Experimentation, Patents, Knowledge Spillovers and Market Incentives

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Abstract

This work addresses the question of what is the optimal patent length in a \( n \) firm R&D race with unknown hazard rate, knowledge spillovers and variable effort under general post innovation market structures. Explicit analytical solution for symmetric equilibrium strategies are given. We show that longer patents increase research intensity by increasing the competition within firms. When post innovation competition is in differentiated products, the increase in patent length is costly as it reduces firms revenues by conceding a legal monopoly longer. These trade-offs lead to the optimal patent policy to have finite duration were post innovation markets with higher degree of product differentiation have a shorter optimal length. An increase in knowledge spillovers have a two way effect in the equilibrium of the game. On one hand, projects are pursued longer as the marginal value of the project increases; on the other, they tend to decrease the equilibrium effort as it tends to exacerbate the benefits of free riding from the competition. In the aggregate, positive spillovers tend to speed up the experimentation process only in markets in which short patents and high degree of product differentiation. As a consequence, it is also observed that markets with higher spillovers should have longer optimal patents to offset the diminish in speed.

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1 Introduction

One of the main goals of a patent system is to incentivise firms to develop new ideas by conceding them the legal right of monopolistically profit from their invention for a, ex-ante determined, period of time. This concession is made under the premise that free entry will dissipate any potential profit that firms could archive. Therefore, since innovation is a costly activity, if a legal monopoly is not granted all the incentives to develop a new idea will fade away. Even though this assumption is applicable to many industries it is not particularly suitable in scenarios where post innovation competition is limited. This limitation could be due to branding issues, specific or costly investments necessary for production or imitation of the innovation, capacity constraint, or situations in which limited amount of human capital necessary to imitate such activity is available. In a world without perfect competition in the post innovation market the way that firms compete may play an important role in the induced incentives to innovate. One of the main goals of this paper is to study how post innovation market structure shapes the competition in a R&D race where firms are working on being the first on developing a new technology.

The concession of a legal monopoly is not innocuous from a welfare point of view as it not only sacrifices social welfare in the post-innovation market, it also precludes other firms to use the innovation for the commercialization of other products that may or may not directly compete in the market of the patentee’s product. Despite of the clear connection between incentives in the R&D stage and post innovation competition, most of the work on patents has focused on the issues separately. That is, previous work has either studied equilibria in R&D races for a given patent structure disregarding the question of how patents and post innovation competition may affect equilibrium and social outcomes; or have reduced the development stage to a simplified constraint which a social planner has to meet when maximizing the social welfare with respect to a patent system. The second main goal of this paper is to provide an interesting and parsimonious framework that allows for an aggregate analysis between the incentives for innovation and welfare analysis.

In concrete, this paper analyses the equilibrium behavior of a $n$ firm R&D race with unknown hazard rate, knowledge spillovers and variable research effort under general post innovation market competition, for a different patent length. One of the main advantages of the formulation provided is that allows to find explicit analytic solution for the firms equilibrium strategies, enabling to provide clean and sharper comparative statics and, more importantly, characterize the determinants of the optimal patent policy.

By using an unknown hazard rate we attempt to capture an environment in which there is “if” uncertainty –firms do not know whether the project they are undertaking is feasi-
ble or not– and “when” uncertainty –conditional on the project being feasible, the arrival process of a breakthrough is random.– These two uncertainties induce a “learning by doing” environment in which firms get less optimistic about the feasibility of the project as time goes through and no success is observed. The purpose to study an R&D race with unknown hazard rate is that it induces interesting dynamics in which firms may reduce their efforts while they get bad news and get discourage from the project. From an efficiency point of view, this assumption implies that efficiency is characterized by a pace of research intensity for each belief and for how much projects are pursued before dropping them.

In this work a patent is uniquely characterized by the length of time that the legal monopoly is granted. Payoffs, on the other hand, depends on both the patent system and patent length. While the patent protection is active the patent holder will enjoy of monopoly profits whereas the unsuccessful firms will obtain a normalized payoff of zero. Once the patent protection is over, firms will oligopolistically compete and obtain an amount of profits that will be function of how differentiated the products are and which kind of competition the firms have. These dimensions will create a trade off between the intensity of competition and market structure, while longer patents will exacerbate competition increasing the R&D intensity, they also will induce inefficient competition in the post innovation market by precluding new firms to adopt the innovation; the more differentiated the firms are, the more it will harm welfare in both private and social terms.

Another aspect of the proposed model is that it embeds positive and negative knowledge spillovers. For the former feature we allow that a firm’s ability to succeed and learn is not only function of the firm’s private investments, it also depends on the opponents efforts. This modeling assumption we tries to capture that the firms’ learning process in the race is not independent but complementary as the firms may improve their knowledge by intentionally sharing information. Instances of information sharing are making public research reports, interactions with a third party (like universities or research institutions) that may spread out the findings, or by simply hiring workers from the competition. Also, firms may learn about the feasibility of the project by observing their rivals’ effort and whether their succeeded or not. This ability to learn from the competition is called negative knowledge spillover as it induces firms to free ride by lowering own efforts. The profitability of this strategy depends, however, on two interacting elements of the race. First, on the patent length, the shorter the patent length the more a firm may benefit from free riding as there is a lower cost of losing the race. Secondly, it depends on market structure, if competition when no patent is active is strong then there is less incentives to free ride from the competition’s successes. As with every externality, this spillovers make the private value and the social value of effort to
differ generating inefficiencies in both the intensity of effort and for how long the projects are pursued.

Finally, we study two assumption with about of the firms ability to observe their opponents efforts. There two motivations on doing this. Theoretically, when firms can observe their opponents effort the incentives to free ride are higher than a situation in which effort is not observable since in the former case all inferences that the firm makes are base on facts and not conjectures (as in the latter) increasing the certainty on the information. This distinction may be relevant when designing competing teams developing a technology within a firm. Methodologically, is convenient to study the unobservable action case first. This scenario has cleaner intuitions that are applicable to the observable case but lie hidden in the latter.

The following is a brief summary of the paper’s results. After constructing the efficient benchmark for efficiency and patent optimality (Proposition 1 and 2) we solve for the unique symmetric equilibrium when actions are not observable (Proposition 3) which we show to be generically inefficient whenever spillovers exists (Corollary 1). Then we turn to the properties of equilibrium and show that under weak conditions stronger patents intensifies competition speeding up the experimentation process (Proposition 5). On the other hand, we find out that positive knowledge spillovers have three effects, they: increase the marginal benefit of effort, increase the arrival rate of breakthroughs and increase the benefits to free ride from the competition. Proposition 6 shows that the first effect tends to increase how much the projects are pursed by lowering the quitting belief, it also shows that an increase in the positive spillover may lead to reduce the speed of learning in the industry and this effect is exacerbated by the stronger patent systems. Following, equilibrium behavior under observable actions is analyzed where we reproduce most of the aforementioned results. In the observable effort scenario there are two results that are distinctive from the unobservable case. Using best responses to the opponents effort we can prove that under observable action all Markov equilibria (not only symmetric) are generically inefficient (Proposition 7). Also, since the observability of actions creates certainty about the current belief, we have stronger incentives to free ride and, as a consequence, lower equilibrium efforts (Proposition 9). Finally, we use numerical analysis to show that when the innovation adopters compete in sufficiently differentiated markets the optimal patent is interior as there exists trade-offs between maximizing the experimentation speed and the total revenue. We also observe that market with higher spillovers should have longer patents to offset the diminish in speed.

Part of the main contribution of this paper belongs to the R&D race literature. Most of the early papers like Lee and Wilde (1980), Loury (1979) and Reinganum (1982) focus to
the case when the hazard rate is known so there is no learning throughout the race. More recent papers include learning into their modeling, for instance Doraszelski (2003) study races where the probability of a success depends on the stock of knowledge; Choi (1991) study an exponential arrival process where the firm’s decision is binary, i.e. whether to invest or not; and Malueg and Tsutsui (1997) study continuous effort in a specification that does not allow to have analytical solution precluding all the analysis about patent strength, spillovers and post innovation competition. In another form of learning Harris and Vickers (1987) study races with several stages where firms know in which stage of the development process each opponents is.

From the optimal patent literature this paper relates to Gilbert and Shapiro (1990) and Klemperer (1990). The former studies patent length and breadth when a known amount of profits has to be assigned to the patentee in order to produce the innovation. The latter study similar issues incorporating the possibility of different forms of product differentiation in the post innovation market. Both of this papers suggest that, under certain conditions, an infinitely lived patent with a small breath is optimal. In contrast Gallini (1992) suggest that when imitation is possible but costly longer patents increase the incentives to invent around the patent and commercialize the product. Denicolò (1996) study optimal patent length in the context of an R&D race with known hazard rate; unlike the present work, he takes the optimal intensity as given and studies if there is a patent strength that matches the given intensity. As we shall see later, the optimal intensity depends on both the patent structure and post innovation competition, both elements not fully incorporated in his analysis.

As far the author knows this is the first paper in to study the effects of knowledge spillovers in a patent race. Most of the theoretical literature focuses on the development of cost saving technologies as introduced by D’Aspremont and Jacquemin (1988) (see also Suzumura (1992) and Kamien, Muller, and Zang (1992)). In both situations, this literature and the present work, spillovers lead to pursue projects more; however, in the context of a race we will find that generically this done at a lower experimentation pace. Knowledge spillover has been largely documented in empirical work, notably Jaffe, Trajtenberg, and Henderson (1993) relates spillovers to geographic distance and existence of research institutions, like universities, at a close range. This empirical finding is highly consistent with Proposition 6 and the existence of innovation clusters like Silicon Valley (or the lack of such instances in the pharmaceutical sector), as the proposition relates the benefits of positive spillovers with short patents, like in the software industry.

This paper uses as a tool the literature of strategic experimentation (see Bolton and Harris (1999), Keller, Rady, and Cripps (2005)); which studies, trough the analysis of a two-
armed bandit problem, the trade-off between taking a risky action with unknown probability of success (corresponding to the innovative activity in this paper) and a safe action (which correspond to no innovate at all.) Closely related is Bonatti and Hörner (2011) which study the incentives within a team to collaborate in the completion of an uncertain task. In terms of this paper, their work could be thought as a deeper study of the scenario in which full positive spillover exists and no patent is granted. This works also relates to Acemoglu, Bimpikis, and Ozdaglar (2011) who study innovation game in which a sequence of firms decide whether to innovate or copy previous innovations. They assume instantaneous learning about the feasibility of the project and establish that a patent systems with compulsory licensing prescribe, in equilibrium, higher probability of innovation than without patents. In a similar fashion, we find stronger patents lead to higher research intensity in environment in which continuous actions and smooth learning is allowed, and relate it to the optimality of a patent system for different post innovation market competition.

2 The Model

Consider an economy with $n$ identical research firms (denoted by $i = 1, \ldots, n$) competing in a R&D race. Each firm maximizes the present discounted value of their profits. Time is continuous and all firms discount the future at a rate $r > 0$.

Firms compete in being the first on realizing a particular innovation. The value of winning the R&D race depends on the underlying patent system and the post-innovation market competition. Let $\hat{\pi}_n \geq 0$ be the flow profits that a firm gets when there are $n$ firms competing in the post-innovation market. When firm $i$ is the first firm succeeding in developing the innovation she is granted a patent that gives her the right to exclude others from using her innovation for $T \in \mathbb{R}_+ \cup \{\infty\}$ periods. That is, the payoff of the winner of the R&D race consist in getting monopoly profits for $T$ periods and oligopolistic profits afterwards, i.e. $V^W = \pi_1 \sigma + \pi_n (1 - \sigma)$ where $\sigma = 1 - \exp(-rT)$ and $\pi_n = \hat{\pi}_n/r$ for all $n$; whereas the payoff off all other (non-winning) firms is given by $V^L = \pi_n (1 - \sigma)$.\footnote{We have normalized the payoff of losing firms when the protection of patent is active to zero. Section X (to be added) discusses the inherence of this assumption on the results of the model.} The length of the patent is regarded as the strength of the patent system, we say that a patent is infinitely strong when the first firm making a breakthrough is granted a monopoly to commercialize the innovation forever ($T = \infty$); on the other hand we say that no patent exists ($T = 0$) if all the firms are able to commercialize the innovation as soon any firm succeeds. Making use that the increasing mapping $T \mapsto \sigma$ is a bijection we work with $\sigma \in [0, 1]$ in which $\sigma = 1$
and $\sigma = 0$ respectively represent the infinitely strong and the weakest patent system.

The relationship between profits and the number of firms commercializing the innovation depends on the market structure. In general it is assumed that the profit function $\pi_n$ is weakly decreasing in the number of firms on the market; however, the amount of the decrease is left open being the key characteristic to define the market structure. Most, if not all, of the literature has focused on the strong substitutes case in which $\pi_1 > n\pi_n$. This condition generally applies to models where the competition in the post-innovation market is through homogenous goods; this assumption could accurately describe competition in sectors like the development of pharmaceutical goods or the development of chemical products in which the entrance of new firm increases the supply and competition of the product in a homogeneous good market. On the other hand, when the inequality is reversed, there is evidence that there exists some degree of product differentiation in the post-innovation market competition. This would correspond to cases where same technology or idea has different applications, like the software industry and technologic gadgets business in general; or the exists branding issues that makes competition not direct like in sport clothing. A useful way to summarize the information contained in the market structure is the substitutability ratio defined as $\Pi \equiv \pi_1/\pi_n \in [1, \infty)$. The higher the ratio, the less differentiated the markets in which the innovation is commercialized; and the less costly, in term of total revenue, to grant a monopoly for a given length of time. For instance, in the limit, when goods are perfect substitutes and symmetric firms compete a la Bertrand we have for all $n \geq 2 \pi_n$ goes to zero and the substitutability ratio $\Pi$ goes to infinity indicating that only a monopoly may deliver enough rents to incentivise innovation in this type of markets. On the other extreme, when we have perfectly differentiated markets, we have that each firm is a monopoly in its market implying $\pi_n = \pi_1$ or $\Pi = 1$ and a legal monopoly is not necessary to provide rents to induce innovation.

The feasibility of the project is ex-ante not known by the firms; however, there is a common prior assigning probability $p^o$ to the event that the project can be developed. In that case, the probability that firm $i$ obtains a breakthrough depends on the effort that all firms incur. More precisely, at every instant of time each firm chooses an effort level $u_{i,t} \in [0, 1]$ at a cost of $c > 0$ per unit of effort. Given a profile of efforts $u_t = (u_{1,t}, \ldots, u_{n,t})$ agent $i$ obtains a breakthrough with instantaneous probability equal to $f_i(u_t) \equiv u_{i,t} + I_iAu_{-i,t}$ for some $A \in [0, 1]$, where $I_i$ is an indicator function that takes the value of 1 if firm $i$ exerts positive effort and 0 otherwise, and $u_{-i,t} \equiv \sum_{j \neq i} u_{j,t}$. The coefficient $A$ is assumed to be exogenously given and it aims to capture the degree of positive knowledge spillover in the industry. It represents the flow of information among the firms that help in the development of the
innovation, this information may emerge from hiring people of the competition, circulation of papers or just sharing information intentionally. In the model the higher $A$ the more synergic the industry is and the more firms benefit from one another efforts. The indicator function aims to capture that research spillovers can be exploited only if firm $i$ is actually pursuing the project. The range of $A$ is chosen to capture situations that goes from: no spillovers exists ($A = 0$); and the competition effort’s is as effective than own’s effort at the moment of obtaining a the probability of a breakthrough ($A = 1$), this last case could correspond to competing teams within a firm.

It is assumed that firms’ experimentation are statistically independent from one another, meaning that at every instant of time each firm obtains a random draw from an exponential distribution with hazard rate $f_i(u_t)$ if the project is successful and with rate 0 otherwise. We study two assumptions about the observability of effort; in the case that effort is observable, all firms have the same information at every instant of time and therefore there is common belief $p_t$ that the firms assign to the feasibility of the project. In the second scenario effort is unobservable; thus, a firm belief will depend on the conjecture that she makes about the opponents’ effort at each instant of time. This implies that the path of beliefs is only commonly known to all players in equilibrium. Let $u_t$ be any profile of strategies –known or conjectured– at instant $t$, using Bayes’ rule it can be shown that beliefs evolve according to

$$ dp_t = -p_t (1 - p_t) \lambda_a v_t dt $$

(1)

with $p_0 = p^0$, where $a = \# \{ i : u_{i,t} > 0 \}$ is the number of firms that exert positive effort for the given profile $u_t$ and $\lambda_a v_t$ is the speed of learning in which $\lambda_a = 1 + A (a - 1)$ could be understood as the arrival parameter of an exponential distribution with $a$ active players and $v_t \equiv \sum_i u_{i,t}$ is the total effort. The firms ability to learn from their opponents actions and the potential profits they can make from a rival’s success when patents are not too strong generates what is called negative knowledge spillovers. These spillovers induce firms to free ride and to lower their efforts throughout the competition.

1 To see why the number of active players $a$ is relevant consider an scenario where the total effort is 1. Split equally among the 3 players delivers a speed of learning of $1 + 2A$. If instead only one player exerts all the effort, the speed of learning is 1. Even dough in a symmetric equilibrium $a = n$, the distinction is relevant in asymmetric ones.

2 Equation (1) is derived in the following way: given $p_t$ and the profile of efforts $u_t$, Bayes’ rule determines that at the instant $t + dt$ the belief is

$$ p_{t+dt} = \frac{p_t e^{-\sum_i f_i(u_t)}}{1 - p_t + p_t e^{-\sum_i f_i(u_t)}}. $$

Subtracting $p_t$ and diving by $dt$ on both sides, taking limits, and using $\sum_i f_i(u_t) = \lambda_a v_t$ gives the result.
Strategies at instant $t$ are a mapping from the game history $h_t$ to an effort level. Since the game stops when the first success occurs, relevant histories contain own effort in the past and other’s effort in the case of observable action. A strategy is called Markov when effort is a function of the current belief only. This paper focus on finding Markov Perfect equilibria, that is equilibrium at which after any possible history is a best reply to play a Markov strategy given a Markov strategy from the opponents.

Fix the strength of the patent $\sigma$. Given a profile of strategies $\{u_t\}_{t=0}^\infty$, the expected payoff of firm $i$, in today’s terms, is determined by

$$
 r \int_0^\infty \left( p_t (f_i(u_t) V^W + f_{-i}(u_t) V^L) - cu_{i,t} \right) e^{-\int_0^t (r + p_s \lambda_{a_s} v_s) ds} dt \tag{2}
$$

where $p_t$ evolves according to (1) and $f_{-i}(u_t) \equiv \sum_{j \neq i} f_j(u_t)$. In words, $p_t (f_i(u_t) V^W + f_{-i}(u_t) V^L) - cu_{i,t}$ represents the expected instantaneous payoff of firm $i$ when a success occurs at instant $t$; $p_t f_i(u_t) V^W$ is the expected payoff when $i$ is the succeeding firm; $p_t f_{-i}(u_t) V^L$ is the expected revenue when firm $i$ loses, and $cu_{i,t}$ is the instantaneous cost of effort. Finally, the term $\exp \left\{ -\int_0^t (r + p_s \lambda_{a_s} v_s) ds \right\}$ corresponds to the probability that no firm has succeed by time $t$ and the discount factor.

Before going in to the strategic analysis, is useful to study the efficient benchmark to our model. This will allow us to understand more the inefficiencies that the race generates and how different patent systems may overcome them.

3 Optimal Patent in Cooperative Environment

This section analyses firm behavior in an environment in which firms can coordinate efforts but they still compete in both the patent race and in the ex-post innovation market. This is of special interest for various reasons. First, it serves as a fair benchmark for efficiency as it assumes away potential benefits that the firms may obtain by coordinating/colluding in the post innovation competition, it ignores potential gain or loses that may exists if the $n$ firm were owners of the patent as it would be in a Research Joint Venture (RJV). Also, this corresponds exactly to the problem that a benevolent planner would face if he can just decide firm’s effort but does not know the feasibility of the project and cannot control firms competition behavior after a breakthrough is obtained.

The analysis is started by noting that the law of motion (1) implies that the probability
of no breakthrough occurring by time $t$ can be expressed as

$$
\exp \left\{ - \int_0^t p_s \sum_i f_i(u_s) \, ds \right\} = \frac{(1 - p^t)}{(1 - p_t)}.
$$

Therefore, the cooperative problem can be written as maximizing the average payoff of the firms or

$$
\max_{\{v_t, a_t\}_{t=0}^{\infty}} r \int_0^\infty \int n^{-1} [p_t \lambda_n, \alpha - c] v_t \frac{(1 - p^t)}{(1 - p_t)} e^{-rt} \, dt.
$$

subject to the law of motion of beliefs (1), where $\alpha = V^W + (n - 1) V^L$, $v_t \in [0, a_t]$ is the sum of firms’ effort and $a_t$ corresponds to the number of firms that exert positive effort at instant $t$. To solve this problem it is first noted that for a given profile of effort $u$ such that not all firms are active ($a < n$) and total effort given by $k$, it is always profitable for the firms to redistribute efforts using a profile $u'$ such that all firms are active ($a' = n$) and total effort is identical $\sum u_i = k$ as this exercise strictly increases the payoff by increasing up the speed of the experimentation process and keeping all costs constant. Thus, any profile of strategies that is profit maximizing will have the property that all firms will be simultaneously active whenever effort is exerted, i.e. $a_t = n$ for all $t$ in which $v_t > 0$, and the problem reduces to choose an aggregate effort $v_t \in [0, n]$ at each instant of time. Due to the flow payoffs being linear on $v_t$ the solution to this problem is a cutoff strategy in which firms exert maximal effort while the stage payoff is positive, i.e. $v_t = n$ if $p_t \geq p_C$ where $p_C \equiv c/\alpha \lambda_n$, and $v_t = 0$ otherwise.

With the previous information we can state and prove the following proposition.

**Proposition 1** In the $n$-firm cooperative problem, firms exert maximal effort until a cutoff belief of $p_C \equiv c/\alpha \lambda_n$ and no effort bellow that threshold. The value function $V^C(p)$ for the $n$ firms is given by

$$
V^C(p) = \frac{\mu}{\mu + 1} \left( p \lambda_n \alpha - c + \frac{c}{\mu} (1 - p) \left( \frac{\Omega(p)}{\Omega(p_C)} \right)^\mu - 1 \right)
$$

where $\mu = r/n \lambda_n$ when $p > p_C$, and $V^C(p) = 0$ otherwise.

In the cooperative equilibrium firms internalize all spillovers that may exist. First, they incorporate into their investment decisions that the social marginal benefit of effort is greater than the private benefit. As we will see later on, this externality is not internalized in strategic environments implying that, in equilibrium, projects will be pursued less than what is efficient. Under cooperation negative spillovers are also internalized: thus, no free

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4To see this note that equation (1) can be rewritten as $-p_s \sum_i f_i(u_s) \, ds = dp/(1 - p_t)$. The equality is obtained by replacing the expression in the integral and solving.
riding exists implying that projects in these environments are pursued at a faster speed compared to the strategic situations.

Given the firms’ equilibrium strategy we can easily solve the differential equation driving the law of motion of the beliefs (1) in the case that no success has occur by time $t$,

$$p_t = \frac{p^0}{p^0 + (1 - p^0)e^{\alpha \lambda_n t}}$$

using the the cut-off belief $p_C$ it can be shown that, if no success occur, the firms will become pessimistic enough so that the R&D race a will eventually stop having a maximal length $L$ given by\footnote{To write the length as a function of the belief that we want to reach is going to be useful for equilibrium characterization below.}

$$L(p_C) = \frac{1}{n\lambda_n} \ln \left( \frac{\Omega(p_C)}{\Omega(p^0)} \right)$$

where $\Omega(p) = (1 - p)/p$ is the odds ratio of the beliefs on the feasibility of the project. It is not hard to show that the length of the R&D race has the expected comparative statics: through the effect in $p_C$ it can be seen that $L$ is decreasing in $c$ and increasing in $V^W$ and $V^L$ so the race last longer if the net payoff are higher. On the other hand, an increase on $A$ leads a to a reduction in $L$ meaning that if we speed up the arrival process the faster we arrive to the cut-off belief.

It is worth noting here that the cut of belief $p_C$ is a function of both the patent strength and the ex-post market competition (both elements affecting through $\alpha$); also, the project is undertaken only if $p_C < p^0$, which has as a necessary condition $\alpha \lambda_n > c$ –the average stage payoff when the project is feasible with certainty has to be greater than the average flow cost. In order to study what incentives a patent length induce beyond those incentives to guarantee the pursue of a project, we assume that all projects are developed regardless of the patent length. Concretely, it is assumed that $c < \min \{ \pi_1, \lambda_n \pi_n \}$ which is stronger than what is needed for project engagement in the cooperative case but precisely what is needed in the strategic environments.

**Proposition 2** In a cooperative environment, knowledge spillovers are irrelevant in the determination of the firms’ optimal patent. The firms’ optimal patent depends on the market structure only. In particular, no patent ($\sigma = 0$) is optimal if $n > \Pi$ and an infinitely strong patent ($\sigma = 1$) otherwise.

**Proof.** Replacing the optimal strategy in to (3) delivers

$$r (1 - p^0) \int_0^{p_C} \frac{1}{1 - p_t} (p_t \lambda_n \alpha - c) e^{-\pi t} dt$$
where $p_t$ is given by (4). Analyzing the first derivative of the previous expression with respect to $\sigma$ we obtain

$$r (1 - p^o) \int_0^{pc} \frac{p_t}{1 - p_t} \lambda_n \pi_n (\Pi - n)$$

since the integrand has always the same sign the derivative is always positive if $\Pi > n$ implying $\sigma = 1$ optimal or negative when $\Pi < n$ implying $\sigma = 0$. This delivers the result. ■

Since in cooperative environments all spillovers are internalized the optimal patent from the firms’ point of view depends on market structure only. In particular, if the competition in the post innovation market decreases the total revenue a monopoly will be granted. Otherwise, when the total revenues are increasing in the number of firms, no protection for the innovation is needed.

**Example** Suppose that the inverse demand function for firm $i$ is $P_i(Q) = a - b(q_i + \gamma Q_{-i})$ where $\gamma \in [0, 1]$ is the degree of product substitution. Under zero marginal cost of production in a symmetric equilibrium the ratio $\Pi$ is given by $(1 + \gamma (n - 1)/2)^2$. Then $\gamma > \gamma^*$ infinitely lived patents are optimal and no patents are optimal if $\gamma < \gamma^*$ where

$$\gamma^* = \left( \sqrt{4(n - 1) + 1} - 1 \right) / 2(n - 1)$$

which is decreasing on $n$.

### 4 Equilibrium with Unobservable Effort

This section studies how firms make their R&D investment decisions when they are unable to observe the competition’s efforts. In this scenario firms’ beliefs depend on their own effort and what they think their opponents are playing; therefore, in principle, firms may have different beliefs during the path of play. To find equilibrium behavior we study the firms incentives at any instant of time to postpone effort to the next instant. In concrete terms, in order for the firm to exert interior effort, $u_{i,t} \in (0, 1)$, at every instant of time $[t, t + dt]$, given the firm’s beliefs and conjectures about the opponents investments, she has to be indifferent between exerting its last unit of effort at that instant of time or to substitute it to the next instant $[t + dt, t + 2dt]$. In different words, we use a similar methodology to the construction of mixed strategy equilibrium in any normal form game.

To start the analysis of effort substitution we take second order approximation of the following problem: Given firm $i$’s beliefs and conjectures to their opponents action, she solves at time $t$

$$V_{i,t} \equiv \max_{u_i(t)} \int_t^{\infty} \left( p_r (V^W f_i (u_r) + V^L f_{-i} (u_r)) - c u_{i,\tau} \right) e^{- \int_0^\tau (r + p_s \lambda_n u_s) ds} d\tau$$
subject to (1). From here we will be able to build the best response of a firm to his opponents conjectured behavior. In equilibrium this best response is consistent with the conjectured behavior of others and we will have to show that the proposed behavior satisfies both necessary and sufficient conditions for being optimal.\footnote{A second order approximation is used instead of the traditional a first order approach because, as it will be noted below, the first order terms will cancel out implying that only second order effects matter to determine equilibrium behavior.}

Fix \( t \) and let \( u_i, u_{-i}, v \) and \( p \) be the effort, the conjectured opponents effort, aggregate effort and the belief at the instant of time \([t, t + dt]\); and let \( u'_i, u'_{-i}, v' \) and \( p' \) be correspondent terms at the instant \([t + dt, t + 2dt]\). Up to a second order the probability that firm \( i \) obtains a breakthrough at instant \([t, t + dt]\) when the profile of effort is \( u \) and the current belief is \( p \) is given by \( P_i (p, u) \equiv pf_i (u) dt - p^2 f_i^2 (u) dt^2 / 2 \). At this point we make the assumption that at the instant \([t, t + 2dt]\) the number of active firms is fixed and equal to \( n \), thus we just write \( \lambda \) instead of \( \lambda_n \) to simplify notation; as we will see later this assumption is not restrictive in a symmetric equilibrium where by construction all firms will be active simultaneously. With this information we can approximate \( V_{t,t} \) by

\[
V_{i,t} = P_i (p, u) V^W + P_{-i} (p, u) V^L - cu_i dt + \left( 1 - (r + p\lambda v) dt + (r + p\lambda v)^2 dt^2 / 2 \right) V_{t+dt}.
\]

Where \( P_{-i} (p, u) \equiv \sum_{j \neq i} P_j (p, u) \). In the same way we can approximate the term \( V_{t+dt} \) to find

\[
V_{i,t} = P_i (p, u) V^W + P_{-i} (p, u) V^L - cu_i dt + \left( 1 - (r + p\lambda v) dt + (r + p\lambda v)^2 dt^2 / 2 \right) \times \left[ P_i (p', u') V^W + P_{-i} (p', u') V^L - cu'_i dt + \left( 1 - (r + p'\lambda v') dt + (r + p'\lambda v')^2 dt^2 / 2 \right) V_{t+2dt} \right].
\]

At this point is key to notice that as the span of time \( dt \) tends to zero, to decrease effort during \([t, t + dt]\) by a sufficiently small \( \varepsilon \) and to increase it by the same amount during \([t + dt, t + 2dt]\) leads to the same terminal belief at \( t + 2dt \) as if no substitution of effort where performed\footnote{To see this recall the law of motion of belief (1) and notice that it implies: \( p'' = p - p (1 - p) \lambda_v v dt \). Approximating \( p'' \), as a function of \( p \) and effort, up to first order terms and assuming that the number of actives firm remains constant it is found that \( p'' = p - p (1 - p) \lambda (v + v') dt \). Since \( v + v' \) is constant regardless of how much effort is substituted, the terminal belief is independent of what happens in this two instants. Similar logic applies in a second order analysis.}. This property allow us to isolate these two instants of time and restrict the analysis only to the trade-offs that substituting effort generates between these two instances without worrying about what is going to happen to the continuation value of the game \( V_{t+2dt} \).
By using the law of motion of belief \( p' = p - p(1-p) \nu dt \) we can substitute in \( V_{i,t} \) and express all beliefs in terms of \( p \). Then, to express the incentives of substituting effort \( \varepsilon \) today and performing it tomorrow we differentiate \( V_{i,t} \) with respect to \( \varepsilon \) using the fact that 
\[-du_i/d\varepsilon = du'_i/d\varepsilon = 1. \]
After expanding the expression and taking the derivative it is not hard to show that all the first order terms cancel out leaving second terms only, thus the incentives for substitution are given by

\[
\frac{1}{d^2} \frac{dV_{i,t}}{d\varepsilon} = \frac{1}{dt^2} \left( \frac{dV_{i,t}}{du_i} \frac{du_i}{d\varepsilon} + \frac{dV_{i,t}}{du'_i} \frac{du'_i}{d\varepsilon} \right) = \frac{1}{dt^2} \left( -\frac{dV_{i,t}}{du_i} + \frac{dV_{i,t}}{du'_i} \right)
\]

where

\[
-\frac{1}{dt^2} \frac{dV_{i,t}}{du_i} = p^2 \left( (f_i + \lambda f'_i) V^W + (Af_{-i} + \lambda f'_{-i}) V^L \right) - p\lambda (2r + p\lambda (u + u')) V_{i,t+2dt} \\
+ p (1-p) \lambda \left( (f_i V^W + f'_{-i} V^L) - \lambda u' V_{i,t+2dt} \right) - p\lambda u'_c \\
\]

\[
\frac{1}{dt^2} \frac{dV_{i,t}}{du'_i} = -p^2 \left( f'_i V^W + Af'_{-i} V^L \right) + p (1-p) \lambda \left( \lambda V_{i,t+2dt} - \beta \right) u + p\lambda (r + p\lambda u') V_{i,t+2dt} \\
+ (r + p\lambda u) (c + p (\lambda V_{i,t+2dt} - \beta)).
\]

In the previous expression \( \beta \equiv \sigma \pi_1 + \lambda (1-\sigma) \pi_n \) represents the marginal benefit of individual effort. An increase of \( u_{i,t} \), lead to an increase of the probability of winning at a linear rate and increasing the expected payoff by \( \sigma \pi_1 + (1-\sigma) \pi_n \); also, it increases the probability that any of the \( n-1 \) opponents obtain a breakthrough at a rate \( A \) increasing the payoff by \( (1-\sigma) \pi_n \). The conjunction of these two effects sum up to \( \beta \).

To go further we conjecture that \( u_{i,t} \) is a continuous function of \( t \) for all firms \( i \) (this conjecture has to be verified at the end); so that, in the limit when \( dt \to 0 \) and \( \varepsilon \to 0 \), we obtain that \( u_i = u'_i \) for all \( i \) and expression (6) reduces to

\[
\frac{1}{dt^2} \frac{dV_{i,t}}{d\varepsilon} = p\lambda u_{-i} (c - (1-A) \sigma \pi_1) - r (p\beta - c)
\]

This equation admits a simple interpretation. The only benefit of postponing effort is that at a rate \( p\lambda u_{-i} \) the competition obtains a breakthrough and the firm saves the postponed cost \( c \). On the other hand, there are two cost of postponing effort. First, at a rate \( p\lambda u_{-i} \) the game finishes and the decrease of today’s effort leads to a decrease in the probability of being the winning firm by a rate of one which leads to a decrease of payoff \( \sigma \pi_1 \) (the term \( (1-\sigma) \pi_n \) is obtained by firm \( i \) independently of she winning the race or not). However, the decrease of today’s effort also decreases the opponents’ probability of winning at a rate \( A \), thus the net cost of postponing effort is \( (1-A) \sigma \pi_1 \) at a rate of \( p\lambda u_{-i} \); thus, the term \( \gamma \equiv c - (1-A) \sigma \pi_1 \) is the net benefit of free riding. Finally, there is a cost of postponing the effort that decreases the continuation value of the game at a rate \( r \), this decrease corresponds
to the net marginal benefit of today’s effort composed by the expected marginal benefit $p\beta$ minus its marginal cost $c$.

Equation (7) expresses the trade-off of substituting effort as a function of firms beliefs and conjectured strategies for the opponents. If it is not equal to zero we would have a corner solution and the firm will exert full effort if the derivative is negative and no effort if positive. Observe that the term $p\beta - c$ must be positive in order to be profitable for the firm to exert effort as no firm would undertake the project if their marginal benefit from effort $p\beta$ does not exceed its marginal cost $c$; indeed it will be shown later that the belief $p^* \equiv c/\beta$ is the last belief in which effort is performed. On the other hand, the term $\gamma$ may be positive or negative depending on the patent strength $\sigma$. If the patent is sufficiently strong, i.e. $\sigma > \bar{\sigma} \equiv c/ (1 - A) \pi_1$, $\gamma$ would be negative and so would equation (7); that is, to postpone effort decreases the value of the project regardless of the firm’s beliefs and conjectures about the opponents. Hence, firms will try to speed up the experimentation process and to exert full effort until the belief $p^*$ is reached. Therefore, a necessary (but not sufficient) condition have interior effort is $\gamma > 0$ which holds if and only if patents are sufficiently weak (i.e. $\sigma < \bar{\sigma}$).

To move further we impose symmetry. In an symmetric equilibrium we must have that the firms’ conjecture are consistent and the same, that is $u_i = u_j$ for all firms $i$ and $j$. Symmetry plus $\gamma > 0$ allow us to get our conjectured equilibrium effort

$$u_i^* (p) \equiv \min \left\{ \frac{r}{\lambda (n-1)} \frac{(p\beta - c)}{p\gamma}, 1 \right\}$$

at which the min is operator is imposed as $\gamma > 0$ does not guarantee that $u_{i,t} \leq 1$ for all $p$. However, since $p\beta - c$ goes to zero when $p$ goes to $p^*$ there must exists $\bar{p}$ such that $u_i^* (p) \in [0, 1)$ for all $p \in [p^*, \bar{p})$. Observe that the equilibrium effort is a continuos function of $p$; therefore, by the law of motion of beliefs (1), $u_t$ is a continuous function of $t$ and our conjecture used to solve for an equilibrium is satisfied.

**Proposition 3** There exists a unique symmetric equilibrium for the R&D race with unobservable actions. If the patent system is sufficiently strong, i.e. $\sigma \geq \bar{\sigma} \equiv c/ (1 - A) \pi_1$, firms exert full effort until the cutoff belief $p^* = c/\beta$ is reached. If the patent system is sufficiently weak, i.e. $\sigma < \bar{\sigma}$, firms exert the continuously decreasing effort

$$u_{i,t}^* \equiv \begin{cases} 1 & \text{if } t \in [0, L (\bar{p})] \\ \frac{r(\beta-c)}{\lambda (n-1)\gamma} \left( 1 + p^* \frac{1-\bar{p}}{\bar{p}-p^*} e^{\frac{r(\beta-c)}{\gamma} (t-L (\bar{p}))} \right)^{-1} & \text{if } t > L (\bar{p}) \end{cases}$$

where $L (p)$ is given by equation (5) and $\bar{p} \equiv rc/ (r\beta - \lambda (n-1)\gamma) > 0$. 

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As it can be appreciated in equation (3) in the interior effort case firms pursue the projects forever as they decrease they effort at a faster speed than the required by the Bayes update to reach the cutoff belief \( p^* \). Beliefs can also be pinned down in equilibrium. For beliefs or patents system in which unitary effort is performed, beliefs are given at time \( t \) by equation (4); otherwise, we can use (8) to solve the differential equation driving the law of motion of the beliefs (1) and get for \( t \geq L(\tilde{p}) \)

\[
p_t = \left(1 + p^* \frac{1 - \tilde{p}}{\tilde{p} - p^*} e^{\frac{r(\beta - 1)}{\gamma} (t - L(\tilde{p}))}\right) \left(1 + \frac{1 - \tilde{p}}{\tilde{p} - p^*} e^{\frac{r(\beta - 1)}{\gamma} (t - L(\tilde{p}))}\right)^{-1}
\]

which converges to \( p^* \) when \( t \) goes to infinity.

If a firm where to deviate from equilibrium by exerting more (less) effort at any instant of time, the firm would be more pessimistic (optimistic) than her opponents. Then the equilibrium strategy would prescribe to this firm to exert less (more) effort, if possible, than her opponents and eventually catching up with the beliefs of their rivals returning to the equilibrium path. This specifies the equilibrium strategy at any other subgame.

The symmetric equilibrium carries two sources of inefficiencies. If there are positive spillovers, \( A > 0 \), the private benefit of effort is lower than the social benefit, implying that projects are going to be pursed less than optimally. Also, the incentive to free ride given by the incentive to save costs from the experimentation process (see interpretation of equation (7)) causes that firm tend to experiment at a lower pace when patents are weak. These observations are summarized in the next corollary of Proposition 3.

**Corollary 1** The symmetric equilibrium with unobservable action is inefficient unless there is no positive spillovers \( A = 0 \) and both the optimal and the implemented patent strength is equal to one.

**Proof.** In the symmetric equilibrium are two sources of inefficiencies, the experimentation does not last long enough and the speed is not high enough. The latter effect proves the result for \( \sigma < \tilde{\sigma} \); in the former case we have that \( p_C < p^* \) if and only if \( \sigma A \pi_1 + (1 - \sigma) \lambda \pi_n \) which is positive unless \( A = 0 \) and \( \sigma = \sigma^* = 1 \).

**4.1 Comparative Statics**

This section studies how equilibrium dynamics are affected by changes in the parameters of the model. We shall see that payoffs and discount parameters tend to have the expected comparative statics. However, we will see that positive spillovers do not have a clear cut effect as there are many forces interacting with \( A \); also, we will find that strong patents have
social benefits that lie beyond generating enough profits for innovation to occur, they speed up the experimentation process.

**Proposition 4** In a symmetric equilibrium an increase in payoff ($\pi_1$ or $\pi_n$), or a decrease in costs $c$ makes the firms to pursue the projects further ($p^*$ decreases), full effort is performed for longer time ($\bar{p}$ decreases) and more effort is performed at each belief ($u^*_i(p)$ goes up at each $p$ where effort is interior). An increase in $r$ has no effect in $p^*$ and decreases and increases $\bar{p}$ and $u^*_i(p)$ respectively.

As expected, if the value of the project increases, all measures of research intensity go up. On the other hand, when firms are less patient, they value present innovation more than in the future; therefore, rushing the R&D process and increasing their effort intensity; however, the cutoff belief $p^*$ remains unchanged as the marginal cost and benefit of effort remain unaltered. Figure 1(a) and 1(b) show how effort evolves through time for different
values of \( \pi_2 \) and patent lengths. As we can see, for each length \( \sigma \), effort has a higher intensity in the case where \( \pi_2 \) is higher.

For the next proposition it is convenient to recall the substitutability ratio \( \Pi = \frac{\pi_1}{\pi_n} \).

**Proposition 5** In a symmetric equilibrium, necessary and sufficient conditions to an increase in the patent strength to induce:

1. A decrease in \( p^* \) is \( \Pi > \lambda \).
2. A decrease in \( \bar{p} \) is \( \Pi > \lambda/ (1 + \lambda(1 - A)(n - 1)r^{-1}) \).
3. An increase in \( u_i^* (p) \) is \( \Pi > \lambda/ (1 + (1 - A)(p\beta - c)(p\gamma)^{-1}) \).

In other words, \( \Pi > \lambda \) is a sufficient condition for an increase of the patent length to induce higher research intensity and to pursue projects longer.

When the expected marginal benefit of having a monopoly \( (p\beta = p\pi_1) \) is higher than that of full competition \( (p\beta = p\lambda) \), i.e. when \( \Pi > \lambda \), a stronger patent turns the R&D race more and more in to a “the winner takes it all” game. In that case, losing under a stronger patent means a less favorable position and firms will be induced to increase their effort in order to become the winner of the race. In other words, stronger patents lead to more intense competition leading to a faster innovation process. Figure 1 (a), (b) and (c) show how effort evolves through time for different set of parameters and patent strength, in the three cases it can be appreciated that intensity increases with the patent length.

Recall the speed of the experimentation process at \( t \) which corresponds to \( \lambda v_t \) that is the conjunction of the arrival rate \( \lambda \) and the total effort \( v_t = \sum_i u_{i,t} \).

**Proposition 6** In a symmetric equilibrium with unobservable effort an increase of the positive spillover \( A \) leads to pursue the projects further (a decrease in \( p^* \)) whenever \( \sigma \neq 1 \) and:

1. as \( p \to p^* \) to an increase effort and speed for all \( \sigma < 1 \)
2. if \( \sigma = 0 \), to an increase in effort and speed for all \( p \in [p^*, \bar{p}] \) and a increase in \( \bar{p} \);
3. if \( \sigma \to \bar{\sigma} \), a decrease in effort, speed for all \( p \in [p^*, \bar{p}] \) and a increase in \( \bar{p} \), iff

\[
\Pi > \frac{1}{\lambda} \left( \frac{1 - \bar{\sigma}}{\bar{\sigma}} \right);
\]

4. higher \( \sigma \) induce more incentives to reduce the speed whenever effort increases with \( \sigma \).
The first statement of proposition 6 is fairly intuitive; when the race is not of the form “the winner takes it all” –whenever $\sigma \neq 1$,– an increase of the positive spillover $A$ increases the private benefit of effort by increasing the probability of others developing the innovation, thus, leading firms to pursue projects further in case of no innovation occurring. In this sense, positive spillover increases the amount of R&D as in the literature that started D’Aspremont and Jacquemin (1988).

The effect of an increase of the positive spillover $A$ in the equilibrium effort is not clear cut as there are three forces interacting in different directions. In an interior equilibrium, firms must have no incentives to postpone effort at any instant of time, this occurs when equation (7) is equal to zero and from which it can be appreciated that an increase in $A$ leads to an increase in the benefit of postponing effort by increasing the arrival rate of the experimentation process $\lambda$ and increasing the net benefit of the postponed effort $\gamma$. Intuitively, the arrival rate $\lambda$ is a substitute of current effort; thus, an increase in $\lambda$ induces firms to diminish their efforts. Similarly, the value of $\gamma$ is the benefit of free riding, when it goes up firms have more incentives to free ride from the competition’s effort, decreasing it own.

In contrast, an increase in $A$ also leads to an increase in the lost of continuation value of postponing effort through an increase in $\beta$. In order to avoid this loss firms have incentives to increase current effort. Both, the effects through $\gamma$ and $\beta$, depends on the patent strength. Under the weakest patent, the effect through $\gamma$ is minimal as the benefit of free riding does not depend on $A$, and the effect through $\beta$ is maximal as no matter who win the firms share the benefits. Therefore, when $\sigma$ is close to zero the incentives to increase effort are maximal. The contrasting result is obtained when $\sigma$ approaches the maximum patent strength in which interior effort is performed.

The speed of the experimentation process at time $t$ can be written as $n\lambda u_t^e$, thus a change in $A$ is the composition of two effects: the increase in the arrival rate $\lambda$ and the effect in effort. Perhaps surprisingly, an increase in the knowledge spillover may lead to a reduction of the pace of experimentation. One the main points of Proposition 6 is that positive knowledge spillovers speed up the experimentation process in markets where the innovation is rewarded with shorter patents, and slows down otherwise. Empirical findings have linked distance with spillovers, jointly with the previous proposition would imply that only markets with weak patents and a high degree of product substitutability are suitable for clustering. Figure 2 shows graphically the effect of the positive spillover in the equilibrium effort for each belief under the case of $\sigma = 0$ and $\sigma = 3/20$, as shown in the proposition in the former case an increase in the positive spillover uniformly increases the equilibrium effort. In the latter case,
less effort is exerted for higher belief and higher effort for lower beliefs. Figures 1(a), and 1(c) show how this effort intensity evolves through time for different value of the spillover and patents, it can be appreciated that the net effect of an increase of $A$ generically reduces effort intensity when $\sigma > 0$. Moreover, the reduction increases with patent strength. Figures 1(d) shows how for a given patent strength changes in $A$ reduces effort intensity.

5  Equilibrium under Observable Effort

5.1 Preliminaries

To start the analysis we assume that at each instant of time $t$, the profile of strategies $\{u_{i,s}\}_{s=t}^{\infty}$ is fixed and known to firm $i$. Then, we are able to maximize $i$’s value of the project with respect to her own action and construct firm’s $i$ best response to her opponents investments. Firm’s $i$ problem at any instant $\tau$, for an arbitrary belief $p$ and a predetermined patent strength $\sigma$, is given by

$$V_{i,t}(p) \equiv \max_{u_{i}} r \int_{t}^{\infty} (p_{\tau} (V^{W} f_{i}(u_{\tau}) + V^{L} f_{-i}(u_{\tau})) - cu_{i,\tau}) e^{-\int_{0}^{\tau}(r+ps_{\lambda}s)ds} dt$$

subject to (1) and $p_{t} = p$. Taking a first order approximation of $V_{i,t}(p)$ the problem can be rewritten, for each $t$, as

$$V_{i}(p_{t}) = \max_{u_{i}} r \left(p_{t} (V^{W} f_{i}(u_{t}) + V^{L} f_{-i}(u_{t})) - cu_{i,t}\right) dt + (1 - (r + p\lambda a_{v_{t}}) dt) V_{i}(p_{t+dt}).$$
Assuming that $V_i(p)$ is a continuous differentiable function of $p$, we can express $V_i(p_{t+dt})$ as $V_i(p_t) + V'_i(p_t)\, dp_t$ to replace in $V_i(p_t)$ which after simplifications it can be rearranged in to the following linear problem

$$rV_t(p_t) = u_{-i,t} \left( E(a_t, p_t) - C(V_i(p_t), a_t, p_t) \right) + \max_{u_i} \left( B(a_t, p_t) - C(V_i(p_t), a_t, p_t) \right)$$

where $u_{-i,t}$ is the sum of the efforts of all firms but $i$ at instant $t$; $B(a, p) = r(p\beta_a - c)$, where $\beta_a = \pi_1\sigma + \lambda_a\pi_n (1 - \sigma)$, is the gain in expected net benefit derived from private effort of a breakthrough at the current instant of time; $C(V(p), a, p) = p\lambda_a(V(p) + (1 - p)V'(p))$ is the expected change in the continuation value of the project due to the update in the belief when experimentation leads to no breakthrough, and $E(a, p) = r(\pi_1\sigma\mathbb{I}[u > 0] + \lambda_a\pi_n (1 - \sigma))$ is the payoff externality of a breakthrough from other firms’ effort.

### 5.2 Best Responses

The formulation presented in (10) is linear in firm $i$’s effort. This linearity allow us to characterize the best responses as a function of the value of the problem

$$u_{i,t} \begin{cases} 
= 0 & \text{if } B(a_t, p_t) < C(V_i(p_t), a_t, p_t) \\
\in [0, 1] & \text{if } B(a_t, p_t) = C(V_i(p_t), a_t, p_t) \\
= 1 & \text{if } B(a_t, p_t) > C(V_i(p_t), a_t, p_t)
\end{cases} .$$

(11)

Equivalently, substituting $C(V(p), a, p)$ by $B(a, p)$ in $E(a, p) - C(V(p), a, p)$ and ignoring the second term in (10), the best response can be expressed in terms of parameters, beliefs and opponents’ effort as

$$u_{i,t} \begin{cases} 
= 0 & \text{if } V_i(p_t) < u_{-i,t}\gamma(p) \\
\in [0, 1] & \text{if } V_i(p_t) = u_{-i,t}\gamma(p) \\
= 1 & \text{if } V_i(p_t) > u_{-i,t}\gamma(p)
\end{cases}$$

(12)

where $\gamma(p) = c - p\sigma\pi_1 (1 - A)$ is the free riding externality for the case with perfect action observability. That is, we can characterize the best response of firm $i$, for a given belief $p$, strength $\sigma$ and facing the total effort of the opponents $u_{-i,t}$ by identifying wether the value of exerting effort $V_i(p_t)$ is above or below the benefit of free riding $u_{-i,t}\gamma(p)$. Compare this to the benefit of free ringing with unobservable actions which in current terms can be written as $p_t u_{-it}\gamma(1)$. Due to the effort observability, the benefit of free-riding is uniformly higher for each possible belief.

Recall Proposition 2 which stated that the efficient equilibrium is to exert full effort until the cutoff belief $p_C = c/\max \{\pi_1, n\pi_n\}$. By analyzing the best responses we can generalize it by showing that all patent system are inefficient if $A > 0$ or if $\sigma < 1$. 

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Proposition 7 Under observable actions. When positive spillover exists all Markov perfect equilibria are inefficient. When no spillovers exists all Markov Perfect equilibria are inefficient whenever the strength is finite (σ < 1).

Proof. Since firms have the possibility to quit experimentation we have \( V_i(p) \geq 0 \). By our continuity assumption, \( V_i(p) \) goes continuously to zero when \( p \) is sufficiently low. The term \( u_{-i}(p) \) is positive if and only if \( u_{-i} > 0 \) and \( p < p^i \equiv c/\pi_1 \sigma (1 - A) \). If the current belief \( p \) is above the cooperative cutoff \( p_C \) and \( u_{-i} \neq n - 1 \) the equilibrium would be inefficient as we don’t have full effort. Thus, only relevant case corresponds to \( u_{-i} = n - 1 \). Observe that \( p_C < p^i \) as \( \max \{ \pi_1, n\pi_n \} > \pi_1 \sigma (1 - A) \) whenever \( A > 0 \), or \( A = 0 \) and \( \sigma < 1 \). Since \( V_i(p) \) has to be less than the average cooperative payoff \( V_C(p) \), it has to reach zero at belief \( p^{**} \geq p_C \). If \( p^{**} > p_C \) the result follows as no full effort is performed until \( p_C \), if \( p^{**} = p_C \) there must exists \( p \in (p_C, p^i) \) such that \( V(p) < (n - 1) \gamma(p) \) as \( \gamma(p_C) > 0 \). Thus by (12) not full effort is exerted and the equilibrium is inefficient.

The previous proposition is stronger than Corollary 1. For the aforementioned assumptions it says that all Markov perfect equilibria are inefficient, not only the symmetric ones. When \( \sigma = 1 \) and \( A = 0 \) it can be easily shown that the previous argument extends to scenarios in which products are sufficiently differentiated so the weakest patent is socially optimal, i.e. \( \Pi < n \), since this would imply that \( p_C < p^i \) for all \( \sigma \) and the proof would go through. Even though, due to the generality of the proof, is less clear, the main forces driving the result are again that the firms have either incentives to free ride from their competitors’ efforts (negative spillovers) and that they do not internalize the positive externality of their effort on others driving firm to pursue projects less than efficiently (positive spillovers). The only candidate for an efficient equilibrium is the situation in which we have strong substitutes \( \Pi > n \). As before we will show that the symmetric equilibrium is efficient in this case.

5.3 Solution to Payoff Functions

On intervals where firm \( i \) has a unique best response we can explicitly solve the value function up to a constant of integration which is pinned down in equilibrium. Let \( W(p) \) and \( F(p) \) respectively be the value function when firm \( i \) exert full effort \( (u_i = 1) \) and free rides \( (u_i = 0) \) given that her opponents play \( u_{-i} \), then using (10) we find the following differential equations for each scenario

\[
W'(p) + \frac{r + p\lambda_u}{p(1 - p)\lambda_u}W(p) = \frac{r(pH - c)}{p(1 - p)\lambda_u}
\]

\[
F'(p) + \frac{r + p\lambda_u}{p(1 - p)\lambda_u}F(p) = \frac{r(1 - \sigma)\pi_n}{(1 - p)}
\]
where \( v = u_{-i} + u_i \) and \( H = (\beta_a + u_{-i} (A\sigma \pi_1 + \lambda_a (1 - \sigma) \pi_n)) \). Both equations can be solved implying the following value function to each case

\[
W(p) = \frac{1}{r + \lambda_a v} \left( r (pH - c) - c (1 - p) \lambda_a v \right) + K (1 - p) \Omega (p) \frac{r}{\lambda_a v} \tag{13}
\]

\[
F(p) = \frac{u r (1 - \sigma) \pi_n \lambda_a}{r + \lambda_a v} + \hat{K} (1 - p) \Omega (p) \frac{r}{\lambda_a v} \tag{14}
\]

where \( K \) and \( \hat{K} \) are constants of integration. It is worth noting here that the value of \( a \) and \( v \) differ among these two value functions as in the first case firm \( i \) is active and exerting full effort whereas in the latter firm \( i \) is free rides.

Finally it is possible to characterize the value function when firm \( i \) has an interior solution. According to (12) an interior solution exists if and only if \( B(a, p) = C(V(p), a, p) \), or equivalently

\[
I' (p) + \frac{1}{1 - p} I(p) = \frac{B(a, p)}{p (1 - p) \lambda_a}. \tag{15}
\]

Solving first order ordinary differential equation (15) the following value function is obtained

\[
I(p) = \frac{r}{\lambda_a} (\beta_a - c + c (1 - p) \ln (\Omega (p))) + \hat{K} (1 - p) \tag{16}
\]

where \( \hat{K} \) is the constant of integration to be determined in equilibrium.

5.4 Symmetric Equilibrium

To find a symmetric equilibrium we start by looking for an interior solution. To do so we take (16) and impose standard value matching and smooth pasting conditions \( (I(p^{**}) = 0 \) and \( I'(p^{**}) = 0 \) where \( p^{**} \) is the beliefs at which firm \( i \) stop experimenting) to find that the value function is

\[
S(p) = \frac{r}{\lambda} \left[ p \beta - c + c (1 - p) \ln \left( \frac{\Omega (p)}{\Omega (p^{**})} \right) \right]
\]

and \( p^{**} = p^* = c/\beta \). Thus, with and without observable effort, projects are pursued the same. This result is intuitive as, given a belief, the observability of others’ effort does not affect the marginal benefit and cost of effort. Finally, using (12) and imposing symmetry among firms, the symmetric equilibrium effort is obtained

\[
u_{i}^{**} (p) = \min \{ S(p) / (n - 1) \gamma (p), 1 \}.
\]

The next proposition, proved in the appendix, formalize the previous findings.

**Proposition 8** Regardless of the patent system and the market structure, the \( n \)-firm R&D race with observable effort has a unique symmetric equilibrium with the common belief \( p \)
as state variable. There are two threshold $p^*$ and $\tilde{p}$ such that: For beliefs $p \in [p^*, \tilde{p}]$ a continuously decreasing effort

$$u_i^{**}(p) = \frac{1}{(n-1)\gamma(p)} S(p)$$

where $\gamma(p) \equiv c - p\sigma (1-A)\pi_1$ and

$$S(p) = \frac{r}{\lambda} \left[ p_\beta - c + c(1-p) \ln \left( \frac{\Omega(p)}{\Omega(p^*)} \right) \right],$$

is performed. No effort is performed for $p < p^*$, which implies $S(p) = 0$, and full effort is performed for $p > \tilde{p}$ in which case the value function is given by

$$S(p) = p \frac{r\alpha + cn}{n(\mu + 1)} - c + K \frac{(1-p)}{1-\tilde{p}} \left( \frac{\Omega(p)}{\Omega(\tilde{p})} \right)^\mu$$

with $K = (n-1)\gamma(\tilde{p}) + c - \tilde{p} \frac{r\alpha + cn}{n(\mu + 1)}$.

The upper threshold is given by $\tilde{p} = \min \{ p^*, 1 \}$ where $u^S(p^*) = 1$ and the lower threshold by $p^* = c/\beta$.

The main features of the equilibrium with observable action are similar to those when effort is not observable. Effort is a continuously decreasing function of the belief. However, there is a key difference with respect unobservable effort case, here if $\sigma < 1$ there is always a belief in which effort is interior and decreasing. This contrasts our previous findings that $\sigma > \bar{\sigma}$ was sufficient to guarantee full effort throughout the race. It is possible to go further and show the following relation between efforts in the two scenarios.

**Proposition 9** When the symmetric equilibrium effort is observable and interior, it is strictly lower than the symmetric equilibrium with unobservable effort.

**Proof.** The difference between equilibrium efforts is given by

$$u_i^*(p) - u_i^{**}(p) = \frac{r}{\lambda n (n-1)} \frac{c(1-p)}{\gamma(p)} \left[ \frac{p_\beta - c}{p_\gamma} - \ln \left( \frac{\Omega(p)}{\Omega(p^*)} \right) \right].$$

It is enough to prove that that the term in brackets, call it $g(p)$, is positive for all $p \geq p^*$. This follows from the observation that $g(p^*) = 0$ and $g'(p) = (c(1-p) + \gamma p) / (\gamma p^2 (1-p))$ which is positive whenever the comparison is relevant, i.e. $\gamma > 0$. ■

As corollary of the previous proposition we can conclude that effort will last forever in the observable case as well. The effect that drives this result is that in the observable case firms are more certain of what are the gains of free riding of the competition, this certainty
makes the value of free riding to go up implying both the uniformly less effort and that all patent with strength lower than \( \sigma < 1 \) can lead to interior effort for sufficiently low beliefs.

In the case that \( \sigma = 1 \) and no positive spillover exists we can prove the efficiency of the infinitely strong patent when is cooperatively optimal to have \( \sigma = 1 \). This statement is a simple corollary of Proposition 8.

**Corollary 2** In a symmetric equilibrium with observable effort, if the patent strength is maximal and if no positive spillover exists firms exert full effort for all belief. Moreover, if \( \Pi > n \), this equilibrium is efficient.

**Proof.** Suppose \( \sigma = 1 \) and \( A = 0 \) then highest belief such that \( \gamma (p) \) is positive is \( p^\dagger = c/\pi_1 = p^* \). Thus for all \( p > p^\dagger = p^* \) we have \( \gamma (p) < 0 \) and by (12) we can conclude that \( u_i = 1 \) and full effort is performed. In the case that \( \Pi > n \), \( p^* = p_C \) and this behavior is also efficient. ■

## 5.5 Comparative Statics

The standard comparative statics with respect to the parameters \( r \), \( c \), \( \pi_1 \) and \( \pi_n \) have the expected effects, and its proof is similar to the proof of Proposition 4, thus is going to be omitted. We will focus on the effect of strengthening the patent system and changes in the positive spillovers.

**Proposition 10** In the symmetric equilibrium with observable effort and for beliefs that effort is interior \( (p \in (p^*, \bar{p})) \), \( \Pi > \lambda \) is sufficient condition for an increase in the strength \( \sigma \) to:

1. increase the value of the project \( S(p) \) and the effort exerted by the firms \( u_i^*(p) \);
2. decrease the full effort belief \( \bar{p} \) and the shut down belief \( p^* \).

For the daggered (\( \dagger \)) items the condition is also necessary.

As before, an increase of the patent length induces an increase of all the measures of research intensity. The intuition is clear, the stronger the patent the more is at play, thus in equilibrium firms increase their efforts to lower their risk to lose. Figure 3(a) shows how effort evolves through time, the stronger the patent the higher the intensity of the competition speeding up the experimentation process at the beginning of the race.

**Proposition 11** In a symmetric equilibrium with observable effort an increase of the positive spillover \( A \) leads to pursue the projects further (a decrease in \( p^* \)) whenever \( \sigma \neq 1 \) and:
Figure 3: Effort through time across different spillovers and patent strengths when 
\[ r = \frac{1}{20}, \quad c = \frac{1}{5}, \quad n = 2, \quad \pi_1 = 1 \text{ and } \pi_2 = 3/5. \]

1. as \( p \to p^* \) to an increase effort and speed for all \( \sigma < 1 \)
2. If \( \sigma = 0 \), to an increase on the speed for all \( p \in [p^*, \bar{p}) \).
3. if \( \sigma = 1 \), a decrease in effort and speed for all \( p \in [p^*, \bar{p}) \);

As before the effect of an increase of the positive spillover is ambiguous. However, as Proposition 9 shows, the incentive to free ride is higher than in the unobservable case. As a consequence, under observable effort, the firms tend to lower their efforts more than the unobservable case and is no longer possible to show that effort increases for short patents regardless the belief. Figure 3(b) shows how effort evolves through time for different values of the spillover for a given patent strength, as can be observed the higher the spillover the lower the intensity of experimentation.

6 Optimal Patent

In the previous sections we have characterized the equilibrium strategy and understood the role of stronger patents in the intensity of the R&D competition. This section characterizes the optimal patent as a function of private benefits; that is, we ignore Consumers Surpluses (CS) and focus on private surplus only. We discuss how to incorporate CS affects the forthcoming results bellow. The following proposition is immediate from the results in previous sections.
Proposition 12 If $\Pi \geq n$, the optimal patent is $\sigma = 1$ independently of the observability of efforts.

Proof. When $\Pi > n$ Monopoly is more efficient in terms of profits that $n$ firms competing, thus $\sigma = 1$ lead to higher social ($\alpha$) and private benefits ($\beta$). Moreover, the speed of experimentation is maximal as $\Pi \geq n \geq \lambda$, the sufficient condition $\sigma$ to increase the speed in Propositions 5 and 10. ■

When competition in the post innovation market is in sufficiently differentiated products, it is optimal to set the patent length at its maximum. This result resembles to those in the classical literature about patents (see Lee and Wilde (1980) and Loury (1979)).

To proceed further we study the problem that a social planner will face when $n$ symmetric firms compete in a R&D race playing the symmetric equilibrium and his only tool for policy is the patent strength $\sigma$, i.e.

$$\max_{\sigma \in [0,1]} \int_0^\infty \hat{u}_t (p_t \lambda_n \alpha - c) \frac{1}{1 - p_t} e^{-rt} dt$$

subject to (1) where $\hat{u}_t$ is the symmetric strategy equilibrium (observable or unobservable) at time $t$ and $\alpha = \sigma \pi_1 + (1 - \sigma) n \pi_n$ is the private surplus. Observe that this problem is similar to the cooperative problem (3) but now the social planner takes the action as given and instead he chooses the patent length. Define $q_t \equiv p_t / (1 - p_t)$ to be the inverse of the odds ratio at $t$, differentiating (18) with respect to $\sigma$ using Leibniz integral rule we obtain

$$\pi_n \lambda_n (\Pi - n) \int_0^\infty \hat{u}_t q_t e^{-rt} dt + \int_0^\infty \frac{\partial \hat{u}_t}{\partial \sigma} \left( \frac{p_t (\lambda_n \alpha - c)}{1 - p_t} \right) e^{-rt} dt$$

where $\hat{p}$ is the last belief where full effort is performed and $L (\hat{p})$ is the length of time while full effort is performed. For simplicity, the previous expression ignores the second order effects of $\sigma$ in $p_t$ through $\hat{u}_t$. Equation (19) makes clear the trade-offs that the planner faces. On one hand, when $\Pi < n$, to increase $\sigma$ reduces the total profit of the race; on the other, whenever $\Pi \geq \lambda$, it increases the experimentation pace (propositions 5 and 10). Unfortunately we cannot solve (19) to get an analytical solution, so we study the patent policy numerically. Figure 4 shows how social welfare for different spillovers $A$ and substitutability ratio. On black the unobservable case is depicted. Since efforts are uniformly higher for the unobservable case we can see that, in terms of welfare, the observability of actions is detrimental for social welfare. In the unobservable case we had that when $\sigma > \bar{\sigma}$ no interior effort existed, thus only the first term of equation (19) is relevant indicating that an increase in the patent length diminishes social welfare whenever $\Pi < n$. When $\sigma \in [0, \bar{\sigma}]$ the social welfare appears to be a convex function of $\sigma$ which leads to the following result.
Figure 4: Patent strength versus Social Welfare for different spillovers degrees $A$ and Substitutability Ratio $\Pi$ when $r = \frac{1}{20}$, $c = \frac{1}{5}$, $n = 2$, and $\pi_1 = 1$. In black case with unobservable effort and in purple the observable case.
Result When $\Pi < n$, the optimal patent in the unobservable case $\sigma^* \in \{0, \bar{\sigma}\}$. In particular,

$$\Pi < \frac{\lambda_n}{1 + (1 - A)(p\beta_n - c)(p\gamma)^{-1}}$$

is a sufficient condition for $\sigma^* = 0$.

For the observable case the behavior reacts in similar fashion, however the interior optimal patent is not analytically identified. In both cases we see that an increase of the positive spillover $A$ shifts the optimal patent to the right, i.e. markets with higher spillover should have stronger patents. This result that is in principle unintuitive can be explained as follows: When an increase in the positive spillover $A$ from an optimal interior patent $\sigma^*$ occurs, the cost of having a strong patent decreases as the development speed diminishes. This decrease in the cost of delaying calls for stronger patents.

Result When the optimal patent $\sigma^*$ is interior, an increase in the knowledge spillover $A$ leads to an increase in $\sigma^*$.

Finally, it can be appreciated that an increase in $A$ leads to an increase in in social welfare for almost all $\sigma$.

7 Future Extensions

There are many extensions that are not part of this draft and are planned to be added in future versions of this work.

7.1 Changes in $n$

To analyze the impact of increasing the post innovation competition is not direct as we have as an extra factor to consider how $\pi_{n+1}$ relates with $\pi_n$. In principle an increase in $n$ increases competition and decreases profits when no patent is active. Also, it should speed up the experimentation process and increase the aggregate spillovers in the economy.

Conjecture 3 An increase in $n$ is beneficial for the R&D when patents are weak and post innovation competition occurs in differentiated products.
7.2 First to File Patent

A particular case of the team problem is the single agent case corresponding to a patent system in which the firm that is first in filing an idea gets the patent independently of her had developed the innovation. This scenario precludes competition in the race and corresponds to the new patent system that the US will start implementing in September 2013.

Corollary 4 In the single agent scenario, there is a cut-off belief $p_S = c/\pi_1$ such that below this belief the firm exerts no effort and above it effort is maximal. The value function $V_S$ is given by

$$V_S(p) = \frac{r}{r+1} \left[ \frac{c}{r} (1-p) \left( \frac{\Omega(p)}{\Omega(p_C)} \right)^r - 1 \right].$$

When the number of competitors in the race varies, there is no simple way to compare the expected cost of product development as in the fixed number case.

Conjecture 5 A first to file system incurs in more costs than fist to invent for sufficiently large $\lambda_n$. In particular, is more costly for all $\lambda_n > 1$ if and only if $\ln (\Omega(p_0)/\Omega(c/\pi_1)) + c/\pi_1 > 1$.

7.3 $M$ stages to develop the innovation

In the current setting the innovation took one breakthrough to be develop. In many applications there are $M$ well defined stages to develop a complete commercializable innovation. In an scenario in which the stage of the development is observable but cannot be copied:

Conjecture 6 The effect of the competition moving forward to the next step is two-fold: Firms left behind speed up the experimentation process due to the rivals success in that stage lead them to be more optimistic about the feasibility of the project. If no success occur for sufficiently long time, firms left behind quit earlier than they would if no firm had taken advantage due that is too costly to catch up.

7.4 Asymmetric Equilibrium

We limited our analysis to symmetric equilibrium. The following statements are conjecture about the robustness of the results

Conjecture 7 Under a infinitely strong patent, regardless the observability of the efforts, there is no asymmetric equilibrium.
Conjecture 8  The more efficient asymmetric equilibrium, regardless the observability of the efforts, has \( p^* \) as a quitting point.
A Omitted Proofs

A.1 Proof of Proposition 1.

Let $V^C(t, p)$ be the value function representing the sum of the payoff of the $n$ firms when maximal effort is exerted

$$V^C(t, p) := \int_t^\infty r [p_s \lambda_n \alpha - c] \frac{(1 - p_t)}{(1 - p_s)} e^{-rs} ds, \text{ with } p_t = p.$$ 

Start by observing that $V^C(s, p) = r (1 - p) e^{-rs} v(p)$ where

$$v(p) = \int_0^\infty \frac{p_s \lambda_n \alpha - c}{(1 - p_s)} e^{-rs} ds$$

is a function that does not depend on $t$. We solve this problem through the Hamilton-Jacobi-Bellman (HJB) equation: $-V^C_t(s, p) = g(p, s) + dp V^C_p(s, p)$ where $g(p, s)$ is the flow payoff of $V^C$, and $V^C_t, V^C_p$ are the derivatives of $V^C$ with respect the first and second argument respectively. The HJB method assumes that the solution of $V^C(t, p)$ is continuous and differentiable, we check that conjecture later. It can be shown that the HJB equation reduces to the following first order differential equation

$$rv = \frac{1}{1 - p} (p \alpha \lambda_n - c) - p(1 - p) \lambda_n n v'.$$

Solving and imposing value matching $v(p_C) = 0$ the continuous differentiable solution

$$v(p) = \frac{\mu}{r(\mu + 1)} \left( \frac{(p \alpha \lambda_n - c)}{(1 - p)} + \frac{c}{\mu} \left( \frac{\Omega(p)}{\Omega(p_T)} \right)^\mu - 1 \right)$$

is obtained, here $\Omega(p) = (1 - p)/p$ and $\mu = r/n \lambda_n$. Replacing back for $V^C(t, p)$ implies the result.

A.2 Proof of Proposition 3

By construction, in the conjectured interior strategies, firms do not have incentives to deviate. Thus, we just have to show that under those strategies firms maximize profits. We use Pontryagin’s maximum principle to show that the proposed strategies satisfy necessary conditions for a maximum and to establish uniqueness. Then we prove sufficiency by using Arrow’s sufficiency theorem. Before proceeding with the proof it is convenient to rewrite the objective function

$$r \int_0^\infty (p_t (s \pi_1(u_{i,t} + A_{-i,t}) + (1 - s) \pi_n \lambda v_t) - cu_{i,t}) \frac{1 - p_0}{1 - p_t} e^{-rt} dt.$$
Since we look for a symmetric equilibrium we use $\lambda$ instead of $\lambda_n$ and neglect all indicator functions. Using the law of motion for belief we substitute in $-\dot{p}_t/(1-p_t) = p_t \lambda u_t$ and $u_{i,t} = -[\dot{p}_t/\lambda p_t (1-p_t) + u_{-i,t}]$ to the previous objective function and eliminating irrelevant terms (those that do not include beliefs) we obtain

$$\int_0^\infty \left( -\frac{\dot{p}_t}{(1-p_t)^2} \left( (1-\sigma) \pi_n + \frac{\pi_1}{\lambda} - \frac{c}{\lambda p_t} \right) + \frac{(c - p_t (1-A) \sigma \pi_1)}{1-p_t} u_{-i,t} \right) e^{-rt} dt.$$

Integrating by parts, with respect to $t$, all the terms that contain $\dot{p}_t$ we get

$$\int_0^\infty \left( \frac{rc}{\lambda} \ln \frac{p_t}{1-p_t} - \frac{1}{1-p_t} \left( r \left( (1-\sigma) \pi_n + \frac{\pi_1}{\lambda} - \frac{c}{\lambda} \right) + (c - p_t (1-A) \sigma \pi_1) u_{-i,t} \right) \right) e^{-rt} dt.$$

After defining $\omega_t = \ln \Omega(p_t)$, observing that $1+\exp \omega_t = (1-p_t)^{-1}$ and eliminating irrelevant terms the objective can be written as

$$\int_0^\infty (rc \omega_t + \lambda cu_{-i,t} + e^{\omega_t} (r (c - \beta) + \lambda \gamma u_{-i,t}) e^{-rt} dt.$$

We want to maximize the previous equation subject to the law of motion $\dot{\omega}_t = \lambda (u_{i,t} + u_{-i,t})$. To do this define the Hamiltonian

$$H_i (u_{i,t}, \omega_t, \mu_{i,t}) = rc \omega_t + \lambda cu_{-i,t} + e^{\omega_t} (r (c - \beta) + \lambda \gamma u_{-i,t})$$

$$+ \mu_{i,t} \lambda (u_{i,t} + u_{-i,t})$$

(20)

and use Pontryagin’s maximum principle.

**Necessity:** Let $(u^*_t, \omega^*_t)$ be the optimal profile of effort and the induced path for the state variable. The necessary condition for a maximum are that, for each individual, there exist a function $\mu_{i,t}$ such that:

1. The maximum principle holds: $H_i (u^*_t, \omega^*_t, \mu_{i,t}) \geq H_i (u_{i,t}, \omega^*_t, \mu_{i,t})$ for all $u_{i,t}$. Which under our formulation is equivalent to

$$\mu_{i,t} \lambda (u^*_t + u_{-i,t}) \geq \mu_{i,t} \lambda (u_{i,t} + u_{-i,t}) \text{ for all } u_{i,t}.$$  (21)

2. Coestate motion: $\dot{\mu}_{i,t} = \partial H_i / \partial \omega_t$ or in our terms

$$\dot{\mu}_{i,t} = rc + e^{\omega_t} (r (c - \beta) + \lambda \gamma u_{-i,t})$$

(22)

3. Transversality condition: $\lim_{t \to \infty} \mu_{i,t} (\omega^*_t - \omega_t) \leq 0$ for all path $\omega_t$. 33
Let \( \mu_t \equiv (\mu_{1,t}, \ldots, \mu_{n,t}) \). To prove necessity we make a guess based in (8) and show that satisfies conditions 1 to 3. Define \( \mu_{i,0} = K (e^{-n\lambda L(\bar{p})} - 1) - rcL(\bar{p}) \) for each firm \( i \) where \( K \equiv \Omega (p^0) (r(c - \beta) + \lambda \gamma (n - 1)) (n\lambda)^{-1} \), \( \bar{p} \) is the last belief in which full effort is exerted and \( L(\bar{p}) \) is the period of time that firms exert full effort if no breakthrough occurs (see equation (5)).

If \( L(\bar{p}) = 0 \) then \( \mu_0 = 0 \). Moreover, using or conjecture (8) to compute \( u_{-i,t} = r(p_t \beta - c) / (p_t \lambda \gamma) \) and replace in to (22) we can show that \( \dot{\mu}_{i,t} = 0 \) for all \( t \); thus, \( \mu_{i,t} = 0 \) for all \( i \) and \( t \), and It is straight forward to show that the transversality condition hold and (21) is trivially maximized. Therefore when \( L(\bar{p}) = 0 \) the proposed strategy satisfies the necessary condition to be an equilibrium. Solving \( \dot{\omega}_t \) and replacing back to find an expression for \( p_t \) we get

\[
p_t = \left( 1 + p^* \frac{1 - p^0}{p^0 - p^*} e^{\frac{n}{\gamma}(r(\beta_n - c) - t)} \right) \left( 1 + \frac{1 - p^0}{p^0 - p^*} e^{\frac{n}{\gamma}r(\beta_n - c)} \right)
\]

and replacing in our conjecture

\[
u_{i,t} = \frac{r(\beta_n - c) - t}{\lambda_n (n - 1) \gamma} \left( 1 + p^* \frac{1 - p^0}{p^0 - p^*} e^{\frac{n}{\gamma}(r(\beta_n - c) - t)} \right)^{-1}.
\]

In the case that \( L(\bar{p}) > 0 \), we have that for \( t \in [0, L(\bar{p})] \) the individual effort is one; thus, \( \omega_t = \Omega (p^0) e^{-n\lambda t} \). The only relevant case is when \( \gamma > 0 \) form which we conclude that \( r(c - \beta/p_t) > (n - 1) \lambda \gamma \) (otherwise \( u_{i,t} < 1 \) thus

\[
\dot{\mu}_{i,t} = r(c + e^{\lambda t}) (r(c - \beta) + \lambda \gamma (n - 1)) < r(c + e^{\lambda t}) (r(c - \beta) + r(\beta - c/p_t)) = 0
\]

and the coestate variable decreases for \( t \in [0, L(\bar{p})] \). Moreover, we can obtain \( \mu_{i,t} \) explicitly by solving (22) and imposing our conjectured initial condition \( \mu_{i,0} \)

\[
\mu_{i,t} = K (e^{-n\lambda L(\bar{p})} - e^{-n\lambda t}) - rc (L(\bar{p}) - t)
\]

which is positive and reaches zero at \( t = L(\bar{p}) \). After this point, effort is interior and \( \mu_{i,t} = \dot{\mu}_{i,t} = 0 \) by the same analysis as before. Therefore, for \( t \in [0, L(\bar{p})] \) since \( \mu_{i,t} > 0 \) the only effort that satisfies (21) is \( u_{i,t} = 1 \). For \( t \geq L(\bar{p}) \) or conjecture satisfies the optima condition by the same arguments as in the case \( L(\bar{p}) \). Solving \( \dot{\omega}_t \) and replacing back to find an expression for \( p_t \) we get that \( t \geq L(\bar{p}) \)

\[
p_t = \left( 1 + p^* \frac{1 - \bar{p}}{\bar{p} - p^*} e^{\frac{n}{\gamma}(t - L(\bar{p}))} \right) \left( 1 + \frac{1 - \bar{p}}{\bar{p} - p^*} e^{\frac{n}{\gamma}(t - L(\bar{p}))} \right)
\]

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and replacing in our conjecture
\[ u_{i,t} = \frac{r (\beta_n - c)}{\lambda_n (n - 1) \gamma} \left( 1 + p^* \frac{1 - \tilde{p}}{\tilde{p} - p^*} e^{\frac{-r(\beta_n - c)(t-L)}{\gamma}} \right)^{-1}. \]

To find \( \tilde{p} \) we use (8) and solve \( u (\tilde{p}) = 1. \)

**Sufficiency:** Let \( u^*_{i,t} (t, \omega_t, \mu_{i,t}) \) be an optimal solution of (20) and let \( H^* (t, \omega_t, \mu_{i,t}) \equiv H \left( u^*_{i,t} (t, \omega_t, \mu_{i,t}), \omega_t, \mu_{i,t} \right). \) Using Arrow’s sufficiency theorem its enough to show that \( H^* (t, \omega_t, \mu_{i,t}) \) is concave in \( \omega_t. \) When \( u_{i,t} \) is interior, thus \( \gamma > 0, \) we showed that \( \mu_{i,t} = 0. \) Replacing the candidate for a maximum in to equation (20) it reduces to
\[ H \left( t, \omega_t, \mu_{i,t} \right) = cr \left( \frac{\beta - c}{\gamma} - \frac{c}{\gamma} e^{-\omega_t} + \omega_t \right) \]

which by analyzing its first and second derivatives can be readily seen concave. Suppose that effort is interior, i.e. \( u_{i,t} = 1. \) Using (23) it is not hard to show that
\[ \mu_{i,t} = -e^{\omega_t} (r (c - \beta) + \lambda \gamma (n - 1)) (n \lambda)^{-1} + Ke^{-n\lambda L (\tilde{p})} - rc (L (\tilde{p}) - t) \]

Replacing the coestate variable and effort in to equation (20) we obtain
\[ H \left( t, \omega_t, \mu_{i,t} \right) = \lambda \left( c (n - 1) + nKe^{-n\lambda L (\tilde{p})} \right) + rc (\omega_t - \lambda n (L (\tilde{p}) - t)) \]

which is linear function of \( \omega_t; \) thus, concave.

**Uniqueness:** [To type] Use Pontryagin’s maximum principle to show that any other \( \mu_0 \) and effort that satisfies the necessary condition are not symmetric.

### A.3 Proof of Proposition 4

We differentiate \( p^*, \tilde{p} \) and with respect to \( r, c, \pi_1 \) and \( \pi_n \) to get for \( p^* \)
\[ \frac{\partial p^*}{\partial r} = 0, \quad \frac{\partial p^*}{\partial c} = \frac{1}{\beta}, \quad \frac{\partial p^*}{\partial \pi_1} = p^* \frac{\sigma}{\beta}, \quad \frac{\partial p^*}{\partial \pi_n} = p^* \frac{\lambda (1 - \sigma)}{\beta}. \]

Similarly for \( \tilde{p} \)
\[ \frac{\partial \tilde{p}}{\partial r} = -\tilde{p} \frac{\lambda_n (n - 1) \gamma}{\Gamma}, \quad \frac{\partial \tilde{p}}{\partial \pi_1} = -\frac{rc\sigma}{\Gamma^2} (r + \lambda_n (n - 1) (1 - A)), \quad \frac{\partial \tilde{p}}{\partial \pi_n} = -\frac{r^2 c\lambda (1 - \sigma)}{\Gamma^2}, \]
\[ \frac{\partial \tilde{p}}{\partial c} = \frac{r}{\Gamma^2} (r \beta + \lambda_n (n - 1) (1 - A) (1 - \sigma) \pi_1) \]

where \( \Gamma = r \beta - \lambda (n - 1) \gamma > 0 \) for cases in which \( \tilde{p} \) is relevant. Finally, for \( u^*_i (p) \)
\[ \frac{\partial u^*_i}{\partial r} = \frac{p \beta - c}{p \gamma \lambda (n - 1)}, \quad \frac{\partial u^*_i}{\partial \pi_n} = \frac{1 - \sigma}{p \gamma^2 (n - 1)}, \quad \frac{\partial u^*_i}{\partial \pi_1} = \frac{(p + 1 - A) \sigma}{p \gamma^2 \lambda (n - 1)}, \]
\[ \frac{\partial u^*_i}{\partial c} = \frac{(p \beta - c) + \gamma}{p \gamma^2 \lambda (n - 1)} \]

and the result follows.
A.4 Proof of Proposition 5

We differentiate $p^*$, $\bar{p}$ and $u^*_i(p)$ with respect to $\sigma$ to get
\[
\frac{\partial p^*}{\partial \sigma} = -p^* \pi_1 - \lambda_n \tilde{\pi}_n
\]
\[
\frac{\partial \bar{p}}{\partial \sigma} = -\frac{p r}{\Gamma} \left( \pi_1 + \lambda_n (1 - A) (n - 1) r^{-1} - \lambda_n \tilde{\pi}_n \right)
\]
\[
\frac{\partial u^*_i(p)}{\partial \sigma} = \frac{r}{\lambda (n - 1) \gamma} \left( \pi_1 (1 - A) (p \beta_n - c) (\gamma p)^{-1} - \lambda_n \tilde{\pi}_n \right)
\]
since the expressions $u^*_i(p)$ and $\bar{p}$ are relevant only if $\gamma > 0$ and $\Gamma = r \beta - \lambda_n (n - 1) \gamma > 0$ respectively. Hence, conditions 1), 2) and 3) in the proposition follow. We conclude that the experimentation is faster as for each belief when $\Pi > \lambda_n$ as this condition guarantee that firms exert weakly more effort at each possible beliefs, thus speeding up the experimentation process uniformly.

A.5 Proof of Proposition 6

The first statement follows from $\partial p^* / \partial A = -p^* (n - 1) (1 - \sigma) \pi_n / \beta$. For the second part, notice that $\lambda u^*_i$ is sufficient statistics for the speed of the experimentation process. We differentiate effort, speed and $\bar{p}$ with respect to $A$ to get
\[
\frac{\partial u^*(p)}{\partial A} = \frac{1}{\gamma \lambda} \left[ r (1 - \sigma) \pi_n - u^*(p) (\sigma \lambda_n \pi_1 + (n - 1) \gamma) \right]
\]
\[
\frac{\partial (\lambda_n u^*(p))}{\partial A} = \frac{1}{\gamma} \left[ r (1 - \sigma) \pi_n - u^*(p) \sigma \pi_1 \right]
\]
\[
\frac{\partial \bar{p}}{\partial A} = -\frac{p n - 1}{\Gamma} \left[ r (1 - \sigma) \pi_n - \sigma \lambda_n \pi_1 + (n - 1) \gamma \right]
\]
for the first two expressions the sign of the derivative corresponds to the sign of the term in square brackets, for the third is the opposite sign. For claim 1 notice that as $p \to p^*$ we have that $u^*_i(p) \to 0$ so the first derivative is positive. For 2, replacing $\sigma = 0$ delivers $rc/p\lambda$, $p\pi_n$ and $r\pi_n - c(n - 1)$ for the three terms in square brackets. The first two are positive for all $p$ and the third is positive as $\bar{p}(\sigma = 0) = rc / (r\pi_n - c(n - 1))$ is by construction positive. For claim 3 take the limit when $\sigma \to \bar{\sigma}$ and get
\[
\frac{\partial u^*(p)}{\partial A} = \frac{1}{\gamma \lambda} \pi_n [r (1 - \bar{\sigma}) - \bar{\sigma} \lambda_n \Pi], \quad \frac{\partial (\lambda_n u^*(p))}{\partial A} = \lambda \frac{\partial u^*(p)}{\partial A}
\]
\[
\frac{\partial \bar{p}}{\partial A} = -\frac{p (n - 1)}{r \beta - (n - 1) \lambda \gamma} \left[ r (1 - \bar{\sigma}) - \bar{\sigma} \lambda_n \Pi \right]
\]
where we have used that $u^*(p) = 1$ for all $p$ when $\sigma = \bar{\sigma}$, and $\gamma(\bar{\sigma}) = 0$ by definition. The results follows by continuity. For the last claim we need to understand in which direction moves
the term in square brackets for the speed when patent strength increases. Differentiating we get
\[- \left( r \pi_n + \lambda_n \pi_1 \left( \frac{\partial u_i^* (p)}{\partial \sigma} \lambda + u_i^* (p) \right) \right)\]
which is negative whenever Proposition 5 holds.

### A.6 Proof of Proposition 8

We start by proving that for beliefs \( p \in [p^*, \tilde{p}] \) the interior solution provided is an equilibrium. Suppose the opponents play according to (17) that is
\[ u_i^{**} (p) = S (p) / (n - 1) \gamma (p) , \]
replace in (10) and check that when \( B (a, p) = C (V (p), a, p) \) it is indeed a best response to play \( u_i^{**} (p) \) thus an equilibrium. By definition \( u_i^{**} (\tilde{p}) = 1 \), therefore by (17) we get
\[ S (p) = (n - 1) (c - \gamma p) . \]
It is easy to check that \( \partial S / \partial p > 0 \) and that the right hand side is decreasing in \( p \), therefore by the best response correspondence (12) for \( p > \tilde{p} \) an interior solution no longer can hold and we must have \( u_i^{**} (p) = 1 \). Using the solution (13) of the ODE that characterizes full effort and the value matching condition \( S (\tilde{p}) = W (\tilde{p}) \), the corresponding value function is pinned down.

### A.7 Proof of Proposition 10

Analyzing the change in strength in \( S (p) \) and \( p^* \) we obtain
\[ \frac{\partial S}{\partial \sigma} = \frac{r}{\lambda_n} \frac{p \beta_n - c}{\beta_n - c} (\pi_1 - \lambda_n \pi_n) \quad \text{and} \]
\[ \frac{\partial p^*}{\partial \sigma} = - \frac{p^*}{\beta_n} (\pi_1 - \lambda_n \pi_n) \]
which are respectively positive and negative if and only if \( \Pi > \lambda_n \), establishing that the condition is both necessary and sufficient. Using equation (17) can be shown that
\[ \frac{\partial u_i^* (p)}{\partial \sigma} = \frac{1}{(n - 1) \gamma (p)} \left( \frac{\partial S}{\partial \sigma} + S (p) \frac{\pi_1 (1 - A) p}{\gamma (p)} \right) \] (24)
by (12) we know that \( \gamma (p) > 0 \) whenever effort is interior, therefore \( \Pi > \lambda_n \) implies that equation (24) is positive. Finally, we show that the partial derivative of \( \tilde{p} \) w.r.t the strength is
negative by using the implicit function theorem. Define $H(p) = u^*(p) - 1$ so that $H(\bar{p}) = 0$. Then

$$\frac{\partial \bar{p}}{\partial \sigma} = \frac{\partial H(\bar{p})}{\partial \sigma} \frac{\partial \bar{p}}{\partial \sigma}$$

$$= - \frac{\partial u(\bar{p})}{\partial \sigma} \left( \frac{1}{(n-1)\gamma(\bar{p})} \left( \frac{\partial S(\bar{p})}{\partial \bar{p}} + S(\bar{p}) \frac{\gamma}{\gamma(\bar{p})} \right) \right)$$

the denominator is always positive as $S'(p)$, $S(p)$ and $\gamma(\bar{p})$ are all positive. Therefore

$$\text{sign} \left( \frac{\partial \bar{p}}{\partial \sigma} \right) = -\text{sign} \left( \frac{\partial u^*(p)}{\partial \sigma} \right)$$

delivering the result.

### A.8 Proof of Proposition 11

The first statement follows from $\partial p^*/\partial A = -p^*(n-1)(1-\sigma)\pi_n/\beta$ and claim 1 follows directly by the effect in $p^*$ as there are belief with positive effort that did not have effort before the change in $A$. Differentiating equation (17) with respect to $A$ delivers

$$\frac{\partial u^{**}}{\partial A} = \frac{1}{(n-1)\gamma(p)} \left( \frac{\partial S}{\partial A} - S(p) \frac{p\sigma\pi_1}{\gamma(p)} \right)$$

replacing $\partial S/\partial A$ in the previous expression

$$\frac{\partial u^{**}}{\partial A} = \frac{1}{(n-1)\gamma(p)} \left( (n-1) r \left[ p - \frac{1-p}{\Omega(p^*)} \right] (1-\sigma) \pi_n - S(p) \frac{p\sigma\pi_1 + (n-1)\gamma(p)}{\gamma(p)} \right)$$

using equation (17) to replace $S(p)$ we find

$$\frac{\partial u^{**}}{\partial A} = \frac{1}{\gamma(p)} \left( r \left[ p - \frac{1-p}{\Omega(p^*)} \right] (1-\sigma) \pi_n - u^{**}(p) \left( p\sigma\pi_1 + (n-1)\gamma(p) \right) \right).$$

Similarly, for the speed of learning we get

$$\frac{\partial \lambda_n u^{**}_i}{\partial A} = \frac{1}{\gamma(p)} \left( r \left[ p - \frac{1-p}{\Omega(p^*)} \right] (1-\sigma) \pi_n - u(p) p\sigma\pi_1 \right),$$

and observer that the term in square brackets is positive for all $p > p^*$. Since $\gamma(p) > 0$ for all $p \in [p^*, \bar{p}]$ the effect of a change in $A$ is given by the sign of the expression in the round parenthesis. To prove 2 just notice that

$$\frac{\partial (\lambda_n u^{**}_i)}{\partial A} = \frac{1}{c} \left( r \left[ p - \frac{1-p}{\Omega(p^*)} \right] \pi_n \right) > 0.$$

For claim 2 observe that if $\sigma = 1$, then

$$\frac{\partial (\lambda_n u^{**}_i)}{\partial A} = - \frac{1}{\gamma(p)} u(p) p\sigma\pi_1 < 0$$

$$\frac{\partial u^{**}}{\partial A} = - \frac{1}{\gamma(p)} u(p) \left( p\sigma\pi_1 + (n-1) \gamma(p) \right) < 0$$

proving the result.
References


