Equilibrium Asset Pricing and Portfolio Choice in the Presence of both Liquid and Illiquid Markets*

Preliminary version. Comments welcome

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Abstract

I study equilibrium asset prices and portfolio choice in a model where agents hedge endowment risks using two imperfectly correlated assets. The first one is traded on a liquid exchange, whereas the second one is traded on an illiquid over-the-counter (OTC) market. I show the existence and uniqueness of an equilibrium, and obtain semi-analytic expressions for all the relevant quantities, up to first order in the illiquidity level. The illiquidity level on the decentralized market has no impact on the price of the liquid asset, but it can generate additional trading volume. The dynamics of the decentralized market is driven by the distance of the correlation between the two assets from an explicitly given critical correlation. Namely, this distance defines (i) the trading pattern on the OTC market, (ii) the proportion of meetings on the OTC market that result in a trade, (iii) the sign and the size of the impact of illiquidity on the price of the illiquid asset. In particular, the illiquidity premium is not monotonic in the correlation between the assets. I then let agents choose their search intensities, and show that the endogenous liquidity increases in the distance from the critical correlation.

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1 Introduction

Corporate bonds, single-name credit derivatives, and oil forward contracts are three among many financial contracts that are traded over-the-counter (OTC). In particular, trading them requires to search for a suitable counter-party, and to bargain over the exact terms of the transaction. This search friction naturally leads to sub-optimal risk exposure between the randomly arriving trading rounds. Therefore, investors may find it optimal to hedge their illiquid positions by trading liquid instruments with correlated cash flows.\(^1\)

The possibility to trade liquid securities may have a non-trivial effect on the OTC transactions and the corresponding bargaining outcomes. On the individual level, it improves each investor’s outside option by increasing the set of investment opportunities. However, it also improves the outside options of an investor’s trading counter-parties, leading to equilibrium feed-back effects. The goal of this paper is to study the interaction of liquid and illiquid markets and the corresponding liquidity spillover.

To address this question, I propose a model in which investors can share endowment risks by trading two distinct assets. The first one is traded in a frictionless way on an exchange, whereas the second one is traded on an illiquid OTC market. The illiquidity stems both from the time required to contact a potential trading partner, and from the bargaining that defines the terms of the transaction. The OTC market is modelled following Duffie, Gârleanu, and Pedersen [2007], whereas the modelling of the centralized market is based on a standard capital asset pricing model.

Using an original argument, I show the existence and uniqueness of an equilibrium of the model. Furthermore, I characterize all the equilibrium quantities in a semi-explicit form when the expected time required to contact another investor on the OTC market is small. As in Duffie, Gârleanu, and Pedersen [2007], I refer to this expected time as the illiquidity level.

In equilibrium, the price of the liquid asset is independent of the illiquidity level on the OTC market. The reason is that even if changing the illiquidity level modifies the cross-sectional distribution of the demands for the liquid asset, these deviations cancel out at the aggregate level. Despite leaving the price of the liquid asset unchanged, these hedging demands may, under conditions, generate some additional trading volume.

By contrast, the presence of the liquid market has a more significant impact on trading on the OTC market. I show that the exact nature of this impact is determined by the distance of the correlation between the two risky assets (liquid and illiquid) from an explicitly given critical correlation. This critical correlation corresponds to

\(^1\) In the context of the above-mentioned examples, the hedging instruments could be: for the corporate bond, a share in a bond exchange-traded fund; for the single-name credit derivative, a credit index derivative; for the forward contract, an oil future. All these hedging instruments are traded on exchanges and are highly liquid.
the case when OTC trading brings no utility benefits beyond what can be achieved by trading the liquid asset alone. This correlation is critical in several ways. First, certain equilibrium quantities change discontinuously when the correlation between the two assets crosses the critical level. The cross-sectional distribution of types is an example. Second, the distance between the actual and the critical correlations drives the behaviour of most of the equilibrium quantities. The reason is that, intuitively, this distance is a measure of the risk-sharing benefits brought by OTC trading and is therefore crucial for characterizing the surplus from the OTC transactions. In particular, it determines, which pairs of agents benefit from OTC trading and, as a result, the evolution of the distribution of portfolio holdings across the population. In turn, this impacts the average time it takes to find a suitable counter-party and the outside options of the agents participating in bargaining.

In the second half of the paper, I extend my model by letting the investors endogenously choose, for a cost, the frequency at which they contact each other. Their decisions are also largely determined by the distance from the critical correlation. More specifically, a large distance corresponds to large utility benefits, which justifies an intensive, and costly, search.

The distance to the critical correlation is also a useful concept when it comes to describing the impact of illiquidity on the price of the asset traded OTC. Two channels should be distinguished: the sensitivity of the price to the illiquidity level, and the illiquidity level itself.

Regarding the sensitivity, a large distance to the critical correlation corresponds to a large surplus to be shared in bilateral transactions. This makes those agents on the short side of the market (i.e., those that gain most from the trade) willing to make price concessions. This then translates into a large sensitivity of the bargained prices to the illiquidity level, and into a large price impact of the prevalent illiquidity level. Importantly, there is in general no monotone relationship between the distance to the critical correlation and either the correlation or its absolute value. In particular, a higher correlation between the assets does not necessarily leads to a smaller illiquidity premium.

Turning to the level itself, as already mentioned, a large distance to the critical correlation makes a high, and costly, search intensity on the OTC market sustainable, which leads to a low endogenous illiquidity level, which mitigates the price impact of the first channel.

**Related literature** This works builds on the literature considering the general equilibrium impact of market frictions. These frictions can be transaction costs on centralized markets, as in Lo, Mamaysky, and Wang [2004], but also the search and bargaining friction on OTC ones. This second strand started with Duffie, Gärleanu, and Pedersen [2005], and shares modelling features with job market models such as Diamond [1982].

The model the most closely related to mine is in Duffie, Gärleanu, and Pedersen [2007]. They also study bilateral trading in OTC markets with risk averse agents. In
particular, they investigate equilibrium behavior when the illiquidity level is small. In my model, I extend their analysis by introducing the possibility of trading in a liquid market. I derive conditions for equilibrium existence and show that it is always unique. Furthermore, I also let the illiquidity to be endogenously determined by the agents’ choice of search intensities, as in Duffie et al. [2009], which allows me to study endogenous liquidity.

Some other references model an intermittent, and sometimes costly, access to a centralized markets. They include Lagos and Rocheteau [2009] and Gărleanu [2009], both of which propose tractable models with unrestricted holdings, and Duffie [2010].

My focus, however, is not so much on the price impact of market frictions as on how the availability of correlated and liquid instruments modifies this impact.

This work also relates to the literature considering how markets with different structures may coexist. Examples include Pagano [1989], Rust and Hall [2003], Miao [2006], and Vayanos and Wang [2007]. In all of these references, agents intend to execute one trade and, to a large extent, balance the benefits of a better price against a costly search. The investors in my model do not choose one market, though. Quite differently, they are active on both of them, and my purpose is to understand the spillovers they induce. I should also mention Vayanos [1998] and Huang [2003], who propose models where agents simultaneously invest in assets with different illiquidity levels, but where illiquidity is induced by exogenously specified transaction costs.

Finally, I would like to mention a recent paper, Melin [2011], that also analyses a setting where distinct assets trade with different levels of search friction. Two main elements distinguish his setting from mine. First, Melin [2011] borrows the preferences used, for example, in Lagos and Rocheteau [2007], Lagos and Rocheteau [2009], and Afonso and Lagos [2011]. Namely, utility is directly derived from holding the asset. By contrast, I rely on the standard utility specification used in most equilibrium asset pricing models where agents gain utility by consuming from their wealth. The second main difference is that, as in Lagos and Rocheteau [2007], Lagos and Rocheteau [2009], and Gărleanu [2009], Melin [2011] assumes that all OTC trades are always intermediated by risk neutral dealers, who themselves trade in a centralized, frictionless market. In particular, upon meeting, the trade always goes through. By contrast, I assume (as in Duffie et al. [2007]) that investors can directly contact each other. This makes the trading pattern in the OTC market endogenous because, a-priori, it is not clear which agents will be participating in the trade. This creates an equilibrium feed-back loop: a trading pattern determines the cross-sectional distribution of holdings. The latter determines the agents’ utilities, which in turn determine the trading pattern. In order to deal with this complicated feedback effect, I assume that, as in Duffie et al. [2007], the illiquid asset holdings are restricted.

Another example mentioning risk-averse agents is Vayanos and Weill [2008].

In the context of asset pricing with search friction, the reference using such classical preferences include Duffie et al. [2005], Duffie et al. [2007], Weill [2007], Weill [2008], Vayanos and Weill [2008] and Gărleanu [2009].
My paper also relates to the literature investigating the importance of hedging demand as a determinant of illiquidity corrections. In my model, the distance to the critical correlation is a natural measure of this hedging demand. In the context of bond markets, this influence is discussed or documented by, for example, Duffie [1996], Duffie and Singleton [1997], Krishnamurthy [2002], and Graveline and McBrady [2011].

The outline of the paper is as follows. Section 2 introduces the model, Section 3 analyses the individual decisions of the investors, Section 4 describes the cross-sectional characteristics of the population of investors, Section 5 defines an equilibrium of the model, and shows that there always exists a unique equilibrium, Section 6 characterizes all the equilibrium quantities, conditional on a relatively small illiquidity level, Section 7 endogenizes the search intensity on the decentralized market, and Section 8 concludes.

2 Model definition

I need a model in which agents can trade both an illiquid asset OTC, and a distinct, but related, asset on a perfectly liquid exchange. I chose to extend the setting of Duffie, Gărleanu, and Pedersen [2007] by adding a liquid, centralized market.

2.1 Investment opportunities

There are two correlated aggregate risk factors, described by the Brownian motions

\[(B_{c,t}, B_{d,t})_{t \geq 0},\]

\[\rho_{cd} dt \Rightarrow d\langle B_{d,t}, B_{c,t}\rangle.\]

To each risk corresponds an asset, d or c. Their cumulative dividend payouts are given by

\[dD_{c,t} = m_c dt + \sigma_c dB_{c,t},\]
\[dD_{d,t} = m_d dt + \sigma_d dB_{d,t}.\]  \(1\)

These assets are available in net supplies \(S_c\) and \(S_d\), respectively. The indexes \(c\) and \(d\) refer to the markets on which these assets will be traded. Namely, \(c\) stands for centralized, and \(d\) for decentralized.

There is also a risk-free asset, available in perfectly elastic supply, and paying out dividends at the constant rate \(r > 0\). 4

2.2 Investors

There is a continuum of investors, each of whom receives an endowment driven both by the aggregate risk factors and by two types of idiosyncratic shocks.

4 As is customary in models of over-the-counter markets (see Duffie, Gărleanu, and Pedersen [2005], Duffie, Gărleanu, and Pedersen [2007], Vayanos and Wang [2007], and many others), the interest rate is exogenous.
More specifically, $A$ is the set of agents, and there is a probability measure $\mu$ defined on it. Let us choose one particular agent. To this agent corresponds a further Brownian motion, say

$$(Z_t)_{t \geq 0},$$

that is independent of any other process in the model. The cumulative endowment of this agent is then given by

$$d\eta_t = m_\eta \, dt + \alpha_{d,t} \, dB_{d,t} + \alpha_{c,t} \, dB_{c,t} + \alpha_{\eta,t} \, dZ_t,$$

(2)

where the vector

$$\alpha_t \triangleq (\alpha_{d,t} \quad \alpha_{c,t} \quad \alpha_{\eta,t})$$

(3)

of exposures to the various Brownian risk factors also evolves stochastically.

Now, I may equivalently parametrize $\alpha_t$ or the vector

$$\left(\sigma_{\eta,t} \quad \rho_{d,t} \quad \rho_{c,t}\right)$$

(4)

whose components are the volatility of the endowment and its correlations with the aggregate risk factors $B_d$ and $B_c$.\(^5\) I will work with this second, possibly more intuitive, set of variables.

I choose a volatility $\sigma_\eta$ of the endowment that is constant over time, and let the vector

$$\rho_t \triangleq (\rho_{d,t} \quad \rho_{c,t})^*$$

evolve like a time-homogeneous Markov chain jumping back and forth between two (two-dimensional) values.\(^6\) I denote these values by

$$\left(\rho_{1d} \quad \rho_{1c}\right), \left(\rho_{2d} \quad \rho_{2c}\right) \in \mathbb{R}^2,$$

and the generator of the Markov chain by

$$\begin{pmatrix}
-\lambda_{12} & \lambda_{12} \\
\lambda_{21} & -\lambda_{21}
\end{pmatrix}.$$ (5)

The Markov chains are independent across agents.

There are at least two ways of interpreting the shocks in the endowment correlations. First, one may follow Duffie et al. [2005] in a rather literal interpretation. A shock could then represent (i) a large loss incurred by an individual while investing in assets that are

\(^5\) Indeed, the relation between the quantities in (3) and in (4) is given by

$$\begin{cases}
\left(\frac{1}{\sigma_\eta} \frac{d(\eta,t)}{d\eta}\right) = \sigma_\eta^2 &= \alpha_{id}^2 + \alpha_{ic}^2 + 2\rho_{id}\alpha_{id}\alpha_{ic} \\
\left(\frac{1}{\sigma_\eta} \frac{d(\eta,B_d)}{d\eta}\right) = \rho_{id} &= \frac{1}{\sigma_\eta} (\alpha_{id} + \rho_{ic}\alpha_{ic}) \\
\left(\frac{1}{\sigma_\eta} \frac{d(\eta,B_c)}{d\eta}\right) = \rho_{ic} &= \frac{1}{\sigma_\eta} (\rho_{id}\alpha_{id} + \alpha_{ic})
\end{cases}$$

$$\begin{cases}
\alpha_e &= \frac{\sigma_\eta}{1-\rho_{id}^2}\left(\rho_{id} - \rho_{id}\rho_{ic}\right) \\
\alpha_d &= \frac{\sigma_\eta}{1-\rho_{ic}^2}\left(\rho_{ic} - \rho_{id}\rho_{ic}\right) \\
\alpha_{\eta} &= \sigma_\eta^2 \left(1 - \frac{\rho_{id}^2 - 2\rho_{id}\rho_{ic}\rho_{id} + \rho_{ic}^2}{1-\rho_{ic}^2}\right)
\end{cases}.$$ \(^6\)

In particular, both components of the vector of exposures jump together.
not modelled explicitly; (ii) significant inflows or outflows experienced by a fund; (iii) a
significant movement in the inventory of a dealer; (iv) the underwriting by a bank of a
new bond issue. In all of these cases, an idiosyncratic change in risk exposure calls for
a portfolio rebalancing.

Alternatively, one may borrow the interpretation put forward in Cochrane [2005],
and understand that these shocks are an artefact that induces trading, thus allowing to
study the impact of illiquidity on prices and portfolio decisions, while keeping the model
tractable.

Finally, I introduce two notations. Choosing an agent with current endowment corre-
lations indexed by \( i \in \{1, 2\} \), I denote by

\[
\Sigma_i \Delta t = \frac{1}{d} \langle \begin{pmatrix} \eta_t \\ D_{d,t} \\ D_{c,t} \end{pmatrix} \rangle, (\eta_t D_{d,t} D_{c,t}) = \begin{pmatrix} \sigma_\eta^2 & \rho_{id}\sigma_\eta\sigma_d & \rho_{ic}\sigma_\eta\sigma_c \\ \rho_{id}\sigma_\eta\sigma_d & \sigma_d^2 & \rho_{cd}\sigma_d\sigma_c \\ \rho_{ic}\sigma_\eta\sigma_c & \rho_{cd}\sigma_d\sigma_c & \sigma_c^2 \end{pmatrix}, \tag{6}
\]

the matrix of covariations she faces. I also denote by \( \mu_i \triangleq \frac{\lambda_{ii}}{\lambda_{12} + \lambda_{21}} \) \( \tag{7} \)

the stationary cross-sectional proportion of agents whose current endowment correlations
is of type \( i \). \( \tag{8} \)

Before turning to the trading mechanisms, I make the following observation, which
will allow to simplify the exposition.

**Remark 1.** The shocks in this model are independent and identically distributed, as
can be seen from (1) and (2). Moreover, I will only analyse situations where the cross-
sectional distribution of the relevant individual characteristics is constant over time. \( \tag{9} \)

As a result, there will not be any state-variable in this model and, in particular, the
asset prices will also be constant.

### 2.3 Trading mechanisms

Two of the three assets can be traded with barely any constraint. Specifically, the risk-
less asset is traded without delay, cost, or holding restriction. Each agent can also trade
the liquid asset \( c \) on a centralized market. This market can be accessed without delay,
there are no transaction costs, and the only constraint on the holdings is that they must belong to a given range

$$[-K, K],$$

where $K > 0$ is a fixed, large number.\(^{10}\)

Quite differently, $d$ is traded OTC, meaning that completing a trade requires to search for a potential counterparty, and to negotiate the details of the transaction.

The search process is governed by a “random matching” along the lines of those appearing in Duffie, Gárleanu, and Pedersen [2005] or Duffie, Gárleanu, and Pedersen [2007]. Namely, a given investor contacts one of a set $B \subset A$ of investors at the jump times of an idiosyncratic Poisson process whose intensity is proportional to the measure of $B$. Conditional on a meeting, the specific agent who is met is drawn from a uniform distribution on the set $B$. The constant of proportionality is written $\lambda$ and called the meeting intensity of the model. Summing up, agents from two subsets of agents $B, C \subset A$ meet at the rate

$$2\lambda \mu(B) \mu(C),$$

where the factor 2 recall that both the agents in $B$ and those in $C$ are searching - and finding.\(^{11}\) Also, I refer to the opposite of the meeting intensity $\xi = 1/\lambda$ as to the expected search time, or as to the illiquidity level.

Once two agents have met, they negotiate the exact terms of a possible transaction, and agree on the generalized Nash solution. These exact terms could be both the size and the price. In this model, however, I make the common but restrictive assumption that the illiquid holdings can only take two possible values, namely 0 or $\Delta_\theta > 0$.\(^{12}\) But then, the bargaining regarding the size of a transaction is reduced to accepting to trade or not.

Also, as is traditionally assumed, agents that are indifferent between trading and not trading choose to trade. Finally, I assume that the bargaining power is already determined by the illiquid holdings and the endowment correlations.

I add two comments regarding the modelling of the search for counterparties. First, other specifications of the matching technology exist in the financial literature (see, for example, Weill [2008] or Inderst and Müller [2004]). However, two advantages of the specification above are, first, and as already argued in Weill [2008], that it results from an explicitly specified search process, and that the existence of a random matching is, partly, justified by the discrete time results in Duffie and Sun [2011]. Second, the

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\(^{10}\)The constant $K$ is large enough so that it will not bind in equilibrium. Proposition 10 below indicates when the constraint will be binding, and when not. This way of arguing is adopted, for example, in Gárleanu [2009].

\(^{11}\) As in Footnote 8, this statement is intuitive but non-trivial. More specifically, it assumes a certain Law of Large Numbers. See ?? for the rigorous treatment this issue in discrete time.

\(^{12}\) References that relax this assumption include Lagos and Rocheteau [2007], Lagos and Rocheteau [2009], Gárleanu [2009], and Melin [2011], but they model an intermittent access to a centralized market. The unique reference I am aware of that combines bilateral trades and multiple holdings is Afonso and Lagos [2011].
specification in (8) exhibits increasing returns to scale and, as argued in Gavazza [2011] in the context of real assets, this seems to be an intuitively appealing and empirically important feature when modelling markets with search friction.

My second comment concerns the choice of working with a continuum of agents. This choice, made in Duffie et al. [2005], Duffie et al. [2007], Vayanos and Wang [2007], Weill [2007], and Gărleanu [2009], and many more, is sometimes viewed as problematic. The main criticism is that random matching precludes repeated interactions between market participants, whereas actual OTC market are often dominated by a limited number of participants. In such contexts, repeated strategic interactions may be an important determinant of the functioning of the market. This criticism is valid, but there are at least two arguments in favour of the continuum of agents. First, OTC markets can involve significantly more participants than the limited number of the most active ones. For example, Schultz [2001] analyses the market for corporate bond and reports that, even if close to three quarters or the trades involve one of the top 12 dealers, there are more than 600 dealers appearing in the sample. Second, assuming a continuum of agents reduces the strategic interactions to their minimum, which allows to isolate the impact of the the search friction from, say, concerns of reputation or punishment.

2.4 Preferences

All agents share the same preferences and maximize their expected flow of utility from consumption. Their utility function $U$ has a constant coefficient of absolute risk aversion $\gamma > 0$ (CARA utility), meaning that

$$U : c \mapsto -e^{-\gamma x}.$$  

If I pick a certain agent, write $w_0$ for the sum of her initial holdings in cash and in the liquid asset $c$, $\rho$ for the current type of her endowment correlations, and $\theta$ for her current illiquid holdings, then, her behaviour will be dictated by the optimization

$$V (w, \rho, \theta) \equiv \sup_{(\tilde{c}_u)_{u \geq 0}} \mathbb{E}\left[ \int_0^\infty e^{-mu} U (\tilde{c}_u) \, du \bigg| w_0 = w, \rho_0 = \rho, \theta_t = \theta \right],$$  

where the admissible consumption processes satisfy the following budget and admissibility constraints.

(i) **Budget constraint** For any admissible consumption rate process

$$(\tilde{c}_u)_{u \geq 0}$$

there exists an adapted process

$$(\pi_t)_{t \geq 0}$$

taking values in $[-K, K]$, describing the holdings in $c$, and financing the consumption in the sense that the sum $w_t$ of the holdings in $c$ and in the risk-free asset evolve like

$$dw_t = rw_t \, dt - \tilde{c}_t \, dt + d\eta_t + \theta_t \, dD_{dt} + \pi_t \left( dD_{ct} - rP_c \, dt \right) - d\theta_t P_d,$$

where
• $P_c$ is the price at which $c$ trades, and is taken as given;
• $(\theta_t)_{t \geq 0}$ are the holdings in the illiquid asset. It takes value in $\{0, \Delta \theta\}$ and only jumps when another agent is successfully met;
• $P_d$ is the unit price at which, upon meeting someone, $d$ is exchanged.

(ii) Admissibility of the wealth process
The wealth process is so that, for any $T > 0$,

\[ E \left[ \int_0^T \left( e^{-\rho u e^{-r \gamma W_u}} \right)^2 \right] < \infty, \]

and

\[ \lim_{T \to \infty} e^{-\rho T} E \left[ e^{-r \gamma W_T} \right] = 0. \]

The requirements in (ii) exclude pathological wealth processes, e.g. doubling strategies. I use these regularity assumptions in the verification argument.

I conclude the exposition of my model by imposing the following.

Assumption 2. The endowment correlation shocks, as described by (5), and the supply of the illiquid asset $d$ are so that

\[ \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} \neq S_d \neq \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}}. \]

Also, the various correlation are so that

\[ \rho_{1d} - \rho_{2d} \neq \rho_{cd} (\rho_{1c} - \rho_{2c}), \quad \rho_{1c} \neq \rho_{2c}. \]

This assumption prevents the lengthy treatment of non-generic cases.

3 Individual investment, consumption, and bargaining decisions
I analyse the individual strategies using the relevant Hamilton-Jacobi-Bellman (HJB) equations.\(^{13}\)

Let me consider a certain agent. It is characterized by her wealth $w$, her current endowment correlations $\rho_i$, and her illiquid holdings $\theta$. I group these last two properties together, call them the type of my agent, and denote them by $i\theta$. There are thus four possibilities for this type. Namely,

\[ i\theta \in \{1l, 1h, 2l, 2h\} \overset{\Delta}{=} \mathcal{T}, \]

where $l$ refers to “low illiquid holdings”, meaning $0$, and $h$ to “high” ones, meaning $\Delta \theta$.

\(^{13}\)A verification argument justifying this approach is provided in Appendix C.2
The HJB equation corresponding to this agent is intuitively derived from the observation that, along an optimal path \((\pi^*, \theta^*, c^*, w^*)\), the process
\[
\left(\int_0^t e^{-\rho s} U(c_s^*) \, ds + e^{-\rho t} V(w^*, i_t \theta^*_t)\right)_{t \geq 0} = \left(\mathbb{E} \left[\int_0^\infty e^{-\rho s} U(c_s^*) \, ds \bigg| \mathcal{F}_t\right]\right)_{t \geq 0}
\]
must be a martingale. Equating the expected dynamics of this process to zero then yields \(^{14}\)
\[
\rho V(w, i\theta) = \sup_{\tilde{c}, \tilde{\pi}} U(\tilde{c})
\]
\[
+ \frac{\partial V}{\partial w}(w, i\theta) \left( rw - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi} (m_c - rP_c) \right)
\]
\[
+ \frac{1}{2} \frac{\partial^2 V}{\partial w^2}(w, i\theta) \left(1 \theta \tilde{\pi}\right) \Sigma_i \left(1 \theta \tilde{\pi}\right)^*
\]
\[
+ \lambda \left( V(w, i\theta) - V(w, i\theta) \right)
\]
\[
+ 2 \lambda \mathbb{E} \mu(a) \left[ V(w - (\theta a - \theta) P_d^{(w, i\theta):a}, i\theta a) - V(w, i\theta) \right]^+ \].
\]

The left-hand side of the equation refers to the subjective discounting rate of the agent. On the right-hand side, the first line refer to the consumption over the next instant, the second one to the change in the value function due to the drift component of the wealth process, the third one to the volatility of the wealth process, the fourth one to the endowment correlation shocks, the last one to the utility changes resulting from OTC trading. In this last term,
\[
P_d^{(w, i\theta):a}
\]
denotes the price at which an agent with current characteristics \((w, i\theta)\) would be able to trade with another agent \(a \in A\). A priori, there may be as many such prices as there are pairs of agents.

Now, given the structure of the HJB equation (11), I assume the following.\(^{15}\)

**Assumption 3.** The value functions of the agents are given by
\[
V(w, i\theta) = -\exp\{-\alpha (w + a(i\theta) + \bar{a})\},
\]
where the quantities
\[
\alpha \in \mathbb{R}_{>0}, \ a \in \mathbb{R}^4, \ \bar{a} \in \mathbb{R}
\]
are to be characterized.

\(^{14}\) I write the positive and negative parts of a number according to, for \(x \in \mathbb{R}\),
\[
x = [x]^+ - [x]^-. \]
In particular, the “negative part” of a number is positive.

\(^{15}\) This functional form is standard for problems similar to the one at hand. It is used, among others, by Duffie, Gârleanu, and Pedersen [2007], Gârleanu [2009], and Vayanos and Weill [2008].
In the remaining of this section, I characterize the trading on both the decentralized and the centralized markets, the consumption, and the value functions of the agents.

**Proposition 4 (OTC trading).** On the OTC market, investors trade as follows.

1. The decision to trade or not only depends on the types of the agents meeting. And so does the bargained price.

2. A meeting between one agent of types $1h$ and one of type $2l$ results in the sale of the illiquid asset by the former one to the latter one exactly when

   \[ a(1l) - a(1h) - a(2l) + a(2h) \geq 0. \]

3. A meeting between one agent of types $2h$ and one of type $1l$ results in the sale of the illiquid asset by the former one to the latter one exactly when

   \[ a(1l) - a(1h) - a(2l) + a(2h) \leq 0. \]

4. If the meeting between two agents of types $i\theta$ and $\bar{i}\bar{\theta}$, respectively, results in a successful transaction, then, the bargained price $P_d$ is the unique solution to

   \[
   (1 - \eta_{i\theta}) \left( 1 - e^{\alpha(a(i\theta) - a(\overline{i\theta}) - P_d(i\theta - \overline{i\theta}))} \right) = \eta_{\overline{i\theta}} \left( 1 - e^{\alpha(a(\overline{i\theta}) - a(i\theta) - P_d(i\theta - \overline{i\theta}))} \right).
   \]

I add a few comments. First, as a given agent is only interested in the rest of the population as far as they are potential counterparties, and as bargaining outcomes are independent of the wealth, there is no reason to keep track of the distribution of wealth. This explains why the wealth of an agent is not part of her type.

Also, as shown by the second and third statements of the proposition, there are essentially two possible dynamics on the over-the-counter market. To describe the situation in part 2 of the proposition, I will say that 2-agents have the high valuation of the illiquid asset. For the situation in part 3, I will say that 1-agents have the high valuation.

Finally, an explicit expression for the bargained price $P_d$ is available, if a bit cumbersome. Fixing a type $i\theta \in \mathcal{T}$, I define

\[
\epsilon_{i\theta} : \mathbb{R}^4 \to \mathbb{R}
\]

\[
a \mapsto a(\overline{i\theta}) - a(i\theta) + a(\overline{i\theta}) - a(i\theta),
\]

which is a measure of the benefits an $i\theta$-agent may extract from trading. I also define the function

\[
\chi : (0,1) \times \mathbb{R} \to \mathbb{R}
\]

\[
(\eta, \epsilon) \mapsto \frac{2(1 - \eta)}{1 - 2\eta + \sqrt{(2\eta - 1)^2 + 4\eta(1 - \eta)e^{\alpha \epsilon}}} - 1
\]

and, solving (12) for $P_d$ (or, more simply, for $x = \Delta \exp(\alpha \Delta \theta P_d)$), I can deduce that

\[
e^{\alpha(a(i\theta) - a(\overline{i\theta}) - (\overline{i\theta} - \theta)P_d)} - 1 = \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)),
\]
from which I could deduce \( P_d \) itself immediately. \(^{16}\)

I can characterize the other individual decisions as well.

**Proposition 5** (Trading on the centralized market and consumption). Choosing a type \( i\theta \in T \) and a liquid wealth \( w \), the maximizers in the HJB equation (11) are

\[
c(i\theta) = \frac{1}{\gamma} \left( \alpha (w + a(i\theta) + \bar{a}) - \log \left( \frac{\alpha}{\gamma} \right) \right)
\]

and

\[
\pi(i\theta) = \frac{1}{\sigma^2_c} \left( \frac{1}{\alpha} (m_c - rP_c) - \rho ic_i\sigma_c \gamma - \rho cd \sigma_c \sigma_d \theta \right).
\] (16)

Using the previous results, I can characterize the value functions more narrowly.

**Proposition 6** (Value functions). The value functions

\[
V(w, i\theta) = -\exp \left( -\alpha (w + a(i\theta) + \bar{a}) \right),
\]

are so that the constant related to the liquid wealth is

\[
\alpha = r\gamma.
\]

Further, choosing

\[
\bar{a} = \frac{1}{r\gamma} \left( -1 + \frac{\rho}{r} + \gamma m_\eta + \log(r) \right),
\] (17)

taking the cross-sectional distribution of types

\[
\mu \equiv \{\mu(i\theta)\}_{i\theta \in T}
\]

as given, and defining

\[
k(i\theta) \equiv \Delta \theta m_d + \pi(i\theta) (m_c - rP_c) - \frac{1}{2} r\gamma \left( \begin{array}{cc} 1 & \theta \pi(i\theta) \end{array} \right) \Sigma_i \left( \begin{array}{cc} 1 & \theta \pi(i\theta) \end{array} \right)^*,
\] (18)

the type specific constants “\( a(i\theta) \)” are the unique solution to

\[
\forall i\theta \in T : 0 = r a(i\theta) - k(i\theta) + \frac{\lambda_i}{r\gamma} \left( e^{r\gamma (a(i\theta) - a(\bar{i}\theta))} - 1 \right) - \frac{2\lambda}{r\gamma} \mu(\bar{i}\theta) [\chi (\eta_{i\theta}, \epsilon_{i\theta} (a))]^{-1} \Delta F_{i\theta}(\mu, a).
\] (19)

The quantities \( \epsilon_{i\theta}, \chi, \) and \( \pi(i\theta) \), are defined in (13), (14), and (16), respectively.

\(^{16}\) In the symmetric case where \( \eta_{i\theta} = 1/2 \), certain expressions significantly simplify. For example,

\[
\chi \left( \frac{1}{2}, x \right) = \frac{1}{\sqrt{e^x}} - 1.
\]

I will not, however, assume this symmetry. This allows me to distinguish the exogenously specified bargaining powers from the “endogenous” ones, resulting from the endogenous type distribution. See the discussion following Proposition 16.
I add a few words regarding the \( \kappa(i\theta) \)'s. They balance the expected change in liquid wealth (first line of the right hand side of (18)) against the riskiness of this same change, measured by the volatility of the wealth, scaled by the risk aversion (second line of the right hand side of (18)). One may thus interpret

\[
\kappa(i\bar{\theta}) - \kappa(i\theta)
\]

as a measure of the benefits a change in illiquid holdings would bring to an agent of type \( i\theta \). But then, the quantity

\[
\beta \equiv \Delta (\kappa(1l) - \kappa(1h)) + (\kappa(2h) - \kappa(2l))
\]

should indicate whether a trade between a 1h agent and a 2l one is profitable. In particular, recalling Proposition 4, one may expect \( \beta \) and \( \epsilon_{1h}(a) = \epsilon_{2l}(a) \) to have the same sign. As stated in Proposition 10 below, this is indeed the case.

4 Type distribution

In this section, I characterize the stationary distribution of type \( \mu \). This is necessary because it appears in the HJB equations (19).

The type of a given agent changes either because of a shock in her endowment correlations, or because she traded on the decentralized market. As described in Proposition 4, there are two mutually exclusive trade patterns on the decentralized market, depending on which agents have the higher valuation. Investor endogenously choose which trading pattern they follow but, for this section, I assume the following.

Assumption 7. Agents with endowment correlation type 2 buy the illiquid asset.

Now, recalling both the dynamics of the endowment correlations assumed in Section 2.2 and the linear matching technology assumed in Section 2.3, the type distribution \( \mu \) must satisfy the stationary Kolmogorov Forward Equation

\[
\begin{cases}
0 = \dot{\mu}(1l) = 2\lambda \mu(1h)\mu(2l) - \lambda_{12} \mu(1l) + \lambda_{21} \mu(2l) \\
0 = \dot{\mu}(1h) = -2\lambda \mu(1h)\mu(2l) - \lambda_{12} \mu(1h) + \lambda_{21} \mu(2h) \\
0 = \dot{\mu}(2l) = -2\lambda \mu(1h)\mu(2l) - \lambda_{21} \mu(2l) + \lambda_{12} \mu(1l) \\
0 = \dot{\mu}(2h) = 2\lambda \mu(1h)\mu(2l) - \lambda_{21} \mu(2h) + \lambda_{12} \mu(1h)
\end{cases}
\]

(21)

On the right hand side of each equation, the first term refers to trading, and the other ones to endowment shocks. Also, \( \mu \) being a density, it must satisfy both

\[
\mu(1l) + \mu(1h) + \mu(2l) + \mu(2h) = 1
\]

(22)

The terms referring to trading only involves trades between agents with different endowment correlation. However, according to Proposition 4, agents with the same endowment correlations, but different holdings will also trade. However, as such agents will only swap their types, this has no impact on the distribution of types.
and
\[(\mu(1h), \mu(1l), \mu(2h), \mu(2l)) \in \mathbb{R}^4_{\geq 0}.\]  
(23)

Finally, the OTC market has to clear, meaning that every unit of the illiquid asset \(d\) must be held by someone. This is expressed by imposing the condition
\[\Delta_\theta (\mu(1h) + \mu(2h)) = S_d.\]  
(24)

18 As shown in Duffie, Gârleanu, and Pedersen [2005, Proposition 1], the system defining the stationary distribution is well-behaved. I recall their result for convenience.

**Proposition 8.** There exists a unique stationary type distribution that is reached from any initial distribution.

If Assumption 7 fails, and neglecting the non-generic configurations where \(\epsilon_{1h}(a) = 0\), the model will be in a symmetric situation where all statements are still valid, up to a systematic swap of the indexes 1 and 2.

There are thus only two possible stationary distribution. I denote the one arising under Assumption 7 by \(\mu^{1h\rightarrow 2l}\) and the other one by \(\mu^{2h\rightarrow 1l}\). In equilibrium, the vector \(\beta\) selects the appropriate one, in the sense that
\[\mu(a) = 1_{\{\epsilon_{1h}(a) > 0\}} \mu^{1h\rightarrow 2l} + 1_{\{\epsilon_{2h}(a) > 0\}} \mu^{2h\rightarrow 1l},\]  
(26)
where one recalls that \(\epsilon_{1h} = \epsilon_{2l} = -\epsilon_{1l} = -\epsilon_{2h}\).  
19

## 5 Equilibrium

In this section, I make the individual decisions of the agents consistent with the aggregate quantities on which these decisions are based.

For the centralized market, I use a classical Walrasian equilibrium concept. Namely, as seen in Proposition 5, the only aggregate quantity impacting the liquid holdings is the price \(P_c\) of the liquid asset. I thus impose the consistency between the individual and aggregate quantities by requiring the price \(P_c\) to be so that the centralized market clears.

Turning to the OTC market, the decisions to trade or not and, if so, at which price, is dictated by the parametrization
\[a = \{a(i\theta)\}_{i\theta \in T}\]
of the value functions (see Proposition 4).
Now, on the one hand, aggregating the individual trading decisions on the OTC market yields a certain type distribution, as presented in Section 4.

On the other hand, $a$ also depends on the distribution of types across the population. This is clear both at the intuitive and at technical levels. Intuitively, because the utility of an investor searching for a counterparty on an OTC market should depend on the likelihood of finding such a counterparty. Technically, because $a$ is a solution to the HJB equation (19), an equation in which the distribution $\mu$ appears.

I will thus impose the equilibrium condition that the type distribution assumed when writing the HJB equation (19) and the one generated by the solution to (19) are equal. I formalize this discussion as follows.

**Definition 9.** A stationary equilibrium of the model consists of a price $P_c$, liquid holdings $\{\pi(i\theta)\}_{i\theta \in \mathcal{T}}$, a distribution of types $\{\mu(i\theta)\}_{i\theta \in \mathcal{T}}$, and a parametrization $\{a(i\theta)\}_{i\theta \in \mathcal{T}}$ of the value functions so that

- taking the price $P_c$ as given, an agent of type $i\theta$ optimally invests the amount $\{\pi(i\theta)\}_{i\theta \in \mathcal{T}}$ in the liquid asset $c$;
- the centralized market clears, meaning that
  \[ E^{\mu(i\theta)} [\pi(i\theta)] = S_c; \]
- the value functions, as described by $a$, and type distribution $\mu$ are consistent, in the sense that
  \[ \forall i\theta \in \mathcal{T} : 0 = F_{i\theta} (\mu (\epsilon_{1h} (a)), a), \]
  where the functions $F_{i\theta}$ and $\mu(\cdot)$ are defined in (19) and (26), respectively.

As it turns out, the model is well-behaved.

**Proposition 10.** There exists exactly one equilibrium of the model. More specifically, writing

\[ \tilde{\rho}_c \triangleq \mu_1 \rho_1c + \mu_2 \rho_2c, \]

for the average correlation between endowments and the liquid payouts, the equilibrium price of the liquid asset is

\[ P_c = \frac{m_c}{r} - \frac{\gamma}{\sigma_c} (S_c \sigma_c + \sigma_{\eta} \bar{\rho}_c + \rho_{cd} \sigma_d S_d), \]

and the corresponding holdings are

\[ \pi(i\theta) = S_c + \frac{1}{\sigma_c} (\sigma_{\eta} (\bar{\rho}_c - \rho_{1c}) + \rho_{cd} \sigma_d (S_d - \theta)), \quad i\theta \in \mathcal{T}. \]

Regarding the OTC market, and defining the critical correlation of the model to be

\[ \tilde{\rho}_{cd} \triangleq \frac{\rho_{2d} - \rho_{1d}}{\rho_{2c} - \rho_{1c}}, \]

17
2-agents have a high valuation of the illiquid asset $d$ exactly when

$$(\rho_{2c} - \rho_{1c})(\rho_{cd} - \hat{\rho}_{cd}) > 0.$$ 

This last condition is equivalent to $\beta > 0$, where $\beta$ is defined in (20).

On the technical side, this existence and uniqueness result appears to be new. More specifically, in the setting introduced by Duffie, Gârleanu, and Pedersen [2005], the uniquely defined equilibrium quantities are known in closed-form, but was based on risk neutral agents. This setting was then extended by Duffie, Gârleanu, and Pedersen [2007], Vayanos and Weill [2008], and others, to accommodate risk-averse (CARA) agents. In these cases, the authors showed how, asymptotically, the solutions to these models were formally equivalent to the ones encountered in settings with risk-neutral agents. However, the asymptotic analyses involved either a vanishing risk-aversion, or a vanishing heterogeneity of the agents. My argument does not need these assumptions.  

The equilibrium quantities for the centralized market do not depend on the illiquidity on the over-the-counter market, which may appear counter-intuitive. This is however reminiscent of results obtained in, among others, Gârleanu [2009] and Rostek and Weretka [2011]. In both cases, the price of an illiquid asset is independent of its illiquidity level, where illiquidity is understood as an intermittent access to the market in the former case, and as the price impact of transactions in the latter one. The similarities are both at the intuitive and technical levels.

Intuitively, an improved liquidity on the OTC market makes the allocation of the asset $d$ more efficient. This means that more agents hold the illiquid portfolio they intend to keep over a longer period, and fewer of them resort to the centralized market as a mean to hedge an inappropriate illiquid exposure. Now, among those agents using the liquid asset to hedge their inadequate illiquid exposure, those longing to buy the illiquid asset and those longing to sell it probably hold liquid positions that, in some sense, cancel each other. Now, an increased liquidity on the OTC market reduces both types of hedgers. Indeed, it takes one buyer to get rid of one seller. But suppressing a number of approximately cancelling orders need not have any aggregate impact and, in the setting at hand, it has none.

At a technical level, the optimal holdings in the liquid asset, as described in Proposition 5, are linear both in the correlation $\rho_{ic}$ of the endowment with the payouts of $c$, and in the illiquid holdings $\theta$. Now, the cross-sectional average of the endowment correlations is independent of the illiquidity of the decentralized market. Indeed, the correlations define which holdings agents intend to hold, and this is independent of how much time it will actually take to obtain these holdings. Similarly, the cross-sectional average of the illiquid holdings is a matter a market-clearing, and not of illiquidity either. Taking things together, as the optimal holdings in the illiquid asset are linear in its determinants, and as the cross-sectional averages of these determinants are independent.

20 Gârleanu [2009] sketches an existence argument for an alternative model of illiquid market. However, in his setting, prices are Walrasian and not bargained, which modifies the structure of the equilibrium.
of the liquidity level, so is the aggregate demand, and so is the price of the liquid asset $c$.

In general, the equilibrium quantities for the decentralized market are more delicate to manipulate. Namely, the stationary distribution $\mu$ is known in closed-form, but the expressions are rather cumbersome. Also, I cannot characterize the parametrization $a$ of the value functions in closed-form. Certain conclusions can already be drawn, though.

First, the intuitive explanation presented after Proposition 6 is correct, meaning that 2-agents have the high valuation of the illiquid asset exactly when $\beta > 0$. Now, the concept of critical correlation, defined in Proposition 10 above, helps to make this result more intuitive.

This critical correlation,

$$\hat{\rho}_{cd} = \frac{\rho_{2d} - \rho_{1d}}{\rho_{2c} - \rho_{1c}}$$

balances the relative hedging benefits of the two assets, and it is best understood as a threshold indicating whether one should hold long positions in both the liquid and the illiquid assets, or rather have a long exposure to only one of them. For example, if

$$\rho_{cd} > \hat{\rho}_{cd},$$

the two two assets are so correlated that a long exposure to both assets is unattractive. As a result, those agents who have a high valuation of the illiquid asset are those who find the liquid one relatively unattractive, and this lack of attractiveness should be expressed by relatively low endowment hedging benefits. For example, this would mean that, under (30), 2-agents buy the illiquid asset when

$$\rho_{2c} > \rho_{1c}.$$  

This is indeed the result in Proposition 10.

A similar argument can be formulated when

$$\rho_{cd} < \hat{\rho}_{cd}.$$  

In this case, the moderate correlation between the liquid and illiquid assets means that investors will tend to have a positive exposure to both, or none. In particular, 2 agents will buy the illiquid asset if they also find the liquid one attractive, meaning that

$$\rho_{2c} < \rho_{1c}.$$  

Again, this is indeed the result in Proposition 10.

Finally, as may be intuitively derived from Proposition 10, or from the Walrasian limit described in Appendix A, in the limit case where $\rho_{cd} = \hat{\rho}_{cd}$, all agents are actually indifferent between holding the illiquid asset or not. In other words, the illiquid asset
brings no risk sharing benefits on top of what can be achieved by trading the liquid asset only.

Another, more general, observation is that even if illiquidity distorts the value functions and prices, it does not modify an agent’s decision to hold an asset or not. Even in an illiquid market, the fundamentals of the asset guide this decision. 21

Combining (the proofs of) Propositions 8 and Proposition 10 yields the following

**Corollary 11.** The trading volumes on both markets are decreasing in the illiquidity level $\xi = 1/\lambda$.

Having a monotone relation between the expected search time on the OTC market and the trading volume on this same market is comforting, but not obvious. This is not obvious because increasing the meeting intensity $\lambda$, even if it reduces the average time needed to contact another investor, also makes the allocation of the illiquid asset more efficient across the population and, as a result, reduces the proportion of agents willing to trade at all. As it turns out, however, the first effect dominates. This result is comforting because “illiquidity”, as modelled in search models of asset pricing, and “illiquidity”, as measured in those empirical papers that use trading volumes as a proxy for liquidity, are thus consistent. 22 This result is also consistent with the predictions of Gårleanu [2009].

Second, the monotone increasing relation between the volume on the centralized market and the liquidity level on the decentralized one hints toward empirical pitfalls. Indeed, in my setting, the centralized market is perfectly liquid, and this holds no matter how illiquid the over-the-counter market is. In spite of this, an empirical analysis of this model that would use trading volumes as a proxy for liquidity would probably “document” an illiquidity spillover from the OTC market into the centralized one.

Finally, I compare Proposition 11 with a result in Longstaff [2009]. This paper proposes a model where, over a given period, of the two existing assets, only one can be traded, and concludes that illiquidity induces an increase in the trading volume of the liquid asset. This obviously contradicts my conclusion. The origin of this divergence is in the preferences of the agents.

In my model, investor with CARA preferences intend to keep their holdings fixed essentially all the time. A rebalancing is only triggered by a preference shock, or by a wish to adjust the liquid holdings as the result of a change in the illiquid ones. But then, making bilateral transactions more difficult makes the trade motives even less frequent, and reduces trading volumes.

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21 As may be checked in Appendix A, the residual risk sharing benefits $\beta$ also define which agents buy the asset $d$ in a related, Walrasian setting.

22 With bid-ask spreads and market depth, trading volume is one of the quantities on which most empirical studies of illiquidity are based. For example, Amihud [2002] and Acharya and Pedersen [2005] combine market depth and trading volume, and use returns normalized by volume as a measure of illiquidity.
Quite differently, in Longstaff [2009], investors have a constant relative risk aversion (CRRA) and would like to constantly rebalance their holdings in both assets. Now, if trading in one of them is impeded, this is compensated by trading the other one more intensively, which induces the trading volume increase.

Actually, I can make more detailed statements regarding the trading on the centralized market.

**Corollary 12.** The trading volume on the centralized market can be split between those trades generated by agents hedging an inappropriate illiquid exposure, and those generated by agents who, having completed a trade OTC, resolve their hedge and adopt a permanent portfolio. The transactions of the first kind amount to

\[
V_{hedge} = \frac{1}{2} (\lambda_{12} \mu(1l) + \lambda_{21} \mu(2h)) \frac{\sigma_n}{\sigma_c} |\rho_{1c} - \rho_{2c}|
\]

per unit of time, and is increasing in the illiquidity level, whereas those of the second time amount to

\[
V_{unhedge} = \frac{1}{2} (\lambda_{12} \mu(1h) + \lambda_{21} \mu(2l)) \frac{\sigma_n}{\sigma_c} |\rho_{1c} - \rho_{2c}| + \lambda \mu(1h) \mu(2l) |\rho_{cd}| \frac{\sigma_d}{\sigma_c} \Delta \theta
\]

per unit of time, and are decreasing in the illiquidity level.

This first corollary raises the question of whether the process of building up and liquidating a hedge to the inadequate illiquid exposure actually generates some additional trading volume on the centralized market, or simply makes agents spread the execution of their overall orders over time. This is answered in the next corollary.

**Corollary 13.** Assuming that an investor is hit by a correlation shock, and is able to rebalance her illiquid holding before being hit by another shock, two patterns are possible, and characterized as follows.

- **In each case, the investor trades the liquid asset both at the time of the correlation shock and at the time of the OTC trade.**

  - 1. **Overshooting and reversal:** If the critical and actual correlations between the assets are so that

    \[
    \rho_{cd} \in [\min \{\hat{\rho}_{cd}, 0\}, \max \{\hat{\rho}_{cd}, 0\}],
    \]

    then, the two orders have opposite signs.

  - 2. **Gradual adjustment:** Otherwise, the two orders have the same sign.

This last result illustrates how the search friction on the OTC market can generate some additional, or “excessive”, trading on the centralized market. Intuitively, this arises because agents first attempt to mimicry with the liquid asset the illiquid exposure they intend to achieve, but reverse this trade once the illiquid asset becomes available.
6 Equilibrium with a small illiquidity level

An investor will only bother to enter an illiquid market if she expect to amortize the costly process of building up, and liquidating, a position over a reasonably long holding period. This intuition is formalized in Vayanos and Wang [2007] within a search model of asset pricing, and goes back to Amihud and Mendelson [1986] for a setting with exogenous transaction costs.

This suggests that I may assume the illiquidity level $\xi = 1/\lambda$ to be small relatively to $1/\lambda_{12}$ and $1/\lambda_{21}$, which are the average times (continuously) spent with a high or a low valuation.

Now, a small expected search time should correspond to a more efficient allocation of the illiquid asset, meaning that more of it is held by agents with a high valuation. One may thus guess that the equilibrium is strongly determined by whether or not the supply can meet the demand generated by investors with a high valuation. This is actually the case, and this justifies the following assumption. It will be assumed for the remaining of the derivations, and statements.

**Assumption 14.** Agents with endowment correlation 2 have a high valuation of the illiquid asset. Recalling Proposition 10, I may equivalently assume that $(\rho_{2c} - \rho_{1c})(\rho_{cd} - \hat{\rho}_{cd})$ or $\beta$ is positive.

Second, there is a surplus of the illiquid asset, meaning that

\[ s \triangleq \frac{S_d}{\Delta g} - \mu_{2*} > 0. \]  \hspace{1cm} (32)

In other words, the supply of the illiquid asset is larger than what can be held by the group of agents having a high valuation of the illiquid asset.

I call a shortage of the illiquid asset a situation where, instead of (32), the reverse inequality holds. \(^{23}\)

I can now characterize the equilibrium quantities related to the decentralized market.

**Proposition 15.** The equilibrium density is so that

\[
\begin{pmatrix}
\mu(1l) \\
\mu(1h) \\
\mu(2l) \\
\mu(2h)
\end{pmatrix} = \mu W + \frac{1}{\lambda} \delta_{\mu} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + o \left( \frac{1}{\lambda} \right) 14, \]  \hspace{1cm} (33)

\(^{23}\) Assumption 14 causes no loss of generality, and only allows me to focus on one of four very similar cases. More specifically, given the expressions derived under Assumption 14, obtaining those for a situation where 1-agents have the high valuation only amounts to switching the appropriate indexes. Characterizing the situations with a shortage is barely more complicated. Indeed, a shortage of the illiquid asset may be seen as a surplus of “the ability not to hold the illiquid asset”. This “asset” is is net supply $\Delta g - S_d$. 

22
where the limit value and sensitivity are

\[
\mu^W \triangleq \begin{pmatrix}
\frac{1 - S_d}{\vartheta} \\
\frac{S_d}{\vartheta} \\
0 \\
\mu_2^\star
\end{pmatrix}, \quad \delta_\mu \triangleq \frac{\lambda_{12}}{2} \frac{1 - \frac{S_d}{\vartheta}}{s},
\]

respectively, and where \(1_4 \in \mathbb{R}^4\) is the vector whose components all equal 1.

The asymptotic expressions above can be understood intuitively. The common absolute value of the four components of the first order correction reflects the functioning of the decentralized market. Namely, every time one potential buyer and one potential seller meet, a transaction occur, and they become one satisfied holder of the asset \(d\), and one satisfied non-holder.

Keeping this and the assumed surplus of \(d\) in mind, a very liquid decentralized market will “run out of buyers”. This can be recognized in the asymptotic value \(\mu(2l)\) being zero.

Another consequence of the rigid relation between the various types of agents is that a surplus of the illiquid asset exactly corresponds to having a buyers’ market, meaning that there are more sellers than buyers. Similarly, a shortage of the illiquid asset corresponds to having a sellers’ market. As a result, I will refer to 1h-agents as to those on the long side of the market, and to 2l-ones as to those on the short one.

The expressions for the value functions are as follows.

**Proposition 16.** The equilibrium parametrization of the value functions is so that

\[
\begin{pmatrix}
a(1l) \\
a(1h) \\
a(2l) \\
a(2h)
\end{pmatrix} = a^W + \frac{1}{\lambda} \delta_a \begin{pmatrix}
- (1 + \tau) \frac{\lambda_{12}}{r} \nu_{12} \\
\tau \left( r + \frac{\lambda_{12}}{r} \frac{1}{\nu_{12}} \right) \\
- (1 + \tau) \frac{\lambda_{21}}{r} \nu_{12} \\
\tau \left( r + \frac{\lambda_{21}}{r} \frac{1}{\nu_{12}} \right)
\end{pmatrix} + o \left( \frac{1}{\lambda} \right) 1_4,
\]

where the limit value is

\[
a^W \triangleq -\frac{1}{r} \begin{pmatrix}
\frac{\lambda_{12}}{r} (\nu_{12} - 1) & - \kappa(1l) \\
\frac{\lambda_{12}}{r} (\nu_{12} - 1) & - \kappa(1h) \\
\frac{\lambda_{21}}{r} \left( \frac{1}{\nu_{12}} - 1 \right) & - \kappa(2l) + \beta \\
\frac{\lambda_{21}}{r} \left( \frac{1}{\nu_{12}} - 1 \right) & - \kappa(2h)
\end{pmatrix},
\]

with the quantity

\[
\nu_{12} \triangleq \lim_{\lambda \to \infty} e^{a(1h) - a(2h)} = \lim_{\lambda \to \infty} e^{a(1l) - a(2l)}
\]

\footnote{The rigid relation between the proportions of the four types holds in general, and not only in first order in the illiquidity level. This can be verified in the proof of Proposition 8}
begin the unique solution to

\[ r \log (\nu_{12}) + \frac{\lambda_{12}}{r\gamma} \nu_{12} - \frac{\lambda_{21}}{r\gamma} \frac{1}{\nu_{12}} (\Delta) \equiv \kappa(2h) - \kappa(1h) + \frac{\lambda_{12}}{r\gamma} - \frac{\lambda_{21}}{r\gamma}. \]  

(35)

Also, the “\( \kappa(i\theta) \)”s are defined in (18), the scaling constant in the first order correction is

\[ \delta_a \Delta \frac{\beta}{2\eta_{12}}. \]

and \( \tau \) is defined as

\[ \tau \Delta \frac{\eta_{1h}}{r + \frac{\lambda_{12}}{r\gamma} \nu_{12} + \frac{\lambda_{21}}{r\gamma} \frac{1}{\nu_{12}}}. \]

The expressions in Proposition 16 can also be understood intuitively. Before elaborating on this, I would like to evaluate the validity of the asymptotic approximation. To this aim, in Figure 1, I plot both the asymptotic approximations derived from Proposition 16 (and from Proposition 18 below), and the results of a direct numerical resolution of the problem based on the HJB equations 19. The calibration of the model is described in Table 1.

At least for the chosen calibration, the first order approximation gives a good qualitative description of the equilibrium quantities even for relatively low meeting intensities. Quantitatively, for example, the relative error for the price \( P_d \) is below 1% as soon as the meeting intensity is over 30. For comparison, the calibration in Duffie et al. [2007] uses a meeting intensity of 625.

This, at least loosely, justifies an analysis of the equilibrium based on its asymptotic description. This also justifies the following definitions.

In the last two propositions, the expressions (33) and (34) propose the additive decomposition of the equilibrium quantities \( \mu \) and \( a \) into three parts. I call the first two of them the Walrasian values\(^{25}\) and the illiquidity corrections, respectively.

I will now focus on the illiquidity corrections, which are my main interest.

One should first realize that the asymptotic impact of the endowment correlation shocks, meaning

\[ \nu_{12} = e^{a_W(1l) - a_W(2l)} = e^{a_W(1h) - a_W(2h)}. \]

is independent of the illiquid holdings.

This clarified, we can look at the various expressions with more detail. Why is there an illiquidity correction at all? In a competitive market, as the supply \( S_d \) of the illiquid asset is greater than the demand from agents with high valuation, the equilibrium price should be so that agents with a low valuation are indifferent between holding the asset

\(^{25}\) Indeed, consistently with the results in Duffie, Gărleanu, and Pedersen [2005], the equilibrium converges towards a Walrasian one when the search friction vanishes. The Walrasian equilibrium is explicitly derived in Appendix A.
and not holding it. In the jargon of bargaining, the price of the asset \( d \) reaches the reservation value of a low valuation seller, and the entire trade surplus is hoarded by the buyers. In a search-and-bargaining market, however, the bargaining results in a split of the trade surplus. In particular, high valuation buyers make price concessions, and this translates into the negative sensitivity

\[
\lim_{\xi \to 0} \frac{\partial}{\partial \xi} a(2l) = -\delta_a (1 + \tau) (r + \lambda_{12} \nu_{12}) < 0.
\]

Now, illiquidity makes the decentralized market less favourable not only to agents trying to build up their illiquid position, but also to those agents who may want to do so in the future. Namely 1l-agents know that, at some point, a preference shock will induce them to start searching for and, eventually, buy the illiquid asset. The price concession they will then have to make already depresses their value function. This said, this payment happening further in the future, its impact is more heavily discounted, and the utility change is smaller than for 2l-agents. More specifically,

\[
\lim_{\xi \to 0} \frac{\partial}{\partial \xi} a(1l) > \frac{1}{1 + \frac{r}{\lambda_{12} \nu_{12}}} < 1.
\]

Without surprise, the higher the interest rate, the less are 1l-agents impacted by the price concession they will make at some point in the future. In a similar way, the more violent or frequent are the utility shocks, in the sense that

\[
\lambda_{12} \nu_{12}
\]

is large, the more urgent will be the trade motive, the larger will be the price concession, and the larger is the impact of illiquidity on utility, even for 1l-agents.

The sharing of the surplus conceded by the 2l-agents must benefit their counterparties, meaning 1h-agents, and those that may become their counterparties, meaning 2h-agents. This is the case, and the illiquidity correction for both 1h and 2h agents are checked to be negative.

I conclude this discussion by looking at the determinants of the “utility transfer” conceded by the short side of the market. Inspection of the expression (34) in Proposition 16 suggests

\[
\delta_a \tau = \left( \frac{\eta_{1h}}{\eta_{2l}} \right) \left( \frac{\delta_s}{s} \right) \left( \frac{\beta}{r + \lambda_{12} \nu_{12} + \lambda_{21} \nu_{12}} \right)
\]

as a measure of this transfer. The three terms in the product refer to the bargaining powers, to the “endogenous bargaining powers”, and to the “relative risk-sharing benefits”.

The first term measures the relative bargaining powers. The stronger are people on the short side at bargaining (meaning, the higher is \( \eta_{2l} \)), the least they will have to give away, and the smaller the impact of illiquidity.
The second term is the ratio of the sensitivity of the distribution and the surplus of the illiquid asset. The more sensitive is the type distribution to the expected search time, the stronger is the impact of illiquidity, and the stronger is the ability of the long side of the market to extract some surplus from the short one. However, if the imbalance between demand and supply for the illiquid asset is very large, even in a very illiquid market buyers will easily find sellers, whereas sellers will struggle to find buyers. In particular, illiquidity will not change the outside options by very much, and the illiquidity corrections will be modest. As this term also relates to the ability of the long side to extract some surplus from the short side, I call it the endogenous bargaining power of the long side.

The last term measures how important the trade is. The numerator
\[ \beta = r \gamma \Delta \sigma_d \sigma_{\eta} (\rho_{2c} - \rho_{1c}) (\rho_{cd} - \hat{\rho}_{cd}) \]

involves both the distance to the critical correlation, which measures how well an OTC trade can redistribute exposures, the risk aversion, and volatilities. This numerator appears to be a natural measure of how important the execution of a bilateral trade is and, quite naturally, the more important is the trade, the larger is the price concession. The denominator puts these benefits into perspective, though. Indeed, it is the sum of the interest rate and the rates of utility change due to preference shocks. It may thus be interpreted as a measure of the value function changes naturally occurring in the economy. Now, should these changes be large as compared to the utility benefits resulting from decentralized trading, this latter activity would be of minor interest, agents on the short side of the market would not sacrifice much to ensure the proper execution of the bilateral trades, and the impact of illiquidity would be small.

Another way of looking at the impact of illiquidity is to consider the benefits from OTC trading instead of the entire value functions.

Namely, for an agent of type \( \bar{\eta} \in \mathcal{T} \), the expected benefits from bilateral trading, can be measured by the term
\[ (2 \lambda \mu (\bar{\theta})) \left[ \chi (\eta_{\bar{\eta}}, \epsilon_{\bar{\eta}} (a)) \right]^- \]

appearing in the HJB equation (19). This is the product of the rate at which potential counterparties are met with a measure of the change in the value function resulting from this meeting.

**Proposition 17.** The expected benefits from bilateral trading are so that

\[
\begin{aligned}
\frac{2 \lambda \mu (2h)}{r \gamma} & \left[ \chi (\eta_{1l}, \epsilon_{1l} (a)) \right]^- = 0 \\
\frac{2 \lambda \mu (2l)}{r \gamma} & \left[ \chi (\eta_{1h}, \epsilon_{1h} (a)) \right]^- = \beta \left( \frac{1}{\chi} \frac{\eta_{2l} \delta_{\mu}}{s} + o \left( \frac{1}{\chi} \right) \right) \\
\frac{2 \lambda \mu (1h)}{r \gamma} & \left[ \chi (\eta_{2l}, \epsilon_{2l} (a)) \right]^- = \beta \left( 1 - \frac{1}{\chi} \left( \frac{\eta_{2l} \delta_{\mu}}{s} + \frac{r + \lambda_{2l} \nu_{12} + \lambda_{2l} \nu_{12}}{2 \lambda_{2l} \nu_{12}} \right) + o \left( \frac{1}{\chi} \right) \right) \\
\frac{2 \lambda \mu (1l)}{r \gamma} & \left[ \chi (\eta_{2h}, \epsilon_{2h} (a)) \right]^- = 0
\end{aligned}
\]
In this last proposition, we can again see how the search and bargaining friction impacts the various agents. Looking at an agent on the short side of the market, a 1h-agent in our case, we can decompose the expected utility benefits from over-the-counter trading into the sum of three terms. First, we again have the measure $\beta$ of the importance of the bilateral trade. This recalls both the asymptotic level of the surplus to be shared, and that it entirely extracted by the short side of the market.

The second term,

$$-rac{1}{\lambda} \beta \eta_{1h} \delta_{\mu} \frac{r}{\eta_{2l} s}$$

is the product of the illiquidity level, $\beta$, the ratio of the exogenous bargaining powers, and the endogenous bargaining powers of the long side of the market. It represents the concession that agents on the long side accords to their counterparties and also appears, with an opposite sign, in the expression referring to the 1h-agents.

The third and last term,

$$r + \frac{\lambda}{\gamma} \nu_{12} + \frac{\lambda}{\gamma} \nu_{12} \frac{1}{2\eta_{2l}s},$$

recalls that the search friction does not only induces utility transfers but also, by making the allocation process less efficient, causes aggregate utility losses. These losses must be borne by the agents on the short side of the market. Indeed, they are, asymptotically, the only beneficiaries from bilateral trading and, as such, the only ones that can see their utility benefits from OTC trading decrease.

I can also characterize the price negotiated on the over-the-counter market.

**Corollary 18.** The price of the illiquid asset is so that

$$P_d = P_d^W + \frac{1}{\lambda} \delta_a r \frac{\eta_{2l}}{r} + \frac{2\delta_{\mu} \eta_{1h} + \frac{\lambda}{\gamma} \nu_{12}}{r \gamma \Delta_{\theta}} + o \left( \frac{1}{\lambda} \right)$$

where

$$P_d^W \Delta \frac{\kappa(1h) - \kappa(1l)}{r \Delta_{\theta}} = \frac{m_d}{r} - \gamma \sigma_d \left\{ \begin{array}{c} \sigma_d (S_d + \left( \frac{\Delta_{\theta}}{2} - S_d \right) (1 - \rho_{cd}^2)) \\ + \sigma_c (\rho_{cd} S_c) \\ + \sigma_\eta (\rho_{1d} - \rho_{cd} (\rho_{1c} - \bar{\rho}_c)) \end{array} \right\}.$$

Here again, I distinguish a Walrasian part and an illiquidity correction. The Walrasian price is the sum of the risk-neutral valuation $m_d/r$ and of three risk discounts. In the first discount, the presence of both $S_d$ and $\Delta_{\theta}$ is reminiscent of the constraint on the size of the holdings (either 0 or $\Delta_{\theta}$), whereas the factor $(1 - \rho_{cd}^2)$ hints toward the irrelevance of this constraint when the two assets become increasingly correlated. Also, looking at the third discount suggests that the average endowment correlation are maybe less important than $\rho_{1c}$ and $\rho_{1d}$, which are the endowment correlations of the marginal buyers.
Regarding the illiquidity correction, the expressions, and interpretations, are similar to their analogues for the value functions (presented in Proposition 16) and for the expected trade benefits (presented in Proposition 17). I however note the following.

**Proposition 19.** As long as the correlation $\rho_{cd}$ between the two assets is close enough to the critical correlation $\hat{\rho}_{cd}$, the absolute value of the first order illiquidity correction is increasing in $|\rho_{cd} - \hat{\rho}_{cd}|$.

This is illustrated in Figure 3. There, I let the correlation between the two risky payouts vary and plot the illiquidity corrections in the first panel, and the type of the high valuation agents in the second. In the first panel, for now, only the dotted blue line labelled “$\lambda_{ls} = \lambda_{ss} = \lambda_L$” is relevant, and corresponds to a setting where the illiquidity level is $1/\lambda^2_L$.

As can be seen in the second panel, 2-agents buy the illiquid asset as long as $\rho_{cd} < \hat{\rho}_{cd} = 0.4$, and sell it otherwise. In particular, when $\rho_{cd} = \hat{\rho}_{cd}$, all agents are indifferent between buying and not buying the asset and, quite naturally, there is no price impact of illiquidity. However, moving away from this threshold, the interest in completing the bilateral trade grows and, as described in Proposition 19, the negotiated and bargained prices start diverging.

The figure makes clear that the illiquidity correction is neither monotonic in $\rho_{cd}$, nor in $|\rho_{cd}|$. In particular, under certain conditions, increasing the correlation between the liquid and illiquid assets increases the illiquidity correction.

To make this phenomenon more intuitive, let us consider a configuration so that

$$\rho_{1d} > \rho_{2d} \text{ and } \rho_{1c} < \rho_{2c}.$$ 

This suggests a natural “clientele” allocation of the asset. Namely, 2-agents are the natural holders of the illiquid asset, and 1-agents those of the liquid one. Now, the more correlated are the assets, the least diversification benefits there are in holding both of them, and the stronger will be the clientele effect. As a result, the risk-sharing benefits will increase with the correlation, and so will the illiquidity correction conceded by the short side of the market.

The explicit expressions presented in Proposition 18 also allow to relate the search and bargaining friction present on OTC markets to another popular measure of illiquidity, market depth. The depth of a market is the relation between the size of a transaction and the price movement induced by this transaction. As formalized in Kyle [1985], this is the sensitivity of the price to the supply of the asset.

Proposition 18 shows how illiquidity, measured as a search friction, relates to this sensitivity. Indeed, increasing the supply $S_d$ of the illiquid asset causes its Walrasian

---

26 I obtain the bargained price by numerically solving the HJB equations (19), calculate the Walrasian directly, and plot their difference. In particular, these plots are not based on the first order approximations presented in the text.

27 Assumption 14 still holds. In particular, it is still assumed that $\beta > 0$. 

---
price to drop, but also reduces the illiquidity correction. This second effect occurs both via the surplus
\[
S(\Delta) = \frac{S_d}{\Delta \theta} - \mu_{2*}
\]
and via
\[
\delta_{\mu}(\Delta) = \lambda_{12} \frac{1 - \frac{S_d}{\Delta \theta}}{2}
\]
which is the sensitivity of the type distribution on the illiquidity level.\footnote{Regarding the impact of the supply on the sensitivity, as
\[
\frac{\partial}{\partial S_d} \delta_{\mu} = \frac{\partial}{\partial S_d} \left( \frac{\lambda_{12}}{2} \frac{1 - \frac{S_d}{\Delta \theta}}{\frac{S_d}{\Delta \theta} - \mu_{2*}} \right) = -\lambda_{12} \frac{1 - \mu_{2*}}{2} \left( \frac{\frac{S_d}{\Delta \theta} - \mu_{2*}}{\Delta \theta} \right) < 0,
\] from which it will follow that this third channel also tends to decrease the price.}
Now, the larger the illiquidity level $\xi = 1/\lambda$, the more important are these two additional channels. This means that the search friction magnifies the sensitivity of $P_d$, the price of the illiquid asset $d$ to its supply $S_d$. In particular, the illiquidity modelled in asset pricing with search and the one understood by the literature on market depth are consistent.

7 Endogenous search intensity

A significant part of my analysis relies on the relatively large value of the meeting intensity on the OTC market. In spite of the justification of this assumption at the beginning of Section 6, it may still appear problematic.

Indeed, on actual decentralized markets, the rate at which market participants contact each others should be split into two components. On the one hand, there is a technological component. For example, locating counterparties is faster on the foreign-exchange market, where an electronic communication network is available, than on bond markets, where participants have to call each other on the phone. Even on markets sharing the same technology in the strict sense, conventions can significantly speed up or slow down the search process. Let us consider two “phone-markets”. As reported by [Bessembinder and Maxwell, 2008, p.223], the American corporate bond market function very quickly, with phone quotations being good “as long as the breath is warm”. Quite differently, negotiating the sale of certain European corporate loans can be a time-consuming process\footnote{See [Duffie et al., 2007, p. 1888], who cites The Financial Times, November 19, 2003.}

Beyond this technological aspect, market participants also have a significant freedom regarding their “contact rate”. For example, a given dealer chooses how many traders she assigns to a given market, or a certain investor decides with how many dealers she will maintain a regular contact.

Whereas the technological component may be, at least in the short run, taken as exogenous, the second one results from endogenous decisions. Now, as stated in Proposition 16 above, not all market participants benefit from a reduced expected search time
and, as a result, an analysis based on an exogenous and large meeting intensity may look fragile.

This motivates me to make the search intensity endogenous. This will offer a further justification for the asymptotic approximations of Section 6. This will also uncover the two opposing effects that the distance to the critical correlation has on the price impact of illiquidity. Namely, if this distance is large, prices are very sensitive to illiquidity level. At the same time, however, the level itself will be low, which will mitigate the impact of the first channel.

7.1 Costly search technology

Agents can now balance the costs and benefits resulting from their presence on the decentralized market. Namely, each agent chooses a search intensity within the range

$$[0, \lambda_H],$$

but search is costly beyond a certain level

$$\lambda_L \in [0, \lambda_H].$$

Namely, for a positive constant $\zeta$, maintaining a search intensity $\bar{\lambda}$ requires to pay a running cost of

$$\zeta \cdot \left[ \bar{\lambda} - \lambda_L \right]^+$$

per unit of time.

Regarding the potential benefits, a higher search intensity makes meetings more likely. I borrow the matching technology used by Duffie, Malamud, and Manso [2009], and assume that a given agent is matched at a rate that is proportional to her own effort, given the effort of the other agents. Put differently, given a group $B$ of agents, all of whom search with an intensity $\lambda_B$, and another group $C$ with a corresponding search intensity $\lambda_C$, then, the rate at which meetings between the two groups occur is

$$2\lambda_B \lambda_C \mu(B) \mu(C).$$

In particular, the search intensities are complements for the matching technology.

I would like to justify choice of the cost structure (36). The lowest level of search an agent can adopt on an OTC market is to wait for a counterparty to contact her. Arguably, this primitive search strategy should still make meetings possible, and be cost-free. Now, the multiplicative matching technology (37) only allows for this if the search is made costless over a certain range. This observation, and the tractability of the model, motivated me to chose the quasi-linear cost structure (36).
7.2 Equilibrium analysis

Up to a few adjustments, most of the arguments and results from Sections 5 and 6 are still valid. To begin with, the HJB equations (11) become

\[ \rho V(w, i\theta) = \sup_{\tilde{c}, \tilde{\pi}, \tilde{\lambda}} U(\tilde{c}) \]

\[ + \frac{\partial V}{\partial w}(w, i\theta) \left( rw - \tilde{c} - \zeta \left[ \tilde{\lambda} - \lambda_L \right]^+ + m_\eta + \theta m_d + \tilde{\pi} (m_e - r P_c) \right) \]

\[ + \frac{1}{2} \frac{\partial^2 V}{\partial w^2}(w, i\theta) \left( 1 \theta \tilde{\pi} \right) \Sigma_i \left( 1 \theta \tilde{\pi} \right)^* \]

\[ + \lambda_{\bar{i}} \left( V(w, i\theta) - V(w, i\theta) \right) \]

\[ + 2 \tilde{\lambda} E^{\mu(a)} \left[ \lambda_a \left( V \left( w - \left( \theta_a - \theta \right) P_d(w,i\theta,a), i\theta_a \right) - V(w, i\theta) \right) \right]^+ . \]

(38)

Again taking Assumption 3 for valid, the search strategy dictated by the HJB equation only depends on the type, and I describe it with the notation

\[ \{\lambda_{i\theta}\}_{i\theta \in T} . \]

(39)

This strategy is characterized by the comparison of \( \zeta \), meaning the unit search costs, with the quantity

\[ \frac{-2\lambda_{i\theta}}{r^{\gamma}} \mu(\tilde{i} \theta) \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) . \]

This second quantity, a version of which already appeared in Proposition 17, measures the expected benefits from decentralized trading.

The exact characterization of the search strategy is as follows.

Proposition 20. The search strategy dictated by the HJB equation (38) is

\[ \lambda_{i\theta} \begin{cases} 
\in [0, \lambda_L] &, \frac{-2\lambda_{i\theta}}{r^{\gamma}} \mu(\tilde{i} \theta) \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) \in (-\infty, 0] \\
\lambda_L &, \frac{-2\lambda_{i\theta}}{r^{\gamma}} \mu(\tilde{i} \theta) \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) \in (0, \zeta) \\
\in [\lambda_L, \lambda_H] &, \frac{-2\lambda_{i\theta}}{r^{\gamma}} \mu(\tilde{i} \theta) \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) = \zeta \\
\lambda_H &, \frac{-2\lambda_{i\theta}}{r^{\gamma}} \mu(\tilde{i} \theta) \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) \in (\zeta, +\infty) 
\end{cases} . \]

(40)

The remaining of the analysis is largely based on combining the HJB equation (38) with Proposition 20. The first observation is that, taking the search intensities “\( \lambda_{i\theta} \)”s as given, the new HJB equation (38) is identical to the original ones, (11), up to the changes

\[ \begin{array}{c}
\text{base setting} \\
\kappa(i\theta) \\
\lambda \\
\end{array} \quad \begin{array}{c}
\text{endogenous search} \\
\kappa(i\theta) - \zeta \left[ \lambda_{i\theta} - \lambda_L \right]^+ \\
\lambda_{1l} \lambda_{2l}, \text{or} \lambda_{1b} \lambda_{2l} \\
\end{array} \]

(41)

In particular, recalling Proposition 10, there is exactly one solution. \(^{30}\)

\(^{30}\) I still assume that Assumption 14 holds. In particular, it is already clear that the meeting intensity will be \( \lambda_{1b} \lambda_{2l} \).
Based on this solution, I derive the optimal search strategy that individual adopt. An equilibrium of the model with endogenous search arises when the optimal individual strategy is identical to the one that was assumed in the first place.

Now, invoking Kakutani Fixed-Point Theorem yields the following.

**Proposition 21.** There exists an equilibrium of the model with endogenous search intensities.

Closed-form expressions are still elusive, and I again focus on equilibria where the illiquidity level is low. This obviously requires to choose the maximal search intensity \( \lambda_H \) large. This time, however, the actual meeting intensity being determined endogenously, I can only encourage agents to adopt a high meeting intensity. I do this by making the search cost \( \zeta \) small.

Now, to formulate my asymptotic results, I make two further assumptions. First, the cost is a function

\[
\zeta = \zeta (\lambda_H)
\]

of the highest possible search intensity. I will not make this dependence explicit in my notations. Second, I restrict my attention to equilibria that are sufficiently regular. Namely, I only consider equilibria so that, for \( i' \in T \), the limits

\[
z_{i'0} \triangleq \lim_{\lambda_H \to \infty} \zeta [\lambda_{i'0} - \lambda_{L}]^+ \in [0, \infty),
\]

and

\[
z_{i'1} \triangleq \lim_{\lambda_H \to \infty} (\lambda_{i'0} \lambda_{i'1}) (\zeta [\lambda_{i'0} - \lambda_{L}]^+ - z_{i'0}) \in [0, \infty]
\]

exist. This means that the search costs can be characterized by first order approximations in the endogenous meeting intensity.

Based on these assumptions, the new HJB equations (38), and the characterization of the optimal individual search strategies in Proposition 20, I derive the following.

**Proposition 22.** When the maximum possible search intensity is let to increase arbitrarily, there exists equilibria where the endogenous meeting intensity also becomes arbitrarily large. These equilibria are as follows.

Agents on the short side of the market always asymptotically search for counterparties with the highest possible intensity: for \( \lambda_H \) large enough,

\[
\lambda_{2l} = \lambda_H.
\]

The equilibrium search strategy of agents on the long side of the market is characterized by the rate at which the search costs \( \zeta \) go to zero. Explicitly,

- if both

\[
\lambda_{2l}^2 \geq \frac{\eta_{1l} \delta_{\mu}}{\eta_{2l} s}
\]
and
\[
\lim_{\lambda_H \to \infty} \zeta \lambda_H \geq \frac{\beta}{\frac{\eta_{1h}}{\eta_{2l}} s} - 1,
\]
then, for \( \lambda_H \) large enough,
\[
\lambda_{1h} = \lambda_L;
\]

- otherwise, if \( \lim_{\lambda_H \to \infty} \zeta \lambda_H > 0 \), then
\[
\lim_{\lambda_H \to \infty} \lambda_{1h} = \frac{1}{2} \left( \lambda_L + \sqrt{\lambda_L^2 + 4 \frac{\eta_{1h} \delta_a}{\eta_{2l} s} \left( 1 + \lim_{\lambda_H \to \infty} \frac{\beta}{r \gamma \zeta \lambda_H} \right)} \right);
\]

- otherwise, if \( \lim_{\lambda_H \to \infty} \zeta \lambda_H^3 > \beta \frac{\eta_{1h} \delta_a}{\eta_{2l} s} \), then
\[
\lambda_{1h, \lambda_H \to \infty} = \sqrt{\frac{1}{\zeta \lambda_H} \sqrt{\frac{\beta \eta_{1h} \delta_a}{\eta_{2l} s}}};
\]

- otherwise, for \( \lambda_H \) large enough,
\[
\lambda_{1h} = \lambda_H.
\]

Summarizing this last proposition, agents on the short side of the market always find it worthwhile to increase the meeting intensity, but those on the long side of the market only do so if the search costs are low enough. Let us consider each type of agents more carefully, beginning with those on the short side of the market.

If we recall the discussion following Proposition 6, the surplus to be shared in a bilateral trade is linked to the dynamics of the value function before and after the trade. These dynamics used to change because of a redistribution of the holdings, but now they also change because, after a successful bilateral trade, agents do not have to pay any search cost any more. Now, in a very liquid OTC market, agents on the short side can extract most of the surplus. Hence, even if a high search intensity makes those agents incur a significant running cost, it also increases by an equivalent quantity the surplus to be shared in a bilateral trade and, maybe paradoxically, their expected benefits from trading. As a result, asymptotically, no search intensity will be too expensive for those agents that are on the short side of the market.

The strategy of the agents on the long-side of the market is more cautious. Namely, as stated in (46), they will never adopt the highest possible search intensity \( \lambda_H \) unless the search cost \( \zeta \) decreases like the inverse of the third power of this highest possible search intensity. Otherwise, they will adopt an intermediate search intensity, meaning that
\[
\lambda_{1h} \in (\lambda_L, \lambda_H).
\]
That some agents may adopt an intermediate search intensity, and that this regime may be robust, deserves a few words. Indeed, the individual choice of an intensity is based on a linear optimization, and one would generally expect a corner solution. Recalling the discussions following Proposition 16 and Proposition 17, agents on the long side of the market benefit from illiquidity. The individual decisions to search with a high intensity thus induces, once aggregated, a decrease in the benefits from trading for the agents on the long side of the market. In the settings described by equations (44) and (45), as long as nobody searches, the expected benefits from bilateral trading justify a high search intensity. However, as soon as everybody searches with that high intensity, the reduced expected benefits do not justify a costly search any more. In equilibrium, agents will thus adopt an intermediate search intensity that exactly balances out costs and benefits.

In Figure 2 I illustrate how, when the search costs are decreased, agents gradually adopt a higher search intensity. I again use the parametrization described in Table 1. We can see how the 2l-agents, those that are on the short side of the market, and extract most of the surplus when bargaining, start searching intensively as soon as the costs are of the order \(10^{-1}\). Their counterparts, the 1h-agents, will however only significantly increase their search intensity when the search costs become significantly smaller, meaning below \(10^{-4}\). This figure also shows how, for a fixed (non-asymptotic) parametrization, even those agents on the short side of the market will not tolerate any search costs.

Finally, Proposition 22 illustrates the ambiguous effect that the distance to the critical correlation, or any of its components, has on the illiquidity correction.

There are two determinants of the illiquidity correction. The first one is the illiquidity level itself, meaning the time it takes to contact a potential counterparty. The second one is the sensitivity of the price to this illiquidity level.

With an endogenous meeting intensity, these two determinants react differently to changes in the correlation between the two assets. On the one hand, increasing the distance to the critical correlation \(\hat{\rho}_{cd}\) makes the bilateral trade more profitable. Via the mechanism described in Proposition 19, this makes the price more sensitive to the illiquidity level, and tends to increase the illiquidity correction.

On the other hand, trades being more profitable, agents may find it worthwhile to incur the cost attached with a more intensive search for counterparties. This increases the endogenous meeting intensity, and thus reduces the illiquidity correction. This was just described in Proposition 22.

I can characterize the magnitude of these two types of changes. Proposition 19 ensures that the sensitivity of the price to the illiquidity level is locally driven by the distance to the critical correlation, meaning by \((\hat{\rho}_{cd} - \hat{\rho}_{cd})\). Actually, both the proof of this proposition and Figure 3 suggest that this dependence is approximately linear. Regarding the endogenous illiquidity level \(1/(\lambda_{1h}^{-1}\lambda_{2l})\), Proposition 22 indicates \(^{31}\) that it is either constant, or proportional to \(1/\sqrt{\rho_{cd} - \hat{\rho}_{cd}}\). In particular, the illiquidity

\(^{31}\) This heuristic argument only considers the asymptotics described in (45). The case described in (44) is similar, but a bit more cumbersome in terms of notations.
correction, which is the product of these two quantities, will then behave either like \( \rho_{cd} - \hat{\rho}_{cd} \), or like \( \sqrt{|\rho_{cd} - \hat{\rho}_{cd}|} \).

In particular, even in the extended model, the illiquidity correction is locally increasing in the distance to the critical correlation, and the endogenous search intensity only has a dampening effect.

I illustrate this mitigating effect in Figure 3. There I compare two cases. First, there is a base case where all agents search with the minimum intensity \( \lambda_L \). This was already mentioned after Proposition 19, and is denoted by “\( \lambda_{ls} = \lambda_{ss} = \lambda_L \)”. In the second specification, agents freely choose their meeting intensities. This case is denoted by “\( \lambda_{ls}, \lambda_{ss} \) endogenous”. We can see how, in both cases, the illiquidity correction is increasing in the distance to the critical correlation and how, with an endogenous search intensity, this correction is reduced.

8 Conclusion

I proposed a model to characterize the mutual influences of an illiquid, OTC market and of a liquid, centralized one when distinct assets are traded on each venue.

I conclude that the illiquidity of the decentralized market impacts both the distribution of demands for the liquid one and its trading volume, but not its price.

Further, the main determinant of the price impact of illiquidity on the OTC market is how far the correlation between the two risky assets is to a certain critical correlation. This critical correlation is so that the illiquid asset brings no utility benefits beyond what can be achieved by trading the liquid asset only. This distance drives, in opposite ways, both the sensitivity of prices to the illiquidity level, and the endogenous determination of this illiquidity level.

In particular, the liquid asset impacts the illiquidity correction by making a sub-optimal illiquid exposure more or less tolerable, but this is not directly linked to being more or less correlated with the illiquid payouts.
Appendices

A  Walrasian equilibrium

The asymptotic price of the illiquid asset is, in an appropriate sense, Walrasian.

Let me thus consider a model identical to the one proposed in the main text, up to the trading mechanism for the asset $d$. The holdings in this asset are still restricted to two possible values, but the trading now takes place on an exchange to which the access is immediate and costless.  

In this new setting, the Walrasian price of the asset $d$ is indeed the asymptotic price stated in Corollary 18. The equilibrium price of the asset $c$ is still given by the expression (28), that I recall in the next statement.

**Proposition 23.** The unique market clearing prices are

$$P^W_c = \frac{m_c}{r} - \gamma \sigma_c \left( S_c \sigma_c + \sigma_q \rho_c + \rho_{cd} \sigma_d S_d \right)$$

and

$$P^W_d = \frac{\kappa(ih) - \kappa(il)}{r \Delta_\theta},$$

(47)

where, in the expression for $P^W_d$, the index $i$ is given by

$$i = \begin{cases} 
1 & \text{if } \beta > 0 \text{ and } S_d / \Delta_\theta > \mu_2, \\
1 & \text{if } \beta < 0 \text{ and } S_d / \Delta_\theta < \mu_1, \\
2 & \text{otherwise} 
\end{cases}$$

and describes the endowment correlation type of the marginal buyers.

**Proof of Proposition 23.** The HJB equation for an agent with endowment correlation $\rho_i$ is now

$$\rho V(w, i) = \sup_{c, \pi, \theta} U(c)$$

$$+ V_w(w, i) \left( rw - c + m_n + \theta (m_d - r P_d) + \pi (m_c - r P_c) \right)$$

$$+ \frac{1}{2} V_{ww}(w, i) \left( 1 - \theta \pi \right) \Sigma_i \left( 1 - \theta \pi \right)^* \Sigma_i$$

$$+ \lambda_{12} (V(w, \tilde{i}) - V(w, i)),$$

(48)

where the optimization is over the domain

$$(c, \pi, \theta) \in \mathbb{R}^2 \times \{0, \Delta_\theta\}.$$ 

The only difference with the corresponding HJB equation (11) appearing in the main text is that the illiquid holdings $\theta$ are not an argument of the value function any more, but is a quantity over which the agent can optimize.

---

32 I also maintain Assumption 2 which excludes certain non-generic parametrizations.
Adapting Assumption 3 to the present context, I assume that
\[ V(w, i) = -\exp\{-r\gamma(w + a(i) + \bar{a})\}, \]
and inject this expression into the HJB equation (48).

Taking the holdings \( \theta \) in \( d \) as given, the optimization defining \( \pi(i\theta) \), the optimal holdings in \( c \), is the same as the one in the original model. These holdings are thus still described by (16). Equating the resulting aggregate demand and the net supply of the asset, and already using the market clearing for the asset \( d \) then, again, yields the expression (28) for the Walrasian price. \footnote{I use the market clearing condition for \( d \), but not the actual holdings decisions.}

Turning to the holdings in \( d \), they are defined by
\[ \theta \in \arg \max_{\tilde{\theta} \in \{0, \Delta \theta\}} \tilde{\theta}(m_d - rP_d) + \pi(i\tilde{\theta})(m_c - rP_c) - \frac{1}{2}r\gamma(1 \cdot \pi(i\tilde{\theta})) \Sigma_i (1 \cdot \pi(i\tilde{\theta})){\ast} \]
\[ = \arg \max_{\tilde{\theta} \in \{0, \Delta \theta\}} \kappa(i\tilde{\theta}) - r\tilde{\theta}P_d, \]
where the "\( \kappa(i\tilde{\theta})"s were defined in (18).

In particular, the large exposure \( \Delta \theta \) is certainly chosen when
\[ \kappa(il) - \kappa(ih) < \frac{\kappa(ih) - \kappa(il)}{r\Delta \theta}, \]
and it is still optimal to choose it with any probability in case of equality.

Now, first assuming that
\[ \beta \overset{(\Delta)}{=} (\kappa(2h) - \kappa(2l)) - (\kappa(1h) - \kappa(1l)) > 0, \]
the resulting aggregate demand for the asset \( d \) is described by
\[ \mathbb{E}^{\mu(\alpha)}[\theta^a] \begin{cases} = \Delta \theta, & P_d < \frac{\kappa(1h) - \kappa(1l)}{r\Delta \theta}, \\ \in [\mu_2 \Delta \theta, \Delta \theta], & P_d = \frac{\kappa(1h) - \kappa(1l)}{r\Delta \theta}, \\ = \mu_2 \Delta \theta, & P_d \in \left(\frac{\kappa(2h) - \kappa(2l)}{r\Delta \theta}, \frac{\kappa(1h) - \kappa(1l)}{r\Delta \theta}\right), \\ \in [0, \mu_2 \Delta \theta], & P_d = \frac{\kappa(2h) - \kappa(2l)}{r\Delta \theta}, \\ = 0, & P_d > \frac{\kappa(2h) - \kappa(2l)}{r\Delta \theta}. \end{cases} \]

Equating this demand with the supply \( S_d \) then yields a Walrasian price \( P^W_d \), and the frequency with which the large holdings are chosen by indifferent agents. Following the same steps when \( \beta < 0 \), and summarizing, yields (47).
B Technical prerequisites

B.1 Technical results

Lemma 24. Let \( G : X \times Y \to \mathbb{R} \) be a continuous function of two variables so that

(i) for any \( x \in X \) there exists exactly one \( f(x) \in Y \) so that \( G(x, f(x)) = 0 \); and

(ii) if \( \{x_n\}_{n \geq 0} \subset X \) is a convergent sequence, then the sequence \( \{f(x_n)\}_{n \geq 0} \subset Y \) admits a convergent subsequence.

Then, the function \( f \) is continuous.

Proof. The proof is by contradiction. Let me assume there exists a real number \( \epsilon > 0 \), a sequence \( \{x_n\}_{n \geq 0} \subset X \), and a point \( x \in X \) so that

\[
\lim_{n \to \infty} x_n = x
\]

and, for any \( n \geq 0 \),

\[
|f(x_n) - f(x)| > \epsilon. \tag{49}
\]

Then, maybe choosing a subsequence, the hypothesis (ii) above ensures the existence of a certain \( \hat{y} \in Y \) so that

\[
\lim_{n \to \infty} f(x_n) = \hat{y}.
\]

But then, \( G \) being continuous,

\[
0 = \lim_{n \to \infty} G(x_n, f(x_n)) = G(x, \hat{y}).
\]

From which, by hypothesis (i)

\[
\hat{y} = f(y) = \lim_{n \to \infty} f(x_n),
\]

which contradicts (49).

\[\square\]

Lemma 25. Consider a smooth map \( H : \Omega \to \Omega \) for some \( \Omega \subset \mathbb{R}^d \). If for any \( i = 1, \ldots, d \), there exists a \( \eta < 1 \) so that

\[
\sum_{j=1}^{d} \left| \frac{\partial H_i}{\partial x_j} \right| \leq \eta,
\]

then \( H \) is a contraction in \( l_\infty \) and has a unique fixed point.

Proof of Lemma 25. Fix \( x_1, x_2 \in \Omega \) and define, for \( t \in [0,1] \),

\[
x(t) \triangleq x_1 + t(x_2 - x_1).
\]
Then, for any $i \in \{1, \ldots, d\}$,

$$
|H_i(x_2) - H_i(x_1)| = \left| \int_0^1 \sum_j \frac{\partial H_i}{\partial x_j}(x(t))(x_2^j - x_1^j) \, dt \right|
\leq \int_0^1 \sum_j \left| \frac{\partial H_i}{\partial x_j}(x(t)) \right| |x_2^j - x_1^j| \, dt
\leq \sum_j (\partial_j H) \max_j |x_2^j - x_1^j| \int_0^1 \, dt
\leq \eta \|x^2 - x^1\|_{l_{\infty}}.
$$

The last claim follows from the Contraction Mapping Theorem (see [Stokey and Lucas, 1989, Theorem 3.2, p.50]).

### B.2 A general existence and uniqueness result

I begin with the formulation of a general problem. I would like to find those $\beta \equiv (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ so that, for any $k \in \{1, \ldots, d\}$,

$$
0 = r\beta_k + \sum_{j \neq k} \kappa_{kj} e^{\beta_k - \beta_j} + c_k \triangleq F_k(\beta),
$$

where $K_k \equiv (\kappa_{kj}) \geq 0$ and $c \equiv (c_k)$ are known constants. Let $\beta_-k$ be the vector of $\beta$ without $\beta_k$.

The first lemma introduces the objects of interest.

**Lemma 26.** For any $k \in \{1, \ldots, d\}$, there exists a unique real analytic (and hence smooth) function

$$
G_k = G_k(\beta_-k, K_k, c_k)
$$

such that $\beta = G_k(\beta_-k, K_k, c_k)$ is the unique solution to

$$
r\beta + \sum_{j \neq k} \kappa_{kj} e^{\beta - \beta_j} + c_k = 0.
$$

$G_k$ is monotone increasing in the components of $\beta_-k$, and monotone decreasing in $\kappa_{kj}$ and $c_k$ for all $j \neq k$.

Now, the functions $G_k$ define a contraction.

**Proposition 27.** Fix $K = (K_k)$ and $c = (c_k)$ and consider the map

$$
G = (G_k)_k : \mathbb{R}^d \rightarrow \mathbb{R}^d.
$$

Then, this map is a contraction on a compact set $\Omega$ in the $l_{\infty}$-norm and has a unique fixed point $\beta^* = \beta^*(K, c)$. This fixed point is the unique solution to (50). It is monotone decreasing in the components of $K$ and $c$. 39
The proof of Proposition 27 is based on an application of Lemma 25.

**Proof of Proposition 27.** I first show that $G$ maps a compact set into itself. Let me choose two real numbers $L < U$, and assume that for any $k \in \{1, \ldots, d\},$

$$\beta_k \in [L, U]^{d-1}.$$  

For a given $k$, let me further define two functions, $F^L_k$ and $F^U_k$, that bound the function $F_k$ defined in (50). Namely,

$$r \beta + \sum_{j \neq k} \kappa_{kj} e^{\beta_j - U} + c_k \overset{\Delta}{=} F^L_k(\beta) \leq F_k(\beta) \leq F^U_k(\beta) \overset{\Delta}{=} r \beta + \sum_{j \neq k} \kappa_{kj} e^{\beta_j - L} + c_k.$$  

Now, due to the monotonicity of $F_k(\cdot, \beta_{-k})$, if

$$0 \leq F^L_k(U) = rU + \sum_{j \neq k} \kappa_{jk} + c_k \overset{(51)}{=}$$

and

$$0 \geq F^U_k(L) = rL + \sum_{j \neq k} \kappa_{jk} + c_k \overset{(52)}{=}$$

then

$$G_k(\beta_{-k}) \in [L, U].$$  

But both (51) and (52) will hold for all $k \in \{1, \ldots, d\}$ as soon as

$$U \geq \max_{k \in \{1, \ldots, d\}} \frac{-1}{r} \left( \sum_{j \neq k} \kappa_{jk} + c_k \right)$$

and

$$L \leq \min_{k \in \{1, \ldots, d\}} \frac{-1}{r} \left( \sum_{j \neq k} \kappa_{jk} + c_k \right).$$  

Now, by the Implicit Function Theorem,

$$\frac{\partial G_k(\beta_{-k})}{\partial \beta_j} = \frac{\kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}}{r + \sum_{j \neq k} \kappa_{kj} e^{G_k(\beta_{-k}) - \beta_j}},$$

which can be bounded strictly below 1, uniformly in $\beta_{-k} \in [L, U]^{d-1}$, for $L$ and $U$ chosen as above. But then, Lemma 25 ensures the existence and uniqueness of a fixed point on $[L, U]^d$. Finally, as $-L$ and $U$ can be chosen arbitrarily large, the existence and uniqueness on $\mathbb{R}^d$ hold.

Monotonicity follows because

$$\beta^* = \lim_{n \to \infty} G^n(\beta_0)$$

for any fixed $\beta_0$ and $F$ is monotone. □
Proof of Proposition 4. Let $a$ be an agent with type $i_a \theta$ and wealth $w_a$. She met another agent $b$. Clearly, no trade will be possible unless the holdings of $b$ are $\bar{\theta}$. I denote the two other characteristics of $b$ by $i_b$ and $w_b$.

There is a surplus for $a$ and $b$ to share if

$$\emptyset \neq \left\{ \tilde{P} : \begin{array}{c} V(w_a - (\bar{\theta} - \theta) \tilde{P}, i_a \bar{\theta}) \\ V(w_b - (\theta - \tilde{\theta}) \tilde{P}, i_b \tilde{\theta}) \end{array} \geq \begin{array}{c} V(w_a, i_a \theta) \\ V(w_b, i_b \theta) \end{array} \right\}. $$

Under Assumption (3), this is equivalent to

$$\emptyset \neq \mathcal{P} \overset{\Delta}{=} \left\{ \tilde{P} : a(i_a \bar{\theta}) - a(i_a \theta) \geq \tilde{P} (\bar{\theta} - \theta) \geq a(i_b \tilde{\theta}) - a(i_b \theta) \right\},$$

or to

$$a(i\bar{\theta}) - a(i \theta) + a(j \theta) - a(j \tilde{\theta}) \geq 0.$$

This proves the first three statements.

Now, if there actually is a surplus to share, the outcome of the bargaining is given by the Nash bargaining solution. Namely, $a$ and $b$ trade the asset at the price $P_d$ so that

$$P_d = \arg \max_{\tilde{P} \in \mathcal{P}} \left( V(w_a - \tilde{P}(\bar{\theta} - \theta), i_a \bar{\theta}) - V(w_a, i_a \theta) \right)^{\eta_a \theta} \cdot \left( V(w_b - \tilde{P}(\theta - \tilde{\theta}), i_b \tilde{\theta}) - V(w_b, i_b \theta) \right)^{1-\eta_a \theta}.$$

Unless $\mathcal{P}$ is reduced to a single point, in which case the solution of the optimization is trivial, the first order condition characterize the point of maximum $P_d$ as the solution to

$$\eta_a \theta \frac{\partial_w (V(w_a - P_d(\bar{\theta} - \theta), i_a \bar{\theta}))}{V(w_a + P_d(\theta - \tilde{\theta}), i_a, \theta) - V(w_a, i_a \theta)} = (1 - \eta_a \theta) \frac{\partial_w (V(w_b - P_d(\theta - \tilde{\theta}), i_b \theta))}{V(w_b - P_d(\theta - \tilde{\theta}), i_b \theta) - V(w_b, i_b \theta)}$$

which, with Assumption 3, becomes (12).

Proof of Proposition 5. The first order necessary condition for the maximization over the consumption rate is

$$\gamma e^{-\gamma c} - \frac{\partial V}{\partial w}(w, i, \theta) = 0.$$

Recalling the Assumption 3, and solving for $c$ yields a unique candidate $c(i \theta)$ which, by concavity of the objective function, is a point of maximum.

A similar argument yields the optimal liquid holdings $\pi(i \theta)$.
Proof of Proposition 6. Starting from the HJB equation (11), picking a type \( i\theta \in T \), using Proposition (4) to transform the expected value into a deterministic quantity, Proposition 5 to express the optimal consumption, and normalizing by \( r\gamma V(w, i, \theta) \), I obtain

\[
0 = r - \rho - r\log(r) + r^2\gamma \bar{a} - r\gamma m_{\eta} \\
+ \frac{\alpha}{r\gamma} \left( \frac{\alpha}{\gamma} - r \right) w \\
+ ra(i\theta) - \kappa(i, \theta) \\
+ \frac{\lambda i i}{r\gamma} \left( e^{\beta(i, \theta) - \beta(i, \theta)} - 1 \right) \\
+ \frac{2\lambda \mu i i}{r\gamma} \min \left\{ \frac{2(1 - \eta \theta)}{1 - 2\eta \theta + Y_{i\theta}(\epsilon_{i\theta}(\beta)) - 1}, 0 \right\}.
\]

(54)

Now, I can set the constant term as

\[
\bar{a} = \frac{1}{r\gamma} \left( -1 + \frac{\rho}{r} + \log(r) + \gamma m_{\eta} \right),
\]

which makes the first line of the right hand side equal zero. Also, as the equation above must hold for any value of the liquid holdings \( w \), it must be that

\[
\alpha \left( r - \frac{\alpha}{\gamma} \right) = 0.
\]

Discarding the possibility of a value function that would be independent of the liquid holdings, this imposes that \( \alpha = r\gamma \), and makes the second line of the right hand side equal to zero. Taking these two observations into account yields the system (19).

It remains to show that this equation admits exactly one solution. I split my argument into four steps.

**Step 1** I first rearrange the four equations described in (19) into just two. Namely, defining the variables

\[
\Delta_h \triangleq a(1h) - a(2h)
\]

and

\[
\Delta_l \triangleq a(2l) - a(1l),
\]

and taking the corresponding differences in the HJB equations (19) ensures that

\[
0 = r\Delta_l - \kappa(2l) + \kappa(1l) + \frac{\lambda 21}{r\gamma} \left( e^{\Delta_l} - 1 \right) - \frac{\lambda 12}{r\gamma} \left( e^{-\Delta_l} - 1 \right) \\
+ \frac{2\lambda}{r\gamma} \left( \mu(1h) \left[ \chi(\eta 2l, -\Delta_l - \Delta_h) \right] - \mu(2h) \left[ \chi(\eta 1l, \Delta_l + \Delta_h) \right] \right).
\]

(57)

\[
\triangleq F_l(\Delta_l, \Delta_h)
\]

42
and
\[ 0 = r \Delta_h - \kappa(1h) + \kappa(2h) + \frac{\lambda_{12}}{r\gamma} (e^{\Delta_h} - 1) - \frac{\lambda_{21}}{r\gamma} (e^{-\Delta_h} - 1) \]
\[ + \frac{2\lambda}{r\gamma} [\mu(2l) [\chi(\eta_{1h}, -\Delta_l - \Delta_h)] - \mu(1l) [\chi(\eta_{2h}, \Delta_l + \Delta_h)]] \]
\[ \Delta F_h(\Delta_l, \Delta_h). \] (58)

Inspection ensures that, for any \( \Delta_h \), the function \( F_l(\cdot, \Delta_h) \) is strictly increasing with a range equal to the entire real line. It also ensures that, for any given \( \Delta_l \), the function \( F_l(\Delta_l, \cdot) \) is strictly increasing, but with the bounded range
\[ r \Delta_l - \kappa(2l) + \kappa(1l) + \frac{\lambda_{21}}{r\gamma} (e^{\Delta_l} - 1) - \frac{\lambda_{12}}{r\gamma} (e^{-\Delta_l} - 1) + \frac{2\lambda}{r\gamma} [-\mu(1h), \mu(2h)]. \] (59)

Similar properties hold for \( F_l \).

**Step 2**  Given these properties of \( F_l \) and \( F_h \), I can define the functions
\[ \Phi_l, \Phi_h : \mathbb{R} \to \mathbb{R} \]
by requiring that, for any \( x \in \mathbb{R} \),
\[ 0 = F_l(\Phi_l(x), x) = F_h(x, \Phi_h(x)). \] (60)

The monotonicity properties also ensure that both \( \Phi_l \) and \( \Phi_h \) are decreasing. I show two more properties of these functions.

First, these functions decrease relatively slowly. Namely, for any choice of \( x \in \mathbb{R} \) and \( y \in \mathbb{R}_{>0} \), it follows from (58) and (60) that
\[ F_l(\Phi_l(x) - y, x + y) < F_l(\Phi_l(x), x) = 0 = F_l(\Phi_l(x + y), x + y), \]
meaning that
\[ y + \Phi_l(x + y) - \Phi_l(x) > 0. \]
As a result, the function
\[ x \mapsto x + \Phi_l(x) \] (61)
is increasing and, by a similar argument, so is
\[ x \mapsto x + \Phi_h(x). \]

Second, their range is compact. Let me first consider \( \Phi_h \). Recalling the bounded range described by (59), I may write, for any pair \( (\Delta_l, \Delta_h) \in \mathbb{R}^2 \),
\[ F_l^L(\Delta_l) \leq F_l(\Delta_l, \Delta_h) \leq F_l^U(\Delta_l), \]
where I defined
\[ F_l^L(x) = rx - \kappa(2l) + \kappa(1l) + \frac{\lambda_{21}}{r\gamma} (e^x - 1) - \frac{\lambda_{12}}{r\gamma} (e^{-x} - 1) - \frac{2\lambda}{r\gamma} \mu(1h) \]
and

\[ F_t^U(x) \overset{\Delta}{=} rx - \kappa(2l) + \kappa(l) + \frac{\lambda_{21}}{r^\gamma}(e^x - 1) - \frac{\lambda_{12}}{r^\gamma}(e^{-x} - 1) + \frac{2\lambda}{r^\gamma} \mu(2h) \]

Inspection now ensures that for \( b_{U,l} \) large enough, \( F_t^L(b_{U,l}) \geq 0 \), and that for \( b_{L,l} \) small enough, \( F_t^U(b_{L,l}) \leq 0 \). But then, for any \( \Delta_h \),

\[ F_t(b_{L,l}, \Delta_h) \leq F_t^U(b_{L,l}) \leq 0 \leq F_t^L(b_{U,l}) \leq F_t(b_{U,l}, \Delta_l). \]

Keeping the monotonicity and continuity of \( F_t \) in mind, this implies that \( \Phi_l(\Delta_h) \in [b_{L,l}, b_{U,l}] \), and thus that

\[ \Phi_l(\mathbb{R}) \subset [b_{L,l}, b_{U,l}] \].

A similar argument formulated with \( F_h \) would yield two other constants \( b_{L,h} \) and \( b_{U,h} \) so that

\[ \Phi_h(\mathbb{R}) \subset [b_{L,h}, b_{U,h}] \].

In particular, if I define

\[ \Omega \overset{\Delta}{=} [b_{L,h} \wedge b_{L,l}, b_{U,h} \vee b_{U,l}] \],

then

\[ \Phi(\Omega) \overset{\Delta}{=} (\Phi_l, \Phi_h)(\Omega) \subset \Omega \times \Omega. \]

**Step 3** I now show that \( \Phi \) is a contraction, and admits a unique fixed point. First, keeping in mind the boundedness of the ranges of \( \Phi_l \) and \( \Phi_h \), Lemma 25 ensures that these functions are continuous.

Second, if I choose \( x \in \Omega \) so that

\[ \Phi_l(x) + x \not= 0, \quad (62) \]

then, an application of the Implicit Function Theorem based on the relation (60) ensures that \( \Phi_l \) is differentiable at \( x \), with derivative given by

\[
\Phi_l'(x) = - \frac{\frac{\partial F_t}{\partial \Delta_l} (\Phi_l(x), x)}{\frac{\partial F_t}{\partial \Delta_l} (\Phi_l(x), x)} = - \frac{2\lambda \left( \begin{array}{c} 1_{\{-\Phi_l(x) - x > 0\}} \mu(1h)(-1) \frac{\partial \chi}{\partial \epsilon} (\eta_{2l}, -\Phi_l(x) - x) \\ -1_{\{-\Phi_l(x) - x > 0\}} \mu(2h) \frac{\partial \chi}{\partial \epsilon} (\eta_{1l}, \Phi_l(x) + x) \end{array} \right)}{r + \lambda_{12} e^x + \lambda_{21} e^{-x} + 2\lambda \left( \begin{array}{c} 1_{\{-\Phi_l(x) - x > 0\}} \mu(1h)(-1) \frac{\partial \chi}{\partial \epsilon} (\eta_{2l}, -\Phi_l(x) - x) \\ -1_{\{-\Phi_l(x) - x > 0\}} \mu(2h) \frac{\partial \chi}{\partial \epsilon} (\eta_{1l}, \Phi_l(x) + x) \end{array} \right)}. \]

Now, one checks that the numerator is positive, bounded on \( \Omega \), and also appears in the denominator. Further, the second part of the denominator,

\[ r + \lambda_{12} e^x + \lambda_{21} e^{-x}, \]

44
is also positive, and bounded on $\Omega$. But then, there exists a constant $C \in (0, 1)$, independent of the choice of $x$, so that

$$|\Phi'(x)| < C.$$ 

Finally, remembering the monotonicity of the function in (61), there is at most one point in $\Omega$ where (62) does not hold and where, as a result, $\Phi_t$ is not differentiable.

Summing up, the restriction of $\Phi_t$ to $\Omega$ maps a compact into itself, is continuous, is differentiable everywhere but possibly at one point, and has a derivative whose absolute value that is bounded strictly below one.

This argument can be adapted for $\Phi_h$, and Lemma 25 then ensures that $\Phi$ admits a unique fixed point $(\Delta_l^*, \Delta_h^*)$ in $\Omega$. Finally, as the range of $\Phi$ is already contained in $\Omega$, this is the unique fixed point over $\mathbb{R}^2$.

**Step 4** Finally, given the fixed point of $\Phi$, the solution $\beta$ to the HJB equations (19) can be recovered. For example,

$$\beta(1h) = -\frac{1}{\rho} \left( -\kappa(1h) + \lambda_{12} (e^{\Delta_h^*} - 1) + 2\lambda \mu (2l) \right),$$ 

In particular, there is exactly one solution to the system of HJB equations (19). $\square$

**C.2 Verification argument**

I intend to show that the HJB equations (11) actually describe an optimal behaviour.

Being more specific, on the one hand, a given agent with wealth $w$ and type $i\theta$ maximizes

$$V(w, i\theta) \equiv \mathbb{E} \left[ \int_0^\infty e^{-\rho t} (-e^{-\gamma W_t}) \, ds \bigg| w_0 = w, i_0\theta_0 = i\theta \right],$$ 

where

- the budget constraint
  $$dw_t = rw_t \, dt - \delta_t \, dt + d\eta_t + \theta_t \, dD_t + \pi_t (dD_t - rP_c \, dt) - P_{dt} \, d\theta_t$$
  holds for a liquid holding process taking values in $[-K, K]$, with $K$ positive and large$^{34}$;
- the price $P_d$ is the outcome of a bargaining with another agent;
- for any $T > 0$,
  $$\mathbb{E}^{w, i\theta} \left[ \int_0^T (e^{-\rho u} e^{-\gamma W_u})^2 \, du \right] < +\infty$$ 

and
  $$\lim_{T \to \infty} e^{-\rho T} \mathbb{E}^{w, i\theta} \left[ e^{-\gamma W_T} \right] = 0.$$ 

$^{34}$See footnote 10 for conditions on $K$. 

45
On the other hand, the HJB equation for the problem above is
\[
\rho V(w, i\theta) = \sup_{\tilde{c}, \tilde{\pi}} U(\tilde{c})
\]
\[
+ \frac{\partial V(w, i\theta)}{\partial w} (r w - \tilde{c} + m_\eta + \theta m_d + \tilde{\pi} (m_c - r P_c))
\]
\[
+ \frac{1}{2} \frac{\partial^2 V(w, i\theta)}{\partial w^2} (w, i\theta) \left( \begin{array}{cc} 1 & \theta \\ \theta & \tilde{\pi} \end{array} \right) \Sigma_i \left( 1 \begin{array}{c} 1 \\ \theta \end{array} \right) \tilde{\pi}^* (w, i\theta)
\]
\[
+ \lambda_{ii} (V(w, \tilde{i}\theta) - V(w, i\theta))
\]
\[
+ 2\lambda E_{\mu(a)} \left[ V(w - (\theta_a - \theta) P_d^{(w,i\theta);\mu}; \tilde{\pi}_a) - V(w, i\theta) \right]^+.
\]

and developments in Section 3 show that there exists exactly one choice of \(a \in \mathbb{R}^4, \delta \in \mathbb{R}\), so that \(\tilde{V}(w, i\theta) = -\exp(-r\gamma(w + a(i\theta) + \tilde{a}))\) is a solution to (67) (\(\tilde{a}\) was defined in (17)). It remains to show that, actually, the candidate value function \(\tilde{V}\) is optimal for the problem (64). I show that, under conditions, this is the case.

**Proposition 28. If the risk-free rate \(r > 0\) is small enough, and if the meeting intensity \(\lambda\) on the OTC market is large enough, then the solution \(\tilde{V}\) to the HJB equations (67) and the associated consumption and investment strategies are optimal.**

**Proof of Proposition 28.** My argument comprises four steps.

- in Lemma 29 I show that no admissible strategy can achieve an expected utility higher than \(\tilde{V}\);
- in Lemma 30 I show that under certain conditions, the strategy dictated by \(\tilde{V}\) is admissible;
- in Lemma 31 I show that, when \(r\) is small enough and \(\lambda\) large enough, the conditions of Lemma 30 are satisfied;
- in Lemma 32 I show that the expected utility \(\tilde{V}\) can be achieved.

I first show that \(\tilde{V}\) represents an upper bound on the attainable expected utilities.

**Lemma 29. If all the agents believe that their value function is given by \(\tilde{V}\), then, for any admissible consumption strategy \(c\) financed by the trading strategy \(\pi\),

\[\tilde{V}(w, i\theta) \geq E_{\mu,i\theta} \left[ \int_{0}^{\infty} e^{-\rho u} U(c_u) \, du \right].\]

**Proof.** First note that the beliefs regarding the value functions will already fix the outcome of the Nash bargaining, meaning that both the price \(P_d\) of the illiquid asset and the cross-sectional distribution of types \(\mu\) are fixed.

46
Let me choose an admissible consumption strategy \( c \) financed by the trading strategy \( \pi \), and a time \( T > 0 \). Recalling the budget constraint and the definition of “\( \alpha_t \)” in (3),

\[
E \left[ \int_0^T e^{-\rho u} U(c_u) \, du + e^{-\rho T} \tilde{V}(w_T, i_T, \theta_T) \right] = E \left[ \int_0^T e^{-\rho u} U(c_u) \, du + \tilde{V}(w_0, i_0, \theta_0) + \int_0^T d \left( e^{-\rho u} \tilde{V}(w_u, i_u, \theta_u) \right) \right]
\]

\[
= \tilde{V}(w_0, i_0, \theta_0) + \int_0^T e^{-\rho u} U(c_u) \, du + \int_0^T e^{-\rho u} d \left( \tilde{V}(w_u, i_u, \theta_u) \right)
\]

\[
\begin{pmatrix}
U(c_u) \, du \\
- \rho \tilde{V}(w_u, i_u, \theta_u) \, du \\
+ \frac{\partial \tilde{V}}{\partial w}(w_u, i_u, \theta_u) \left( (rw_u - c_u) \, du \\
+ \theta_u \, dW_u \\
+ \eta_u \, dD_{\eta u} \\
+ \alpha_u(i_u) \, d\alpha_d(i_u) + \theta_u \sigma_d \\
+ \pi_u \Sigma \left( \theta_u \pi_u \right)^* \, d\pi_u \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tilde{V}(w_0, i_0, \theta_0) \\
- \rho e^{-\rho u} \tilde{V}(w_u, i_u, \theta_u) \, du \\
+ \frac{\partial \tilde{V}}{\partial w}(w_u, i_u, \theta_u) \left( (rw_u - c_u) \, du \\
+ \theta_u \, dW_u \\
+ \eta_u \, dD_{\eta u} \\
+ \alpha_u(i_u) \, d\alpha_d(i_u) + \theta_u \sigma_d \\
+ \pi_u \Sigma \left( \theta_u \pi_u \right)^* \, d\pi_u \\
\end{pmatrix}
\]

\[
\Delta \equiv (\star).
\]

Now, defining

\[
K_1 \equiv (r\gamma)^2 \sup_{j_0, j_\sigma \in \mathcal{T}} \int_{\Gamma} e^{-2(\beta(j_\sigma) + \delta)} \left( 1 \sigma \tilde{\pi} \right) \Sigma_j \left( 1 \sigma \tilde{\pi} \right)^* \in \mathbb{R},
\]

\[
\pi \in [-K, K]
\]

47
and recalling the admissibility condition (66) on \((c, \pi)\), I may write
\[
E \left[ \left( \int_0^t e^{-\rho u} \frac{\partial \tilde{V}}{\partial w}(w_u, i_u \theta_u) \left( \begin{array}{c} \alpha_d(i_u) \\ \alpha_d(i_u) + \theta_u \sigma_d \\ \alpha_d(i_u) + \pi_u \sigma_c \end{array} \right) \cdot \left( \begin{array}{c} dZ_u \\ dB_{d,u} \\ dB_{c,u} \end{array} \right) \right)^2 \right]
\]
\[
= E \left[ \left( \int_0^t e^{-\rho u} \frac{\partial \tilde{V}}{\partial w}(w_u, i_u \theta_u) \right)^2 \left( \begin{array}{c} \theta_u \\ \pi_u \end{array} \right) \Sigma_i \left( \begin{array}{c} \theta_u \\ \pi_u \end{array} \right)^* \right] du
\]
\[
\leq K_1 E \left[ \int_0^t e^{-2(\rho u + r \gamma w)} du \right]
\]
\[< \infty.\]

In particular, in (*), the stochastic integrals against the Brownian motions are true martingales, and their expected values equal zero.

I now turn to the stochastic integrals against the Poisson processes. Keeping in mind the admissibility condition (65),
\[
\int_0^t \left| \tilde{V}(w_u, i_u \theta_u) - \tilde{V}(w_u, i_u \theta_u) \right| du
\]
\[
\leq \sup_{i \in T} e^{-r \gamma (a(i_u \theta_u) + \bar{a})} - e^{-r \gamma (a(i_u \theta_u) + \bar{a})} \int_0^t e^{-r \gamma w} du
\]
\[< \infty.\]

But then, using a classical result (see, for example, Brémaud [1981][Lemma C4, p.235]),
\[
E \left[ \int_0^T e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \right] dN_{u,i}
\]
\[
= E \left[ \int_0^T e^{-\rho u} \lambda u \tilde{V}(w_u, i_u \theta_u) \right] du.
\]

Similarly,
\[
E \left[ \int_0^T e^{-\rho u} \max \left\{ 0, \tilde{V}(w_u, i_u \theta_u) \right\} dN_{u,i}
\]
\[
= E \left[ \int_0^T e^{-\rho u} 2 \lambda u \tilde{V}(w_u, i_u \theta_u) \right] du.
\]
I may thus write

\[
(\ast) = \mathbb{E} \left[ \begin{array}{c}
\tilde{V}(w_0, i_0 \theta_0) \\
+ \int_0^T e^{-\rho u} \left( -\rho e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \\
- \rho e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \\
+ \partial_\theta \tilde{V}(w_u, i_u \theta_u) \left( r w_u - c_u + m_q + \theta_u m_d + \pi_u (m_c - r P_c) \right) \\
+ \lambda_i \tilde{V}(w_u, i_u \theta_u) - \tilde{V}(w_u, i_u \theta_u) \\
+ 2\lambda \mu (i_u \tilde{\theta}_u) \max \left\{ 0, \tilde{V}(w_u - (\tilde{\theta}_u - \theta_u) P_d, i_u \theta_u) \right\} \\
\right) \right] du
\]

\[
\tilde{V}(w_0, i_0 \theta_0) \leq \mathbb{E} \left[ \begin{array}{c}
\tilde{V}(w_0, i_0 \theta_0) \\
+ \int_0^T e^{-\rho u} \sup_{\tilde{\pi}, \tilde{\theta}} \left( -\rho e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \\
- \rho e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \\
+ \partial_\theta \tilde{V}(w_u, i_u \theta_u) \left( r w_u - \tilde{\theta} + m_q + \theta_u m_d + \tilde{\pi} (m_c - r P_c) \right) \\
+ \lambda_i \tilde{V}(w_u, i_u \theta_u) - \tilde{V}(w_u, i_u \theta_u) \\
+ 2\lambda \mu (i_u \tilde{\theta}_u) \max \left\{ 0, \tilde{V}(w_u - (\tilde{\theta}_u - \theta_u) P_d, i_u \theta_u) \right\} \\
\right) \right] du
\]

\[
= \tilde{V}(w_0, i_0 \theta_0).
\]

Taking things together, this means that, for any \( T > 0, \)

\[
\tilde{V}(w_0, i_0 \theta_0) \geq \mathbb{E} \left[ \int_0^T e^{-\rho u} U(c_u) \ du \right] + e^{-\rho T} \mathbb{E} \left[ \tilde{V}(w_T, i_T \theta_T) \right].
\]

Letting \( T \) become arbitrarily large in this last expression, recalling the admissibility condition (66) satisfied by the strategy \((c, \pi),\) and realizing that the process

\[(a(i_t \theta_t))_{t \geq 0}\]

can only take one of four finite values, yields

\[
\tilde{V}(w_0, i_0 \theta_0) \geq \lim_{t \to \infty} \mathbb{E} \left[ \int_0^t e^{-\rho u} U(c_u) \ du \right] + e^{-\rho t} \mathbb{E} \left[ \tilde{V}(w_t, i_t \theta_t) \right]
\]

\[
\geq \mathbb{E} \left[ \int_0^\infty e^{-\rho u} U(c_u) \ du \right] + \lim_{t \to \infty} e^{-\rho t} \mathbb{E} \left[ -e^{-r \gamma w_t} \sup_{\tilde{\theta} \in T} e^{-r \gamma (a(i_t \theta_t)+\tilde{\theta})} \right]
\]

\[
= \mathbb{E} \left[ \int_0^\infty e^{-\rho u} U(c_u) \ du \right].
\]
As the consumption and trading strategies were arbitrary, this concludes.

I now propose a condition under which the strategy dictated by the HJB equations is admissible.

**Lemma 30.** Assuming that, for any \(i\theta \in \mathcal{T}\),

\[
ra(i\theta) - \kappa(i\theta) \leq \frac{1}{\gamma},
\]

then, the strategy \((\hat{c}_t, \hat{\pi}_t)\) dictated by the optimization in the HJB equation is admissible.

**Proof.** The candidate strategy must satisfy two admissibility properties. The first one is (65), meaning

\[
E \left[ \int_0^T \left( e^{-\rho u} e^{-r\gamma \hat{w}_u} \right)^2 \, du \right] < \infty.
\]

Now, from Proposition 5, the optimal consumption policy is

\[
\hat{c}(i\theta, w) = r(w + a(i\theta) + \bar{a}) - \frac{1}{\gamma} \log(r),
\]

and the resulting wealth dynamics are

\[
d\hat{w}_t = \left( -r(a(i\theta) + \bar{a}) + \frac{1}{\gamma} \log(r) + \theta_t dD_{d,t} + \hat{\pi}_t (dD_{d,t} - rP_d dt) - P_d d\theta_t. \right.
\]

I may thus write

\[
\hat{w}_t - w_0 = \int_0^t \left( -r(a(i\theta) + \bar{a}) + \frac{1}{\gamma} \log(r) + \theta_t dD_{d,t} + \hat{\pi}_t (dD_{d,t} - rP_d dt) - P_d d\theta_t. \right.
\]

In particular, recalling that the Brownian motions and Poisson processes are independent, and defining, for \(t \geq 0\),

\[
m_t \triangleq \int_0^t \left( -r(a(i\theta) + \bar{a}) + \frac{1}{\gamma} \log(r) + \theta_t dD_{d,t} + \hat{\pi}_t (dD_{d,t} - rP_d dt) - P_d d\theta_t. \right.
\]

and

\[
s_t^2 \triangleq \int_0^t \left( 1 \theta_t \hat{\pi}(i_u \theta_u) \right) \Sigma_i \left( \frac{1}{\hat{\pi}(i_u \theta_u)} \right) \, du,
\]

I know that the distribution of the wealth conditional on the history of the correlation shocks and OTC trades is

\[
\mathcal{L} \left( \hat{w}_t \mid (i_u \theta_u)_{0 \leq u \leq t} \right) = \mathcal{N} \left( m_t, s_t^2 \right).
\]
Further, for \( t \geq 0 \), and defining the two constants

\[
K_2 = \min_{i\theta \in T} \{ -ra(i\theta) + \theta m_d + \pi (m_c - rP_c) \}
\]

and

\[
K_3 = \sup_{i\theta \in T} \{ 1 - (i_u \theta_u) \Sigma_i \left( \frac{1}{\theta_u} \hat{\pi} (i_u \theta_u) \right) \},
\]

I can write both

\[
m_t \geq t \left( K_2 + \frac{1}{\gamma} \log(r) - r\bar{a} + m_\eta \right) - |P_d| \Delta_{\theta_t}
\]

and

\[
s_t^2 \leq tK_3.
\]

As a result,

\[
\mathbb{E} \left[ \int_0^T \left( e^{-\rho u} e^{-r\gamma \hat{w}_u} \right)^2 \, du \right] = \int_0^T e^{-2\rho u} \mathbb{E} \left[ e^{-2r\gamma \hat{w}_u} \right] \, du
\]

\[
= \int_0^T e^{-2\rho u} \mathbb{E} \left[ e^{-2r\gamma m_u + 2(r\gamma)^2 s_t^2} \right] \, du
\]

\[
\leq \int_0^T e^{-2\rho u - 2r\gamma u \left( K_2 + \frac{1}{\gamma} \log(r) - r\bar{a} + m_\eta \right) + 2r\gamma |P_d| \Delta_{\theta_t} + 2u(r\gamma)^2 K_3} \, du
\]

\[
\leq e^{2r\gamma |P_d| \Delta_{\theta_t} \int_0^T e^{-2u \left( \rho + r\gamma \left( K_2 + \frac{1}{\gamma} \log(r) - r\bar{a} + m_\eta \right) - (r\gamma)^2 K_3 \right)} \, du
\]

\[
< \infty.
\]

I must still show that the candidate policy satisfies the transversality condition (66), meaning that

\[
\lim_{T \to \infty} e^{-\rho T} \mathbb{E} \left[ e^{-r\gamma \hat{w}_T} \right] = 0.
\]

The argument is similar to the one in the first part. Namely, for a given \( T > 0 \),

\[
e^{-\rho T} \mathbb{E} \left[ e^{-r\gamma \hat{w}_T} \right] = e^{-\rho T} \mathbb{E} \left[ e^{-r\gamma \hat{w}_T \mid (i_s \theta_s)_{0 \leq s \leq t}} \right]
\]

\[
= \mathbb{E} \left[ e^{-\rho T - r\gamma m_T + \frac{1}{\gamma} (r\gamma)^2 s_T^2} \right],
\]

Now, recalling the definition

\[
\bar{a} = \frac{1}{r\gamma} \left( -1 + \frac{\rho}{r} + \gamma m_\eta + \log(r) \right)
\]
from (19),
\[
- \rho T - r \gamma m_T + \frac{1}{2} (r \gamma)^2 s_T^2 = \int_0^T \left( -\rho - r \gamma \left( \frac{1}{2} \log(r) - r (a(i_u \theta_u) + \bar{a}) + m_u + \theta_u m_d + \bar{\pi}_u (m_c - r P_u) \right) + \frac{1}{2} (r \gamma)^2 \left( 1 \ \theta_u \ \bar{\pi}(i_u \theta_u) \right) \Sigma_i (1 \ \theta_u \ \bar{\pi}(i_u \theta_u)^*) \right) \, du \\
- r \gamma P_d (\theta_t - \theta_0) = \int_0^T r \gamma \left( -\frac{1}{\gamma} + ra(i_u \theta_u) - \kappa(i_u \theta_u) \right) \, du - r \gamma P_d (\theta_t - \theta_0). \tag{70}
\]

Now, from (68), there exists a \( \epsilon > 0 \) so that
\[
\int_0^T \left( -\frac{1}{\gamma} + ra(i_u \theta_u) - \kappa(i_u \theta_u) \right) \, du \leq \int_0^T -\epsilon \, du = -\epsilon T. \tag{71}
\]

Finally, combining (69), (70), and (71),
\[
0 \leq \lim_{T \to \infty} e^{-\rho T} E \left[ e^{-r \gamma \bar{w}_T} \right] \leq \lim_{T \to \infty} e^{r \gamma |P_d| \Delta \theta} e^{-r \gamma \epsilon T} = 0,
\]
as stated.

The condition (68) in Lemma ?? is not really practical, if only because it involves the vector \( a \), that is not known in closed form. The next lemma however states that it holds in certain limit cases.

**Lemma 31.** Assuming \( r \) small enough and \( \lambda \) large enough, the strategy dictated by the HJB equations, meaning the one described in Proposition 5, is admissible.

**Proof.** I show that Assumption (68) holds, and conclude using Lemma 30.

From the HJB equations (19),
\[
0 = ra(i \theta) - \kappa(i \theta) + \lambda_{\bar{i}} \left( e^{r \gamma(a(i \theta) - a(\bar{i} \theta))} - 1 \right) - 2 \lambda \mu(\bar{i} \bar{\theta}) [\chi(\eta_{\bar{i} \theta}, \epsilon_{i \theta}(a))]^- \\
\geq ra(i \theta) - \kappa(i \theta) + \lambda_{\bar{i}} r \gamma (a(i \theta) - a(\bar{i} \theta)) - 2 \lambda \mu(\bar{i} \bar{\theta}) [\chi(\eta_{\bar{i} \theta}, \epsilon_{i \theta}(a))]^-.
\]

Rearranging, and using the same argument a second time,
\[
a(i \theta) \leq \frac{1}{r + \lambda_{\bar{i}} r \gamma} \left( \lambda_{\bar{i}} r \gamma a(\bar{i} \theta) + \kappa(i \theta) + 2 \lambda \mu(\bar{i} \bar{\theta}) [\chi(\eta_{\bar{i} \theta}, \epsilon_{i \theta}(a))]^- \right) \leq \frac{1}{r + \lambda_{\bar{i}} r \gamma} \left( \lambda_{\bar{i}} r \gamma a(\bar{i} \theta) + \kappa(i \theta) + 2 \lambda \mu(\bar{i} \bar{\theta}) [\chi(\eta_{\bar{i} \theta}, \epsilon_{i \theta}(a))]^- \right).
\]

52
As a result,

\[
a(i\theta) \leq \frac{1}{r + \lambda i r^\gamma} \left( \frac{\lambda_i r^\gamma}{r + \lambda_i r^\gamma} \left( \begin{array}{l} \kappa(i\theta) + 2\lambda\mu(i\theta) \left[ \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) \right] - \\
+ \kappa(i\theta) + 2\lambda\mu(i\theta) \left[ \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)) \right] - 
\end{array} \right) \right)
\]

Now, combining this last bound with the definition of the “\(\kappa(i\theta)\)"s and the asymptotic behaviours of the expected benefits from OTC trading, I may, for example, write

\[
ra(1h) - \kappa(1h) \leq r \left( \frac{2S_c \sigma_c (\rho_{1,c} - \rho_{2,c})}{\gamma} \right) \left( \frac{-\sigma_y (\mu_{2*} - \mu_{1*}) (\rho_{1,c} - \rho_{2,c})^2}{\gamma} \right) \left( \frac{2\sigma_d (S_d (\rho_{1,c} - \rho_{2,c}) \rho_{c,d} + \Delta \Pi)}{r + \lambda_1 + \lambda_2} \right) + o\left( \frac{1}{\lambda} \right).
\]

Now, the first term on the right hand side goes to zero with \(r\). In particular, if \(r\) is chosen small enough for this same quantity to be smaller than \(1/\gamma\), one may then choose \(\lambda\) large enough for the right hand side of this last equation to be smaller than \(1/\gamma\) as well. Similar arguments regarding the agent types 1l, 2l, and 2h show that, if \(r\) is small enough and \(\lambda\) large enough (the bound may depend on \(r\)), (68) holds, and the candidate strategy is admissible. □

The last step is to show that the beliefs are rational. In other words, I must show that the strategy dictated by \(\tilde{V}\) and the HJB equations indeed generates an expected utility from consumption equal to \(\tilde{V}\).

**Lemma 32.** Assuming that (68) holds, and writing \((\hat{c}, \hat{\pi})\) for the strategy dictated by the HJB equations, then

\[
\tilde{V}(w, i\theta) = \mathbb{E} \left[ \int_0^T e^{-\rho_u} U(\hat{c}_u) \right] w_0 = w, i_0 = i\theta
\]

**Proof.** Thanks to the admissibility of the candidate policy, first, the process

\[
\left( \int_0^t e^{-\rho_u} U(c_u) \, du + e^{-\rho u} \tilde{V}(w_u, i_u \theta_u) \right)_{t \geq 0}
\]

is a martingale and, second,

\[
\lim_{T \to \infty} e^{-\rho T} \mathbb{E} \left[ -e^{-r \gamma w_T} \right] = 0.
\]

One may then conclude that

\[
\tilde{V}(w_0, i_0 \theta_0) = \mathbb{E} \left[ \int_0^T e^{-\rho_u} U(\hat{c}_u) \, du \right],
\]

meaning that the expected utility \(\tilde{V}\) can be achieved with an admissible strategy. □

This concludes the proof of Proposition 28. □
C.3 Proofs for Section 4

Proof of Proposition 8. The proof is presented in Duffie et al. [2005]. I only give a partial sketch to introduce certain notations.

There are three linear relations linking the components of a stationary distribution \( \mu \). They follow from the stationary distribution of endowment correlation types presented in (7), and from the market clearing condition (24), and are

\[
\begin{align*}
\mu(1l) + \mu(1h) &= \mu_1, \\
\mu(2l) + \mu(2h) &= \mu_2, \\
\mu(1h) + \mu(2h) &= \frac{S_d}{\Delta \theta}.
\end{align*}
\]  

(72)

One can then use these equations to express one of the flow conditions (21) as an equation in, say, \( \mu(2l) \) only. This yields the quadratic equation

\[
0 = \mu(2l)^2 + b(\xi)\mu(2l) + c(\xi) \triangleq Q(\mu(2l), \xi),
\]

where

\[
b(\xi) \triangleq \frac{S_d}{\Delta \theta} - \mu_2 + \xi \frac{\lambda_{12} + \lambda_{21}}{2},
\]

\[
c(\xi) \triangleq -\xi \frac{\lambda_{12}}{2} \left( 1 - \frac{S_d}{\Delta \theta} \right).
\]

Solving this equation already characterize a unique candidate. \( \Box \)

I will use the following results when proving Corollary 11 and Proposition 15. They follow from the characterization (73).

**Lemma 33.** The sensitivity of the stationary cross-sectional distribution of types to the illiquidity level is so that

\[
\frac{\partial}{\partial \xi} \mu(1h) = \frac{\partial}{\partial \xi} \mu(2l) = -\frac{\partial}{\partial \xi} \mu(1l) = -\frac{\partial}{\partial \xi} \mu(2h) = \frac{-\mu(2l) \frac{\lambda_{12} + \lambda_{21}}{2} + \frac{\lambda_{12}}{2}}{\mu(1h) + \mu(2l) + \xi \frac{\lambda_{12} + \lambda_{21}}{2}},
\]

(74)

which is positive. Also,

\[
\frac{\partial}{\partial \xi} (\lambda \mu(1h) \mu(2l)) = \frac{1}{\xi^2} \frac{\mu(2l) \mu(1h) \frac{1}{2 \xi} (\lambda_{12} + \lambda_{21})}{\mu(2l) + \mu(1h) + \xi \frac{\lambda_{12} + \lambda_{21}}{2}},
\]

(75)

which is negative. Finally, for \( i \theta \in \{1h, 2l\} \),

\[
\frac{\partial}{\partial \xi} (\lambda \mu(i \theta))
\]

is negative as well.
Proof of Lemma 33. For the first statement, the sensitivity of \( \mu(2l) \) on the illiquidity level follows from an application of the Implicit Function Theorem. The relation between the various sensitivities then follows from (72).

Now, recalling Equations (21) and (24), I deduce from (74) that

\[
\frac{\partial}{\partial \xi} \mu(1h) = \frac{\partial}{\partial \xi} \mu(2l) = \frac{\lambda \mu(1h) \mu(2l)}{\mu(1h) + \mu(2l) + \xi \frac{\lambda_{12} + \lambda_{21}}{2}},
\]

which is positive. A direct calculation then yields (75). Finally the last sensitivity follows from the elementary observation that, if the product of two positive functions is increasing, and if the first term in the product is decreasing, then the second one must be increasing. \( \square \)

Based on the previous Lemma, I can, conditionally on the trading pattern, characterize the behaviour of the type distribution when the illiquidity level becomes small. This will be used in the proofs of Proposition 10 and of Proposition 15.

**Lemma 34.** Under Assumption 7, meaning under the assumption that agents with endowment correlation type 2 buy the illiquid asset, and further assuming that

\[
\frac{S_d}{\Delta \theta} - \mu_{2*} > 0,
\]

then, the distribution of types is so that

\[
\begin{pmatrix}
\mu(1l) \\
\mu(1h) \\
\mu(2l) \\
\mu(2h)
\end{pmatrix} = \begin{pmatrix}
1 - \frac{S_d}{\Delta \theta} \\
\frac{1}{2} \left( \frac{S_d}{\Delta \theta} - \mu_{2*} \right) + \frac{\lambda_{12}}{\lambda} - \frac{S_d}{\Delta \theta} - \mu_{2*}
\end{pmatrix} + \begin{pmatrix}
-1 \\
1 \\
1 \\
-1
\end{pmatrix}
\]

where \( 1_4 \in \mathbb{R}^4 \) is the vector whose components all equal 1.

**Proof of Proposition 34.** From the proof of Proposition 8,

\[
\mu(2l, \lambda) = \frac{1}{2} \left( -b(\lambda) + \sqrt{b(\lambda)^2 - 4c(\lambda)} \right).
\]

Now, as

\[
\lim_{\lambda \to \infty} b(\lambda) = \frac{S_d}{\Delta \theta} - \mu_{2*},
\]

which I assumed to be positive, and

\[
\lim_{\lambda \to \infty} c(\lambda) = 0,
\]

it follows that

\[
\lim_{\lambda \to \infty} \mu(2l, \lambda) = 0.
\]

Recalling (72), this yields the asymptotic distribution.
Now, using the previous lemma yields
\[
\partial_\xi \mu(2l) = \frac{-\mu(2l) \lambda_{12} \lambda_{21}}{2\mu(2l) + \frac{s_d}{\Delta \theta}} - \mu_{2\bullet} + \xi \frac{\lambda_{12} \lambda_{21}}{2} \frac{(1 - S_d \Delta \theta)}{s_d - \mu_{2\bullet}},
\]
and (77).

\[\text{C.4 Proofs for Section 5}\]

\[\text{Proof of Proposition 10.}\] Keeping Proposition 5 in mind, the equilibrium condition for the centralized market becomes
\[
S_c = E^\mu(i\theta) [\pi(i\theta)] = \frac{1}{\sigma_c^2} \left( \frac{1}{r} (m_c - r P_c) - \sigma_c \sigma_c E^\mu(i\theta) [\rho_{1c}] - \rho_{cd} \sigma_c E^\mu(i\theta) [\theta] \right).
\]
Now, realizing that
\[
E^\mu(i\theta) [\rho_{1c}] = \mu_{1\bullet} \rho_{1c} + \mu_{2\bullet} \rho_{2c}
\]
is independent of the trading on the OTC market, and that, thanks to the market clearing condition (24), so is
\[
E^\mu(i\theta) [\rho_{1c}] = S_d,
\]
I can already solve for the equilibrium price, which yields (28). Then, combining this last result and the characterization (16) of the optimal liquid holdings yields the expression (29) in the statement.

I now turn to the OTC market. The existence and uniqueness follows from two elementary observations. First, the value function of a, say, 1h-agents is only impacted by \( \mu \) via \( \mu(2l) \) and only as long as \( \epsilon_{1h}(a) > 0 \). Otherwise, 1h-agents have no intention to trade and, as a result, no interest in knowing how often a counterparty may be met. In mathematical terms, this reads
\[
\mu(a, 2l) [\chi(\eta_{i\theta}, \epsilon_{i\theta}(a))]^- = \mu(a, 2l) 1_{\{\epsilon_{1h}(a) > 0\}} \chi(\eta_{i\theta}, \epsilon_{i\theta}(a))
\]
\[
= \mu_{1h-2l}^{(2l)} 1_{\{\epsilon_{1h}(a) > 0\}} \chi(\eta_{i\theta}, \epsilon_{i\theta}(a))
\]
\[
= \mu_{1h-2l}^{(2l)} [\chi(\eta_{i\theta}, \epsilon_{i\theta}(a))]^-.
\]
In particular, this means that the equilibrium system (27) is equivalent to
\[
\forall i\theta \in \mathcal{T} : 0 = F_{i\theta} (\hat{\mu}, a),
\]
where the vector \( \hat{\mu} \) is given by
\[
\hat{\mu} = \left( \begin{array}{c}
\mu_{2h-1l}^{(1l)} \\
\mu_{1h-2l}^{(1h)} \\
\mu_{1h-2l}^{(2l)} \\
\mu_{2h-1l}^{(2h)}
\end{array} \right)
\]
This vector does not depend on \( a \) but does not, in general, define a density any more.
The second observation is that the proof of Proposition 6 remains valid when the components of \( \mu \) are only positive numbers, and do not necessarily sum up to one. As a result, there is exactly one solution to (78), which shows the uniqueness and existence of an equilibrium.

I must still characterize the ordering of the valuations of the illiquid asset \( d \) or, equivalently, characterize the trading pattern on the OTC market. To do so I first characterize the ordering when the OTC market becomes arbitrarily liquid, and then show that this ordering is maintained at any illiquidity level. The actual argument is articulated around three claims.

**Claim 1** I first show that an equilibrium \( a \) of the model can be bounded by constants that are independent of the illiquidity level.

**Proof of Claim 1.** Let \( \{\lambda_n\}_{n \geq 0} \) be a sequence of intensities be given, and let \( \{a_n\}_{n \geq 0} \) be the corresponding sequence of equilibria, meaning that

\[
\forall n, i\theta : F_{i\theta} (\hat{\mu}(\lambda_n), a_n) = 0. \tag{79}
\]

Let me assume, for the sake of contradiction, that there is an agent type \( i\theta \) so that the sequence \( \{a_n(i\theta)\}_{n \geq 0} \) is unbounded. I first assume it is unbounded below, meaning that, maybe up to taking a subsequence,

\[
\lim_{n \to \infty} a_n(i\theta) = -\infty, \tag{80}
\]

From equation (79), and recalling the definition of \( F_{i\theta} \) in (54),

\[
\frac{\lambda_{i\theta}}{r^\gamma} \left( e^{a_n(i\theta)} - a_n(i\theta) - 1 \right) = -r a_n(i\theta) + \kappa(i\theta) + \frac{2\lambda_n}{r^\gamma} \hat{\mu}(\lambda_n, \overline{i\theta}) [\chi(\eta_i, \epsilon_{i\theta}(a))] - . \tag{81}
\]

But, recalling (80), the left hand side of (81) is bounded below by a sequence that grows arbitrarily. As a result,

\[
\lim_{n \to \infty} a_n(i\theta) - a_n(i\theta) = +\infty. \tag{82}
\]

and, recalling (80) one more time,

\[
\lim_{n \to \infty} a_n(i\theta) = -\infty. \tag{83}
\]

Now, if (82) follows from (80), from (83) I can conclude that

\[
\lim_{n \to \infty} a_n(i\theta) - a_n(i\theta) = +\infty. \tag{84}
\]

In particular, both (82) and (84) follow from (80), which is impossible. There is thus no sequence of equilibria that is unbounded below.
It remains to see whether a sequence of equilibria can be unbounded above. Let me assume that, maybe choosing a subsequence,

$$\lim_{n \to \infty} a_n(1h) = +\infty.$$  \hfill (85)

Before pursuing the argument I note that, assuming an agent of type $i\theta$ does not trade in equilibrium, it follows from (79) that

$$0 = ra_n(i\theta) - \kappa(i\theta) + \frac{\lambda i}{r\gamma} \left(e^{a_n(i\theta) - a_n(i\theta) - 1}\right) - 2\lambda_n \tilde{\mu}(\lambda, i\tilde{i}) \left[\chi(\eta_{i\theta}, \epsilon_{i\theta}(a))\right] -$$

$$= ra_n(i\theta) - \kappa(i\theta) + \frac{\lambda i}{r\gamma} \left(e^{a_n(i\theta) - a_n(i\theta) - 1}\right)$$

$$\geq ra_n(i\theta) - \kappa(i\theta) - \frac{\lambda i}{r\gamma}$$

In other words, I have an a priori upper bound on $\beta_n(i\theta)$. Namely,

$$\frac{1}{r} \left(\kappa(i\theta) + \frac{\lambda_i}{r\gamma}\right) \geq a_n(i\theta), \hfill (86)$$

Now, two further cases must be distinguished, depending on whether $1h$-agents are willing to trade or not. Maybe choosing a further subsequence, I assume that $1h$-agents never trade. In this case, combining (86) and (85) yields

$$\frac{1}{r} \left(\kappa(1h) + \frac{\lambda_{12}}{r\gamma}\right) \geq \lim_{n \to \infty} a_n(1h) = +\infty,$$

which is a contradiction. The only possibility left is thus for the agents with type $1h$ are willing to trade. I can thus assume that, for any $n \geq 0$,

$$a_n(1l) - a_n(1h) - a_n(2l) + a_n(2h) \geq 0.$$ 

Using (86) for the two types of agent that do not trade, meaning $1l$ and $2h$, then yields

$$\frac{1}{r} \left(\kappa(1l) + \kappa(2h) + \frac{\lambda_{12}}{r\gamma} + \frac{\lambda_{21}}{r\gamma}\right) \geq a_n(1h) + a_n(2l).$$

From this last inequality and (85) I deduce that

$$\lim_{n \to \infty} a_n(2l) = -\infty,$$

which will, by the first part of this proof, lead to a contradiction.

To sum up, there are no circumstances under which an unbounded sequence of equilibria can be found. \hfill \square

This first claim is needed when proving the second one, which follows.
Claim 2  For a sufficiently large meeting intensity $\lambda$, the corresponding equilibrium $a(\lambda)$ is so that
\[
\epsilon_{1h}(a) > 0
\]
exactly when $\Pi > 0$.

Proof of Claim 2. Let me choose a sequence $\{\lambda_n\}_{n \geq 0}$ of meeting intensities so that
\[
\lim_{n \to \infty} \lambda_n = +\infty.
\]
By Claim 1, there exists two constants $L < U$ so that
\[
\forall n : a_n \in [L, U]^4
\]
I can thus choose a convergent subsequence, and call the limit $a_\infty$. Maybe choosing a further subsequence, I assume that
\[
\forall n : \epsilon_{1h}(a_n) = a_n(2h) - a_n(2l) + a_n(1l) - a_n(1h) \geq 0.
\]
In other words, all along the sequence of intensities, and in the limit, agents with endowment correlations type 2 have the high valuation of the illiquid asset.

Under this assumption the HJB equations (19) defining $a_n$ become
\[
\begin{align*}
0 &= r a_n(1l) - \kappa(1l) + \frac{\lambda_1}{\gamma} (e^{a_n(1l)} - a_n(2l) - 1) \\
0 &= r a_n(1h) - \kappa(1h) + \frac{\lambda_2}{\gamma} (e^{a_n(1h)} - a_n(2h) - 1) + 2\lambda_n \hat{\mu}(\lambda_n, 2l) \chi(\eta_{2l}, \epsilon_{2l}(a)) \\
0 &= r a_n(2l) - \kappa(2l) + \frac{\lambda_1}{\gamma} (e^{a_n(2l)} - a_n(1l) - 1) + 2\lambda_n \hat{\mu}(\lambda_n, 1h) \chi(\eta_{1h}, \epsilon_{1h}(a)) \\
0 &= r a_n(2h) - \kappa(2h) + \frac{\lambda_2}{\gamma} (e^{a_n(2h)} - a_n(1h) - 1).
\end{align*}
\]
At this stage, I will consider the asymptotic behaviour of the stationary type distribution, which requires to distinguish two cases.

(a) Surplus I first assume a surplus of the illiquid asset, meaning that
\[
\frac{S_d}{\Delta_\theta} - \mu_{2*} > 0.
\]
In this case, it is known from Lemma 77 that
\[
\lim_{n \to \infty} \hat{\mu}(\lambda_n, 1h) = \frac{S_d}{\Delta_\theta} - \mu_{2*} > 0,
\]
which implies that
\[
\lim_{n \to \infty} \lambda_n \hat{\mu}(a_n, 1h) = \infty.
\]
Now, as stated in (87), the equilibria are bounded, and they also satisfy the HJB equations (89). But this is only compatible with (91) if
\[
\lim_{n \to \infty} \chi(\eta_{2l}, \epsilon_{2l}(a)) = 0.
\]

59
Recalling the definition of “$\chi$” in (14), this is equivalent to
\[ a_{\infty}(1l) - a_{\infty}(1h) = a_{\infty}(2l) - a_{\infty}(2h). \tag{92} \]

But then, as Lemma 34 ensures that
\[ \lim_{n \to \infty} \lambda_n \mu_n(2l) = \frac{\lambda_1}{2} \frac{1 - \frac{S_d}{\Delta \vartheta}}{\frac{S_d}{\Delta \vartheta} - \mu_2}. \]

letting $n$ go to $+\infty$ in (89) yields
\[
\begin{cases}
0 = \rho a_{\infty}(1l) - \kappa(1l) + \frac{\lambda_1}{\gamma} \left( e^{a_{\infty}(1l) - a_{\infty}(2l)} - 1 \right) \\
0 = \rho a_{\infty}(1h) - \kappa(1h) + \frac{\lambda_1}{\gamma} \left( e^{a_{\infty}(1h) - a_{\infty}(2h)} - 1 \right) \\
0 = \rho a_{\infty}(2l) - \kappa(2l) + \frac{\lambda_1}{\gamma} \left( e^{a_{\infty}(2l) - a_{\infty}(1l)} - 1 \right) \\
0 = \rho a_{\infty}(2h) - \kappa(2h) + \frac{\lambda_1}{\gamma} \left( e^{a_{\infty}(2h) - a_{\infty}(1h)} - 1 \right)
\end{cases}
\tag{93}
\]

Now, subtracting the second and third equations from the sum of the first and fourth ones in (95), and then repeatedly using (92), yields
\[ \kappa(1l) - \kappa(1h) + \kappa(2h) - \kappa(2l) = - \lim_{n \to \infty} 2\lambda_n \mu_n \left( \lambda_n, 1h \right) \chi(\eta_{2l}, \epsilon_{2l}(a_n)) \tag{94} \]

I draw two conclusions from this last equality. First, combining it with (93) yields
\[
\begin{cases}
0 = \rho a_{\infty}(1l) - \kappa(1l) + \frac{\lambda_1}{\gamma} \left( e^{\beta(1l) - \beta(2l)} - 1 \right) \\
0 = \rho a_{\infty}(1h) - \kappa(1h) + \frac{\lambda_1}{\gamma} \left( e^{\beta(1h) - \beta(2h)} - 1 \right) \\
0 = \rho a_{\infty}(2l) - \kappa(2l) + \frac{\lambda_1}{\gamma} \left( e^{\beta(2l) - \beta(1l)} - 1 \right) \\
0 = \rho a_{\infty}(2h) - \kappa(2h) + \frac{\lambda_1}{\gamma} \left( e^{\beta(2h) - \beta(1h)} - 1 \right)
\end{cases}
\tag{95}
\]

Now, this system precisely fits in the structure assumed in Section B.2 and, as a result, Proposition 27 ensures the uniqueness of the asymptotic equilibrium $\beta_{\infty}$.

Second, (94) is only compatible with the assumption (88) as long as
\[ \kappa(1l) - \kappa(1h) + \kappa(2h) - \kappa(2l) \geq 0, \]

which, up to a few algebraic manipulations, is equivalent to
\[ \left( 1 - \rho_{cd} \right) \cdot \left( \rho_{1d} - \rho_{2d} \right) > 0. \tag{96} \]

The case of an equality is excluded by Assumption 2. I must still consider the other case.

**(b) Shortage** In case (90) does not hold, meaning that
\[ \frac{S_d}{\Delta \vartheta} - \mu_{2*} < 0, \]

\[ ^{35} \text{Assumption 2 precludes an equality.} \]
an argument similar to the one I just presented makes sure that the limit $a_\infty$ is also uniquely defined and, again, is only compatible with (88) as longs as (96) holds.

As is readily checked, assuming the reverse inequality in (88) would also give a unique candidate for $a_\infty$, but this time require that (96) also holds with a reverse inequality.

Summing up, if (96) holds, then the sequence of equilibria converges and is so that, for $n$ large enough, $\epsilon_{1h}(a_n) > 0$. Otherwise, the sequence converges as well but, for $n$ large enough, $\epsilon_{2h}(a_n) < 0$.

I have now characterized which trades are implemented when the meeting intensity is sufficiently large. The last step is to show that the trading pattern cannot be reverted by an increasing illiquidity level.

**Claim 3** The surplus to be shared in bilateral trades is differentiable and decreasing in the meeting intensity. In other words, for $i\theta \in T$, if $\epsilon_{i\theta}(a(\lambda)) > 0$, then,

$$\frac{\partial}{\partial \lambda} \epsilon_{i\theta}(a(\lambda)) < 0$$

where, in particular, the derivative exists.

**Proof of Claim 3.** I will argue, without loss of generality, under the assumption that $\epsilon_{1h}(a) (\Delta) = a(1l) - a(1h) - a(2l) + a(2h) (\Delta) = -\Delta_h - \Delta_l > 0$, meaning that the 2-agents have the high valuation. From the proof of Proposition 6, I know that for any given $\lambda$, the pair $\Delta (\Delta) = (\Delta_h, \Delta_l)$ is the unique solution to the system

$$0 = F(\Delta, \lambda) (\Delta) \Rightarrow \begin{cases} 0 = F_h (\Delta_h, \Delta_l, \lambda) \\ 0 = F_l (\Delta_l, \Delta_h, \lambda) \end{cases},$$

where the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ is implicitly defined in the last equation. Now, under the above assumption regarding the high valuation agents, I can write

$$\det (D_\Delta F (\Delta, \lambda)) = \det \begin{pmatrix} r + \lambda_1 e^{\Delta_h} + \lambda_2 e^{-\Delta_h} & -2\lambda \mu (2l) \frac{\partial \chi}{\partial \epsilon} (\eta_{h}, -\Delta_l - \Delta_h) \\ -2\lambda \mu (2l) \frac{\partial \chi}{\partial \epsilon} (\eta_{l}, -\Delta_l - \Delta_h) & r + \lambda_1 e^{\Delta_l} + \lambda_2 e^{-\Delta_l} \\ -2\lambda \mu (1h) \frac{\partial \chi}{\partial \epsilon} (\eta_{2l}, -\Delta_l - \Delta_h) & -2\lambda \mu (1l) \frac{\partial \chi}{\partial \epsilon} (\eta_{2l}, -\Delta_l - \Delta_h) \end{pmatrix} (98)$$

Recalling from the definition (14) that $\chi$ is decreasing in its second argument, this last quantity is positive, which justifies an application of the Implicit Function Theorem.
This ensures that $\Delta$ is, locally, a differentiable function $\Delta(\lambda)$ of the meeting intensity, with derivative

$$\partial_\lambda \Delta(\lambda) = - (D_\Delta F(\Delta, \lambda))^{-1} D_\lambda F(\Delta, \lambda)$$

$$= \frac{-1}{\det (D_\Delta F)} \left( \begin{array}{cc} \frac{\partial F_l}{\partial \Delta_l} & \frac{\partial F_h}{\partial \Delta_l} \\ -\frac{\partial F_l}{\partial \Delta_h} & -\frac{\partial F_h}{\partial \Delta_h} \end{array} \right) \left( \begin{array}{c} \frac{\partial F_l}{\partial \lambda} \\ \frac{\partial F_h}{\partial \lambda} \end{array} \right).$$

But then,

$$\frac{\partial}{\partial \lambda} (\Delta_l + \Delta_h)$$

$$= \frac{-1}{\det (D_\Delta F(\Delta, \lambda))} \left( \begin{array}{c} \frac{\partial F_l}{\partial \Delta_l} - \frac{\partial F_l}{\partial \Delta_h} \frac{\partial F_h}{\partial \Delta_l} + \frac{\partial F_h}{\partial \Delta_h} - \frac{\partial F_h}{\partial \Delta_h} \frac{\partial F_l}{\partial \Delta_l} \end{array} \right)$$

$$= \frac{-1}{\det (D_\Delta F(\Delta, \lambda))} \left( \begin{array}{c} (r + \lambda_{12} e^{\Delta_l} + \lambda_{12} e^{-\Delta_l}) \chi(\eta_{1h}, -\Delta_l - \Delta_h) 2\partial_\lambda (\lambda \mu(\lambda, 2l)) \\ + (r + \lambda_{12} e^{\Delta_l} + \lambda_{21} e^{-\Delta_l}) \chi(\eta_{2l}, -\Delta_l - \Delta_h) 2\partial_\lambda (\lambda \mu(\lambda, 1h)) \end{array} \right).$$

With (97), both $\chi(\eta_{1h}, -\Delta_l - \Delta_h)$ and $\chi(\eta_{2l}, -\Delta_l - \Delta_h)$ are negative. As a result,

$$\partial_\lambda (\Delta_h(\lambda) + \Delta_l(\lambda)) > 0$$

or, equivalently,

$$\partial_\lambda (-\Delta_h(\lambda) - \Delta_l(\lambda)) < 0$$

which proves the claim.

I can finally conclude the proof of Proposition 10. Indeed, assuming that $\Pi > 0$, Claim 2 ensures that, if the meeting intensity $\lambda$ is larger than a certain threshold $\bar{\lambda}$, then, $\epsilon_{1h}(\beta(\lambda)) > 0$, meaning that 2-agents have the high valuation. But then, Claim 3 ensures that decreasing $\lambda$ increases $\epsilon_{1h}(\beta(\lambda))$. In particular, 2-agents still have the high valuation for any value of the meeting intensity. The case where $\Pi < 0$ is similar.

**Proof of Proposition 11.** Without loss of generality, I assume that 2-agents have the high valuation of the illiquid asset.

Regarding the OTC market, as the transaction size is fixed, the trading volume is proportional to

$$2\lambda \mu(1h)\mu(2l),$$

meaning to the meeting intensity between 1h and 2l agents. From Lemma 33 this quantity is increasing in the meeting intensity.

Turning to the centralized market, each of the six possible type changes

<table>
<thead>
<tr>
<th>type change</th>
<th>intensity</th>
<th>triggered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1l \rightarrow 2l$</td>
<td>$\lambda_{12} \mu(1l)$</td>
<td>correlation shock</td>
</tr>
<tr>
<td>$1h \rightarrow 2h$</td>
<td>$\lambda_{12} \mu(1h)$</td>
<td>correlation shock</td>
</tr>
<tr>
<td>$1h \rightarrow 1l$</td>
<td>$2\lambda \mu(1h)\mu(2l)$</td>
<td>OTC trade</td>
</tr>
<tr>
<td>$2l \rightarrow 2h$</td>
<td>$2\lambda \mu(1h)\mu(2l)$</td>
<td>OTC trade</td>
</tr>
<tr>
<td>$2l \rightarrow 1l$</td>
<td>$\lambda_{21} \mu(2l)$</td>
<td>correlation shock</td>
</tr>
<tr>
<td>$2h \rightarrow 1h$</td>
<td>$\lambda_{21} \mu(2h)$</td>
<td>correlation shock</td>
</tr>
</tbody>
</table>
induces a portfolio adjustment on the centralized market. Recalling the expressions (29) for the liquid holdings in equilibrium, the volume exchanged on the centralized market per unit of time is thus

\[
\text{Vol} = \frac{1}{2} \left\{ \begin{array}{c}
\lambda_{12} \mu(1l) & |\pi(1l) - \pi(2l)| \\
\lambda_{12} \mu(1h) & |\pi(1h) - \pi(2h)| \\
2 \lambda_{12} \mu(1h) \mu(2l) & |\pi(1h) - \pi(1l)| \\
\lambda_{21} \mu(2l) & |\pi(2l) - \pi(1l)| \\
\lambda_{21} \mu(2h) & |\pi(2h) - \pi(1h)| \\
\end{array} \right\}
\]

\[
= \frac{1}{2} \left\{ (\lambda_{12} \mu_{1l} + \lambda_{21} \mu_{2l}) \left| \rho_{cd} \right| \left| \sigma_{c} \right| |\rho_{1c} - \rho_{2c}| + 4 \lambda_{12} \mu(1h) \mu(2l) \left| \frac{\sigma_{d}}{\sigma_{c}} \right| \Delta_{d} \right\}
\]

From the first part, this last quantity is increasing in the meeting intensity. \hfill \Box

**Proof of Corollary 12.** Immediate from the expressions for the equilibrium liquid portfolios “\(\pi(i\theta)\)” in Proposition 10, and the proof of Proposition 11. \hfill \Box

### C.5 Proofs for Section 6

**Proof of Proposition 15.** Immediate from Proposition 10 and Lemma 34. \hfill \Box

**Proof of Proposition 16.** Regarding the asymptotic level of the vector \(a\), I already presented most of the argument while proving Proposition 10. Namely, recalling (92), I can write

\[
\nu_{12} \triangleq \lim_{\xi \to 0} e^{a_{(1h)} - a_{(2h)}} = \lim_{\xi \to 0} e^{a_{(1l)} - a_{(2l)}}.
\]

Taking \(\nu_{12}\) as given, I can solve (95) for \(a\) and obtain

\[
\begin{align*}
\lambda_{12} (\nu_{12} - 1) & \quad -\kappa(1l) \\
\lambda_{12} h & \quad -\kappa(1h) \\
\frac{(\nu_{12} - 1) - \beta \nu_{12}}{1 - \beta} & \quad -\kappa(2l) \\
-\kappa(2h) & \quad -\kappa(2h)
\end{align*}
\]

Regarding \(\nu_{12}\) itself, subtracting the fourth equation in (95) from the second one yields

\[
0 = r \log (\nu_{12}) - \kappa(1h) + \kappa(2h) + \frac{\lambda_{12}}{r \gamma} (\nu_{12} - 1) - \frac{\lambda_{21}}{r \gamma} \left( \frac{1}{\nu_{12}} - 1 \right).
\]

Considering the monotonicity and limiting behaviour of the right hand side, this uniquely characterize \(\nu_{12}\).
I now turn to the sensitivities to the illiquidity level. I first focus on the quantities $\Delta_h$ and $\Delta_l$. They were defined in (55) and (56), during the proof of Proposition 6, and uniquely define the vector $a$, as seen in (63).

Recalling the proof of Proposition 10,

$$\lim_{\xi \to 0} \left( \frac{\partial \xi}{\partial \Delta_h} \right) = \lim_{\xi \to 0} \frac{-1}{\det (D\Delta F)} \left( \begin{array}{cc} \frac{\partial F_1}{\partial \Delta_l} & -\frac{\partial F_1}{\partial \Delta_h} \\ -\frac{\partial F_2}{\partial \Delta_l} & \frac{\partial F_2}{\partial \Delta_h} \end{array} \right),$$

(99)

where the entries of $D\Delta F(\Delta, \xi)$ can in (98), and where

$$\left( \frac{\partial F_h}{\partial \xi} \right) = \left( \begin{array}{c} 2\chi (\eta_1, -\Delta_h - \Delta_l) \frac{\partial}{\partial \xi} \left( \frac{1}{\xi} \mu (2l) \right) \\ 2\chi (\eta_2, -\Delta_h - \Delta_l) \frac{\partial}{\partial \xi} \left( \frac{1}{\xi} \mu (1h) \right) \end{array} \right).$$

Now, deducing from (92) that

$$\lim_{\xi \to 0} \Delta_l = \lim_{\xi \to 0} \Delta_h,$$

verifying that

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \xi} \chi (\eta, \epsilon) = -\eta,$$

and recalling both the asymptotic behaviour of the type distribution described in Proposition 15, and the asymptotic bilateral trade benefits

$$\kappa (1l) - \kappa (1h) + \kappa (2h) - \kappa (2l) = - \lim_{n \to \infty} 2\lambda_n \mu (\lambda_n, 1h) \chi (\eta_2, \epsilon_2 (a_n)),$$

as stated in (94), I can directly obtain the limits in (99). Namely, defining

$$\delta_a = \frac{\beta}{2\eta_2 s},$$

(100)

and

$$\tau = \frac{2\eta_1 h \delta_u}{r + \frac{\lambda_1}{r} \nu_1 + \frac{\lambda_2}{r} \nu_2},$$

they are

$$\lim_{\xi \to 0} \left( \frac{\partial \xi}{\partial \Delta_h} \right) = \left( \begin{array}{c} \delta_a \\ -\delta_a (1 + \tau) \end{array} \right).$$

I can now derive the sensitivities of $a$ from the HJB equation (19). For example, as $a(1h)$ is so that

$$0 = r a(1h) - \kappa (1h) + \lambda_1 \left( e^{\Delta_h} - 1 \right) + 2\lambda \mu (2l) \chi (\eta_1 h, -\Delta_h - \Delta_l).$$

Invoking the Implicit Function Theorem and using the results above yields

$$\lim_{\xi \to 0} \frac{\partial a(1h)}{\partial \xi} = \lim_{\xi \to 0} \left( -\frac{r^2}{\lambda^{12} e^{\Delta_h} \frac{\partial}{\partial \xi} \Delta_h} + 2\frac{\partial}{\partial \xi} \left( \lambda \mu (2l) \chi (\eta_1 h, -\Delta_l - \Delta_h) \right) + 2\lambda \mu (2l) \frac{\partial}{\partial \xi} \left( (\eta_1 h, -\Delta_l - \Delta_h) \right) (-\Delta_h - \Delta_l) \right)$$

64
\[
\begin{align*}
= & -\frac{\delta_a}{r}\left(\frac{\lambda_{12}}{r\gamma}\nu_{12}\tau - 2\eta_{1h}\delta_{\mu}\right)
= & -\frac{\delta_a}{r}\left(\frac{\lambda_{12}}{r\gamma}\nu_{12} - 2\eta_{1h}\delta_{\mu}\frac{1}{\tau}\right)
= & \frac{\delta_a}{r}\left(\tau + \frac{\lambda_{21}}{r\gamma }\nu_{12}\right).
\end{align*}
\]

The derivations for the other components of \(a\) are similar. \(\square\)

**Proof of Proposition 17.** From Propositions 10, agents of types 1l and 2h do not trade. In particular, their expected benefits from bilateral trading are zero.

I first consider those agents on the long side of the market. Under Assumption 14, these are the 1h-agents, and combining the asymptotic behaviour of the distribution of types, as stated in Proposition 15, the one of \(a\), as stated in Proposition 16, and the finite development

\[
\chi(\eta_0, x) = -\eta_0 x + o(x),
\]

yields

\[
2\lambda\mu(2l)\left[\chi(\eta_{1h}, \epsilon_{2l}(a))\right] = \delta_{\mu} (-\eta_{1h}) \delta_{a} + o\left(\frac{1}{\lambda}\right).
\]

Recalling the definition of \(\delta_a\) in (100), this is the expression in the statement.

The case of the agents on the short side of the market requires more care. More specifically, when considering the expected benefits

\[
2\lambda\mu(1h\theta)\left[\chi(1 - \eta_{1h}, \epsilon_{2l}(a))\right],
\]

the rate at which suitable counterparties are met,

\[
2\lambda\mu(1h\theta) = \lambda s + \delta_{\mu} + o(1)
\]

becomes arbitrarily large with an increasing meeting intensity. In particular, a correct approximation of the overall expected benefits from bilateral trading requires a second order approximation of

\[
\left[\chi(1 - \eta_{1h}, \epsilon_{2l}(a))\right],
\]

which itself requires a second order approximation of the trade surplus \(\epsilon_{2l}(a)\).

Now, when proving Proposition 16, I characterized

\[
\lim_{\xi} \frac{\partial}{\partial \xi} (\Delta_h, \Delta_l)
\]

by applying the Implicit Function Theorem to

\[
0 = F(\Delta, \lambda)
\]

and using previously derived asymptotic behaviours. The characterization of

\[
\lim_{\xi} \frac{\partial^2}{\partial \xi^2} (\Delta_h, \Delta_l)
\]

65
is similar, but based on

\[ 0 = \frac{\partial F}{\partial \Delta} (\Delta, \xi) \frac{\partial \Delta}{\partial \xi} + \frac{\partial F}{\partial \xi} (\Delta, \xi). \]

The derivation being more cumbersome, they are best conducted with an appropriate software. This yields

\[ \epsilon_{2l} (a) = - \Delta_h - \Delta_l \]

\[ = \frac{1}{\lambda^2} \left( \delta_a - \frac{1}{\lambda} \left( 2 \delta_{ll} + \frac{2 \eta_{1h} - 2 \eta_{1l}}{2 \eta_{2l}} \right) + r + \nu_1 + \nu_2 \right) + o \left( \frac{1}{\lambda^2} \right). \]

Combining this characterization, Proposition 15, and the second order development

\[ \chi (\eta, x) = -\eta x + \left( -\eta^2 + 2 \eta^3 \right) x^2 + o (x^2), \]

yields

\[ 2 \lambda \mu (2l) \chi (\eta_{1h}, \epsilon_{2l} (a)) = -\beta + \frac{1}{\lambda} \left( 2 \eta_{1h} \delta_{ll} + r + \nu_1 + \nu_2 \right) + o \left( \frac{1}{\lambda} \right). \]

Proof of Corollary 18. Recalling both the proof of Proposition 4 and (92),

\[ P^\infty_d = \lim_{\xi \to 0} P_d = \frac{\beta_{\infty, 1h} - \beta_{\infty, 1l}}{\Delta_\theta} = \frac{\beta_{\infty, 2h} - \beta_{\infty, 2l}}{\Delta_\theta}. \]

Now, it follows from Proposition 16 that

\[ a^W (1h) - a^W (1l) = -\frac{1}{r} (-\kappa(1l) + \kappa(1h)). \]

Combining the two ensures that

\[ P^W = \frac{\kappa(1h) - \kappa(1l)}{r \Delta_\theta}. \]

It remains to characterize the first order correction. Under Assumption 14, the defining equation (12) specializes to

\[ (1 - \eta_{1h}) \left( 1 - e^{r \gamma (a(1l) + P_d \Delta_\theta - a(1h))} \right) = \eta_{1h} \left( 1 - e^{r \gamma (a(2h) + P_d \Delta_\theta - a(2l))} \right). \]

Now, by the Implicit Function Theorem,

\[ \begin{pmatrix} \frac{\partial_a (1l)}{\partial_{a(1h)}} \\ \frac{\partial_a (1h)}{\partial_{a(2l)}} \\ \frac{\partial_a (2l)}{\partial_{a(2h)}} \end{pmatrix} P_d(a) \]

66
\[
\begin{align*}
= - \frac{1}{r \gamma \Delta \theta} \left\{ \eta_{2l} e^{r \gamma (a(1l) + P_d \Delta \theta - a(1h))} \right. \\
&+ \eta_{1h} e^{r \gamma (a(2h) - P_d \Delta \theta - a(2l))} \left. \right\} \\
&\cdot \begin{pmatrix}
- \eta_{2l} e^{r \gamma (a(1l) + P_d \Delta \theta - a(1h))} \\
\eta_{2l} e^{r \gamma (a(1l) + P_d \Delta \theta - a(1h))} \\
- \eta_{1h} e^{r \gamma (a(2h) - P_d \Delta \theta - a(2l))} \\
\eta_{1h} e^{r \gamma (a(2h) - P_d \Delta \theta - a(2l))}
\end{pmatrix}.
\end{align*}
\]

Finally, applying the chain rule, the expressions I just derived, and recalling the asymptotic characterizations of both \(a\) and \(D_{\xi} a\), yields

\[
\lim_{\xi \to 0} \partial_{\xi} P_d (\beta(\xi)) = \lim_{\xi \to 0} \sum_{\theta \in T} (\partial_{\beta(\theta)} P_d (\beta(\xi))) (\partial_{\xi} \beta(\xi, \theta)) = \frac{r \eta_{2l} + 2 \delta \eta_1 h + \frac{\lambda_{12} \nu_{12}}{r \gamma}}{r^2 \gamma \Delta \theta}.
\]

**Proof of Proposition 19.** The illiquidity correction is

\[
1 \lambda \left( \frac{\delta a}{r \gamma \Delta \theta} \right),
\]

where \(\nu_{12}\) is defined in (35). As a result, and recalling the definition of \(\delta a\) in (100), the sign of the dependence between illiquidity correction and the correlation \(\rho_{cd}\) between the two assets is the one of

\[
\frac{\partial}{\partial \rho_{cd}} \left( \Pi \left( r \eta_{2l} + 2 \delta \eta_1 h + \frac{\lambda_{12} \nu_{12}}{r \gamma} \right) \right) = \frac{\Pi \lambda_{12} \frac{\partial \nu_{12}}{\partial \rho_{cd}} + \Pi \frac{\lambda_{12}}{r \gamma} \frac{\partial \nu_{12}}{\partial \rho_{cd}} + \Pi \frac{\lambda_{12}}{r \gamma} \frac{S_d - \Delta \theta}{\nu_{12}} \sigma_d \sigma_\eta (\rho_{1c} - \rho_{2c})}{r \gamma \nu_{12} + \lambda_{12} + \frac{\lambda_{12}}{r \gamma} \nu_{12}}.
\]

\(\Delta \equiv(*)\)

and, as

\[
\frac{\partial \Pi}{\partial \rho_{cd}} = - (\rho_{1c} - \rho_{2c}),
\]

I may write

\[
(*) = \frac{\partial \Pi}{\partial \rho_{cd}} \left( r \eta_{2l} + 2 \delta \eta_1 h + \frac{\lambda_{12} \nu_{12}}{r \gamma} \right) + \Pi \frac{\lambda_{12}}{r \gamma} \frac{\Delta \theta - S_d}{\nu_{12}} \sigma_d \sigma_\eta \left( \frac{1}{\nu_{12}} + \lambda_{12} + \frac{2 \lambda_{12}}{r \gamma} \nu_{12} \right).
\]

The second term in this product is continuous in \(\rho_{cd}\), and converges towards a positive quantity when \(\Pi\) goes towards zero.
In particular, as long as $\Pi$ is not too large, the sign of

$$\frac{\partial}{\partial \rho_{cd}} \Pi$$

defines whether the illiquidity correction is increasing or decreasing. □

C.6 Proofs for section 7

Proof of Proposition 21. The proof consists in constructing a response function that describes how investors individually react to given search intensities. Then, Kakutani Fixed-Point theorem directly yields the conclusion.

First, given a vector of exogenously specified search intensities, I derive a vector describing the value functions of the various agents. In other words, I construct a function

$$\lambda \triangleq \{ \lambda_i(i\theta) \}_{i\theta \in T} \mapsto a \left( \lambda \right) \triangleq \{ a \left( \lambda, i\theta \right) \}_{i\theta \in T},$$

where $a \left( \lambda \right)$ is the solution to

$$\forall i\theta \in T : 0 = ra \left( \lambda, i\theta \right) - \kappa (i\theta) + \lambda \bar{\theta} r_{\gamma} e^{a \left( \lambda, i\theta \right) - a \left( \lambda, \bar{i}\theta \right) - 1} - \frac{2\lambda \bar{\theta} \lambda \bar{\theta} r_{\gamma}}{r_{\gamma}} \chi \left( \eta_i \theta, a \left( \lambda \right) \right).$$

Recalling the proof of Proposition 10, this mapping is well defined. Also, adapting the argument proving the Claim 3 in this same proof would show that the function defined in (101) is differentiable and, as such, continuous.

Second, given the value functions, the surplus to be shared in bilateral meetings are given by the mapping

$$a \left( \Delta \right) = \{ a \left( i\theta \right) \}_{i\theta \in T} \mapsto \epsilon \left( a \left( \Delta \right) \right) \triangleq \{ a \left( i\theta \right) - a \left( \bar{i}\theta \right) + a \left( \bar{i}\theta \right) - a \left( \bar{i}\theta \right) \}_{i\theta \in T},$$

which is continuous as well.

Finally, given these benefits from OTC trading, and keeping Proposition 20 in mind, the optimal search behaviours are described, for $i\theta \in T$, by the correspondence

$$R : \epsilon_i \theta \mapsto \left\{ \begin{array}{ll}
[0, \lambda_L] & , \frac{-2\lambda \bar{\theta} r_{\gamma}}{r_{\gamma}} \mu \left( \bar{i}\theta \right) \chi \left( \eta_i \theta, \epsilon_i \theta \left( a \right) \right) \in \left( -\infty, 0 \right] \\
\{ \lambda_L \} & , \frac{-2\lambda \bar{\theta} r_{\gamma}}{r_{\gamma}} \mu \left( \bar{i}\theta \right) \chi \left( \eta_i \theta, \epsilon_i \theta \left( a \right) \right) \in \left( 0, \zeta \right] \\
[\lambda_L, \lambda_H] & , \frac{-2\lambda \bar{\theta} r_{\gamma}}{r_{\gamma}} \mu \left( \bar{i}\theta \right) \chi \left( \eta_i \theta, \epsilon_i \theta \left( a \right) \right) = \zeta \\
\{ \lambda_H \} & , \frac{-2\lambda \bar{\theta} r_{\gamma}}{r_{\gamma}} \mu \left( \bar{i}\theta \right) \chi \left( \eta_i \theta, \epsilon_i \theta \left( a \right) \right) \in \left( \zeta, +\infty \right)
\end{array} \right..$$

The image of this correspondence is always non-empty, convex, and closed. Also, the correspondence itself is verified to be upper hemicontinuous.

As a result, combining the transformations defined in (101), (102), and (103) yields a correspondence

$$[\lambda_L, \lambda_H]^4 \mapsto P \left( [\lambda_L, \lambda_H]^4 \right)$$
\[ \lambda \mapsto R(\epsilon(a(\lambda))) \]

that satisfies the assumptions of Kakutani Fixed-Point Theorem, and thus admits a fixed point. This fixed-point defines an equilibrium of the model with endogenous search. \( \square \)

So as to prove Proposition 21, I first adapt Proposition 17, which describes the expected benefits from OTC trading, to the setting with endogenous search.

**Lemma 35.** Given the equilibrium search intensities

\[ \{\lambda_{i\theta}\}_{i\theta \in \mathcal{T}}, \]

then, the expected benefits from OTC trading are characterized by

\[
\begin{aligned}
2\lambda_{1h}\lambda_{2l}\lambda_{2l}(2h) & \left[ \chi(\eta_{1l}, \epsilon_{1l}(a)) \right]^- \equiv 0 \\
2\lambda_{1h}\lambda_{2l}\mu(2l) & \left[ \chi(\eta_{1h}, \epsilon_{1h}(a)) \right]^- = (z_{1h,0} + z_{2l,0} + \beta) \cdot \left( \frac{1}{\lambda_{1h,2l}} \eta_{1l} \eta_{2l}^\prime \right) + o \left( \frac{1}{\lambda_{1h,2l}} \right).
\end{aligned}
\]

\[
\begin{aligned}
2\lambda_{1h}\lambda_{2l}\mu(1h) & \left[ \chi(\eta_{2l}, \epsilon_{2l}(a)) \right]^- = z_{1h,0} + z_{2l,0} + \beta + o(1), \\
2\lambda_{1h}\lambda_{2l}\mu(1l) & \left[ \chi(\eta_{2h}, \epsilon_{2h}(a)) \right]^- \equiv 0.
\end{aligned}
\]

**Proof of Lemma 35.** This argument is identical to the one for Proposition 17, up to the changes, for \( i\theta \in \mathcal{T} \),

\[
\begin{array}{c|c}
\text{without search} & \text{with search} \\
\hline
\kappa(i\theta) & \kappa(i\theta) - z_{i\theta,0} - z_{i\theta,1} \lambda_{1h,2l} - o(1) \\
\end{array}
\]

\( \square \)

The expected benefits for the 2l-agents can also be characterized up to first order in the meeting intensity. I do not report the corresponding expressions because they are relatively cumbersome, and I do not need them.

**Proof of Proposition 22.** As stated in Proposition 20, the optimal search strategy for a \( i\theta \)-agent is defined by the comparison of the unit costs \( \zeta \) and the expected benefits from bilateral trading, as measured by

\[
- \frac{2\lambda_{i\theta}\lambda_{2l}\mu(2l)}{r\gamma} \chi(\eta_{i\theta}, \epsilon_{i\theta}(a)).
\]

In particular, \( 2l \)-agents may search with the highest possible intensity, meaning that

\[ \lambda_{2l} = \lambda_H, \] (104)

whenever

\[
\zeta \leq \frac{2\lambda_{1h}\mu(1h)}{r\gamma} \left[ \chi(\eta_{2l}, \epsilon_{2l}(a)) \right]^- \quad \Leftrightarrow \quad \zeta \lambda_{2l} \leq \frac{2\lambda_{2l}\lambda_{1h}\mu(1h)}{r\gamma} \left[ \chi(\eta_{2l}, \epsilon_{2l}(a)) \right]^- . \] (105)
Now, the combination of (104) with the existence of the limit in (43) implies that

\[ \zeta = O \left( \frac{1}{\lambda H} \right). \]

In particular,

\[ \lim_{\lambda H \to \infty} \zeta (\lambda_{2l} - \lambda_L) = \lim_{\lambda H \to \infty} \zeta \lambda_{2l} = z_{2l,0}, \]

and, combining (105) with the asymptotic results from Lemma 35 will imply that

\[ z_{2l,0} \leq z_{1h,0} + z_{2l,0} + \beta. \]

Now, all the quantity in this last expression being positive, this inequality always holds and, when the endogenous liquidity becomes large, 2h-agents always asymptotically search with the intensity

\[ \lambda_{2h} = \lambda_H. \]

Let me now consider the 1h-agents. I will distinguish the cases where they adopt a search intensity that is maximal, minimal, and intermedit ate. First, they adopt the highest possible intensity

\[ \lambda_{1h} = \lambda_H \]

whenever

\[ \zeta \leq \frac{2\lambda_{2l} \mu(2l)}{r \gamma} \chi (\eta_{1h}, \epsilon (a))^{-} \]

\[ \iff \zeta \lambda_{1h} \leq \frac{2\lambda_{1h} \lambda_{2l} \mu(2l)}{r \gamma} [\chi (\eta_{1h}, \epsilon (a))]^{-}. \]

Now, again recalling the expressions in Lemma 35, this last inequality becomes

\[ z_{1h,0} + \frac{1}{\lambda_{1h} \lambda_{2l}} z_{1h,1} + o \left( \frac{1}{\lambda_{1h} \lambda_{2l}} \right) \]

\[ \leq (z_{1h,0} + z_{2l,0} + \beta) \left( \frac{1}{\lambda_{1h} \lambda_{2l}} \eta_{1h} \epsilon \delta \eta_{2l} \delta \mu + o \left( \frac{1}{\lambda_{1h} \lambda_{2l}} \right) \right). \]

This can only hold asymptotically if

\[ z_{1h,0} = \lim_{\lambda H \to \infty} \zeta \lambda_{1h} = 0 \]

and, as under the current assumptions,

\[ \lambda_{1h} = \lambda_{2l} = \lambda_H, \]

this also requires \( z_{2l,0} = 0 \). In this case, (106) becomes

\[ z_{1h,1} \leq \beta \frac{\eta_{1h} \epsilon \delta}{\eta_{2l} \delta \mu} + o \left( \frac{1}{\lambda_{1h} \lambda_{2l}} \right). \]
Recalling the definitions of the “\(z_{i\theta,1}\)’s in (43), it thus appears that, if
\[
 \lim_{\lambda_H \to \infty} r_\gamma \zeta \lambda_H^3 < \beta \frac{\eta_{1h} \delta_\mu}{\eta_{2l} s},
\]
then, the intensities
\[
 \lambda_{1h} = \lambda_{2l} = \lambda_H
\]
define an equilibrium search strategy.

I now consider under which conditions the 1h-agents would, asymptotically, adopt a
minimal search intensity
\[
 \lambda_{1h} = \lambda_L. \tag{107}
\]
Again comparing the costs and benefits of such a search strategy, it appears that the
strategy (107) is only optimal if
\[
 \zeta \geq \frac{2 \lambda_{2l} \mu(2l)}{r_\gamma} \left[ \chi (\eta_{1h}, \epsilon_{1h}(a)) \right]^{-}
\]
\[
 \iff \zeta \lambda_{1h} \geq \frac{2 \lambda_{1h} \lambda_{2l} \mu(2l)}{r_\gamma} \left[ \chi (\eta_{1h}, \epsilon_{1h}(a)) \right]^{-}
\]
and, using both the asymptotic expressions from Lemma 35 and (107), I can derive from
this last optimality condition that
\[
 \zeta \lambda_L \geq (z_{1h,0} + z_{2l,0} + \beta) \frac{1}{\lambda_L \lambda_{2l}} \frac{\eta_{1h} \delta_\mu}{\eta_{2l} s} + o \left( \frac{1}{\lambda_{1h} \lambda_{2l}} \right). \tag{108}
\]
Now, for this minimal search strategy,
\[
 z_{1h,0} = \lim_{\lambda_H \to \infty} \zeta [\lambda_{1h} - \lambda_L]^+ = \lim_{\lambda_H \to \infty} \zeta [\lambda_L - \lambda_L]^+ = 0,
\]
and I deduce from (108) that
\[
 \lambda_{2l}^2 \zeta \lambda_H \geq (z_{2l,0} + \beta) \frac{\eta_{1h} \delta_\mu}{\eta_{2l} s} + o(1)
\]
\[
 \iff \left( \lim_{\lambda_H \to \infty} \zeta \lambda_H \right) \left( \frac{\lambda_{2l}^2}{\eta_{1h} \delta_\mu \eta_{2l} s} - 1 \right) \geq \beta.
\]
This inequality cannot hold unless
\[
 \lambda_{2l}^2 \geq \frac{\eta_{1h} \delta_\mu}{\eta_{2l} s}
\]
and, in this case, it still requires that
\[
 \lim_{\lambda_H \to \infty} \zeta \lambda_H \geq \beta \frac{\lambda_{2l}^2}{\eta_{1h} \delta_\mu \eta_{2l} s} - 1.
\]
71
Finally, 1h-agents only adopt an intermediate search strategy if
\[
\zeta = \frac{2\lambda_2\mu(2l)}{r \gamma} \left[ \chi (\eta_{1h}, \epsilon_{1h} (a)) \right]^{-}
\]
\[
\Leftrightarrow \zeta \lambda_{1h} = \frac{2\lambda_{1h}\lambda_{2l}\mu(2l)}{r \gamma} \left[ \chi (\eta_{1h}, \epsilon_{1h} (a)) \right]^{-}
\]
Now, recalling that \( \zeta = O(1/\lambda_H) \),
\[
\lim_{\lambda_H \to \infty} \zeta \lambda_{1h} = \lim_{\lambda_H \to \infty} \zeta (\lambda_{1h} - \lambda_L) = z_{1h,0} + \frac{1}{\lambda_{1h}\lambda_{2l}} z_{1h,1} + o \left( \frac{1}{\lambda_{1h}\lambda_{2l}} \right)
\]
and the optimality condition becomes
\[
z_{1h,0} + \frac{1}{\lambda_{1h}\lambda_{2l}} z_{1h,1} = \frac{1}{\lambda_{1h}\lambda_{2l} \eta_{1l}} \left( z_{1h,0} + z_{2l,0} + \beta \right) + o \left( \frac{1}{\lambda_{1h}\lambda_{2l}} \right). \tag{109}
\]
This requires both
\[
z_{1h,0} = \lim_{\lambda_H \to \infty} \zeta (\lambda_{1h} - \lambda_L) = 0,
\]
and
\[
z_{1h,1} = \lim_{\lambda_H \to \infty} \zeta \lambda_{1h} (\lambda_{1h} - \lambda_L) = \frac{\eta_{1h} \delta_{\mu}}{\eta_{2l}} s (z_{2l,0} + \beta).
\]
Now, if
\[
z_{2l,0} = \lim_{\lambda_H \to \infty} \zeta \lambda_{1h} > 0,
\]
one derives from (109) that
\[
\lim_{\lambda_H \to \infty} \left( \lambda_{1h}^2 - \lambda_L \lambda_{1h} - \frac{\eta_{1h} \delta_{\mu}}{\eta_{2l}} s \left( 1 + \frac{\beta}{z_{2l,0}} \right) \right) = 0.
\]
Solving for \( \lim_{\lambda_H \to \infty} \lambda_{1h} \), and discarding the negative solution, yields
\[
\lim_{\lambda_H \to \infty} \lambda_{1h} = \frac{1}{2} \left( \lambda_L + \sqrt{\lambda_L^2 + 4 \frac{\eta_{1h} \delta_{\mu}}{\eta_{2l}} s \left( 1 + \frac{\beta}{z_{2l,0}} \right)} \right).
\]
In the alternative case where \( z_{2l,0} = 0 \), I derive from (109) that
\[
\lim_{\lambda_H \to \infty} \zeta \lambda_{1h} (\lambda_{1h} - \lambda_L) = \frac{\beta \eta_{1h} \delta_{\mu}}{\eta_{2l} s}. \tag{110}
\]
Now, as
\[
0 = z_{2l,0} = \lim_{\lambda_H \to \infty} \zeta \lambda_{1h},
\]
it must be that
\[
\lim_{\lambda_H \to \infty} \lambda_{1h} (\lambda_{1h} - \lambda_L) = \lim_{\lambda_H \to \infty} \lambda_{1h} = \infty.
\]
Combining this and (110) gives
\[
\lambda_{1h}^2 \lambda_H \to \infty \lambda_{1h} (\lambda_{1h} - \lambda_L) \lambda_H \to \infty \frac{1}{\zeta \lambda_H \beta \eta_{1h} \delta_{\mu}} \frac{\eta_{1h} \delta_{\mu}}{\eta_{2l} s}
\]
which is equivalent to (45) in the statement.
D Figures

<table>
<thead>
<tr>
<th>notation</th>
<th>calibration</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_d$</td>
<td>0.7</td>
<td>net supply of the illiquid asset $d$</td>
</tr>
<tr>
<td>$S_c$</td>
<td>1</td>
<td>net supply of the liquid asset $c$</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
<td>1</td>
<td>intensity of endowment correlation shocks</td>
</tr>
<tr>
<td>$\lambda_{21}$</td>
<td>2</td>
<td>idem</td>
</tr>
<tr>
<td>$\Delta_\theta$</td>
<td>1</td>
<td>size of illiquid holdings</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>$10^{-5}$</td>
<td>search cost</td>
</tr>
<tr>
<td>$\lambda_L$</td>
<td>8</td>
<td>largest cost-free search intensity</td>
</tr>
<tr>
<td>$\lambda_H$</td>
<td>16</td>
<td>largest possible search intensity</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.3</td>
<td>volatility of endowment</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>0.3</td>
<td>volatility of the payout of $d$</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>0.2</td>
<td>volatility of the payout of $c$</td>
</tr>
<tr>
<td>$\rho_{cd}$</td>
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<td>correlation between the payouts of $c$ and $d$</td>
</tr>
<tr>
<td>$\rho_{1d}$, $\rho_{2d}$</td>
<td>0.2, 0</td>
<td>correlations between the endowments and the payouts of $d$</td>
</tr>
<tr>
<td>$\rho_{1c}$, $\rho_{2c}$</td>
<td>0.25, -0.25</td>
<td>correlations between the endowments and the payouts of $d$</td>
</tr>
<tr>
<td>$m_\eta$</td>
<td>0.05</td>
<td>expected endowment</td>
</tr>
<tr>
<td>$m_d$</td>
<td>0.07</td>
<td>expected payout for $d$</td>
</tr>
<tr>
<td>$m_c$</td>
<td>0.07</td>
<td>expected payout for $c$</td>
</tr>
<tr>
<td>$r$</td>
<td>0.05</td>
<td>risk-free rate</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>7</td>
<td>coefficient of absolute risk-aversion</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.06</td>
<td>subjective discounting rate</td>
</tr>
<tr>
<td>$\eta_{1h}$, $\eta_{2h}$</td>
<td>0.5</td>
<td>bargaining powers</td>
</tr>
</tbody>
</table>

Table 1: Default parameter values. All the plots are based on this parametrization.
Figure 1: Comparison of the first order approximations derived in Section 6 with a direct, numerical resolution of the equilibrium equations. The main equilibrium quantities (the price $P_d$ of the illiquid asset, and the coefficients $a(i\theta)$ defining the value functions) are plotted against the illiquidity level $\xi = 1/\lambda$. 

74
Figure 2: Endogenously chosen search intensities as a function of the search cost. I only plot the intensities for the types corresponding to agents that are willing to trade OTC. The other ones would adopt any level of costless search.
Figure 3: In panel (a), I plot the illiquidity correction, meaning the difference $P_d - P^W_d$ between the price of the illiquid asset and its Walrasian price, as a function of the correlation between the two risky assets. The critical correlation $\hat{\rho}_{cd}$ equals 0.4 in this parametrization. I plot both the case where the meeting intensity is fixed at $\lambda^2_L$ (labelled as $\lambda_{ls} = \lambda_{ss} = \lambda_L$), and the case where agents endogenously choose a meeting intensity between $\lambda_L$ and $\lambda_H^2$ (labelled as $\lambda_{ls}, \lambda_{ss}$ endogenous). In panel (b), I plot the search intensities chosen by the various agent types in the setting with endogenous search of the panel (a).
References


