Optimal Discretionary Monetary Policy in a Micro-Founded Model with a Zero Lower Bound on Nominal Interest Rate*

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First draft: October 2011; This draft: February 2012

Abstract

We study discretionary monetary policy under zero lower bound on nominal interest rates in the presence of overall economic distortion consisting of monopoly and distortionary labor income tax. We solve a fully non-linear micro-founded (MF) model and compare the results with those from the linear-quadratic (LQ) approach. We find that, when the overall economic distortion is large and the bound is binding in both models, unlike the MF model the LQ approach overstates the output losses of discretionary policy relatively to commitment policy. This occurs due to the inaccuracy of the LQ model under discretion. Using the MF model, we also find that, when the economy is near the lower bound, a central bank who pursues a realistically low positive inflation target will cut the interest rate less aggressively than if his objective is to stabilize price level because the likelihood of being put in a liquidity trap in the case of inflation targeting is much smaller than in the case of price stabilization.

JEL classification: C61, E31, E32, E52.

Keywords: New Keynesian, discretionary monetary policy, zero lower bound on nominal interest rate, price dispersion, preference shock, liquidity trap.

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*I am grateful to Jianjun Miao for his advices, guidance and his contribution to the first draft as a coauthor. I thank Bob King for his guidance and patience. I also thank Luigi Paciello, Simon Gilchrist, Alisdair McKay, Leena Rudako, Francois Gourio, David Seymour and participants at the BU Macro Dissertation Workshop, Green Line Macro Meeting for their comments and suggestions. Additional comments and corrections are welcome and appreciated.

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1 Introduction

In this paper, we study the optimal discretionary monetary policy in the presence of zero lower bound on nominal interest rate, relative price dispersion and overall economic distortion that we define as combination of monopolistic distortion and distortionary labor income tax/subsidy. To conduct the research, we solve a fully non-linear micro-founded (MF) rational expectation model that explicitly imposes a zero lower bound on nominal interest rate. We also use the conventional linear-quadratic (LQ) approach to simplify our MF model, then solve for the simplified LQ model and compare the results from both models.

The current zero lower bound literature utilizes the conventional linear-quadratic (LQ) approach developed in Woodford (2001, 2003). The key assumption of this approach is that the overall economic distortion is zero, meaning that there exists an employment subsidy to fully offset the monopolistic distortion. By this assumption, they can simplify the fully nonlinear optimal discretionary problem and avoid computational difficulty. However, the assumption is questionable in reality. McGrattan (1994) reports that distortionary labor income tax ranges from 10-40%, while Diewert and Fox (2008) estimate that the monopolistic markup ranges from 11-44% in some main industries. As a result, the overall economic distortion ranges from 20-60%.

Also, in the LQ framework, the relative price dispersion is eliminated through the first order approximation and is no longer an endogenous state variable. Alvarez, Gonzalez-Rozada, Neumeyer and Beraja (2011) find that the relative price variance is significantly positive when inflation is high. By using the MF model, we are able to study the role of relative price dispersion in the framework of a zero lower bound.

One of the most influential work in the literature of zero lower bound on nominal interest rate is done by Adam and Billi (2007a). In their paper, they compute optimal discretionary monetary policy in a conventional New Keynesian model with an explicit occasionally-
binding non-negativity constraint on the nominal interest rate. The only nonlinearity in
their model is the zero lower bound on nominal interest rate. They show that a large decline
in the natural real interest rate produces more sizable output losses and deflation in their
model than in perfect foresight models. The results come from the reinforcement of the pri-
ivate sector expectation and optimal discretionary policy responding to these expectations.
Since an adverse shock in the natural real rate may cause the lower bound to be reached
in the future, private agents expect lower output and inflation even at times when nom-
inal rates are still positive. Reduced inflation expectation increases expected real interest
rates and put downward pressure on actual output and inflation. The corresponding optimal
discretionary monetary policy will lower the nominal interest rate and, as the result, real
interest rate, causing the lower bound to be reached much earlier than it would be without
the lower bound or in the perfect foresight models. This in turn lowers the future output
expectation and inflation and generates additional downward pressure on the actual values
of these variables.

Nakov (2008) follows the model setup in Adam and Billi (2007a) to study the simple
monetary policy rules. He shows that simple rules perform substantially worse than the
optimal commitment monetary policy conditional on a strong deflationary shock.

The common feature in Adam and Billi (2007a) and Nakov (2008) is that they use
the linear-quadratic approach in which the central bank objective function is quadratically
approximated and all the constraints are log-linearized as in Woodford (2001,2003). The
only non-linearity in their model is zero lower bound on nominal interest rate. Again, for
the LQ model to be a good approximation of the true model, we assume the overall economic
distortion is small, an assumption that is inconsistent with the data. Another drawback of
the LQ model is that the optimal discretionary policy is independent of changes in relative
price dispersion.

Earlier literature in zero lower bound on nominal interest rate includes Jung, Teranishi
and Wantanabe (2005) and Eggertsson and Woodford (2003). In the former paper, the authors also use the linear-quadratic approach under perfect foresight condition. As shown in Adam and Billi (2007a) and Nakov (2008), the perfect foresight in Jung, Teranishi and Wantanabe (2005) is not able to produce sizable output collapse and deflation as in the stochastic model. In the paper by Eggertsson and Woodford (2003), the authors use the LQ framework and assume that the steady state real interest rate follows two-state absorbing Markov process. The zero lower bound is imposed as an initial condition and in the subsequent periods the economy can either stay at this state or jump back to normal steady state. When the economy jumps back to the normal state, it never comes back to the zero lower bound state. Like the perfect foresight LQ approach in Jung, Teranishi and Wantanabe (2005), this approach lacks uncertainty and is less likely to generate binding bound. Besides, Eggertsson and Woodford (2003) prove that quantitative easings do very little in stimulating the economy. In contrast, committing to a path of low nominal interest rates, especially the long-run interest rate, can help to take the economy out of liquidity trap. Also, they suggest that an output-gap adjusted price target can produce almost the same results as keeping a path of low interest rates. However, in order to do so, the policy maker would need a credible commitment policy.

Recently Levin, Lopez-Salido, Nelson and Yun (2010) study the role of commitment policy as forward guidance under different specification of shock processes to the natural interest rate. They concludes that under a large and persistent shocks, the commitment policy is much less effective and that under these kinds of shocks the combination of the forward guidance and other monetary policy measures might be more desirable. Again, they use the perfect foresight LQ approach.

Bodenstein, Hebden and Nunes (2010) address the question about credibility of central banks in committing to a path of low interest rate. They analyze the optimal monetary policy in the conventional LQ framework with central bank imperfect credibility. In their
model, the monetary authority is uncertain about policy to pursue, it can be either discretion or commitment. Like the other models, the only nonlinearity in their model is due to the zero lower bound interest rate.

Departing from the LQ approach, Yun (2005) studies optimal monetary policy under discretion and initial relative price dispersion. He shows that the complete stabilization of the price level is optimal in the absence of initial price dispersion, while optimal inflation targets respond to changes in the level of relative price dispersion in the presence of initial price dispersion. However, he assumes that there is a production subsidy with the size chosen to eliminate the distortion associated with imperfect competition in goods market. Through our baseline model with no lower bound, our paper extends some of the results in Yun (2005) to the case where there are larger distortions.

The paper that is closest to our paper is Anderson, Kim and Yun (2010). They investigate the size of inflationary bias under discretion in the presence of overall economic distortion. They also depart from the conventional linear-quadratic approach by solving a fully non-linear micro-founded model featuring monopolistic competition and Calvo price setting. Our model extends their results by considering the zero lower bound on interest. Beyond the zero lower bound on interest, we model a different type of shock, called preference shock. We believe that this shock is more realistic to cause the zero lower bound to bind.

By solving both the LQ and MF models in the presence of zero lower bound and overall economic distortion, we obtain four sets of main findings. In order to compare the results from both models, we compute log deviations from steady state for some aggregate variables in the MF model including interest rate, inflation and output. We also compute the output gap for the MF model

First, we find that the lower bound of interest rate deviation computed from the MF model is less restrictive than the one in the LQ model if the overall economic distortion presents. This difference increases in the size of overall economic distortion and it is signifi-
cant when the size of the overall economic distortion is large.

Second, in the MF model without relative price dispersion, the central bank can obtain target output and zero inflation immediately if there are no lower bound and no overall economic distortion. This is not the case when the relative price dispersion presents even when we have a fiscal scheme designed to offset monopolistic distortion completely. This cannot be found in the conventional LQ model as the first order linearization eliminates higher moments of relative price. Under zero lower bound and larger relative price dispersion compared to its steady state value, the output loss and inflation deviation are more sizable. However, the increase in the size of output loss is mainly coming from the higher relative price dispersion, which plays a role as a negative shock in productivity as in Yun (2005).

Third, with sufficiently small overall economic distortion and no price dispersion, the MF model and the LQ model produce almost the same results under the zero lower bound on nominal interest rate. Both models produce the same magnitude of output loss and deflation at all states of preference.

The final and most important result is that, given a large overall economic distortion when the zero lower bound is binding, the output loss and inflation decrease in the MF model are significantly smaller than those from the LQ model. This occurs because the lower bound of the log deviation of the interest rate in the LQ model is greater than the actual bound in the MF model. As a result, the bound is more restrictive in the LQ model and is reached earlier than it would in the MF model. This causes the LQ model to overstate the output loss and inflation decrease. Because of the large output loss, optimal discretionary policy results in a more aggressive interest rate cut when using the LQ model. Since, under commitment, the optimal policies are the same in both models. The LQ model overstates the losses from being unable to commit. We also find that, when the economy is near the lower bound, a central bank who pursues a positive inflation target will cut the interest rate less aggressively than if his objective is to stabilize price level because the likelihood of being
put in a liquidity trap in the case of positive inflation targeting is much smaller than in the case of price stabilization.

The remainder of this paper is organized as follows. Section 2 presents the structure of the economy, while Section 3 presents the setup of discretionary monetary policy problem faced by a central bank. Section 4 shows the simplified solution method and algorithm. In section 5, we calibrate the shocks and report our main numerical results. We conclude in Section 6.

2  Model

The model economic structure presented in this paper characterizes some key New Keynesian features, like in Rotemberg and Woodford (1997) and Yun (1996). The economy is populated by infinite number of intermediate goods firms and identical households. The representative household lives and works in perfectly competitive labor and final good markets. The intermediate good firms are identical except their goods prices vary due to infrequent price adjustment.

2.1 Household problem

The representative household maximizes his total expected discounted lifetime utility:

$$\max \ E \sum_{t=0}^{\infty} \left( \prod_{k=0}^{t-1} \beta_k \right) \left( \frac{C_t^{1-\gamma}}{1-\gamma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right)$$

subject to the budget constraint:

$$C_t + B_t = (1 - \tau_w) w_t N_t + B_{t-1} \frac{R_{t-1}}{\Pi_t} + \int_0^1 D_t(i) di + T_t \quad (1)$$

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where $C, N$ are composite consumption and total labor, $B, D, T$ denote real bond, dividend and lump sum transfer, $R, \Pi$ are nominal interest rate and inflation respectively, $w$ is real wage, $\tau_w$ is the labor income tax, $\gamma, \eta, \chi$ are risk aversion, inverse wage elasticity of labor and steady state labor determining parameters, $\beta$ is the stochastic preference that follows an AR(1) process with a steady state value $\overline{\beta}$:

$$\ln (\beta_t) = (1 - \rho_\beta) \ln (\overline{\beta}) + \rho_\beta \ln (\beta_{t-1}) + \varepsilon_{\beta,t}$$

$$\beta_0 = 1, \beta_1 \text{ is given.}$$

The optimal choices of the household must satisfy the following condition.

$$E_t \left[ \beta_t \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{R_t}{\Pi_{t+1}} \right) \right] = 1 \quad (3)$$

$$\frac{\chi N_t^\eta}{C_t^{-\gamma}} = (1 - \tau_w) w_t \quad (4)$$

The first condition shows the marginal intertemporal trade-off between today and tomorrow consumption. The second condition is the marginal trade-off between working and consuming.

The stochastic preference is a reasonable short-cut that allows the zero-lower bound to be reached.\(^1\) Intuitively, as the uncertainty about the future increases, households would like to save more today for the future. In other words, the time discount is high. The excessive saving from the households drives down the interest rate to zero lower bound. The conventional technology shock is not able to cause the bound to bind realistically in the New

\(^1\)Eggertsson and Woodford (2010) incorporate household debt limit and deleveraging to drive the nominal interest rate to zero lower bound. Hall (2011) models excessive capital stock as the reason for the nominal interest rate to be pinned at zero. Curdia and Woodford (2009) model a shock to the wedge between the deposit and lending rate as a driving force.
Keynesian model. The reason is that we need to have a very big positive technology shock to generate massive saving that can drive the nominal interest rate to the zero lower bound. We did not observe this type of shocks before the onset of the last crisis.

2.2 Firms problems

2.2.1 Final good producer

The final good producer is operating in the perfectly competitive market. He produces the consumption good by aggregating variety of differentiated goods using a CES technology. His problem is to maximize his contemporaneous profit.

$$\max P_t Y_t - \int P_t (i) Y_t (i) \, di$$

subject to

$$Y_t = \left( \int_0^1 Y_t (i)^{ \frac{ \varepsilon - 1}{ \varepsilon} } \, di \right)^{ \frac{1}{ \varepsilon - 1} }$$

where $y_{it}$ is the input of intermediate good $i \in [0, 1]$ and $\varepsilon$ is the elasticity of substitution between differentiated goods.

The optimal decision of the final good producer gives rise to the demand for the intermediate good $i$:

$$Y_t (i) = \left( \frac{P_t (i)}{P_t} \right)^{ - \varepsilon } Y_t$$

where $P_t$ is the price index:

$$P_t = \left( \int P_t (i)^{1-\varepsilon} \, di \right)^{ \frac{1}{1-\varepsilon} }$$

2.2.2 Intermediate goods producers

The intermediate good producers are operating in the monopolistically competitive market. A firm’s objective is to maximize its total discounted flows of profits. Each period a firm
keeps its previous price with probability \((\theta)\) and resets its price with probability \((1 - \theta)\). The firm’s problem is given below:

\[
\max E_t^i \sum_{j=0}^{\infty} \left( \prod_{k=0}^{j-1} \beta_k \right) \left( \frac{C_{t+j}}{C_t} \right)^{-\gamma} D_{t+j} (i) \tag{9}
\]

subject to

\[
D_{t+j} (i) = \frac{P_{t+j} (i)}{P_{t+j}} Y_{t+j} (i) - w_{t+j} N_{t+j} (i) \tag{10}
\]

\[
P_{t+j+1} (i) = \begin{cases} 
P^*_t (i) & \text{with prob } 1 - \theta \\
P_{t+j} (i) & \text{with prob } \theta 
\end{cases} \tag{11}
\]

\[
Y_t (i) = A_t N_t (i) \tag{12}
\]

where \(A_t\) is technology shock that follows an AR(1) process:

\[
\ln (A_{t+1}) = \rho_A \ln (A_t) + \varepsilon_{A,t+1} \tag{13}
\]

Given its price \(P_t (i)\) and demand \(Y_t (i)\), the firm \(i\) chooses to labor that

\[
\max \left\{ \frac{P_t (i)}{P_t} Y_t (i) - w_t N_t (i) \right\} \tag{14}
\]

Satisfying its demand in equation (7)

Let \(\varphi_t (i)\) is the Lagrange multiplier with respect to the demand. The first order condition give us the same marginal cost for all firms, \(\varphi_t\):

\[
\varphi_t = \varphi_t (i) = \frac{w_t}{A_t} \tag{15}
\]
Whenever a firm has a chance to reset price, it chooses the new price to solve:

\[
Max_{P_t(i)} \left\{ \sum_{j=0}^{\infty} \theta^j \left( \prod_{k=0}^{j-1} \beta_k \right) \left( \frac{C_{t+j}}{C_t} \right)^{-\gamma} \left[ \frac{P_t(i)}{P_{t+j}} - \varphi_{t+j} \right] Y_{t+j}(i) \right\}
\]  

subject to equation (7) and \( \varphi_t \) is the marginal cost. The optimal price, \( P_t^* \), satisfies:

\[
\frac{P_t^*}{P_t} = \frac{\left( \frac{\varphi_t}{\varepsilon-1} \right) \sum_{j=0}^{\infty} \theta^j \left( \prod_{k=0}^{j-1} \beta_k \right) C_{t+j}^{-\gamma} \left( \frac{P_{t+j}}{P_t} \right)^{\varepsilon} Y_{t+j} \varphi_{t+j}}{\sum_{j=0}^{\infty} \theta^j \left( \prod_{k=0}^{j-1} \beta_k \right) C_{t+j}^{-\gamma} \left( \frac{P_{t+j}}{P_t} \right)^{\varepsilon-1} Y_{t+j}}
\]  

or we can rewrite the optimal price in the following recursive forms

\[
p_t^* = \frac{S_t}{F_t}
\]  

\[
F_t = C_t^{-\gamma} Y_t + \theta E_t \left[ \beta_t \Pi_{i+1} F_{t+1} \right]
\]  

\[
S_t = \left( \frac{\varepsilon}{\varepsilon-1} \right) C_t^{-\gamma} Y_t \frac{w_t}{A_t} + \theta E_t \left[ \beta_t \Pi_{i+1} S_{t+1} \right]
\]  

where \( p_t^* \) is relative optimal price, \( S_t, F_t \) are auxiliary variables.

Whenever a firm sets its price, it takes into account the current and expected future demand, marginal cost, price level as well as preference.

### 2.3 Aggregate conditions

Aggregate output satisfies:

\[
Y_t = \frac{AN_t}{\Delta_t}
\]  

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where the relative price dispersion, $\Delta_t$, is defined:

$$\Delta_t = \int \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$  \hspace{1cm} (22)$$

or in a recursive form:

$$\Delta_t = \theta \Pi_t \Delta_{t-1} + (1 - \theta) (p_t^*)^{-\varepsilon}.$$ 

We write the price level (8) in a recursive form and dividing both sides by $P_t$ to obtain the relative optimal price:

$$p_t^* = \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}}.$$

(23)

Plugging the relative optimal price in the price dispersion equation (22) we have:

$$\Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{1-\varepsilon}} + \theta \Pi_t \Delta_{t-1}$$

(24)

### 2.4 Competitive Equilibrium

A competitive equilibrium consists the path of $\{C_t, N_t, R_t, w_t, p_t^*, \Pi_t, \Delta_t\}$ satisfying the following competitive equilibrium conditions:

1. **Euler equation**

   $$E_t \left[ \beta_t \left( \frac{R_t}{\Pi_{t+1}} \right) \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] = 1$$

2. **Labor efficiency condition**

   $$\frac{\chi N_t^n}{C_t^{-\gamma}} = (1 - \tau_w) w_t$$
(3) Optimal price adjustment

\[ p_t^* = \frac{S_t}{F_t} \]

\[
F_t = C_t^{-\gamma}Y_t + \theta E_t \left[ \beta_t \Pi_{t+1|t} F_{t+1} \right] \\
S_t = \frac{\chi C_t N_t^\eta}{(1 - \Phi) A_t} + \theta \beta_t E_t \left[ \Pi_{t+1|t}^\varepsilon S_{t+1} \right]
\]

where we define the overall economic distortion (\( \Phi \)) as the combination of monopoly and distortionary labor income tax/subsidy. See our detailed discussion on this metric below.

(4) Inflation equation

\[ 1 = (1 - \theta) \left( \frac{S_t}{F_t} \right)^{1-\varepsilon} + \theta \left( \frac{1}{\Pi_t} \right)^{1-\varepsilon} \]

(5) Resource constraint and market clearing conditions

\[ Y_t = C_t = N_t A_t (\Delta_t)^{-1} \]

\[ \Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\gamma}} + \theta \Pi_t \Delta_{t-1} \]

2.5 Overall economic distortion

In this section we discuss the overall economic distortion in detail because it is the key parameter in our model. We define the overall economic distortion (\( \Phi \)) as below:

\[ \Phi = 1 - (1 - \tau_w) \cdot (1 - \varepsilon^{-1}) \]  

(25)

To understand more about the meaning of this notation, let us consider an economy
with flexible price. In this economy, the marginal cost, $\varphi$, is equal to inverse of markup (or $(1 - \varepsilon^{-1})$). From equation (4) and equation (15), we have the equilibrium flexible-price output:

$$Y_t^f = N^* \cdot (1 - \Phi)^{\frac{1}{\eta + \gamma}} \cdot \frac{1 + \eta}{\eta + \gamma} \cdot A_t$$

or the efficient output:

$$Y_t^* = N^* \cdot A_t^{\frac{1 + \eta}{\eta + \gamma}}$$

where $N^*$ is the long-run efficient output of the economy where sticky price, tax/subsidy and monopoly do not exists and no shocks occur. The percentage deviation of the flexible-price output from the efficient output equals:

$$\left(\frac{Y_t^f - Y_t^*}{Y_t^*}\right) \cdot 100 \approx -\frac{1}{\eta + \gamma} \cdot \Phi \cdot 100$$

The larger the overall economic distortion, the smaller the flexible-price output compared to the efficient output. When the overall economic distortion is zero, or there exists a labor income subsidy to fully offset the monopolistic distortion, the flexible-price output is equal to the efficient output. It is important to be clear that the overall economic distortion is zero does not mean that there is not any distortion. Instead, it means that we can obtain the efficient output level even in the presence of monopolistic distortion and distortionary labor income subsidy.

### 3 Discretionary optimal policy under zero lower bound

The central bank takes the expectations of economic agents as given and maximizes the representative household’s discounted utility subject to the optimality conditions from mar-
ket participants, the aggregate conditions, the law of motion for the state variables and the explicit zero lower bound on nominal interest rate. In the discretionary problem, the optimal choice of aggregate variables should be given in the form of time-invariant policy function of state. By discretion, the private expectations are given and the central bank does not have power to manipulate the expectations.

The problem can be stated in the form of Bellman equation:

\[ V(\Delta_{t-1}, A_t, \beta_t) = \max_{\{R_t, C_t, N_t, S_t, F_t, \Pi_t, \Delta_t\}} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\eta}}{1+\eta} + \beta_t E_t V(\Delta_t, A_{t+1}, \beta_{t+1}) \right\} \]  

subject to

(i) Competitive equilibrium conditions.

(ii) Motion equations for state variables.

\[ \Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{-1}}{1 - \theta} \right)^{\frac{\varepsilon_t}{\theta}} + \theta \Pi_t^\varepsilon \Delta_{t-1} \]

\[ \ln(\beta_t) = (1 - \rho_{\beta}) \ln(\bar{\beta}) + \rho_{\beta} \ln(\beta_{t-1}) + \varepsilon_{\beta t} \]

\[ \ln A_t = \rho_A \ln A_{t-1} + \varepsilon_{A, t} \]

(iii) Zero lower bound on nominal interest rate.

\[ R_t = 1 + r_t \geq 1 \]

The current zero lower bound literature of optimal monetary policy utilizes the conventional linear-quadratic approach developed in Woodford (2001, 2003) to simplify and solve the above system. In this framework the central bank objective function is quadrati-
cally approximated and all the constraints and law of motion are log-linearly approximated. As widely known, using the log-linear approximation will eliminate the endogenous state variable, relative price dispersion (Δ), so that we can not study the role of initial relative price dispersion in determining the steady state values as well as the dynamics of aggregate variables under the optimal discretionary monetary policy with a zero lower bound.

The solution of the above nonlinear system is called Markovian invariant policy function of the state, \( s_t = (\Delta_{t-1}, A_t, \beta_t) \), where \( \Delta_{t-1} \) is an endogenous state and \( A_t, \beta_t \) are exogenous ones. In the paper, we solve the above fully-nonlinear micro-founded (MF) model using a global method called projection method. We also use the linear-quadratic (LQ) approach to simplify the MF model and solve the LQ model using the same method. (see Appendix for how we solve the MF model using projection method.)

4 Results

4.1 Parameter calibration

We calibrate the steady state quarterly time discount factor \( \beta \) to be 0.993 like in Nakov (2008), corresponding to quarterly real nominal interest rate of 2.8%. The relative risk aversion and inverse wage elasticity of labor are chosen to be 1, while the market power is calibrated to be 11 corresponding to 10% markup as in Anderson, Kim and Yun (2010). The relative risk aversion is also the inverse real interest rate elasticity of output. When the relative risk aversion becomes smaller, the elasticity of output with respect to real interest rate becomes larger, leading to a larger decrease in output if the real interest rate would not be lower due to zero lower bound. When the relative risk aversion is big, the market participants as well as the monetary authority dislike the liquidity trap and output losses more, leading to more aggressive cut of nominal interest rate when the economy is near zero

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lower bound. In this paper we calibrate the risk aversion to be 1, which is commonly used value in the New Keynesian literature.

The probability that a firm keeps its price unchanged each quarter, \( \theta \), is also an important parameter. If \( \theta \) is small, firms know that they will have more opportunity to reset their prices in the future. Therefore, when they have a chance to reset a price today, they will put more weight on current real marginal cost or output gap. If the current output gap declines due a bad demand shock and zero lower bound is binding, the firm will set lower prices today, leading to lower inflation. The benchmark probability that a firm keeps its price unchanged each quarter, \( \theta \), is chosen to be 0.75 like in Anderson, Kim and Yun (2010) or on average firms keep their prices for 4 quarters.

The overall economic distortion, \( \Phi \), is calibrated to be 0.091 or \( 1/\varepsilon \). It means that there is only monopolistic distortion while the labor income tax is 0. Note that in the current zero lower bound literature, this metric is assumed to be 0 for the linear quadratic approach to work accurately. According to Diewert and Fox (2008), the gross markup ranges from 1.11 to 1.44 for some main industries. While according to McGrattan (1994), the distortionary labor income tax ranges from 10 – 40\%. Therefore the overall economic distortion ranges from 0.20 – 0.60. As we shown below, the higher value of \( \Phi \) only makes the LQ model more inaccurate.

Technology shock persistence and standard deviation are calibrated to be 0.95 and 1.6\% quarterly. The variation of the shock is a bit smaller than the value in Anderson, Kim and Yun (2010) but still realistic in the real business cycle literature. However, it is shown not to be a force to realistically drive the economy to the zero lower bound situation.

The hard part is to calibrate preference shock. One approach to calibrate preference shock is based on the estimation of the natural interest rate shock in Adam and Billi (2007a) and technology shock in the literature (see Appendix for details). However, the natural rate shock usually includes the government spending shock that we do not include in our model.
for simplicity. Therefore, we calibrate the persistence and variation of preference shock, as in the Table 1, to reconcile with Adam and Billi (2007a), Woodford (2003) and Nakov (2008).

Table 1. Benchmark parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Economic interpretation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Quarterly discount factor</td>
<td>0.993</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Constant relative risk aversion</td>
<td>1</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Inverse wage elasticity of labor</td>
<td>1</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Market power</td>
<td>11</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Probability that a firm keeps its price unchanged each quarter</td>
<td>0.75</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>AR-coefficient of technology shock</td>
<td>0.95</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>S.d. of technology innovations (quarterly percent)</td>
<td>1.6</td>
</tr>
<tr>
<td>$\rho_\beta$</td>
<td>AR-coefficient of preference shock</td>
<td>0.6</td>
</tr>
<tr>
<td>$\sigma_\beta$</td>
<td>S.d. of preference innovation (quarterly percent)</td>
<td>0.4</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Overall economic distortion</td>
<td>0.091</td>
</tr>
<tr>
<td>$N^*$</td>
<td>Long-run efficient output/labor</td>
<td>1/3</td>
</tr>
</tbody>
</table>

4.2 Steady state

The steady state depends on the overall economic distortion ($\Phi$). With a subsidy designated to fully offset the monopolistic distortion, the overall economic distortion is zero. It is not difficult to compute the steady state values of inflation, interest rate and output under discretion. In this efficient case, the steady state inflation and gross interest rate are zero and $1/\beta$ respectively. However, in the case of some positive overall economic distortion, it is difficult to compute the steady state values because it is related to unknown optimal policy functions. We need to solve for those functions before finding the steady state values.

To illustrate the impact of nonlinearity, we compute the steady state inflation and interest rate in the LQ and MF models. The inflation here is the same the definition of inflationary
bias, which is defined as log difference between the steady state inflation with distortion and the efficient steady state counterpart, which is zero. The results are presented in Figure 1 and similar to the results from Anderson, Kim and Yun (2010). This is a clear evidence that there is an impact of the nonlinearity on the steady state values of inflation, interest rate when the overall economic distortion is large.

More importantly, Figure 1 shows the steady state interest rates as increasing functions of the size of overall economic distortion. When the size of the overall economic distortion increases, the gap between the two interest rates increases exponentially. In other words, the LQ model can not generate accurate approximation for the actual steady state interest rate as in the MF model when a large overall economic distortion presents. In the benchmark calibration as discussed above, we choose the overall economic distortion ($\Phi$) conservatively to be 0.091. Even with this conservative value, Figure 1 shows that the LQ steady state interest rate is smaller than the actual value by 0.6 percentage point.
As we know, the steady state of the interest rate is used to define the bound in the LQ model as following:

\[
i_t = (\log R_t - \log \overline{R}) \times 100 \times 4
\]
\[
i_t \geq -\log \overline{R} \times 100 \times 4
\]

Using the steady state interest, one can compute the bound for the log deviation of interest rate for the LQ model and the MF model as the function of the size of overall economic distortion as presented in Figure 2.\(^2\) When the size of overall economic distortion is large, the bound in the LQ model is seriously inaccurate compared to the actual bound in the MF model. Surprisingly, it is not difficulty to prove that, under commitment, the bounds in the two models are the same and equal to \(-2.8\%\) (or \(-\log(\beta) \times 100 \times 4\)) regardless the size of overall economic distortion.\(^3\)

\(^2\)We converted the results in the MF model to log deviation from steady state. The conversion allows us to compare the results between the two models.

\(^3\)See Ngo (2011) or Schmitt-Grohe and Uribe (2010) for the proof.
Being greater than the actual one, the bound in the LQ model is more restrictive and can be reached more likely. Therefore, given the same uncertainty of preference shock causing the bounds to bind in both models, the LQ model generates more sizeable output loss than the MF model. We will see this more clearly in the next section.

4.3 Optimal output, inflation and interest rate policy under preference shock

In order to compare the results of the two models, we compute annualized log deviation of gross interest rate from the steady state value ($i_t$), log deviation of gross inflation from steady state value ($\pi_t$), log deviation of the gross natural real rate from its steady state value ($r^n_t$) and output gap ($x_t$) in the MF model as below:
\[ i_t = (\log R_t - \log \bar{R}) \times 100 \times 4 \]

\[ x_t = \left( \hat{Y}_t - \hat{Y}_t^f \right) \times 100 \times 4 = \left( \left( \log Y_t - \log \bar{Y} \right) - \left( \log Y_t^f - \log \bar{Y}^f \right) \right) \times 100 \times 4 \]

\[ \pi_t = (\log \Pi_t - \log \bar{\Pi}) \times 100 \times 4 \]

\[ r^n_t = (\log \bar{\beta} - \log \beta_t) \times 100 \times 4 \]

where \( R, Y, Y^f, \bar{\Pi} \) are steady state interest rate, sticky-price output, flexible-price output and inflation respectively, \( r^n_t \) is the log deviation of natural real rate from it steady state value due to a change in preference. The natural real rate is defined as the real interest rate needed to attain the flexible-price output in case of no ZLB. When there is a positive preference shock, households tend to save more and consume less, resulting in a smaller output. To prevent the households from saving too much and to restore consumption and output, we need a lower nature real rate. If there would not be zero lower bound on nominal interest rate, the actual real interest rate could be adjusted to be the same with the natural real rate, and the central bank can obtain the flexible-price output. However, in case where the nominal rate is bounded from below, a big positive preference shock causes the ZLB to bind. As the result, the actual real rate will be larger than the natural real rate because the nominal interest rate can not be negative, resulting in a sizable decrease in output compared to the flexible-price output.

Note that under discretion the steady state value associated with zero inflation exists in the LQ model when the overall economic distortion (\( \Phi \)) is zero. When \( \Phi > 0 \), we impute the
steady state values for the LQ model by solving the model using the global method.

For comparison, we experiment with different levels of overall economic distortion and relative price dispersion.

### 4.3.1 Zero overall economic distortion

Zero overall economic distortion ($\Phi = 0$) means that there exits labor income subsidy designed to fully offset the monopolistic distortion. The economy could attain the first best output level if prices were adjusted flexibly. Figure 3 shows the policy function at each value of preference and at steady state technology shock and relative price dispersion.

![Figure 3. Optimal policy without overall economic distortion](image)

Note on figure 3: $r^n$ is a change in natural real rate from its steady state, or $(\log \beta - \log \beta_s) \times 100 \times 4$; $i$ and $\pi$ are log deviation of gross interest rate and inflation from their steady state values; $x$ is output gap; $r^n, i, \pi, x$ are annualized; technology ($A$) and relative price dispersion ($\Delta$) are kept at steady state values of 1; the overall economic distortion ($\Phi$) is 0.
In the paper we only plot the optimal policy with respect to a positive preference shock, or a decline in the natural real rate. The reason is that a negative preference shock does not allow the nominal interest rate to reach the lower bound. A negative preference shock leads to an increase in the natural real rate. The central bank can increase the nominal interest rate without limit to equalize the natural real rate and the expected real rate given any expectation.

Without lower bound and initial price dispersion, the central bank can obtain target efficient output and price stabilization immediately in both LQ and MF models under any positive shock to preference (or decline in the natural real rate), as shown by the solid lines. This occurs because the central bank can reduce the nominal interest rate, and as a result the real interest rate, one for one with the decline in natural real rate (or positive change in preference).

When the nominal interest rate is bounded from below, a big preference shock causes the natural real rate to decrease sharply. Given any expected inflation, even though the central bank lower the nominal interest rate to zero, the real interest rate is still larger than the natural real rate. The zero lower bound is reached and output gap falls. In the framework of monopoly, the fall in output results in a decline in the optimal price and price level, and as a result the inflation rate. Some prices are adjusted downward right away, some will be adjusted in future due to sticky price assumption. The optimal policy are presented by the dotted and dashed lines in Figure 3.

From Figure 3, when the economy is at a state close to the lower bound, the central bank is more aggressive in cutting the policy rate and, as a result, the zero lower bounds are reached earlier that they would otherwise. The slope of the interest rate policy function is steeper than in the no bound case. The slope in fact depends on the risk aversion parameter. A larger risk averse leads to more aggressive cut because the central bank as well as households dislike the zero lower bound and output losses. This result is well-known in the literature,
like Adam and Billi (2007a) and Nakov (2008). The aggressiveness in cutting interest rate and the early binding comes from the complementarity between private expectation and the interest rate policy responding to the expectation and preference shocks. When a big enough positive preference shock (or negative shock to the natural real rate) occurs, the economy is put near the lower bound and the central bank has to cut the nominal interest rate. In this situation, it is highly likely that another bad shock may occur and put the economy into a liquidity trap with deflation and sizable output loss. As a result, the expected deflation occurs and puts more downward pressure on extra interest rate cut. This reinforcement causes the lower bound to be reached earlier than in the case without bound.

Adam and Billi (2007a) ask whether a full nonlinear model might cause their policy function to be different in the framework of zero lower bound. By solving a MF model, we is able to show that full nonlinearity generates very small difference between the two models if there is neither overall economic distortion nor relative price dispersion. The optimal policy are very similar in the two models. The finding is robust to parameters and nature of shocks. The reason is that, when $\Phi = 0$, there is no difference in the lower bound in both models regardless of parameters and nature of shocks, as in Figure 2. Therefore, without overall economic distortion and price dispersion, the conventional linear-quadratic approach works well.

To investigate the role of price dispersion under the lower bound on nominal interest rate, we plot the optimal policy function using different levels of relative price dispersion as in Figure 4. The central bank can not obtain the target output gap and inflation in any state of preference (or natural real rate). From equation (21), when relative price dispersion increases, aggregate output and inflation fall. More interesting, the monetary authority keeps the optimal interest rate policies the same regardless relative price dispersion. The results can not be found in the conventional LQ model since there is no endogenous state such as relative price dispersion in the LQ model. Note that most of additional output losses
is due to a higher relative price dispersion or more inefficiency of the economy. Yun (2005) considers the relative price dispersion as an endogenous technology.

![Figure 4. Optimal policy with different relative price dispersion levels](image)

Note on figure 4: $r^n$ is a change in natural real rate from its steady state, or $(\log \beta_3 - \log \beta_1) \times 100 \times 4$; $i$ and $\pi$ are log deviation of gross interest rate and inflation from their steady state values; $x$ is output gap; $r^n, i, \pi, x$ are annualized; technology is kept at steady state values of 1; the overall economic distortion ($\Phi$) is 0.

Figure 5 shows the impulse responses due to a shock to initial relative price dispersion. We can imagine a situation where at time 0 the economy is at steady state with 2.1% inflation and 1% relative price dispersion annually. This case corresponds to an economy with only monopolistic distortion and no distortionary labor income tax/subsidy (or the overall economic distortion is 0.091 the case we will consider later). Then at time 0, the government declares suddenly that it will subsidy employment to fully offset monopolistic markup from next period. In other words, the overall economic distortion is reduced from
0.091 to 0. The change in tax regime causes the new steady state values to change too. The new steady state relative price dispersion is smaller than the one the economy inherits from the last tax regime. Therefore, the relative price dispersion keeps falling over time and converts to the new steady state value. Inflation target responds to changes in the level of relative price dispersion. The dynamics of optimal inflation, output gap and interest rate are presented by the solid lines in Figure 5. The results are similar to Yun (2005). Also, Yun (2005) states that deflation takes place under the optimal monetary policy during transition periods if the measure of relative price dispersion takes an initial value greater than one. The statement is right in this case. However, we prove later that it is not always true when a large overall economic distortion exists.

![Figure 5. Impulse response with a shock to relative price dispersion](image)

**Note on figure 5:** $i$ and $\pi$ are log deviation of gross interest rate and inflation from their steady state values; $x$ is output gap; $i, \pi, x$ are annualized; technology ($A$) and time discount ($\beta$) are kept at steady state values of 1 and 0.993 respectively; the overall economic distortion
is 0.

4.3.2 Positive overall economic distortion

In this section we investigate the economy with only monopolistic distortion and no dis-
tortionary labor income tax/subsidy. In other words, the overall economic distortion \((\Phi)\) is 0.091. As we discuss above, this value is bigger than zero but still quite small compared to the empirical estimates which range from 20 – 60%. Figure 6 shows the optimal policy where the relative price dispersion is chosen to be at the steady state value of 1.0026 and no technology shock.

In general, the results from the MF model are quite different from those from the LQ model. The output gap and inflation declines in the LQ model are much larger than those in the MF model given any state where the lower bounds are binding in both models. For example, when the time discount is 5\% higher than the steady state value (or the natural real rate is 5\% smaller than the its steady state), the bounds are reached in both models.\(^4\) The output gap falls by 1\% and inflation declines by 0.2\% in the LQ model, compared to 0.25\% and 0.05\% in the MF model.

Figure 6 also shows that when the economy is near the zero lower bound of the LQ model, the interest rate is cut more aggressively in the LQ model. Also, the nominal interest rate in the LQ model reaches the lower bound earlier than in the MF model. The results come from the fact that with a larger overall economic distortion the bound in the LQ model is much more restrictive than the actual bound in the MF model. Due to the reinforcement between the private expectation and the interest rate policy responding to it, the bound in the LQ model is reached earlier than it would otherwise as in the MF model, causing larger output losses and inflation decrease compared to those in the MF model.

\(^{4}\)The shock is slightly above three standard deviations.
Figure 6. Optimal policy with overall economic distortion

Note on figure 6: \( r^n \) is a change in natural real rate from its steady state, or \( (\log \beta - \log \beta_t) \times 100 \times 4 \); \( i \) and \( \pi \) are log deviation of gross interest rate and inflation from their steady state values; \( x \) is output gap; \( r^n, i, \pi, x \) are annualized; technology (A) and relative price dispersion (\( \Delta \)) are kept at steady state values of 1 and 1.0026 respectively; the overall economic distortion (\( \Phi \)) is 0.091.

It is interesting to compare the results in this case with those from the case of zero overall economic distortion for only true MF model. With a positive overall economic distortion, the central bank no longer targets price stabilization or zero inflation. In steady state, it targets a positive level of inflation, about 2% in this case. As Krugman (1998) points out, when the central bank targets positive inflation, the zero nominal interest rate and liquidity trap are less likely to occur. In the first case of price level target, a shock of 1.5 standard deviations (or 2.4% annually) can cause the nominal interest rate to reach zero. In the second case of 2% inflation target, in order for the economy to hit the zero bound, a much more severe
shock is needed, about 4.8% or 3 standard deviations.

Unlike the case of zero overall economic distortion, zero lower bound and liquidity trap in this case are not necessarily associated with deflation. For example, at the preference of 5.2% higher than the steady state value (or at the natural real rate 5.2% lower than its steady state rate), output loss is 0.75% that is associated with −0.1% decline in gross inflation from its steady state value. With the steady state inflation of 2.1%, inflation rate associated with this shock is around 2%.

More importantly, when the economy is near the zero lower bound, the central bank is more aggressive in cutting nominal interest rate if it targets price stabilization than if it pursues a positive inflation target. When the central bank pursues a positive inflation targeting policy, it requires an extreme shock to drive the economy to be near the zero bound. Because the preference is mean reverting process, it is not likely that there is another shock to drive the economy to the liquidity trap that generates output loss and disinflation. As the result, the expected disinflation is extremely small and it is not necessary for the central bank to cut the nominal interest rate further. This might explain why the Fed was not very aggressive in cutting the target federal funds rates in the period right after the 2007 financial crisis. Also, the implication for policy is that when the economy just exists the liquidity trap, there is not necessary for the central bank to commit to extended period of zero nominal interest rate.

In contrast, when the central bank objective is to stabilize price level, like in the case of zero overall economic distortion. When the economy is near the zero bound, it is likely that there is another bad shock to put the economy in the liquidity trap that generates sizable output losses and deflation. The expected deflation is large and puts more downward pressure on cutting the nominal interest rate. The reinforcement between the private expectation and optimal policy causes a aggressive cut of nominal interest rate and the zero lower bound hits earlier as we see in Figure 3.
To extend the results from Yun (2005), we investigate the role of relative price dispersion and report the results in Figure 7. When the price dispersion is greater than the steady state price dispersion, the output gap falls further, which is associated with much more smaller inflation compared to the target value. When the relative price dispersion is smaller than the steady state value, the output is higher. Again, the relative price dispersion plays a role of endogenous technology in the aggregate production function. The higher the price dispersion, the lower the technology and lower output. Therefore, the additional output loss or gain is due to the decrease or increase in endogenous technology. Also, we see in Figure 7 that the central bank keeps the optimal policy rate unchanged regardless relative price dispersion. This policy contrasts the central bank’s response to an exogenous technology shock, where it adjusts interest rate policy one for one according to the New Keynesian IS equation.

Figure 7. Optimal policy with different relative price levels

Note on figure 7: \( r^n \) is a change in natural real rate from its steady state, or \((\log \beta - \log \beta_0)\) *
100 * 4; $i$ and $\pi$ are log deviation of gross interest rate and inflation from their steady state values; $x$ is output gap; $\pi^n, i, \pi, x$ are annualized; technology is kept at steady state values of 1; the overall economic distortion ($\Phi$) is 0.091.

4.3.3 Aggregation dynamics under a change in tax regime

In this section we show the dynamics of the economy under a change in labor income tax/subsidy regime. As we show below, a change in the tax regime is similar to a shock to initial level of relative price dispersion. We analyze the dynamics of aggregate variables under two scenarios.

In the first scenario, at time 0 the economy is at steady state with zero inflation and the initial relative price dispersion of 1. This steady state is associated with zero overall economic distortion or labor income subsidy. Then the government declares that the employment subsidy will be abolished from time 1. The new tax regime corresponds with the overall economic distortion of 0.091 and new steady state relative price dispersion of 1.0026 (approximately 1% annually). Therefore, the initial relative price of 1 is smaller than the new steady state value of 1.0026. The dynamics of price dispersion, output gap, inflation and interest rate under optimal monetary policy are presented as the dotted lines in Figure 8. In this case the accelerating inflation takes place during the transition period. These results can not be produced in Yun (2005).

In the second scenario, the economy is initially at steady state with a positive inflation of 2.5% and relative price dispersion of 1.0035 (approximately 1.4% annually). This steady state corresponds to 1% of labor income tax and the overall economic distortion of 0.1. Abruptly, the government reduces the labor income tax from 1% to 0% in the next period, causing the overall economic distortion to decline from 0.1 to 0.091. The initial relative price dispersion of 1.0035 is greater than the new steady state of 1.0026. The dynamics of aggregate variables are presented by the solid lines in Figure 8. Disinflation takes place during the transition
period while the relative price dispersion declines over time to new steady state value of 1.0026 (or 1% annually). On impact, the output is lower than the steady state output because the initial relative price dispersion is higher than the new steady state one.

![Figure 8. Impulse response due to shocks to relative price dispersion](image)

Note on figure 8: $i$ and $\pi$ are log deviation of gross interest rate and inflation from their steady state values; $x$ is output gap; $i, \pi, x$ are annualized; technology ($A$) and time discount ($\beta$) are kept at steady state values of 1 and 0.993 respectively; the overall economic distortion ($\Phi$) is 0.091.

5 Conclusion

We have shown that by solving the fully non-linear MF model, we are able to compute optimal policy, steady state values as well as dynamics of output, interest rate and inflation in the presence of zero lower bound on nominal interest rate and positive overall economic
distortion that we define as the combination of monopolistic distortion and labor income tax/subsidy. We also use the conventional LQ approach to simplify the MF model, to solve this simplified LQ model and to compare the results from both models.

We find that the MF model generates more accurate results when the overall economic distortion is large enough. Specifically, the difference of steady state interest rates between the MF and LQ model increases in the size of overall economic distortion, causing the lower bound of interest rate deviation in the LQ model to be biased upward to the actual bound in the MF model. The bias is serious when the size of the overall economic distortion becomes large.

The bound of log deviation of interest rate in the LQ model is greater than the actual bound as in the MF model when the overall economic distortion is large. With the more restrictive bound, the interest rate deviation reaches the lower bound earlier in the LQ model than in the MF model, causing larger output loss and inflation decrease compared to those coming from the MF model whenever the lower bounds are binding in the two models. We find that, when the economy is near the lower bound, the interest rate cut in the LQ model is more aggressive than it would otherwise due to a more restrictive bound and the reinforcement between private expectation and optimal interest rate policy responding to it.

The bounds in the MF and LQ models under commitment are the same regardless the size of overall economic distortion. Therefore, the LQ approach overstates the relative losses of discretion to commitment policy in the presence of a large overall economic distortion, mainly due to its inaccuracy under discretion.

We also find that, when the economy is near the lower bound, a central bank who pursues a positive inflation target will cut the interest rate less aggressively than if his objective is to stabilize price level because the likelihood of being put in a liquidity trap in the case of inflation targeting is much smaller than in the case of price stabilization.

By solving the MF model, we are also able to show that the output loss is much more
sizable in the presence of initial relative price dispersion, which is greater than the steady state value. The additional decline in output gap is mostly caused by a higher inefficiency of the economy due to a higher level of relative price dispersion than the steady state value. The monetary authority does not change the optimal interest rate policy when there is only change in relative price dispersion.

References


6 Appendix

6.1 Discretionary optimal policy under zero lower bound

\[ V(\Delta_{t-1}, A_t, \beta_t) = \max_{R_t, C_t, N_t, S_t, F_t, \Pi_t, \Delta_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\eta}}{1+\eta} + \beta_t E_t V(\Delta_t, A_{t+1}, \beta_{t+1}) \right\} \]

Subject to

\[ \frac{C_t^{-\gamma}}{R_t} = \beta_t E_t \left[ \frac{C_{t+1}^{-\gamma}}{\Pi_{t+1}} \right] = \beta_t E_t \left[ Z_1 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \]

\[ F_t - C_t^{-\gamma+1} = \theta \beta_t E_t \left[ \Pi_{t+1}^{\frac{\gamma-1}{\gamma+1}} F_{t+1} \right] = \theta \beta_t E_t \left[ Z_2 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \]

\[ S_t - \frac{\chi C_t N_t^\eta}{(1-\Phi) A_t} = \theta \beta_t E_t \left[ \Pi_{t+1}^{\frac{\gamma-1}{\gamma+1}} S_{t+1} \right] = \theta \beta_t E_t \left[ Z_3 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \]

\[ \Delta_t = (1-\theta) \left( \frac{1-\theta \Pi_t^{\frac{\gamma-1}{\gamma+1}}}{1-\theta} \right) + \theta \Pi_t^\gamma \Delta_{t-1} \]

\[ S_t = F_t \left( \frac{1-\theta \Pi_t^{\frac{\gamma-1}{\gamma+1}}}{1-\theta} \right) + \theta \Pi_t^{\gamma} \Delta_{t-1} \]

\[ C_t = N_t A_t (\Delta_t)^{-1} \]

\[ R_t = 1 + r_t \geq 1 \]

\[ \ln (\beta_t) = (1-\rho) \ln (\beta) + \rho \ln (\beta_{t-1}) + \varepsilon_t \]

\[ \ln A_t = \rho A \ln A_{t-1} + \varepsilon_{A,t} \]

We rewrite the problem in the Lagrangian form as follow:

\[ L = \frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\eta}}{1+\eta} + \beta_t E_t V(\Delta_t, A_{t+1}, \beta_{t+1}) \]

\[ -\lambda_1 \left[ \frac{C_t^{-\gamma}}{R_t} - \beta_t E_t \left[ Z_1 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \right] \]

\[ -\lambda_2 \left[ F_t - C_t^{-\gamma+1} - \theta \beta_t E_t \left[ Z_2 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \right] \]

\[ -\lambda_3 \left[ S_t - \frac{\chi C_t N_t^\eta}{(1-\Phi) A_t} - \theta \beta_t E_t \left[ Z_3 (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \right] \]
The first order conditions:

\[ R_t : \lambda_{1t} C_t^{-\gamma} R_t^{-2} \leq 0 \text{ with equality if } R_t > 1 \]

\[ C_t : 0 = C_t^{-\gamma} + \lambda_{1t} \gamma C_t^{-\gamma - 1} / R_t + (1 - \gamma) \lambda_{2t} C_t^{-\gamma} + \lambda_{3t} \frac{\chi N_t^{\eta}}{(1 - \Phi) A_t} - \lambda_{6t} \]

\[ N_t : 0 = -\chi N_t^{\eta} + \lambda_{3t} \frac{\chi \eta C_t N_t^{\eta - 1}}{(1 - \Phi) A_t} + \lambda_{6t} A_t \Delta_t^{-1}, \]

\[ F_t : 0 = -\lambda_{2t} + \lambda_{5t} \left( \frac{1 - \theta \Pi_t^{\varepsilon - 1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \]

\[ S_t : 0 = -\lambda_{3t} - \lambda_{5t} \]

\[ \Pi_t : 0 = -\lambda_{4t} \varepsilon \left( \frac{1 - \theta \Pi_t^{\varepsilon - 1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \theta \Pi_t^{\varepsilon - 2} + \lambda_{4t} \varepsilon \theta \Pi_t^{\varepsilon - 1} \Delta_t^{-1} \]

\[ + \lambda_{5t} F_t \left( \frac{1 - \theta \Pi_t^{\varepsilon - 1}}{1 - \theta} \right)^{\frac{1}{\varepsilon}} \theta \Pi_t^{\varepsilon - 2} \]

\[ \Delta_t : 0 = \beta_t E_t V_{\Delta} (\Delta_t, A_{t+1}, \beta_{t+1}) + \lambda_{1t} \beta_t E_t \left[ Z_{1\Delta} (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \]

\[ + \lambda_{2t} \beta_t E_t \left[ Z_{2\Delta} (\Delta_t, A_{t+1}, \beta_{t+1}) \right] + \lambda_{3t} \beta_t E_t \left[ Z_{3\Delta} (\Delta_t, A_{t+1}, \beta_{t+1}) \right] \]

\[- \lambda_{4t} - \lambda_{6t} N_t A_t \Delta_t^{-2} \]

and the envelope theorem:

\[ V_{\Delta}(\Delta_{t-1}, A_t, \beta_t) = \lambda_{4t} \theta \Pi_t^{\varepsilon} \]

where \( Z_{i\Delta} \) denotes partial derivative of \( Z_i \) with respect to \( \Delta \). Simplifying and combining with equilibrium conditions, we obtain 13 equations with 13 variables \((R, C, N, S, F, \Pi, \Delta, \lambda_1, ..., \lambda_6)'\)

(1) \( R_t : 0 = \max(\lambda_{1t} C_t^{-\gamma} R_t^{-2}, 1 - R_t) \)

(2) \( C_t : 0 = C_t^{-\gamma} + \lambda_{1t} \gamma C_t^{-\gamma - 1} / R_t + (1 - \gamma) \lambda_{2t} C_t^{-\gamma} + \lambda_{3t} \frac{\chi N_t^{\eta}}{(1 - \Phi) A_t} - \lambda_{6t} \)
where

\[ Z_1 (\Delta_t, A_{t+1}, \beta_{t+1}) = \frac{C_i^{\gamma}}{\Pi_{t+1}} \]

\[ Z_2 (\Delta_t, A_{t+1}, \beta_{t+1}) = \frac{\Pi_{t+1}^{\gamma}}{F_{t+1}} \]

\[ Z_3 (\Delta_t, A_{t+1}, \beta_{t+1}) = \frac{\Pi_{t+1}^{\gamma}}{S_{t+1}} \]

\[ Z_4 (\Delta_t, A_{t+1}, \beta_{t+1}) = \frac{\Pi_{t+1}^{\gamma}}{A_{t+1}} \]

The solution of the above nonlinear system is the function of the state, \( s_t = (\Delta_{t-1}, A_t, \beta_t) \), where \( \Delta_{t-1} \) is the endogenous and \( A_t, \beta_t \) are the exogenous states with the following law of


\[ \Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^e - 1}{1 - \theta} \right)^{\frac{\varepsilon}{\theta}} + \theta \Pi_t^e \Delta_{t-1} \]

\[ \ln (A_{t+1}) = \rho_A \ln (A_t) + \varepsilon_{A_{t+1}} \]

\[ \ln (\beta_{t+1}) = (1 - \rho_\beta) \ln (\beta_t) + \rho_\beta \ln (\beta_t) + \varepsilon_{\beta_{t+1}} \]

### 6.2 Solution method

We rewrite the above 13 functional equations with 13 unknown policy functions in a more compact form:

\[ f(s, X(s), E[Z(X(s'))], E[Z_{\Delta}(X(s'))]) = 0 \]

Here \( f : R^{3+13+4+4} \rightarrow R^{13} \) is the equilibrium relationship.

where

\( s = (\Delta, A, \beta) \) is the current state of the economy

\( X(s) = (R(s), C(s), N(s), S(s), F(s), \Pi(s), \Delta(s), \lambda_1(s), ..., \lambda_6(s))' \) is the policy function we need to solve, \( X : R^3 \rightarrow R^{13} \).

\( s' \) is next period state that evolves according to the following motion equation:

\[ s' = g(s, X(s), \varepsilon) = \begin{bmatrix} \Delta' = (1 - \theta) \left( \frac{1 - \theta \Pi(s)^e - 1}{1 - \theta} \right)^{\frac{\varepsilon}{\theta}} + \theta \Pi(s)^e \Delta \\ A' = A^{\rho_A} \exp(\varepsilon_A) \\ \beta' = \beta^{(1 - \rho_\beta)} \beta^{\rho_\beta} \exp(\varepsilon_\beta) \end{bmatrix} \]

\[ \varepsilon = \begin{bmatrix} \varepsilon_A \\ \varepsilon_\beta \end{bmatrix} \] is the 2-vector of i.i.d technology and preference shocks.

\[ Z(X(s')) = \begin{pmatrix} Z_1(X(s')) = \frac{C(s')^{\gamma}}{\Pi(s')} \\ Z_2(X(s')) = \Pi(s')^{\varepsilon - 1} F(s') \\ Z_3(X(s')) = \Pi(s')^{\varepsilon} S(s') \\ Z_4(X(s')) = \Pi(s')^{\varepsilon} \lambda_4(s') \\ Z_{\Delta}(X(s')) = \begin{pmatrix} Z_{1\Delta}(X(s')) = -\frac{\varepsilon C(s')^{\gamma - 1} C_{\Delta}(s')}{\Pi(s')} - \frac{C(s')^{\gamma - 1} \Pi_{\Delta}(s')}{\Pi(s')^2} \\ Z_{2\Delta}(X(s')) = (\varepsilon - 1) \Pi(s')^{\varepsilon - 2} \Pi_{\Delta}(s') F(s') + \Pi(s')^{\varepsilon - 1} F_{\Delta}(s') \\ Z_{3\Delta}(X(s')) = \varepsilon \Pi(s')^{\varepsilon - 1} \Pi_{\Delta}(s') S(s') + \Pi(s')^{\varepsilon} S_{\Delta}(s') \\ Z_{4\Delta}(X(s')) = \varepsilon \Pi(s')^{\varepsilon - 1} \Pi_{\Delta}(s') \lambda_4(s') + \Pi(s')^{\varepsilon} \lambda_{4\Delta}(s') \end{pmatrix} \]

We solve the above equilibrium relationship using a projection method called the collocation method. Below is the simplified algorithm:

**Step 1:** Define the space of the approximating functions and collocation nodes

\( S = (S_1, ..., S_N) \), where \( N = N_\Delta \times N_A \times N_\beta \) and \( (N_\Delta \times N_A \times N_\beta) \) is the polynomial degrees in each dimension of the space. In this paper, we use cubic spline method where \( N_\Delta \times N_A \times N_\beta \) are number of collocations nodes along each state dimension.

\[ X(s) = (\phi(s)\theta_R, \phi(s)\theta_C, \phi(s)\theta_N, \phi(s)\theta_F, \phi(s)\theta_S, \phi(s)\theta_\Pi, \phi(s)\theta_\Delta, \phi(s)\theta_{\lambda_1}, ..., \phi(s)\theta_{\lambda_6})' \]

or \( X(s) = \phi(s) \Theta \)

where
• \( \phi(s) \) is a 1 \( \times \) \( N \) matrix of cubic spline basis functions evaluated at state \( s \in S = (S_1, ..., S_N) \).

• \( \Theta = (\theta_R; \theta_C; \theta_N; \theta_F; \theta_S; \theta_\Pi; \theta_\Delta; \theta_\lambda_1; ...; \theta_\lambda_6) \) is \( N \times 13 \) coefficient matrix that we want to approximate.

Step 2: initialize the coefficient matrix \( \Theta^0 \), and set up stopping rules.

Step 3: at each iteration \( j \) we have a corresponding \( \Theta^j \), implement the following substep:

1. At each collocation node \( s_i \), \( s_i \in \{S_1..S_N\} \) : compute \( E[Z(X(s'))], E[Z_\Delta(X(s'))] \):
   - \( E[Z(X(s'))] = \sum_j w_j [Z(g(s, X(s'), e_j))] \)
   - \( E[Z_\Delta(X(s'))] = \sum_j w_j [Z_\Delta(g(s, X(s'), e_j))] = 0 \)

2. Solve for \( X(s_i) \) s.t. \( f(s_i, X(s_i), E[Z(X(s'))], E[Z_\Delta(X(s'))]) = 0 \), we solve this complementarity problem using Newton method with user "analytical" Jacobian matrix \( f_X \).

Step 4: update \( \Theta^{j+1} = \Phi^{-1} \Theta^j \), where \( \Phi = (\phi(s_1), ..., \phi(s_N))' \).

Step 5: check the stopping rules, if not satisfied go to Step 3, otherwise go to Step 6.

Step 6: report results.

There is another way to solve for the policy functions. We can define the residual function, \( r(s, \Theta) = f(s, X(s), E[Z(X(s'))], E[Z_\Delta(X(s'))]) \) and use Newton’s method to solve \( r(s, \Theta) = 0 \) by updating \( \Theta^{j+1} = \Theta^j - \alpha \{r_\Theta(s, \Theta^j)\}^{-1} r(s, \Theta^j) \), where \( r_\Theta(s, \Theta^j) \) is user "analytical" Jacobian matrix. We tried this method but it is extremely slow due to the inverse of a huge \( (N \times 13) \times (N \times 13) \) matrix, for example \( (N_d, N_A, N_\beta) = (11, 15, 35) \) implies \( N = 5775 \).

6.3 Calibration of preference shock based on the LQ approximation literature

From Ngo (2011):

\[
\begin{align*}
x_t &= E_t x_{t+1} - \frac{1}{\gamma} \left[ \hat{R}_t - E_t \hat{\Pi}_{t+1} \right] + g_t \\
g_t &= \frac{1}{\gamma} \left[ -\hat{\beta}_t + \gamma \left( \frac{1+\eta}{\eta+\gamma} \right) (\rho_A - 1) \hat{A}_t \right] = \frac{1}{\gamma} \hat{R}_t^n
\end{align*}
\]

where \( \hat{R}_t^n \) is log deviation of natural interest rate from steady state in the world of no zero lower bound.

Let \( \psi = \gamma \left( \frac{1+\eta}{\eta+\gamma} \right) (\rho_A - 1) \), we have:
\[ g_t = \frac{1}{\gamma} \left[ -\beta_t + \gamma \left( \frac{1 + \eta}{\eta + \gamma} \right) (\rho_A - 1) \hat{A}_t \right] \]
\[ = \frac{1}{\gamma} \left[ - (\rho_\beta \beta_{t-1} + \varepsilon_{\beta,t}) + \psi \left( \rho_A \hat{A}_{t-1} + \varepsilon_{A,t} \right) \right] \]
\[ = \frac{1}{\gamma} \left[ -\rho_\beta \beta_{t-1} + \psi \rho_A \hat{A}_{t-1} - \varepsilon_{\beta,t} + \psi \varepsilon_{A,t} \right] \]

Note that:

\[ g_t = \rho_g g_{t-1} + \varepsilon_{gt} \]

\( \rho_g, \sigma_g \) are known

Therefore, we can calibrate \( \sigma_\beta^2 \) and \( \rho_\beta \) as below:

\[ \sigma_g^2 = \left( \frac{1}{\gamma} \right)^2 \left( \sigma_\beta^2 + \psi^2 \sigma_A^2 \right) \]
\[ \sigma_\beta^2 = \gamma^2 \sigma_g^2 - \psi^2 \sigma_A^2 \]

\[ \rho_g = \frac{Cov(g_t, g_{t-1})}{\sigma_g^2} \]
\[ Cov(g_t, g_{t-1}) = E [g_t g_{t-1}] \]
\[ = E \left[ \frac{1}{\gamma} \left[ -\rho_\beta \beta_{t-1} + \psi \rho_A \hat{A}_{t-1} - \varepsilon_{\beta,t} + \psi \varepsilon_{A,t} \right] \right] \]
\[ = \frac{1}{\gamma^2} \left[ \rho_\beta \sigma_\beta^2 + \psi^2 \rho_A \sigma_A^2 \right] \]
\[ \rho_\beta = \frac{\rho_g \sigma_g^2 \gamma^2 - \psi^2 \rho_A \sigma_A^2}{\sigma_\beta^2} \]

In the literature, i.e. Anderson, Kim and Yun (JEDC 2010):
\[ \rho_A = 0.95 \]
\[ \sigma_A = 0.016 = \frac{0.005}{1 - \rho_A^2} \]

6.3.1 If we use Adam and Billi (JME 2007):

\[ \frac{1}{\gamma} = 6.25 \]
\[ \eta = 0.47 \]
\[ \sigma_g = 0.01524 \]
\( \rho_\beta = 0.8 \)

Then, we can compute \( \psi, \sigma_\beta \) and \( \rho_\beta \):
\[
\psi = -0.0187
\]
\[
\sigma_\beta = 0.0024
\]
\[
\rho_\beta = 0.7977
\]

6.3.2 If we use Nakov (IJCB 2008):

\( \frac{1}{\gamma} = 0.25 \)
\( \eta = 0.47 \)
\( \sigma_\gamma = 0.0372/4 \)
\( \rho_\gamma = 0.65 \)

Then, we can compute \( \psi, \sigma_\beta \) and \( \rho_\beta \):
\[
\psi = -0.08
\]
\[
\sigma_\beta = 0.0342
\]
\[
\rho_\beta = 0.806
\]