of Step 2: It is easy to see that if $\delta = 0$ then Ex-ante Dominance is satisfied. To show the other way, let $B = \{l, r\}$. Then Ex-ante Dominance shows that $A \sim B$. Therefore $\delta U(\alpha^* f^* + (1 - \alpha^*) g^*) + (1 - \delta) U(f^*) = V(A) = V(B) = u(l, r) = U(f^*)$, so that $\delta = 0$. \qed 

A.3 Proof of Proposition 3

Suppose that $\succ$ and $\succ'$ are respectively represented by $(\delta, C, u)$ and $(\delta', C', u')$, where $C$ and $C'$ are nondegenerate. Let $V$ and $V'$ be the representations defined with $(\delta, C, u)$ and $(\delta', C', u')$, respectively. Let $U$ and $U'$ be the restrictions of $V$ and $V'$ on $F$.

Step 1: (i) implies (ii).

Proof of Step 1: Suppose that $\succ$ exhibits a stronger preference for randomization than $\succ'$. Since $u$ and $u'$ are unique up to positive affine transformation, we can normalize $u$ and $u'$ by $u(z_+) = 1 = u'(z_+)$ and $u(z_-) = -1 = u'(z_-)$, without loss of generality. Then $u = u'$ and a straightforward argument shows $U = U'$, so that $C = C'$.

In the following we will show $\delta \geq \delta'$.

Since $C$ is nondegenerate, there exist $f^*, g^* \in F$ such that $\frac{1}{2} f^* + \frac{1}{2} g^* \succ f^* \sim g^*$. Since $U = U'$, $\frac{1}{2} f^* + \frac{1}{2} g^* \succ f^* \sim g^*$. Since $U(\alpha f^* + (1 - \alpha) g^*)$ is continuous with respect to $\alpha$, there exists $\alpha^* \in [0, 1]$ such that $U'(\alpha^* f^* + (1 - \alpha^*) g^*) = U(\alpha^* f^* + (1 - \alpha^*) g^*) \geq U(\alpha f^* + (1 - \alpha) g^*) = U'(\alpha f^* + (1 - \alpha) g^*)$ for all $\alpha \in [0, 1]$. It follows that

$$V'\{f^*, g^*\} = \delta' U'(\alpha^* f^* + (1 - \alpha^*) g^*) + (1 - \delta') U'(f^*) = \delta' U'(\alpha^* f^* + (1 - \alpha^*) g^*) + (1 - \delta') U'(f^*) = u'(\delta' l_0, f^* + (1 - \alpha^*) g^* + (1 - \delta') f^*).$$

\footnote{Let $l_0 = -\frac{1}{2} \delta_{z_+} + \frac{1}{2} \delta_{z_-}$, so that $u(l_0) = 0 = u'(l_0)$. Suppose to the contrary that $U \neq U'$. Then without loss of generality assume that there exists $f \in F$ such that $U(f) > U'(f)$. Moreover by the constant linearity, without loss of generality assume $1 > U'(f) > 0$. Fix a positive number $\varepsilon$ such that $\varepsilon < \min\{U(f) - U'(f), 1 - U'(f)\}$. Define $l = (U'(f) + \varepsilon) \delta_{z_+} + (1 - U'(f) - \varepsilon) l_0$. Then $U(l) = U'(l) = U'(f) + \varepsilon < U(f)$. Then, $l \succ f$ but $f \succ l$. This is a contradiction. Hence $U = U'$, so that $C = C'$.}
Let \( l^* = \delta' l_{\alpha^* f^* + (1 - \alpha^*) g^*} + (1 - \delta') l_{\bar{f}^*} \). Therefore \( \{ f^*, g^* \} \sim l^* \). Then \( \delta' U'(\alpha^* f^* + (1 - \alpha^*) g^*) + (1 - \delta') U'(f^*) = u(l^*) \). Since \( \succeq \) exhibits a stronger preference for randomization than \( \succeq' \), then \( \{ f^*, g^* \} \succeq l^* \). Hence, \( \delta U(\alpha^* f^* + (1 - \alpha^*) g^*) + (1 - \delta) U(f^*) \geq u(l^*) \). Therefore
\[
\delta \geq \frac{u(l^*) - U(f^*)}{U(\alpha^* f^* + (1 - \alpha^*) g^*) - U(f^*)} = \frac{u'(l^*) - U'(f^*)}{U'(\alpha^* f^* + (1 - \alpha^*) g^*) - U'(f^*)} = \delta',
\]
where the equality holds because \( u = u' \) and \( U = U' \).

\( \square \)

**Step 2:** (ii) implies (i).

**Proof of Step 2:** Suppose that \( \delta \geq \delta' \), \( C = C' \), and there exist \( \alpha > 0 \), \( \beta \in \mathbb{R} \) such that \( u = \alpha u' + \beta \). Then \( U = \alpha U' + \beta \). Fix any \( A \in \mathcal{A} \) and \( l \in \Delta(Z) \) such that \( A \succeq' l \) to show \( A \succeq l \).

Let \( P^* \) be the optimal randomization among \( A \) for \( \succeq' \). We will show the result, in the following two exhaustive cases.

**Case 1:** \( U'(\int_{\mathcal{G}} f dP^*(f)) = \int_{\mathcal{G}} U'(f) dP^*(f) \). Then \( V'(A) = U'(\int_{\mathcal{G}} f dP^*(f)) \geq U'(l) \). Since \( U = \alpha U' + \beta \), \( V(A) = U(\int_{\mathcal{G}} f dP^*(f)) \geq U(l) \), as desired.

**Case 2:** \( U'(\int_{\mathcal{G}} f dP^*(f)) \neq \int_{\mathcal{G}} U'(f) dP^*(f) \). Since \( U = \alpha U' + \beta \), \( U(\int_{\mathcal{G}} f dP^*(f)) \neq \int_{\mathcal{G}} U(f) dP^*(f) \). Therefore,
\[
\delta \geq \delta'
\]
\[
\geq \frac{U'(l) - \int_{\mathcal{G}} U'(f) dP^*(f)}{U'(\int_{\mathcal{G}} f dP^*(f)) - \int_{\mathcal{G}} U'(f) dP^*(f)} \quad (\because \delta U'(\int_{\mathcal{G}} f dP^*(f)) + (1 - \delta') \int_{\mathcal{G}} U'(f) dP^*(f) = V'(A) \geq U'(l))
\]
\[
= \frac{U(l) - \int_{\mathcal{G}} U(f) dP^*(f)}{U(\int_{\mathcal{G}} f dP^*(f)) - \int_{\mathcal{G}} U(f) dP^*(f)} \quad (\because U = \alpha U' + \beta)
\]

Hence \( V(A) \equiv \delta U(\int_{\mathcal{G}} f dP^*(f)) + (1 - \delta) \int_{\mathcal{G}} U(f) dP^*(f) \geq U(l) \), as desired. \( \square \)

### B Proof of Theorem

First we will show the necessity of the axioms. We will show the representation satisfies two key axioms, Certainty Strategic Rationality and Dominance. To show Certainty Strategic Rational-
ity, assume \( l \succcurlyeq B \). Then \( U(l) \geq \delta U(\int_{\mathcal{F}} f dP(f)) + (1 - \delta) \int_{\mathcal{F}} U(f)dP(f) \) for all \( P \in \Delta(B) \). Since \( U(\alpha f + (1 - \alpha)l) = \alpha U(f) + (1 - \alpha)U(l) \) for all \( f \in \mathcal{F} \), \( l \in \Delta(Z) \), and \( \alpha \in [0, 1] \), this implies \( U(l) \geq \delta U(\int_{\mathcal{F}} f dP(f)) + (1 - \delta) \int_{\mathcal{F}} U(f)dP(f) \) for all \( P \in \Delta(l \cup B) \). Therefore \( U(l) \geq V(l \cup B) \), so that \( l \succcurlyeq l \cup B \).

To show Dominance define \( W(P) = \delta U(\int_{\mathcal{F}} f dP(f)) + (1 - \delta) \int_{\mathcal{F}} U(f)dP(f) \). Then \( V(A) \equiv \max_{P \in \Delta(A)} W(P) \). Fix \( A, B \in \mathcal{A} \) and suppose that for any \( Q \in \Delta(B) \) there exists \( P \in \Delta(A) \) such that \( P \) ex-ante and ex-post dominates \( Q \), so that \( U(\int_{\mathcal{F}} f dP(f)) \geq U(\int_{\mathcal{F}} g dQ(g)) \) and \( \int_{\mathcal{F}} U(f)dP(f) \geq \int_{\mathcal{F}} U(g)dQ(g) \). Since \( \delta \in [0, 1] \), for any \( Q \in \Delta(B) \) there exists \( P \in \Delta(A) \) such that \( W(P) \equiv \delta U(\int_{\mathcal{F}} f dP(f)) + (1 - \delta) \int_{\mathcal{F}} U(f)dP(f) \geq \delta U(\int_{\mathcal{F}} g dQ(g)) + (1 - \delta) \int_{\mathcal{F}} U(g)dQ(g) \equiv W(Q) \). Therefore \( V(A) \equiv \max_{P \in \Delta(A)} W(P) \geq \max_{Q \in \Delta(B)} W(Q) \equiv V(B) \), so that \( A \succcurlyeq B \).

In the following we will show the sufficiency. Suppose that \( \succcurlyeq \) satisfies the axioms in the theorem. By Gilboa and Schmeidler (1989), there exists a nonempty convex set \( C \subset \Delta(S) \) and a mixture linear utility function \( u : \Delta(Z) \to \mathbb{R} \) such that \( \succcurlyeq \) on \( \mathcal{F} \) is represented by

\[
U(f) = \min_{\mu \in C} \int_{S} u(f(s))d\mu(s).
\]

For all \( P \in \Delta(\mathcal{F}) \), define

\[
v_1(P) = U(\int_{\mathcal{F}} f dP(f)),
\]

\[
v_2(P) = \int_{\mathcal{F}} U(f)dP(f),
\]

and \( v(P) = (v_1(P), v_2(P)) \).

**Lemma 1** For all \( f, g \in \mathcal{F} \), \( l \in \Delta(Z) \), and \( \alpha \in [0, 1] \),

(i) \( v_1(\alpha f \oplus (1 - \alpha)g) \geq \alpha v_1(f) + (1 - \alpha)v_1(g) \),

(ii) \( v_2(\alpha f \oplus (1 - \alpha)g) = \alpha v_2(f) + (1 - \alpha)v_2(g) \),

(iii) \( v_1(\alpha f \oplus (1 - \alpha)l) = \alpha U(f) + (1 - \alpha)u(l) = v_2(\alpha f \oplus (1 - \alpha)l) \).

**Proof of Lemma 1:** To show (i) choose \( f, g \in \mathcal{F} \) and \( \alpha \in [0, 1] \). Let \( \mu^* \in \arg \min_{\mu \in C} \int_{S} u(\alpha f(s) +
\( (1 - \alpha)g(s) d\mu(s) \). Then

\[
\begin{align*}
\nu_1(\alpha f \oplus (1 - \alpha)g) &= \int_S u(\alpha f(s) + (1 - \alpha)g(s)) d\mu^*(s) \\
&= \alpha \int_S u(f(s)) d\mu^*(s) + (1 - \alpha) \int_S u(g(s)) d\mu^*(s) \\
&\geq \alpha \min_{\mu \in C} \int_S u(f(s)) d\mu(s) + (1 - \alpha) \min_{\mu \in C} \int_S u(g(s)) d\mu(s) \\
&= \alpha \nu_1(f) + (1 - \alpha) \nu_1(g).
\end{align*}
\]

The results (ii) and (iii) are easy. \( \blacksquare \)

For all \( A \in \mathcal{A} \), define

\[
A^* = \{(v_1(P), v_2(P)) \in \mathbb{R}^2 | P \in \Delta(A)\}. \tag{5}
\]

By Lemma 1 (i) and (ii), \( \nu_1 \geq \nu_2 \) for all \( v \in A^* \). Since \( \Delta(A) \) is compact and \( v \) is continuous, \( A^* \) is compact.\(^8\) Define

\[
\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} A^*.
\]

Define a binary relation \( \succeq^* \) on \( \mathcal{A}^* \) by

\[
A^* \succeq^* B^* \iff A \succeq B,
\]

where \( A^* = v(A) \) and \( B^* = v(B) \).

In the following we will show that there exists \( \delta \in [0, 1] \) such that

\[
A^* \succeq^* B^* \iff \max_{v \in A^*} \delta \nu_1 + (1 - \delta) \nu_2 \geq \max_{v \in B^*} \delta \nu_1 + (1 - \delta) \nu_2 \tag{7}
\]

\(^8\)Since \( S \) and \( Z \) are finite, the set of acts are compact subset of \( (|Z| - 1)|S| \) dimensional Euclidean space. By definition, \( A \) is closed, so that \( \Delta(A) \) is compact. To see \( v \) is continuous, note that with the Euclidean metric, for all \( \mu \in C, U_\mu(f) = \int_S u(f(s)) d\mu(s) \) is continuous with respect to \( f \) (Since \( |S| \) is finite, the integral can be written as the summation). Moreover since \( |Z| \) is finite, \( U_\mu(f) \) is bounded. Therefore with the weak convergence topology, by definition, \( V_\mu(P) = \int_Z U_\mu(f) dP(f) \) is continuous with respect to \( P \). Finally since the set of prior is compact under the product topology, Berge’s maximum theorem shows that \( \nu_1(P) = \min_{\mu \in C} V_\mu(P) \) is continuous with respect to \( P \). In the same way, we can show \( \nu_2(P) \) is continuous as well. Therefore \( v(P) = (\nu_1(P), \nu_2(P)) \) is continuous with respect to \( P \).
Then it follows that

\[ A \succeq B \iff A^* \succeq^* B^* \quad (\because (6)) \]

\[ \iff \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2 \geq \max_{v \in B^*} \delta v_1 + (1 - \delta) v_2 \quad (\because (7)) \]

\[ \iff \max_{P \in \Delta(A)} \delta v_1(P) + (1 - \delta) v_2(P) \geq \max_{Q \in \Delta(B)} \delta v_1(Q) + (1 - \delta) v_2(Q) \quad (\because (5)) \]

Finally the definitions of \( v_1 \) and \( v_2 \) show the representation in the theorem.

First we will show the properties of \( \succeq^* \) those are inherited from \( \succeq \). In contrast to \( \succeq \), \( \succeq^* \) satisfies Indifference to Randomization. This is the main advantage of working with \( \succeq^* \).

We define mixtures among \( A^* \) as follows: for all \( \alpha \in [0, 1] \) and \( A^*, B^* \in A^* \),

\[ \alpha A^* + (1 - \alpha) B^* = \{ \alpha v + (1 - \alpha) u \in \mathbb{R}^2 \mid u \in A^* \text{ and } v \in B^* \} \]

**Definition:** \( A^* \) dominates \( B^* \) if for any \( v \in B^* \) there exists \( u \in A^* \) such that \( u_1 \geq v_1 \) and \( u_2 \geq v_2 \).

**Definition:** \( \succeq^* \) is said to satisfy

(i) **Certainty Monotonicity** if

\[ (c, c) \succeq^* (c', c') \iff c \geq c', \]

(ii) **Certainty Independence** if

\[ A^* \succeq^* B^* \iff \alpha A^* + (1 - \alpha) (c, c) \succeq^* \alpha B^* + (1 - \alpha) (c, c), \]

(iii) **Dominance** if

\[ A^* \text{ dominates } B^* \Rightarrow A^* \succeq^* B^*, \]

(iv) **Preference for Flexibility** if

\[ A^* \supset B^* \Rightarrow A^* \succeq^* B^*, \]

(v) **Certainty Strategic Rationality** if

\[ (c, c) \succeq^* B^* \Rightarrow (c, c) \sim^* \bigcup_{\alpha \in [0, 1]} \left( \alpha (c, c) + (1 - \alpha) B^* \right), \]

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(vi) Indifference to Randomization if 

\[ A^* \sim^* \text{co}(A^*), \]

where co\((A^*)\) is the convex hull of \(A^*\).

**Lemma 2** \(\succeq^*\) is a well-defined weak order that satisfies Certainty Monotonicity, Certainty Independence, Dominance, Preference for Flexibility, Certainty Strategic Rationality, and Indifference to Randomization.

**Proof of Lemma 2:**

**Step 1:** \(\succeq^*\) satisfies Dominance and Preference for Flexibility.

**Proof of Step 1:** Obviously Dominance implies Preference for Flexibility. To show Dominance, choose any \(A^*, B^* \in \mathcal{A}^*\) such that \(A^*\) dominates \(B^*\). Then there exist \(A, B \in \mathcal{A}\) such that 

\[ v(A) = A^* \text{ and } v(B) = B^*. \]

Moreover for all \(Q = \beta_1g_1 + \cdots + \beta_mg_m \in \Delta(B)\), there exists 

\[ P = \alpha_1f_1 + \cdots + \alpha_nf_n \in \Delta(A), U(\alpha_1f_1 + \cdots + \alpha_nf_n) = v_1(P) \geq v_1(Q) = U(\beta_1g_1 + \cdots + \beta_mg_m) \]

and 
\[ U(\alpha_1l_1 + \cdots + \alpha_nl_n) = v_2(P) \geq v_2(Q) = U(\beta_1l_1 + \cdots + \beta_ml_m), \]

so that \(P\) ex post as well as ex ante dominates \(Q\). Therefore \(A\) dominates \(B\). Hence Dominance shows that \(A \succeq B\), so that 

\[ A^* \succeq^* B^*. \]

**Step 2:** \(\succeq^*\) is a well defined weak order.

**Proof of Step 2:** By Preference for Flexibility, \(A^* = B^*\) implies \(A \sim^* B\). Hence \(\succeq^*\) is well defined. Since \(\succeq\) is weak order, \(\succeq^*\) is weak order.

**Step 3:** \(\succeq^*\) satisfies Certainty Monotonicity.

**Proof of Step 3:** Choose any \(c, c' \in u(\Delta(Z))\). Then, there exist \(l, l' \in \Delta(Z)\) such that 

\[ c = u(l) \text{ and } c' = u(l'). \]

Hence 

\[ (c, c) \succeq^* (c', c') \iff l \succeq l' \iff u(l) \geq u(l') \iff c \geq c'. \]
Step 4: \(\succeq^*\) satisfies Certainty Independence.

Proof of Step 4: Choose any \(A^*, B^* \in \mathcal{A}^*\), \((c, c) \in u(\Delta(Z))^2\), and \(\alpha \in [0, 1]\). Then \(A, B \in \mathcal{A}\) and \(l \in \Delta(Z)\) such that \(v(A) = A^*\), \(v(B) = B^*\), and \(u(l) = c\). Therefore

\[
A^* \succeq^* B^* \iff A \succeq B \\
\iff \alpha A + (1 - \alpha)l \succeq \alpha B + (1 - \alpha)l \\
\iff \alpha A^* + (1 - \alpha)(c, c) \succeq^* B^* + (1 - \alpha)(c, c). \quad (\because \text{Lemma 2})
\]

\[
\square
\]

Step 5: \(\succeq^*\) satisfies Certainty Strategic Rationality.

Proof of Step 5: Choose any \(c \in u(\Delta(Z))\) and \(B^* \in \mathcal{A}^*\) such that \((c, c) \succeq^* B^*\). By definition, \(c = u(l)\) and \(B^* = v(B)\) and for some \(l \in \Delta(Z)\) and \(B \in \mathcal{A}\). Then \(l \succeq B\). Hence Certainty Strategic Rationality shows \(l \sim l \cup B\), so that \((c, c) \sim^* (l \cup B)^*\). Since

\[
(l \cup B)^* = \{v(P) \in \mathbb{R}^2 | P \in \Delta(l \cup B)\}
\]

\[
= \{\alpha(c, c) + (1 - \alpha)v(P) \in \mathbb{R}^2 | P \in \Delta(B), \alpha \in [0, 1]\}
\]

\[
= \bigcup_{\alpha \in [0, 1]} (\alpha(c, c) + (1 - \alpha)B^*),
\]

then \((c, c) \sim^* \bigcup_{\alpha \in [0, 1]} (\alpha(c, c) + (1 - \alpha)B^*)\).

\[
\square
\]

Remember that for all \(f \in \mathcal{F}\), \(l_f \in \Delta(Z)\) is a certainty equivalent of \(f\). For simplicity we introduce the following two notations: For all \(A \in \mathcal{A}\), \(l_A = \{l_f \in \Delta(Z) | f \in A\}\). For all \(P \in \Delta(A)\) such that \(P = \alpha_1 f^1 + \cdots + \alpha_n f^n\), \(l_P = \alpha_1 l_{f^1} + \cdots + \alpha_n l_{f^n}\).

Step 6: \(\succeq^*\) satisfies Indifference to Randomization.

Proof of Step 6: Choose any \(A \in \mathcal{A}\). Dominance shows that \(A \sim A \cup l_A\). Hence \(A^* \sim^* (A \cup l_A)^*\). Therefore it suffices to show \((A \cup l_A)^* = co(A^*)\).
Substep 6.1: \((A \cup l_A)^* \subset co(A^*)\).

Proof of Substep 6.1: Choose any \(v \in (A \cup l_A)^*\) to show \(v \in co(A^*)\). By definition, there exists \(P \in \Delta(A \cup l_A)\) such that \(v(P) = v\). Hence there exist \(f_1, \ldots, f_n \in A, l^1, \ldots, l^m \in l_A\), and \(\alpha_1, \ldots, \alpha_{n+m} \in [0, 1]\) such that \(P = \alpha_1 f_1 + \cdots + \alpha_n f_n + \alpha_{n+1} l^1 + \cdots + \alpha_{n+m} l^m\).

Define \(Q = \sum_{i=1}^{\alpha_1} f_1 + \cdots + \sum_{i=1}^{\alpha_n} f_n\) and \(R = \sum_{i=1}^{\alpha_{n+1}} l^1 + \cdots + \sum_{i=1}^{\alpha_{n+m}} l^m\). Since \(U\) is continuous and \(A\) is compact, there exist \(\overline{f}, \underline{f} \in A\) such that \(U(\overline{f}) \geq U(f) \geq U(\underline{f})\) for all \(f \in A\). Hence \(U(\overline{f}) \geq U(l^i) \geq U(\underline{f})\). There exists \(\beta \in [0, 1]\) such that \(v(R) = \beta v(\overline{f}) + (1 - \beta)v(\underline{f})\). Therefore Lemma 1 shows \(v(P) = (\sum_{i=1}^{\alpha_1} v(f)) + (\sum_{i=n+1}^{\alpha_n} v(f))\).

Define \(v = v(Q) + v(R) = (\sum_{i=1}^{\alpha_1} v(f)) + (\sum_{i=n+1}^{\alpha_n} v(f))\). Since \(Q, \delta_\overline{f}, \delta_\underline{f} \in \Delta(A)\), then \(v(Q), v(\overline{f}), v(\underline{f}) \in A^*\). Hence \(v(P) \in co(A^*)\).

\[
\square
\]

Substep 6.2: \(co(A^*) \subset (A \cup l_A)^*\).

Proof of Substep 6.2: By definition, \(A^* \subset (A \cup l_A)^*\). Hence it suffices to show \((A \cup l_A)^*\) is convex. Choose \(v, v' \in (A \cup l_A)^*\) and \(\alpha \in [0, 1]\) to show \(\alpha v + (1 - \alpha)v' \in (A \cup l_A)^*\). By definition, there exist \(P, Q \in \Delta(A \cup l_A)\) such that \(v(P) = v\) and \(v(Q) = v'\). Lemma 1 shows

\[
v_1(\alpha P \oplus (1 - \alpha)Q) \geq \alpha v_1 + (1 - \alpha)v'_1 \geq v_1(\alpha l_P \oplus (1 - \alpha)l_Q).
\]

There exists \(\lambda \in [0, 1]\) such that \(\alpha v_1 + (1 - \alpha)v'_1 = \lambda v_1(\alpha P \oplus (1 - \alpha)Q) + (1 - \lambda)v_1(\alpha l_P \oplus (1 - \alpha)l_Q)\). In addition, Lemma 1 shows \(\alpha v_2 + (1 - \alpha)v'_2 = (\lambda v_2(\alpha P \oplus (1 - \alpha)Q) + (1 - \lambda)(\alpha l_P \oplus (1 - \alpha)l_Q))\). Since \(P, Q \in \Delta(A)\) and \(l_P, l_Q \in \Delta(l_A)\), then \(\lambda(\alpha P \oplus (1 - \alpha)Q) \oplus (1 - \lambda)(\alpha l_P \oplus (1 - \alpha)l_Q) \in \Delta(A \cup l_A)\). It follows that \(\alpha v + (1 - \alpha)v' \in (A \cup l_A)^*\).

\[
\square
\]

The next lemma characterizes the properties of the certainty equivalent \(c(A^*)\) of \(A^* \in \mathcal{A}^*\). Because of Certainty Monotonicity, the value of \(c(A^*)\) can be interpreted as the utility of \(A^*\).
Lemma 3: There exists a function \( c : \mathcal{A}^* \rightarrow u(\Delta(Z)) \) such that for all \( A^* \in \mathcal{A}^* \), \( c \in u(\Delta(Z)) \), and \( \alpha \in [0, 1] \),

(i) \( A^* \sim^* (c(A^*), c(A^*)) \),

(ii) \( \max_{v \in A^*} v_1 \geq c(A^*) \geq \max_{v \in A^*} v_2 \),

(iii) \( c(\alpha A^* + (1 - \alpha)(c, c)) = \alpha c(A^*) + (1 - \alpha)c \).

Proof of Lemma 3: Fix \( A^* \in \mathcal{A}^* \). Since \( A^* \) is compact, \( \max_{v \in A^*} v_1 \) exists. By definition, \( v(A) = A^* \) for some \( A \in \mathcal{A} \). Hence there exists \( P \in \Delta(A) \) such that \( v_1(P) = \max_{v \in A^*} v_1 \). By definition, there exist \( f_1, \ldots, f_n \in A \) and \( \alpha_1, \ldots, \alpha_n \in [0, 1] \) such that \( P = \alpha_1 f_1 + \cdots + \alpha_n f_n \).

Define \( f = \alpha_1 f_1 + \cdots + \alpha_n f_n \). Then \( v_1(f) = v_1(P) = \max_{v \in A^*} v_1 \geq v'_1 \) for all \( v' \in A^* \).

In addition \( v_2(f) = v_1(f) = \max_{v \in A^*} v_1 \geq v'_2 \) for all \( v' \in A^* \). Therefore \( v(f) \geq v(P) \) for all \( P \in \Delta(A) \). Thus \( \{f\} \) dominates \( A \). Hence Preference for Flexibility shows \( f \succeq A \). On the other hand, there exists \( g \in A \) such that \( v_2(g) = \max_{v \in A^*} v_2 \). Since \( g \in A \), \( A \) dominates \( \{g\} \). Hence Preference for Flexibility shows \( A \succeq g \).

Therefore \( l_f \sim f \succeq A \succeq g \sim l_g \). Hence by mixture linearity, there exists \( \beta \in [0, 1] \) such that \( A \sim \beta l_f + (1 - \beta)l_g \). Define \( c(A^*) = \beta u(f) + (1 - \beta)u(g) \). Then \( A^* \sim^* (c(A^*), c(A^*)) \) and \( \max_{v \in A^*} v_1 \geq c(A^*) \geq \max_{v \in A^*} v_2 \). Certainty Monotonicity shows that \( c(A^*) \) is unique.

Finally we show (iii). Choose any \( c \in u(\Delta(Z)) \) and \( \alpha \in [0, 1] \). There exists \( l \in \Delta(Z) \) such that \( c = u(l) \). By the argument above, \( A \sim \beta l_f + (1 - \beta)l_g \). Certainty Independence shows that \( \alpha A + (1 - \alpha)l \sim \alpha(\beta l_f + (1 - \beta)l_g) + (1 - \alpha)l \). Hence \( c((\alpha A + (1 - \alpha)l)^*) = \alpha c(A^*) + (1 - \alpha)c \).
Since \( \alpha A^* + (1 - \alpha)(c, c) = (\alpha A + (1 - \alpha)I)^* \), (iii) holds.

The next lemma proves the theorem (with \( \delta = 0 \)) for the case where \( c(A^*) = \max_{v \in A^*} v_2 \) for all \( A^* \in \mathcal{A}^* \).

**Lemma 4** If \( c(A^*) = \max_{v \in A^*} v_2 \) for all \( A^* \in \mathcal{A}^* \) then \( \succcurlyeq \) is represented by

\[
V(A) = \max_{P \in \Delta(A)} \int_{\mathcal{F}} U(f) dP(f).
\]

**Proof of Lemma 4:** Suppose that \( c(A^*) = \max_{v \in A^*} v_2 \) for all \( A^* \in \mathcal{A}^* \). Then,

\[
A \succeq B \iff A^* \succeq B^* \iff c(A^*) \geq c(B^*) \iff \max_{v \in A^*} v_2 \geq \max_{v \in B^*} v_2 \iff \max_{f \in A} U(f) \geq \max_{f \in B} U(f).
\]

Since \( \max_{f \in A} U(f) = \max_{P \in \Delta(A)} \int_{\mathcal{F}} U(f) dP(f) \), the result holds.

In the following we consider the case where \( c(A^*) > \max_{v \in A^*} v_2 \) for some \( A^* \in \mathcal{A}^* \).

**Lemma 5** If \( c(A^*) > \max_{v \in A^*} v_2 \) for some \( A^* \in \mathcal{A}^* \) then there exist \( f^*, g^* \in \mathcal{F} \) such that \( \frac{1}{2} f^* + \frac{1}{2} g^* \succ f^* \sim g^* \).

**Proof of Lemma 5:** Suppose that if \( f^* \sim g^* \) then \( \frac{1}{2} f^* + \frac{1}{2} g^* \sim f^* \). Then \( C = \{\mu\} \) for some \( \mu \in \Delta(S) \). This implies \( v_1(\cdot) = v_2(\cdot) \). Hence, \( \max_{v \in A^*} v_1 = \max_{v \in A^*} v_2 \) for all \( A^* \in \mathcal{A}^* \). Therefore, Lemma 4 (ii) shows that \( c(A^*) = \max_{v \in A^*} v_2 \) for all \( A^* \in \mathcal{A}^* \). This is a contradiction.

For all \( \beta \in [0, 1] \), define

\[
f^\beta = \beta f^* + (1 - \beta)l_f \quad \text{and} \quad g^\beta = \beta g^* + (1 - \beta)l_g.
\]

Define

\[
\alpha^* \in \arg \max_{\alpha \in [0,1]} U(\alpha f^* + (1 - \alpha)g^*).
\]
Since $[0, 1]$ compact and $U(\alpha f^* + (1 - \alpha)g^*)$ is continuous with respect to $\alpha$, such maximizers exist. Since $\frac{1}{2} f^* + \frac{1}{2} g^* \succ f^* \sim g^*$, $\alpha^* \notin \{0, 1\}$. Hence, $U(\alpha^* f^* + (1 - \alpha^*)g^*) > U(f^*) = U(g^*)$.

For simplicity, normalize $u$ by $U(f^*) = 0$ and $U(\alpha^* f^* + (1 - \alpha^*)g^*) = 1$. Then for all $\beta \in [0, 1]$, $U(f^*) = 0 = U(g^*)$ and $U(\alpha^* f^* + (1 - \alpha^*)g^*) = \beta$.

Define

$$L^* = \left\{ \{f^\beta, g^\beta\}^* | \beta \in (0, 1) \right\}.$$

For all $\beta \in (0, 1)$, $\{f^\beta, g^\beta\}^* = \text{co}\{U(f^\beta), U(f^\beta)), (U(\alpha^* f^\beta+(1-\alpha^*)g^\beta), U(f^\beta))\} = \text{co}\{(0, 0), (\beta, 0)\}$.

For all $A^* \in L^*$, define

$$\delta(A^*) = \frac{c(A^*)}{\beta}.$$

The next lemma shows $\delta$ is a constant.

**Lemma 6** There exists $\delta \in [0, 1]$ such that $\delta = \delta(A^*)$ for all $A^* \in L^*$.

**Proof of Lemma 6:** Fix $A^* \in L^*$. Then there exists $A^* = \text{co}\{(0, 0), (\beta, 0)\}$ for some $\beta \in (0, 1]$.

By Lemma 3 (ii), $\beta \geq c(A^*) \geq 0$. Hence, $\delta(A^*) \in [0, 1]$.

Choose any $A^*, B^* \in L^*$ to show $\delta(A^*) = \delta(B^*)$. By definition there exist $\beta, \beta' \in [0, 1]$ such that $A^* = \text{co}\{(0, 0), (\beta, 0)\}$ and $B^* = \text{co}\{(0, 0), (\beta', 0)\}$. Without loss of generality assume $\beta' < \beta$. Define $\alpha = \frac{\beta'}{\beta} < 1$. Then, $B^* = \alpha A^* + (1 - \alpha)(0, 0)$. Therefore Lemma 3 (iii) and the definition of $\delta$ show

$$\delta(\alpha A^* + (1 - \alpha)(0, 0)) = \frac{c(\alpha A^* + (1 - \alpha)(0, 0))}{\alpha \beta + (1 - \alpha)(0, 0)} = \frac{\alpha c(A^*)}{\alpha \beta} = \delta(A^*).$$
The next lemma shows the desired representation in the restricted domain $L^*$. 

**Lemma 7** For all $A^* \in L^*$,

$$c(A^*) = \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2.$$ 

**Proof of Lemma 7:** Choose any $A^* \in L^*$. Then $A^* = \text{co}\{(0, 0), (\beta, 0)\}$ for some $\beta > 0$. Since $\delta \in [0, 1]$, $\max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 = \beta \delta$. By definition, $c(A^*) = \delta \beta$. Hence the result holds. 

Given the representation on $L^*$, we can expand the representation on a set of triangles $T^*$ defined as follows:

For all $\gamma \in [0, 1]$, define $l^\gamma = \gamma l_{*}^* + (1 - \gamma)l_{*}^*$. Then $U(l^\gamma) = \gamma$. Define

$$T^* = \left\{ \{f^3, g^3, l^\gamma\}^* | \beta, \gamma \in [0, 1]\right\}.$$ 

Obviously $L^* \subset T^*$. By the normalization of $u$, for all $\beta, \gamma \in [0, 1]$.

$$\{f^3, g^3, l^\gamma\}^* = \text{co}\{(U(f^3), U(f^3)), (U(\alpha^*f^3 + (1 - \alpha^*)g^3), U(\beta^3)), (U(l^\gamma), U(l^\gamma))\}$$

$$= \text{co}\{(0, 0), (\beta, 0), (\gamma, \gamma)\}.$$

**Lemma 8** For all $A^* \in T^*$,

$$c(A^*) = \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2.$$ 

**Proof of Lemma 8:** Choose any $A^* \in T^*$. Then, $A^* = \text{co}\{(0, 0), (\beta, 0), (\gamma, \gamma)\}$ for some $\beta, \gamma \in [0, 1]$. Define $B^* = \text{co}\{(0, 0), (\beta, 0)\}$.

**Step 1** $c(A^*) \geq \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2$.

**Proof of Step 1:** Since $A^* \supset B^*$, Preference for Flexibility shows $A^* \succ B^*$. Then by Lemma 7, $c(A^*) \geq c(B^*) = \delta \beta$. Moreover by Lemma 3 (ii), $c(A^*) \geq \gamma$. Therefore $c(A^*) \geq \max\{\delta \beta, \gamma\} = \gamma$. Therefore $c(A^*) \geq \max\{\delta \beta, \gamma\} = \gamma$. 

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\[ \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2. \]

\[ \Box \]

**Figure 7: \( A^* \) and \( B^* \) in Lemma 8**

**Step 2:** \( c(A^*) \leq \delta v_1^* + (1 - \delta)v_2^* \). Define \( d = \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 \). Since \( B^* \subset A^* \), \( c(B^*) = \max_{v \in B^*} \delta v_1 + (1 - \delta)v_2 \leq \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 = d \). Therefore \( (d, d) \succ^* B^* \). Hence Certainty Strategic Rationality shows \( (d, d) \sim^* \bigcup_{\alpha \in [0, 1]} (\alpha(d, d) + (1 - \alpha)B^*) \). By Indifference to Randomization \( \bigcup_{\alpha \in [0, 1]} (\alpha(d, d) + (1 - \alpha)B^*) \sim^* \co((d, d) \cup B^*) \). Moreover by definition \( d \geq \gamma \). Therefore \( \co((d, d) \cup B^*) = \co\{(0, 0), (\beta, 0), (d, d)\} \supset \co\{(0, 0), (\beta, 0), (\gamma, \gamma)\} \equiv A^* \). Hence Preference for Flexibility shows \( \co((d, d) \cup B^*) \succ^* A^* \). Therefore \( (d, d) \succ^* A^* \) so that \( d \geq c(A^*) \).

\[ \Box \]

**Lemma 9** If \( c(A^*) > \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 \) for some \( A^* \in \mathcal{A}^* \) then \( \delta > 0 \).

**Proof of Lemma 9:** Suppose that \( c(A^*) > \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 \) for some \( A^* \in \mathcal{A}^* \). Assume to the contrary that \( \delta = 0 \). Then \( c(A^*) > \max_{v \in A^*} v_2 \). Define \( v^* = \arg \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 = \arg \max_{v \in A^*} v_2 \). Define \( B^* = \co\{(v^*_2, v^*_2), (1, v^*_2)\} \). Since \( B^* = v^*_2(1, 1) + (1 - v^*_2)\co\{(0, 0), (1, 0)\} \) and \( v^*_2 \in [0, 1] \), Lemma 3 and 7 show \( c(B^*) = v^*_2 + (1 - v^*_2)\co\{(0, 0), (1, 0)\} = v^*_2 \). Therefore \( c(B^*) = v^*_2 = \max_{v \in A^*} v_2 < c(A^*) \).

Since \( A^* \subset \co\{(0, 0), (1, 0), (1, 1)\} \) and \( v^*_2 = \max_{v \in A^*} v_2 \), \( v_1 \leq 1 \) and \( v_2 \leq v^*_2 \) for all \( v \in A^* \).
Therefore $B^*$ dominates $A^*$. Hence $B^* \succ^* A^*$. This is a contradiction. 

Figure 8: $A^*$ and $B^*$ in Lemma 9

Given the representation on $\mathcal{F}^*$, we can expand the representation on the whole domain $\mathcal{A}^*$.

**Lemma 10** For all $A^* \in \mathcal{A}^*$,

$$c(A^*) = \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2.$$ 

**Proof of Lemma 10**: Choose any $A^* \in \mathcal{A}^*$. Without loss of generality, assume $A^* \subset \text{co}\{(0, 0), (1, 0), (1, 1)\}$.\(^9\) Define $v^* = \arg \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2$.

**Step 1** $c(A^*) \geq \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2$.

**Proof of Step 1**: Since $A$ is compact and $U$ is continuous, there exists $f \in A$ such that $U(f)$ is a strict maximum of $U$. Therefore there exists $\beta \in [0, 1]$ such that $v^*_2 = \beta v_2(f) + (1 - \beta) v_2(f)$. Hence $(v^*_2, v^*_2) \in \text{co}(A^*)$. Therefore $\text{co}(A^* \cup (v^*_2, v^*_2)) = \text{co}(A^*)$. By Indifference to Randomization, $A^* \sim^* \text{co}(A^* \cup (v^*_2, v^*_2))$.

\(^9\) For all $\alpha \in (0, 1)$, define $A^*(\alpha) \equiv \alpha A^* + (1 - \alpha)(\frac{1}{2}, \frac{1}{2})$. There exists small enough $\alpha > 0$ such that $A^*(\alpha) \subset \text{co}\{(0, 0), (1, 0), (1, 1)\}$. Moreover it is easy to see that $c(A^*(\alpha)) = \alpha c(A^*) + (1 - \alpha) \frac{1}{2}$ and $\max_{v \in A^*(\alpha)} \delta v_1 + (1 - \delta) v_2 = \alpha \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2 + (1 - \alpha) \frac{1}{2}$. Hence $c(A^*) = \max_{v \in A^*} \delta v_1 + (1 - \delta) v_2$ if and only if $c(A^*(\alpha)) = \max_{v \in A^*(\alpha)} \delta v_1 + (1 - \delta) v_2$. 

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Define $B^* = \text{co}\{(v_2^*, v_2^*), v^*\} \in \mathcal{A}^*$. Since $v^* \in A^*$, $\text{co}(A^* \cup (v_2^*, v_2^*)) \supset \text{co}\{(v_2^*, v_2^*), v^*\} \equiv B^*$. Therefore Preference for Flexibility shows $\text{co}(A^* \cup (v_2^*, v_2^*)) \succsim^* B^*$. Since $A^* \sim^* \text{co}(A^* \cup (v_2^*, v_2^*))$, we obtain $A^* \succsim^* B^*$. By Certainty Independence,

$$B^* = \frac{v_2^*}{v_1^*}(v_1^*, v_1^*) + (1 - \frac{v_2^*}{v_1^*})\text{co}\{(0, 0), (v_1^*, 0)\}$$

$$\sim^* \frac{v_2^*}{v_1^*}(v_1^*, v_1^*) + (1 - \frac{v_2^*}{v_1^*})(\delta v_1^*, \delta v_1^*)$$

$$= (\delta v_1^* + (1 - \delta)v_2^*, \delta v_1^* + (1 - \delta)v_2^*).$$

It follows that $A^* \succsim^* (\delta v_1^* + (1 - \delta)v_2^*, \delta v_1^* + (1 - \delta)v_2^*)$, so that $c(A^*) \geq \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2$.

\[\square\]

**Figure 9:** $A^*$, $B^*$, and $D^*$ in Lemma 10

**Step 2:** $c(A^*) \leq \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2$.

**Proof of Step 2:** Define $d \equiv \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2$. Without loss of generality, assume $\frac{d}{\delta} < 1$.

Define $D^* = \text{co}\{(0, 0), (\frac{d}{\delta}, 0), (d, d)\}$. Since $\delta > 0$ by Lemma 9, $\frac{d}{\delta} < 1$ is well defined. Since $D^* \in \mathcal{F}^*$, Lemma 8 shows $c(D^*) = \max_{v \in B^*} \delta v_1 + (1 - \delta)v_2 = d$. In the following we will show $A^* \subset D^*$. Then Preference for Flexibility shows $D^* \succsim^* A^*$, which implies $d = c(D^*) \geq c(A^*)$.

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10Since $B^* = \frac{v_2^*}{v_1^*}(v_1^*, v_1^*) + (1 - \frac{v_2^*}{v_1^*})\text{co}\{(0, 0), (v_1^*, 0)\}$ and $\text{co}\{(0, 0), (v_1^*, 0)\} \in \mathcal{A}^*$, it follows that $B^* \in \mathcal{A}^*$.

11For all $\alpha \in [0, 1]$, define $A^*(\alpha) \equiv \alpha A^* + (1 - \alpha)(0, 0)$. Then $c(A^*(\alpha)) = \alpha c(A^*)$ and $\max_{v \in A^*(\alpha)} \delta v_1 + (1 - \delta)v_2 = \alpha(\max_{v \in A^*} \delta v_1 + (1 - \delta)v_2)$. Therefore $c(A^*(\alpha)) \geq \max_{v \in A^*(\alpha)} \delta v_1 + (1 - \delta)v_2$ if and only if $c(A^*) \geq \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2$. Choose $\alpha$ small enough $\alpha(\max_{v \in A^*} \delta v_1 + (1 - \delta)v_2) < \delta$. 

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Choose any \((v_1, v_2) \in A^*\) to show \(v^* \in D^*\). Then 
\[ \delta v_1 + (1 - \delta)v_2 \leq d, \]
so that 
\[ v_1 \leq \frac{d - (1 - \delta)v_2}{\delta}. \]

Define \(\overline{v}_1 = \frac{d - (1 - \delta)v_2}{\delta}\). Since \(v_2 \leq v_1 \leq \overline{v}_1\), there exists \(\lambda \in [0, 1]\) such that 
\[ (v_1, v_2) = \lambda(v_2, v_2) + (1 - \lambda)(\overline{v}_1, v_2). \]

Moreover, \((v_2, v_2) = \frac{v_2}{d}(d, d) + (1 - \frac{v_2}{d})(0, 0)\), and \((\overline{v}_1, v_2) = \frac{v_2}{d}(d, d) + (1 - \frac{v_2}{d})(\frac{d}{\delta}, 0)\).

Therefore \((v_1, v_2) = \frac{v_2}{d}(d, d) + \lambda(1 - \frac{v_2}{d})(0, 0) + (1 - \lambda)(1 - \frac{v_2}{d})(\frac{d}{\delta}, 0) \in \text{co}\{(0, 0), (\frac{d}{\delta}, 0), (d, d)\} \equiv D^*\).

By Certainty Monotonicity and Lemma 10,

\[ A^* \succeq^* B^* \iff c(A^*) \geq c(B^*) \iff \max_{v \in A^*} \delta v_1 + (1 - \delta)v_2 \geq \max_{v \in B^*} \delta v_1 + (1 - \delta)v_2. \]

References


