Collusion Agreements in Auctions:
Design and Execution by an Informed Principal

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Abstract

The standard mechanism approach to mechanism design by an informed principal posits an uninformed, uninterested third party who conducts the mechanism. In auctions where collusion is illegal, collusion agreements are likely to be both designed and executed by the involved parties. When this is the case, the standard approach is inadequate; it neglects potential "information leakages" and minimizes frictions. We model collusion in a second-price auction as a contract-design problem by an informed principal who runs the mechanism herself. Bidders' valuations are allowed to be interdependent and their private signals, affiliated. We show that all equilibria are monotonic and involve partial pooling at the top. In a pure common-value example, we find a first-mover disadvantage: the proposer is worse off than the receiver, though both are better off than without collusion. Outside competition for the bidders can also undermine the gains from collusion.

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1 Introduction

Collusion is a well-documented phenomenon in auctions, one of first-order interest to sellers. In one-shot games, collusion agreements are typically analyzed using standard tools of mechanism design, invoking an uninformed third party that designs and enforces the contract. Even when bidders interact repeatedly in the same auctions and enforcement is provided endogenously (through continuation values), their communication is typically mediated by a "center." Hendricks et al. (2008) are the first to use these tools to analyze the case of interdependent-value auctions, motivated by oil and gas lease auctions.

Joint bidding was legal during the sampling period in Hendricks et al. (2008). However, many such agreements are now illegal and therefore unlikely to be designed and enforced by an impartial party. When collusion agreements are designed and executed by the involved parties, there is room for information leakages that may erode the benefits to collusion.

Problems of mechanism design by privately involved parties are prevalent in Economics, from bilateral or multilateral trade with an informed seller, to contracting with privately informed firms. The literature stems from the seminal work of Myerson (1983) and Maskin and Tirole (1990, 1992). More recent work includes the work of Mylovanov and Kovac (2012); Mylovanov and Troeger (2012). These recent papers characterize the problem of mechanism design by an informed principal, whose information is payoff-irrelevant to the agents (but is relevant in determining her behaviour in the mechanism).

McAfee and McMillan (1992) introduce the “third-party approach” to the design of collusion agreements. To dispel with the third party in the design of the mechanism in collusion, Quesada (2005) introduces the informed-principal approach to a problem of collusion in production delegation.

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1 See, for instance, McAfee and McMillan (1992).
The tools of mechanism design by an informed principal have both parties simultaneously reporting to the mechanism designed by the principal. In other words, the informed party designs a mechanism but then acts as another “agent”; the mechanism is executed or “run” by an uninformed, impartial outsider.

When collusion is illegal, it is unlikely that the agreement can be implemented by an impartial enforcer. This leads the informed parties in charge of the execution as well as of the design of the mechanism. The literature has studied the design problem in detail, but the execution problem has been largely neglected. While execution is not an issue with formal contracts, it is an important aspect of informal contracts and illegal agreements.

In this paper, we look at collusion in a second-price auction (SPA) with interdependent values and affiliated signals as a problem of contract design and execution by an informed principal. As Hendricks et al. (2008) point out, bidders in common-value auctions have an incentive to share information to determine if the asset in question is worth acquiring.

To isolate the effect of information leakage and other frictions from enforcement issues, we analyze a one-shot interaction game. There are two bidders. Bidder 1 approaches bidder 2 with a collusion proposal. The agreement consists in sharing (unilaterally) information in exchange for a transfer and the commitment to staying out of the auction. In this setting, the informational content of bidder 2’s report turns out to be irrelevant; our results thus constitute an extension of Eső and Schummer (2004). The information shared becomes relevant when we introduce outside competition, in the form of a third bidder who is unaware of the possibility of collusion between the other two bidders.

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3We can think of our game as the stage game in a repeated game, where we look at the cooperative interaction of bidders that can (presumably) be enforced by some form of future punishment. However, this extrapolation can only work if we take the roles of (alternatively, the allocation of bargaining power between) the two bidders as fixed. A more profitable collusion scheme might involve role rotating. This question is left for future research.
We identify conditions under which equilibria are monotonic and feature partial pooling at the top. Under some additional technical assumption, we characterize the unique continuous and non-constant equilibrium.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes equilibria. Section 4 analyzes a pure common-value example and computes ex-ante expected payoffs, compared against different benchmarks. Section 5 re-computes these payoff under the presence of outside competition. Section 6 concludes.

2 The model

There are 2 agents, the bidders, as well as a seller. Bidder 1 ("she") is the "informed principal," endowed with the bargaining power to approach bidder 2 ("he") with a collusion proposal. The auction is a sealed-bid second-price auction (SPA).

Before any social interaction, each bidder receives a private signal $S_i$, which we normalize to lie in the interval $[0,1]$. In addition, there is another signal $Z$ which may influence the value of the object but is not observed by either bidder. Let $F(z, s_1, s_2)$ be the joint cumulative distribution function of $(Z, S_1, S_2)$. We assume that $F$ is symmetric in $(s_1, s_2)$, and that all of the variables are affiliated. In addition, we assume that $F(z, s_1, s_2)$ has a continuous density function denoted $f(z, s_1, s_2)$.

Each bidder has a valuation function $V_i = V(Z, S_i, S_{-i})$, where $V$ is nonnegative, continuous, and nondecreasing in all variables. In addition, we assume that all necessary expectations exist and are finite. This formulation encompasses both the private-value ($V_i = S_i$) and common-value ($V_i = Z$) models.

The timing of the game is as follows.

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4The extreme case of independence is allowed.
5The set-up is very similar to that found in Milgrom and Weber (1982).
Stage 0: Nature draws signals $z, s_1$ and $s_2$; bidder $i$ privately observes $s_i$.

Stage 1 (Collusion stage): Bidder 1 has the option of offering a collusion contract $C$ to bidder 2, which consists of a message space $M$ and an outcome function $(\tilde{I}, \tilde{b}_1, \tilde{b}_2) : M \rightarrow \mathbb{R} \times \mathbb{R}_+^2$ that specifies:

- a transfer $\tilde{I}(m)$ from bidder 1 to bidder 2
- bids $(\tilde{b}_1(m), \tilde{b}_2(m))$ that bidder 1 will submit (on behalf of both) in the auction.

Bidder 2 either accepts or rejects the contract. If she accepts, she reports a message $m$ and receives transfer $\tilde{I}(m)$.

Stage 2 (Auction stage): If bidder 2 accepts the contract, she commits to staying out of the auction and bidder 1 executes the agreement. If bidder 2 rejects the contract, both agents bid non-cooperatively.

The payoff to bidder 1 with value $V_1$ who faces a payment of $m_1$ to the auctioneer and who pays bidder 2 a transfer of $t$ is $\mathbb{I}(1 \text{ wins object})V_1 - m_1 - t$. Bidder 2’s payoff, when his value is $V_2$, pays $m_2$ to the auctioneer and receives $t$ from bidder 1 is $\mathbb{I}(2 \text{ wins object})V_2 - m_2 + t$.

3 Characterizing Equilibria

3.1 Noncooperative auction

We begin by solving the auction stage in the case that collusion does not occur or the proposed collusion agreement is rejected. Let $G_1(s_1)$ be any belief for bidder 2 over signal $s_1$, and let $G_2(s_2)$ be any belief for bidder 1 over $s_2$. Define the function $\omega(x, y) = E[V_1 | S_1 = x, S_2 = y]$. Milgrom and Weber (1982) show that $\omega(x, y)$ is nondecreasing in
under our assumptions; like them, we make the non-degeneracy assumption that this function is strictly increasing in $x$, and normalize $\omega$ so that $\omega(0,0) = 0$.

**Proposition 1** (Milgrom and Weber (1982)). The symmetric bidding profile given by $\beta(s_i) = \omega(s_i, s_i)$ is a Bayesian-Nash equilibrium, for any beliefs, in any continuation game following failure to collude.

A shortcoming of this equilibrium, one that simplifies the analysis, is that equilibrium behaviour “partially” disregards information that may be leaked through the proposal: while the outside options with respect to collusion contract $C$,

$$u_i(s_i; C) := \int_0^{s_i} [\omega(s_i, s_{-i}) - \omega(s_{-i}, s_{-i})]dF_{-i}(s_{-i}|C),$$

obviously depend on the updated beliefs, equilibrium bids do not.\(^6\) Therefore, type $s_2$ of agent 2 will accept a transfer $t$ in collusion-proposal $C$ provided that $t \geq u_2(s_2; C)$.\(^7\)

### 3.2 Simplifying bidder 1’s strategies

We can now move back to the collusion proposal/acceptance stage of the game, and simplify bidder 1’s strategies. Recall that a collusion contract $C$ specifies a message space

\(^6\)As Milgrom (1981) shows, with two bidders there is in fact a family of ex-post equilibria, indexed by a strictly increasing continuous real function $f$,

$$\beta_1(s_1) = \frac{s_1 + f^{-1}(s_1)}{2}, \beta_2(s_2) = \frac{s_2 + f(s_2)}{2}.$$

For simplicity, we focus on the symmetric equilibrium where $f$ is the identity function. This is the equilibrium commonly used in the auction literature, and, with 3 or more bidders, Bikhchandani and Riley (1991) show that the symmetric equilibrium is in fact the only equilibrium in strictly increasing and continuous strategies in which all bidders have a positive ex-ante probability of winning. Therefore, comparability of the results with those of the case of 3 bidders is another rationale for focusing on the symmetric equilibrium. In what follows, we will select this equilibrium as the continuation equilibrium.

\(^7\)In the terminology of Cramton and Palfrey (1995), we allow for a form of ratifiability: there is updating of the outside option from the proposal.
and an outcome function \((\tilde{t}, \tilde{b}_1, \tilde{b}_2)\). By the revelation principle, we can focus on direct revelation mechanisms, and set \(M = [0, 1]\). Additionally, if \(C\) is accepted, bidder 1 will clearly submit bids on behalf of herself and bidder 2 such that she wins the object and pays nothing.\(^8\)

With these simplifications, a collusion contract can be reduced to a menu of transfers \(\tilde{t} : [0, 1] \rightarrow \mathbb{R}^{[0,1]}\). Since the signals are non-verifiable, incentive compatibility implies that the function \(\tilde{t}\) must be a constant. In effect, bidder 1 is “bribing” bidder 2 to staying out of the auction. Thus, our game reduces to an extension of Eső and Schummer (2004) that allows for interdependencies in valuations and affiliation of the private signals. A strategy for bidder 1 in the first stage can be identified by a function mapping her signal into transfers (or “bribes”). We will denote this function \(\tilde{t} : [0, 1] \rightarrow \mathbb{R}\), and write \(\tilde{t}(s_1)\) for the transfer that type \(s_1\) offers to bidder 2.

### 3.3 Simplifying bidder 2’s strategies

A strategy for bidder 2 in the first stage of the game can be identified by a (measureable-valued) acceptance correspondence \(A : \mathbb{R} \rightarrow 2^{[0,1]}\), where \(A(t)\) is the (measurable) set of types that accept transfer \(t\), while the types in the complement reject it.

We have the following result.

**Lemma 1.** In any equilibrium, for any \(t \in \mathbb{R}\), the set \(A(t)\) is of the form \([0, \tilde{s}_2(t)]\) for some \(\tilde{s}_2(t) \in [0, 1]\).

**Proof.** Given that bidder 2 observes an offer of \(t\), he will have some beliefs over the types

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\(^8\)Bidder 1 would not get the good for free if the seller sets a positive reserve price. We assume, for simplicity, that reserve prices are 0. However, this is not an innocuous normalization. With positive reserve prices, Claim 1 below is false; the symmetric equilibrium is to bid one’s private signal when this signal is above a cutoff, but this cutoff depends on beliefs. Hence, the equilibrium strategies in the continuation game, and not just the payoffs, depend on the updated beliefs.
of bidder 1 represented by \( F_1(s_1|t, s_2) \). Therefore, he will accept the transfer if and only if:

\[
t \geq \int_0^{s_2} [\omega(s_2, s_1) - \omega(s_1, s_1)]dF_1(s_1|t, s_2).
\]

Since \( \omega \) is nondecreasing in its first argument and the variables are affiliated, the right-hand side (RHS) is nondecreasing in \( s_2 \). Thus, if the inequality holds for some \( s_2 \), it holds for all \( s_2' \leq s_2 \).

Lemma 1 implies that a strategy for bidder 2 can be summarized by a cutoff function \( \bar{s}_2 : \mathbb{R} \to [0, 1] \) where, for any \( t \), all \( s_2 \leq \bar{s}_2(t) \) accept \( t \) and all \( s_2 > \bar{s}_2(t) \) reject it.

### 3.4 (Weak perfect) Bayesian equilibrium

We can write bidder 1’s payoff if she is of type \( s_1 \), offers transfer \( t \), and bidder 2 follows a strategy summarized by cutoff \( \bar{s}_2 \) as:

\[
U_1(s_1, \bar{s}_2, t) = \int_0^{\bar{s}_2} [\omega(s_1, s_2) - t]dF_2(s_2|s_1) + \mathbb{I}(s_1 > \bar{s}_2) \int_{\bar{s}_2}^{s_1} [\omega(s_1, s_2) - \omega(s_2, s_2)]dF(s_2|s_1).
\]

Similarly, we can write bidder 2’s payoff if he is of type \( s_2 \), is offered transfer \( t \), has beliefs over \( s_1 \) represented by \( F_1(\cdot|t) \) and rejects the contract, as:

\[
U_2(s_2, t) = \int_0^{\bar{s}_2} [\omega(s_2, s_1) - \omega(s_1, s_1)]dF_1(s_1|t).
\]

With the fixed continuation equilibrium in the noncooperative auction, we will refer to \((\bar{t}(\cdot), \bar{s}_2(\cdot), F_1(\cdot|\cdot))\) as a (weak perfect) Bayesian equilibrium if:

(i) \( \bar{t}(s_1) \in \arg\max_{s_1} U_1(s_1, \bar{s}_2(t), t) \) for all \( s_1 \).

(ii) \( \bar{s}_2(t) \in \arg\max_{s_2} \mathbb{I}(s_2 \leq \bar{s}_2(t)) + \mathbb{I}(s_2 > \bar{s}_2(t))U_2(s_2, t) \) for all \( s_2, t \).
(iii) for all \( t \), beliefs \( F_1(\cdot|t) \) follow Bayes’ rule where possible.

It may be that \( U_1(s_1, \hat{s}_2, t) \) is actually decreasing in \( \hat{s}_2 \), i.e., some types of bidder 1 may prefer fewer types of bidder 2 to accept. Intuitively, this will happen when bidder 1 has a low estimate for the object’s value, and so less types of bidder 2 accepting means that bidder 1 has to pay the transfer \( t \) to fewer agents. When the object is not very valuable, bidder 1 may prefer to actually lose the object rather than to pay bidder 2 a transfer that is too high. If this is the case, then in fact bidder 1 will prefer that no types of bidder 2 accept.

**Proposition.** For any fixed \( s_1, t \), the function \( U_1(s_1, \hat{s}_2, t) \) is quasiconvex in \( \hat{s}_2 \).

**Proof.** We show that if \( U_1 \) is increasing near \( \hat{s}_{L2} \), then it is also increasing on \((\hat{s}_{L2}, 1]\), which in turn implies that \( U_1 \) is quasiconvex in \( \hat{s}_2 \). Fixing \( s_1 \) and \( t \), define a function \( h \) as

\[
g(\hat{s}_2) = U_1(s_1, \hat{s}_2, t)
\]

\[
= \begin{cases} 
  \int_0^{s_1} \omega(s_1, s_2) dF(s_2|s_1) - \int_0^{\hat{s}_2} t dF(s_2|s_1) - \int_{\hat{s}_2}^{s_1} \omega(s_2, s_2) dF(s_2|s_1), & \hat{s}_2 < s_1 \\
  \int_0^{\hat{s}_2} [\omega(s_1, s_2) - t] dF(s_2|s_1) & \hat{s}_2 \geq s_1
\end{cases}
\]

Note that \( h \) is everywhere continuous, and is everywhere differentiable, except perhaps \( \hat{s}_2 = s_1 \). The derivative of \( h \) is

\[
g'(\hat{s}_2) = \begin{cases} 
  [\omega(\hat{s}_2, \hat{s}_2) - t] f(\hat{s}_2|s_1), & \hat{s}_2 < s_1 \\
  [\omega(s_1, \hat{s}_2) - t] f(\hat{s}_2|s_1) & \hat{s}_2 \geq s_1
\end{cases}
\]

At \( \hat{s}_2 = s_1 \), since the derivative from the left is equal to the derivative from the right,
g'(s_1) does in fact exist and is equal to g'(s_1) = [ω(s_1, s_1) − t]f(s_1 | s_1). Since ω(·, ·) is increasing in both arguments, the result follows. □

Note that it is not true that bidder 1’s utility is increasing (even weakly) in the cutoff of bidder 2. This is intuitively obvious if we consider transfers that are prohibitively high, so that bidder 1 would prefer all types of bidder 2 to reject. More interestingly, it may be that arg min_{s_2} U_1(s_1, s_2, t) is interior. What the proposition does imply is that, fixing s_1 and t, if at some \( \hat{s}_2 \) bidder 1 would prefer less types of bidder 2 to accept, then he prefers bidder 2 to use a cutoff of 0 rather than \( \hat{s}_2 \). This is important because, in equilibrium, bidder 1 can always deviate to offering a transfer of 0, thereby inducing all types of bidder 2 to reject, and so for any t offered in equilibrium, either \( \bar{s}_2(t) = 0 \) or \( \bar{s}_2(t) \) is on the increasing part of \( U_1 \).

### 3.5 Monotonic equilibria

We say that an equilibrium \((\bar{I}(\cdot), \bar{s}_2(\cdot), F_1(\cdot | \cdot))\) is monotonic or nondecreasing if both \( \bar{I} \) and \( \bar{s}_2 \) are nondecreasing. Under the next two assumptions, it turns out that all equilibria are nondecreasing.

(A1) For all \( \underline{s} \) and \( \bar{s} \), the function \( \int_{\underline{s}}^{\bar{s}} \omega(x, y)dF(y | x) \) is nondecreasing in \( x \).

(A2) \( \omega(x, y) - \omega(y, y) \) is nonincreasing in \( y \) for all \( y \leq x \).

To understand (A1), a higher \( s_1 \) has two opposing effects from the perspective of bidder 1: first, it increases the value of the object itself (\( \omega \) increases). However, because of affiliation, it also suggests that bidder 2’s signal is stochastically higher (\( F(s_2 | s_1) \) decreases), which implies a lower chance of winning the auction. Assumption (A1) requires that the first effect outweigh the second. Assumption (A2) implies that a rise in bidder
2’s signal increases bidder 2’s bid in a noncooperative auction more than it increases bidder 1’s value for the object as long as bidder 1 has the higher signal (and vice-versa). Intuitively, both assumptions (A1) and (A2) require bidder $i$’s signal, $s_i$, to influence his valuation for the object more than his opponent’s signal $s_{-i}$ does.

**Theorem 1** (Monotonic equilibria). Assume that both (A1) and (A2) hold. Then, in any equilibrium, both $\bar{t}$ and $\bar{s}_2$ are nondecreasing on the equilibrium path.

**Proof.** See Lemmas 3 to 5 in the Appendix.

These conditions (A1) and (A2) are sufficient for our results, but they seem more than necessary. Nonetheless, they are assumptions commonly featured in the literature. A sufficient condition for (A1) to hold is that $s_1$ and $s_2$ are independent (possibly after conditioning on the common component $z$). Assumption (A2), on the other hand, will hold both in the pure private value and pure common value models, or if the function $\omega(x, y)$ has decreasing differences in $(x, y)$.

**Theorem 2** (Pooling at the top). In any equilibrium, there exists a type $s$ such that $\bar{t}(s_1) = t$ for some fixed $t$ for all $s_1 \in [\underline{s}, 1]$.

**Proof.** Let $(\bar{t}, \bar{s}_2)$ be an equilibrium. For any $t$ in this equilibrium, let $s_L(t) = \inf\{s_1 : \bar{t}(s_1) = t\}$. Also, let $\gamma(t)$ be defined implicitly as the solution to the equation $t = U_2^{\text{rej}}(\gamma(t), t)$, where the function $U_2^{\text{rej}}$ is defined in Lemma 4 in the Appendix. Now, $\bar{s}_2(t) = \min\{\gamma(t), 1\}$. Note there is a subtle difference between $\bar{s}_2(t)$ and $\gamma(t)$, in that we may have $\gamma(t) > 1$. This occurs when all types $s_2 \in [0, 1]$ strictly prefer to accept the offer $t$ rather than reject, and so $\bar{s}_2(t) = 1$. We first note that for any $t > 0$, we have $\gamma(t) > s_L(t)$. To see this, consider a type $\hat{s}_2 = s_L(t)$ who observes an offer of $t > 0$. Bayes’
rule implies that support\(F(s_1|\hat{s}_2, t) \subseteq [s_L(t), 1]\), and so \(U_2^{\text{rej}}(\hat{s}_2, t) = 0\) for any consistent beliefs \(F(s_1|\hat{s}_2, t)\), which in turn implies that \(\gamma(t) > s_L(t)\).

Next we show that this implies there is pooling at the top. If \(\bar{t}(s_1) = 0\) for all \(s_1\), then there is obviously pooling at the top. Thus, assume that \(\bar{t}(s_1) > 0\) for some \(s_1\), and that there is no pooling at the top. Then, by Theorem 1, there exists some \(\epsilon > 0\) such that \(\bar{t}(s_1)\) is continuous on \((1-\epsilon, 1]\) and \(\bar{t}(1-\delta) < \bar{t}(1)\) for all \(\delta \in (0, \epsilon)\). Next, note that \(\gamma(t)\) is also continuous, and this, together with \(\gamma(\bar{t}(1)) > s_L(\bar{t}(1)) = 1\) from the preceding paragraph, imply that \(\gamma(\bar{t}(1-\delta)) > 1\) for all \(\delta > 0\) sufficiently small. This means that \(\bar{s}_2(\bar{t}(1-\delta)) = \min\{\gamma(\bar{t}(1-\delta)), 1\} > 1\) for all \(\delta > 0\) sufficiently small. Thus, type \(s_1 = 1\) can offer \(\bar{t}(1-\delta)\), still get all types of bidder 2 to accept, and pay a lower transfer \((\bar{t}(1-\delta) < \bar{t}(1))\), which is profitable deviation, and we have a contradiction. \(\square\)

The intuition behind this result is clear. If there is separation at the top, then bidder 2 will perfectly observe bidder 1’s type from her transfer when \(s_1 = 1\). After learning that bidder 1 has such a high signal, all types of bidder 2 will strictly prefer colluding than acting non-cooperatively. But this means that bidder 1 can then imitate type \(s_1 = 1-\epsilon\) by lowering his offer a small amount \(\epsilon > 0\) and still get bidder 2 to accept with probability 1. She is thus paying less to bidder 2 while inducing the same action from him, which is clearly a profitable deviation.

### 3.6 A continuous, partially separating equilibrium

We look for a partially separating equilibrium, where bidder 1’s strategy is continuous and perfectly separating on some interval \([0, \bar{s}_1]\) while all \(s_1 > \bar{s}_1\) pool at some transfer \(t\). In the separating region, bidder 2 correctly infers \(s_1\) and we can use local IC (incentive compatibility) constraints to characterize \(\bar{t}\). In particular, bidder 1’s expected utility when she is of type \(s_1\) in the separating region and mimics type \(\hat{s}_1\) also in the
The separating region is:

\[
U_1(s_1, \tilde{s}_2(\tilde{t}(s_1)), \tilde{t}(s_1)) = \int_0^{\tilde{s}_2(\tilde{t}(s_1))} [\omega(s_1, s_2) - \tilde{t}(s_1)]dF(s_2|s_1).
\]  

(1)

Local IC requires that this function be maximized at \( \tilde{s}_1 = s_1 \).

Define \( h(s_1, t) \) implicitly as the solution to:

\[
t = \omega(h(s_1, t), s_1) - \omega(s_1, s_1).
\]

That is, \( h(s_1, t) \) is the type of bidder 2 who is indifferent between accepting and rejecting \( t \) conditional on knowing \( s_1 \). Assuming that \( \omega \) is continuously differentiable and using the implicit function theorem, we find that:

\[
\frac{\partial h}{\partial s_1} = \frac{\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, t), s_1)}{\omega_1(h(s_1, t), s_1)},
\]

\[
\frac{\partial h}{\partial t} = \frac{1}{\omega_1(h(s_1, t), s_1)}.
\]

Considering \( \tilde{t}(s_1) \) as a function of \( s_1 \), we can write the total derivative of \( h \) with respect to \( s_1 \) as

\[
\frac{dh(s_1, \tilde{t}(s_1))}{ds_1} = \tilde{t}'(s_1) + \frac{\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, t), s_1)}{\omega_1(h(s_1, t), s_1)}.
\]

In the separating region, we have \( \tilde{s}_2(\tilde{t}(s_1)) = h(s_1, \tilde{t}(s_1)) \). Local IC gives the following equation:

\[
[\omega(s_1, \tilde{s}_2(\tilde{t}(s_1))) - \tilde{t}(s_1)]f(\tilde{s}_2(\tilde{t}(s_1))|s_1)\frac{d\tilde{s}_2(\tilde{t}(s_1))}{ds_1} - \tilde{t}'(s_1)F(\tilde{s}_2(\tilde{t}(s_1))|s_1) = 0.
\]  

(2)

Plugging the expression for \( d\tilde{s}_2/ds_1 \) into equation (2), we get the following differential
equation that characterizes $\bar{t}$ in the separating region:

$$
\bar{t}'(s_1) = \frac{[\omega(s_1, \bar{s}(\bar{t}(s_1))) - \bar{t}(s_1)] [\omega_1(s_1, s_1) + \omega_2(s_1, 1) - \omega_2(\bar{s}(\bar{t}(s_1)), s_1)] f(\bar{s}(\bar{t}(s_1))) [s_1] - [\omega(s_1, \bar{s}(\bar{t}(s_1))) - \bar{t}(s_1)] f(\bar{s}(\bar{t}(s_1))) [s_1]}{\omega_1(\bar{s}(\bar{t}(s_1)), s_1) F(\bar{s}(\bar{t}(s_1))) [s_1] - [\omega(s_1, \bar{s}(\bar{t}(s_1))) - \bar{t}(s_1)] f(\bar{s}(\bar{t}(s_1))) [s_1]}.
$$

With this derivation, we have the following result.

**Theorem 3** (Differential characterization). For any $s_1$, let $h(s_1, t)$ be the unique function that solves $t = \omega(h(s_1, t), s_1) - \omega(s_1, s_1)$. Suppose $F(\cdot, \cdot)$ is log-concave. In any equilibrium in which $\bar{t}(\cdot)$ is continuous and is not constant, then it is the unique solution to the following differential equation with initial condition $\bar{t}(0) = 0$:

$$
\bar{t}'(s_1) = \begin{cases} 
\frac{[\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] [\omega_1(s_1, s_1) + \omega_2(s_1, 1) - \omega_2(h(s_1, \bar{t}(s_1)), s_1)] f(h(s_1, \bar{t}(s_1))) [s_1]}{\omega_1(h(s_1, \bar{t}(s_1)), s_1) F(h(s_1, \bar{t}(s_1))) [s_1] - [\omega(s_1, h(s_1, \bar{t}(s_1))) - \bar{t}(s_1)] f(h(s_1, \bar{t}(s_1))) [s_1]}, & h(s_1, \bar{t}(s_1)) < 1 \\
0, & \text{otherwise}
\end{cases}
$$

Let $s^*_1$ be such that $h(s^*_1, \bar{t}(s^*_1)) = 1$, and $t^* = \bar{t}(s^*_1)$. In this equilibrium,

$$
\bar{s}_2(t) = \begin{cases} 
h(\bar{t}^{-1}(t), t), & t < t^* \\
1, & t \geq t^*
\end{cases}
$$

**4  Example: Pure common value**

**4.1  Equilibrium**

Consider a pure interdependent-value example, with $V(Y, S_1, S_2) = V(Y, S_2, S_1) = \frac{S_1 + S_2}{2}$, with $S_1$ and $S_2$ distributed independently and uniformly on the interval $[0, 1]$ (the random variable $Y$ is irrelevant for this example). This implies that $\omega(x, y) = \frac{x+y}{2}$. 

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In the separating region, the differential equation characterizing \( \tilde{t} \) becomes:

\[
\tilde{t}'(s_1) = \begin{cases} 
\frac{s_1}{2t(s_1) - s_1}, & s_1 + 2t(s_1) < 1 \\
0, & \text{else}
\end{cases}
\]

It is easy to show that the unique solution to this differential equation is:

\[
\tilde{t}(s_1) = \min \left\{ s_1, \frac{1}{3} \right\}.
\]

Similarly, if bidder 2 perfectly observes \( s_1 \) from the offered transfer, the indifferent type will be \( \tilde{s}_2(t) = 2t + s_1 \). Thus, the equilibrium strategy for bidder 2 is summarized by:

\[
\tilde{s}_2(t) = \min\{3t, 1\}.
\]

### 4.2 Ex-ante payoffs

Conditional on \( s_1 \), bidder 1’s interim-expected payoff is:

\[
\tilde{U}_1(s_1) := U_1(s_1, \tilde{s}_2(\tilde{t}(s_1)), \tilde{t}(s_1)) = \begin{cases} 
\frac{3s_1^2}{4}, & s_1 \leq \frac{1}{3} \\
\frac{s_1}{2} - \frac{1}{12}, & s_1 > 1/3
\end{cases}
\]

Integrating this over \( s_1 \), we find that her ex-ante expected payoff \( \frac{19}{108} \approx 0.176 \). For bidder 2, his interim-expected profit conditional on \( s_2 \) is

\[
\tilde{U}_2(s_2) := E \left[ U_2(s_2, \tilde{t}(s_1)) \right] = \int_0^{\frac{s_2}{3}} \left( \frac{s_2 - s_1}{2} \right) ds_1 + \int_{\frac{s_2}{3}}^1 \min \left\{ s_1, \frac{1}{3} \right\} ds_1.
\]

Integrating over \( s_2 \), his ex-ante expected payoff is \( \frac{11}{36} \approx 0.3056 \). The expected revenue accruing to the seller is \( \frac{1}{54} \approx 0.019 \).
No-collusion benchmark

The first obvious question to ask is how these numbers compare to the purely non-cooperative game in which no collusion is possible. Both bidders are symmetric in this setting, so consider bidder 1:

\[
\int_0^1 \int_0^{s_1'} \frac{s_1' - s_2}{2} ds_2 ds_1' = \frac{1}{12} \approx 0.08333.
\]

The expected revenue to the seller is \(\frac{1}{3}\). Clearly, with collusion, the seller is much worse off while both bidders are better off. However, bidder 2 captures a (significantly) higher portion of the surplus than bidder 1. Thus, bidder 1 suffers a “first-mover (relative) disadvantage.”

McAfee and McMillan (1992)

McAfee and McMillan (1992) provide the canonical “third-party” approach to modeling collusion. They focus on private values because, as they note, with common values, the optimal collusion mechanism with communication is trivial: the cartel allocates the object according to some exogenous rule. Because reports do not affect the resulting allocation, no bidders’ types have any incentive to misreport. With common values, this allocation rule is clearly efficient.\(^{10}\)

A major assumption inherent in this result is that bidders commit to the cartel ex-ante. If bidders have private information when they decide upon collusion, this mechanism may not work. For example, a bidder with a higher signal for the object may prefer to enter the auction on her own, since she knows the object is valuable and desires a greater

\(^{10}\)Communication is needed for the allocation to be efficient if the seller has a reservation value or sets a reserve price; the cartel needs to establish if the object is worth purchasing. When the seller attaches no (use) value to the object or there is no reserve price, the efficient allocation is to always allocate the object to some bidder. This allocation rule can (obviously) be implemented without any communication.
than $1/n$ chance of winning the object from the cartel. We model collusion at the interim stage.

Computing the ex-ante expected surplus of the two agents when the good is allocated according to the flip of a fair coin, we find that they both receive utility of $1/4$. While they find that all agents receive the same ex-ante surplus, we find a non-symmetric distribution. In particular, we find that bidder 1, the proposer, does worse than when the contract is designed before any private information is acquired, while bidder 2 does better. In addition, the McAfee and McMillan (1992) approach to collusion predicts that collusion always occurs and that the colluding bidders are able to extract the entire surplus. In our model, like in Eső and Schummer (2004), collusion may not occur in equilibrium and the seller is able to retain some positive profits.

**Eső and Schummer (2004)**

The private values model of Eső and Schummer (2004) is a special case of our model when $V_i = S_i$ for both bidders. The equilibrium we describe in Theorem 3 specializes to the same equilibrium that Eső and Schummer study. In this equilibrium, bidder 1’s payoff conditional on $s_1$ is:

$$
\tilde{U}_1(s_1) = \begin{cases} 
\frac{3s_1^2}{4}, & s_1 \leq \frac{2}{3} \\
\frac{1}{2} - \frac{2}{3}s_1, & s_1 > \frac{2}{3}
\end{cases}.
$$

Integrating over $s_1$, we find that her ex ante expected payoff is $\frac{13}{54}$. For bidder 2, his expected payoff conditional on $s_2$ is:

$$
\tilde{U}_2(s_2) = \int_0^{\frac{2}{3}} (s_2 - s_1) ds_1 + \int_{\frac{2}{3}}^1 \min\left\{\frac{1}{3}, \frac{s_1}{2}\right\} ds_1.
$$
Integrating over $s_2$ yields $\frac{1}{3}$. The seller’s expected profit is

$$\int_0^3 \int_{3s_1}^{1} \min\{s_1, s_2\} \, ds_2 \, ds_1 = \frac{2}{27}. $$

The total ex-ante surplus if the good were allocated efficiently would be:

$$\int_0^1 \int_0^1 \max\{s_1, s_2\} \, ds_2 \, ds_1 = \frac{2}{3}. $$

Thus, we lose some surplus when bidder 1 has a lower value for the good but buys bidder 2 out of the auction. Again, both bidders do better than without collusion but bidder 2 does relatively better than bidder 1. We find that bidder 2 does not gain quite as much in the private values case as he does in the common values case, with this extra surplus going mostly to the seller and the rest lost to inefficiency.

Since the common value and private value models are not directly comparable (e.g., the total ex-ante surplus in the common value model is 1/2, while in the private value model it is 2/3), Table 1 below gives the proportion of the total surplus accruing to each bidder and the seller in each case.

<table>
<thead>
<tr>
<th>Model</th>
<th>Bidder 1</th>
<th>Bidder 2</th>
<th>Seller</th>
<th>Efficiency Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our Model</td>
<td>0.35</td>
<td>0.61</td>
<td>0.04</td>
<td>0</td>
</tr>
<tr>
<td>Our Model - No collusion</td>
<td>0.16</td>
<td>0.16</td>
<td>0.67</td>
<td>0</td>
</tr>
<tr>
<td>Private Values - No collusion</td>
<td>0.25</td>
<td>0.25</td>
<td>0.50</td>
<td>0</td>
</tr>
<tr>
<td>Eső and Schummer (2004)</td>
<td>0.36</td>
<td>0.50</td>
<td>0.11</td>
<td>0.03</td>
</tr>
<tr>
<td>McAfee and McMillan (1992)</td>
<td>0.50</td>
<td>0.50</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Comparison of ex-ante equilibrium payoffs against different benchmarks.
5 Outside Competition

When the value functions to the bidders include some common value component, there is an additional rationale for information sharing between bidders that is not present when the cartel is all-inclusive. In the previous discussion, the actual content of bidder 2’s signal is irrelevant to bidder 1; all that matters is that bidder 2 stays out of the auction.

To highlight the role of the information-sharing motivation for collusion we now introduce a third bidder to the model who is outside of the “cartel.” To highlight the importance of communication, we specialize to a pure interdependent value setting. In this environment, the information of bidder 2 is now relevant to bidder 1 in updating her beliefs about the value of the object and refining her bid against the third bidder. This will allow us to study how competition from “outside the cartel” affects the distribution of surplus between the bidders and the seller.

We now have 3 bidders, named 1, 2 and 3, and a seller; bidder 1 is again the “informed principal” who can approach agent 2 with a collusion proposal. Bidder 3 (another “he”) is an outsider who is unaware of the possibility of collusion between 1 and 2. For simplicity, we focus on the example from Section 4. The value of the object is now:

\[ V(S_1, S_2, S_3) = \frac{S_1 + S_2 + S_3}{3} \]

The rest of the game is as before. Define:

\[ \omega(x, y) := E[V(S_1, S_2, S_3) | S_1 = x, \max\{S_2, S_3\} = y] = \frac{x}{3} + \frac{y}{2}. \]

The next proposition identifies equilibrium bidding behavior.

**Proposition 2.** Assume that, when bidder 2 accepts the transfer, he reports truthfully to bidder
Then, it is part of a (weak perfect) Bayesian equilibrium for all bidders to bid \( \beta(s_i) = \frac{5}{6}s_i \) when collusion fails and for bidder 1 to bid

\[
\beta_1^*(s_1, s_2) = \min \left\{ \frac{5}{9}(s_1 + s_2), \frac{5}{6} \right\}
\]

following collusion with bidder 2.

**Proof.** Bidder 3 bids:

\[
\beta(s_3) = \omega(s_3, s_3) = \frac{5}{6}s_3
\]

in the auction. Let the updated beliefs about the highest signal out of those of the other bidders be represented by some cdf \( G_i(\cdot|C) \) (bidder 1 will also update her beliefs about \( s_2 \) upon rejection of the contract). Outside options with respect to proposal \( C \) are then:

\[
u_i(s_i;C) := U_i \left( \frac{5}{6}s_i, s_i; C \right) = \int_0^{s_i} \left( s_i - \frac{u}{3} \right) dG_i(u|C).
\]

If the collusion agreement is accepted, bidder 1 learns bidder 2’s signal and faces only bidder 3 in the auction. Her expected payoff in the auction from submitting a bid of \( b \in \mathbb{R}_+ \), \( U_1(b, s_1, s_2) \), is:

\[
U_1(b, s_1, s_2) := \int_0^{\min\{\frac{6}{5}b,1\}} \left( \frac{s_1 + s_2}{3} + \frac{u}{3} - \frac{5}{6}u \right) du
\]

\[
= \int_0^{\min\{\frac{6}{5}b,1\}} \left( \frac{s_1 + s_2}{3} - \frac{u}{2} \right) du.
\]

---

As in section 3, bidder 2 has no reason to lie if he can commit to staying out of the auction. Inability to commit need not be a big problem with only two bidders; it is an equilibrium for bidder 1 to place a bid of 1 in the auction after colluding with bidder 2, effectively discouraging him from ‘sneaking back’ into the auction. Here, however, the problem is more serious. Bidder 1 cannot always discourage bidder 2 without taking a loss herself. Moreover, bidder 2 can manipulate a trusting bidder 1’s bid in the auction and learn about her signal from her offer. Of course, such bad incentives of bidder 2 break down bidder 1’s incentives to disclose information through her proposals in the first place.
Now, for $b < \frac{5}{6}$,

$$U_1'(b, s_1, s_2) = \left(\frac{s_1 + s_2}{3} - \frac{3}{5}b\right) \frac{6}{5}$$

$$U_1''(b, s_1, s_2) = -\frac{18}{25} < 0.$$

The unique best response is:

$$b^*(s_1, s_2) = \frac{5}{9}(s_1 + s_2).$$

We have that $b^*(s_1, s_2) < \frac{5}{6}$ provided $s_1 + s_2 < \frac{3}{2}$. If $s_1 + s_2 \geq \frac{3}{2}$, then $U_1(b, s_1, s_2)$ is strictly increasing in $b$ up to $\frac{5}{6}$, at which point it becomes constant. Hence, a best response $\beta_1^*$ is:

$$\beta_1^*(s_1, s_2) = \min\left\{ \frac{5}{9}(s_1 + s_2), \frac{5}{6} \right\},$$

as claimed.

We next show that a result analogous to Lemma 1 holds in this setting for bidder 2.

**Lemma 2.** In any equilibrium, the set of types of bidder 2 who accept a transfer of $t$ must be of the form $[0, \bar{s}_2(t)]$.

**Proof.** Given that bidder 2 observes an offer of $t$, she will have some beliefs over the types of the highest signal besides his represented by $G(u|t)$. Then, he will accept the transfer if and only if:

$$t \geq \int_0^{s_2} \left(\frac{s_2 - u}{3}\right) dG(u|t) = \frac{s_2}{3} G(s_2|t) - \int_0^{s_2} \frac{u}{3} dG(u|t).$$

We show that the right-hand side is non-decreasing in $s_2$. Integrating the second term
by parts yields:

\[
\int_0^{s_2} \frac{u}{3} dG(u|t) = \left[\frac{u}{3} G(u|t)\right]_0^{s_2} - \int_0^{s_2} \frac{1}{3} G(u|t) du.
\]

Hence, the right-hand side is \(\frac{1}{3} \int_0^{s_2} G(u|t) du\), which is clearly non-decreasing in \(s_2\).

Just as before, bidder 2’s strategy can be summarized by a cutoff such that all types below the cutoff accept the transfer \(t\), and all types above reject. We again look for an equilibrium \((\bar{t}, \bar{s}_2)\) that is partially separating. In the separating region, we can write an equation analogous to Equation (1) for bidder 1’s utility if he is of true type \(s_1\) and he mimics type \(\hat{s}_1\) as:

\[
U_1(s_1, \bar{s}_2(\bar{t}(\hat{s}_1)), \hat{s}_1) = \int_0^{s_2(\bar{t}(\hat{s}_1))} [U^*_1(s_1, s_2) - \bar{t}(\hat{s}_1)] ds_2.
\]

We can again use local IC to characterize \(\bar{t}\) in terms of a differential equation \(\bar{t}'(s_1) = g(s_1, \bar{t}(s_1))\) in the separating region. The equilibrium is qualitatively very similar to that found before. Low types of bidder 1 separate, up to the point where offering more guarantees all types of bidder 2 will accept, beyond which all types pool.

We can also numerically calculate the equilibrium strategy for player 2, and the proportion of the total ex-ante surplus accruing to each of bidders 1, 2, and 3 and the seller. The results are shown in Tables 2 and 3.

Several points of interest arise from these two tables. Comparing the three bidder

<table>
<thead>
<tr>
<th>No Collusion</th>
<th>Bidder 1</th>
<th>Bidder 2</th>
<th>Bidder 3</th>
<th>Seller</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Bidders</td>
<td>0.16</td>
<td>0.16</td>
<td>-</td>
<td>0.67</td>
</tr>
<tr>
<td>3 Bidders</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.83</td>
</tr>
</tbody>
</table>

Table 2: Comparison of ex-ante equilibrium payoffs with 2 versus 3 bidders in the absence of the possibility of collusion.
model with and without collusion, we find that both bidders 1 and 2 are better off with collusion, and more of the gains go to the receiver (bidder 2) than the proposer (bidder 1). The benefits to the cartel come at the expense of the seller and of bidder 3, as expected.

Next we compare the 2 bidder case to the 3 bidder case. Here, we note that with 2 bidders, collusion increased the profits of bidder 2 by 280% and those of bidder 1 by 118%. With three bidders, these increases drop to 150% for bidder 2 and 33% for bidder 1. While some of this loss is due to the presence of bidder 3, most of the lost surplus goes to the seller instead. With 2 bidders, the presence of collusion reduced seller profits by 94%, while with some competition outside of the cartel, collusion among bidders 1 and 2 only decreases seller profits by about 13%.

Some broad lessons can be drawn from this exercise. First, while collusion certainly decreases seller profits, the presence of at least some competition outside of the cartel drastically decrease the effectiveness of the cartel. Second, the first-mover disadvantage prevails with outside competition. If we think of this game as embedded in some larger game in which the proposer is decided endogenously, it may be likely that neither bidder will propose, hoping instead to be the receiver. Thus, while there will be gains from colluding, waiting for the other bidder to take the first step might erode those gains.
6 Conclusion

In auctions in which collusion is illegal and collusion agreements are both designed and executed by the involved parties, the standard approach is inadequate. In this paper, we model collusion in a SPA with interdependent valuations and affiliated signals as a contract-design problem by an informed principal who runs the mechanism herself.

We characterize equilibria in terms of a transfer function and an “acceptance cutoff” and provide sufficient conditions under which these two are non-decreasing on the equilibrium path. Moreover, show that there is always “pooling at the top” in the transfers and identify the differential equation that characterizes the transfer function on the separating region under an additional differentiability assumption, extending the results in Eső and Schummer (2004). It is a matter of ongoing work whether this equilibrium is the unique equilibrium that satisfies D1-type refinements.

We find that collusion between the two bidders can severely harm the seller. However, while both bidders are better off than not colluding, there is a (relative) first-mover disadvantage. If we think of this game as embedded in some larger game in which the proposer is decided endogenously, it may be likely that neither bidder will want to be the first to propose. An interesting extension that is left for future work is to model the allocation of bargaining power in the collusion stage as a war attrition, where the loser makes the proposal.

Moreover, the amount of surplus captured by the colluding bidders at the expense of the seller might be greatly reduced when “outside competition,” in the form of additional bidders outside the ring, is introduced. The generality of this feature of the example studied is another matter of ongoing work.
References


A Proofs

Bidder 1’s expected payoff given she is of type $s_1$, offers transfer $t$, and bidder 2 uses cutoff $\tilde{s}_2$, is:

\[
U_1(s_1, \tilde{s}_2, t) = \int_0^{\tilde{s}_2} [\omega(s_1, s_2) - t]dF(s_2|s_1) + \mathbb{I}(s_1 > \tilde{s}_2) \int_{\tilde{s}_2}^{s_1} [\omega(s_1, s_2) - \omega(s_2, s_2)]dF(s_2|s_1).
\] (4)

Bidder 1’s equilibrium payoff when she is of type $s_1$ and offers a transfer $t$ is given by $U_1(s_1, \tilde{s}_2(t), t)$.

Lemma 3. In any equilibrium, $\tilde{s}_2(t) \geq \inf \{s_1 : \bar{t}(s_1) = t\}$ for all $t > 0$ in the range of $\bar{t}$.

Proof. Let $\underline{s}_1(t)$ denote the infimum in the statement of the lemma. Assume that the opposite inequality is true and consider any $s'_2 \in (\tilde{s}_2(t), \underline{s}_1(t))$. The equilibrium strategy instructs bidder 2 of type $s'_2$ to reject the offer of $t$. However, upon rejection, he learns that bidder 1 is of type at least $\underline{s}_1(t)$. Being bound to lose in the ensuing auction, accepting $t$ is a profitable deviation. \hfill \Box

For the next lemma, we need some additional notation. For any transfer function $\bar{t}$ and any set $T \subseteq \mathbb{R}_+$, let $\bar{t}^{-1}(T)$ be the preimage of $T$ under $\bar{t}$, $\bar{t}^{-1}(T) := \{s_1 : \bar{t}(s_1) \in T\}$. For any two such sets $X, X' \subseteq \mathbb{R}_+$, if $x' > x$ for all $x \in X$ and all $x' \in X'$, we write $X' >_{SSO} X$ ($X'$ is strictly greater than $X$ in the strong set order).

Lemma 4. Let $(\bar{t}, \bar{s}_2)$ represent an equilibrium. If $\bar{t}$ is strictly decreasing on a neighbourhood of point $\hat{s}_1$, then $\bar{s}_2$ must be strictly decreasing on a neighbourhood of $\bar{t}(\hat{s}_1)$. Similarly, if there exists some $t_L, t_H$ both in the range of $\bar{t}$ such that $t_L < t_H$, but $\bar{s}_2(t_L) > \bar{s}_2(t_H)$ (i.e., $\bar{s}_2$ is strictly
decreasing at some points on the equilibrium path), then $\bar{t}^{-1}(\{t_H\}) \not>_{SSO} \bar{t}^{-1}(\{t_L\})$ (i.e., $\bar{t}$ is strictly decreasing).

**Proof.** First, assume that $\bar{s}_2(t_L) > \bar{s}_2(t_H)$ for some $t_H > t_L$ both in the range of $\bar{t}$, but $\bar{t}^{-1}(\{t_H\}) >_{SSO} \bar{t}^{-1}(\{t_L\})$. Let $F(s_1|\hat{s}_2,t)$ be the updated beliefs of bidder 2 over $s_1$, conditional on his signal $\hat{s}_2$ and the observed transfer $t$. $\bar{t}^{-1}(\{t_H\}) >_{SSO} \bar{t}^{-1}(\{t_L\})$ implies that $F(s_1|\hat{s}_2,t_H)$ first-order stochastically dominates $F(s_1|\hat{s}_2,t_L)$ for any $\hat{s}_2$.

Define a function $Rej(\hat{s}_2,t) = \int_0^{\hat{s}_2} [\omega(\hat{s}_2,s_1) - \omega(s_1,s_1)] dF(s_1|\hat{s}_2,t)$ to be the expected payoff to type $\hat{s}_2$ of agent 2 from rejecting a transfer $t$ (note that bidder 2 uses the updated beliefs $F(s_1|\hat{s}_2,t)$). Now $\bar{s}_2(t)$ is defined as the solution to $t = Rej(\bar{s}_2(t),t)$. We have that $Rej(\bar{s}_2,t_H) \leq Rej(\bar{s}_2,t_L)$, by the fact that the integrand is decreasing and $F(s_1|\hat{s}_2,t_H)$ first-order stochastically dominates $F(s_1|\hat{s}_2,t_L)$. Now, consider some type $\hat{s}_2$, and assume that $\hat{s}_2$ accepts an offer of $t_L$, i.e., $t_L \geq Rej(\hat{s}_2,t_L)$. This implies that:

$$t_H > t_L \geq Rej(\hat{s}_2,t_L) \geq Rej(\hat{s}_2,t_H)$$

or $t_H > Rej(\hat{s}_2,t_H)$, i.e., type $\hat{s}_2$ also accepts an offer of $t_H$. However, this is clearly a contradiction to the fact that $\bar{s}_2(t_H) < \bar{s}_2(t_L)$.

For the other direction, assume that $\bar{t}(\cdot)$ is strictly decreasing at $\bar{s}_1$, but that $\bar{s}_2(\cdot)$ is weakly increasing around $\bar{t}(\bar{s}_1)$. The case of separation is ruled out below using local IC constraints, and so we focus on the case of pooling only. First, consider pooling on both sides of $\hat{s}_1$, so that and let $t_L := \bar{t}(\bar{s}_1)$ and $t_H := \lim_{\varepsilon \to 0} \bar{t}(\bar{s}_1 - \varepsilon)$. We have $t_H > t_L$. Now, assume that $\bar{s}_2(t_L) \leq \bar{s}_2(t_H)$. Equality can be immediately ruled out because obviously no type of bidder 1 would offer $t_H$. and so here we only focus on pooling. First, consider pooling on both sides of $\bar{s}_1$, i.e.,

$$\bar{t}(\bar{s}_1) = \begin{cases} t_H & s_1 \in [\bar{s}_L,\bar{s}_1) \\ t_L & s_1 \in [\bar{s}_1,\bar{s}_H] \end{cases}$$

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for some $s_L < \hat{s}_1 < s_H$, and $t_L < t_H$. Towards a contradiction, assume that $\hat{s}_2(t_L) < \hat{s}_2(t_H)$,\(^\text{12}\) and notice that $\hat{s}_2(t_H) > \hat{s}_2(t_L) > \hat{s}_1.$\(^\text{13}\) Thus, the second term in Equation (4) can be ignored for all types of bidder 1 in some interval $[0, \hat{s}_1 + \delta)$ for some $\delta > 0$. This means we can write bidder 1’s utility as $U_1(s_1, s_2(t), t) = \int_0^{\hat{s}_2(t)} \omega(s_1, s_2(t)) - t|dF(s_2|s_1)$. By assumption, we have $U_1(s_L, \hat{s}_2(t_H), t_H) \geq U_1(s_L, \hat{s}_2(t_L), t_L)$ and $U_1(\hat{s}_1 + \epsilon, \hat{s}_2(t_H), t_H) \leq U_1(\hat{s}_1 + \epsilon, \hat{s}_2(t_L), t_L)$. Now, the following string of inequalities is true:

$$
\int_{\hat{s}_2(t_L)}^{\hat{s}_2(t_H)} \omega(s_1 + \epsilon, s_2) dF(s_2|\hat{s}_1 + \epsilon)
\geq \int_{\hat{s}_2(t_L)}^{\hat{s}_2(t_H)} \omega(s_L, s_2) dF(s_2|s_L) \tag{assumption (A1)}
\geq \int_{\hat{s}_2(t_L)}^{\hat{s}_2(t_H)} t_H dF(s_2|s_L) + \int_0^{\hat{s}_2(t_L)} [t_H - t_L] dF(s_2|s_L) \tag{IC for type s_L}
\geq \int_{\hat{s}_2(t_L)}^{\hat{s}_2(t_H)} t_H dF(s_2|\hat{s}_1 + \epsilon) + \int_0^{\hat{s}_2(t_L)} [t_H - t_L] dF(s_2|\hat{s}_1 + \epsilon) \tag{affiliation}
\geq \int_{\hat{s}_2(t_L)}^{\hat{s}_2(t_H)} \omega(s_1 + \epsilon, s_2) dF(s_2|\hat{s}_1 + \epsilon) \tag{IC for type \hat{s}_1 + \epsilon}
$$

Putting together the first and last expressions, we get a contradiction. The case of pooling only to the right of $\hat{s}_1$ can be ruled out using the exact same argument.

Pooling only to the left of $\hat{s}_1$ requires a slight modification. Assume that $\bar{t}(s_1) = t_H$ for all $s_1 \in (\hat{s}_1 - \delta_1, \hat{s}_1)$, $\bar{t}(\hat{s}_1) = t_L$, and $\bar{t}(s_1) > t_L$ for all $s_1 \in (\hat{s}_1, \hat{s}_1 + \delta_2)$ and all $\delta_1, \delta_2 > 0$ sufficiently small. For this case, we use essentially the same argument, only consider a type $\hat{s}_1 - \epsilon$ for $\epsilon \in (0, \delta_1)$. Note by standard arguments, type $\hat{s}_1$ must be indifferent.

\(^{12}\)Equality can be ruled out because then both $t_L$ and $t_H$ would induce the same response from bidder 2, and hence it is obvious that all agents would choose $t_L$.

\(^{13}\)Lemma 3 above only gives $\hat{s}_2(t_L) \geq \hat{s}_1$, but this inequality is actually strict because of pooling at $t_L$. 

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between offering \( t_L \) and \( t_H \).

\[
\int_{\tilde{s}_2(t_H)}^{\tilde{s}_2(t_L)} \omega(\hat{s}_1 - \varepsilon, s_2) dF(s_2|\hat{s}_1 - \varepsilon) \\
\leq \int_{\tilde{s}_2(t_L)}^{\tilde{s}_2(t_H)} \omega(\hat{s}_1, s_2) dF(s_2|\hat{s}_1) \\
= \int_{\tilde{s}_2(t_L)}^{\tilde{s}_2(t_H)} t_H dF(s_2|\hat{s}_1) + \int_{t_H - t_L}^{\tilde{s}_2(t_L)} |t_H - t_L| dF(s_2|\hat{s}_1) \quad \text{(IC for type } \hat{s}_1) \\
< \int_{\tilde{s}_2(t_L)}^{\tilde{s}_2(t_H)} t_H dF(s_2|\hat{s}_1 - \varepsilon) + \int_{0}^{\tilde{s}_2(t_L)} |t_H - t_L| dF(s_2|\hat{s}_1 - \varepsilon) \quad \text{(affiliation)} \\
\leq \int_{\tilde{s}_2(t_L)}^{\tilde{s}_2(t_H)} \omega(\hat{s}_1 - \varepsilon, s_2) dF(s_2|\hat{s}_1 - \varepsilon) \quad \text{(IC for type } \hat{s}_1 - \varepsilon)
\]

Once again, looking at the first and last expressions, we see a contradiction.

Lemma 5. Given an equilibrium transfer function \( \bar{I} \), assume that we can find a point \( s_M \in (0, 1) \) such that, for some \( \varepsilon > 0 \), \( \bar{I}(s_M - \delta) > \bar{I}(s_M + \delta) \) for all \( \delta \in (0, \varepsilon) \). Then: (i) there cannot be pooling to the left of \( s_M \); i.e., there do not exist \( s_L < s_M \) and \( t_L < t_H \) such that

\[
\bar{I}(s_1) = \begin{cases} 
  t_H & s_1 \in [s_L, s_M) \\
  t_L & s_1 = s_M 
\end{cases}
\]

(ii) there cannot be pooling to the right of \( s_M \); i.e., there do not exist \( s_M < \bar{s} \) and \( t_L \) such that:

\[
\bar{I}(s_1) = \begin{cases} 
  \gamma(s_1) & s_1 < s_M \\
  t_L & s_1 \in [s_M, \bar{s}) 
\end{cases}
\]

where \( \gamma(s_1) \) is some function, continuous on \( [s_M - \delta, s_M) \) for some \( \delta > 0 \) sufficiently small, with the property that \( \lim_{s_1 \to s_M} \gamma(s_1) > t_L \).

Proof. Proof of (i). Assume that there exists an equilibrium with \( \bar{I} \) satisfying the property
in the statement. Then, by Lemma 4, we know that \( \bar{s}_2(t_H) < \bar{s}_2(t_L) \). Using Equation (4), incentive compatibility for type \( s_L \) requires

\[
\int_0^{\bar{s}_2(t_H)} [\omega(s_L, s_2) - t_H]dF(s_2|s_L) \geq \int_0^{\bar{s}_2(t_L)} [\omega(s_L, s_2) - t_L]dF(s_2|s_L).
\]

This inequality implies that \( \int_{s_2(t_H)}^{\bar{s}_2(t_L)} [\omega(s_L, s_2) - t_L]dF(s_2|s_L) < 0 \), or \( \frac{\int_{s_2(t_H)}^{\bar{s}_2(t_L)} \omega(s_L, s_2)dF(s_2|s_L)}{F(\bar{s}_2(t_L)|s_L) - F(s_2(t_H)|s_L)} < t_L \). Note that this inequality can be written as:

\[
E_{s_2|s_1}[\omega(s_L, s_2)|\bar{s}_2(t_H) \leq s_2 \leq \bar{s}_2(t_L)] < t_L.
\]

However, we also know that:

\[
E_{s_2|s_1}[\omega(s_L, s_2)|0 \leq s_2 \leq \bar{s}_2(t_H)] \geq t_H > t_L.
\]

Since \( \omega(s_L, s_2) \) is increasing in its 2nd argument, we have:

\[
E_{s_2|s_1}[\omega(s_L, s_2)|\bar{s}_2(t_H) \leq s_2 \leq \bar{s}_2(t_L)] \geq E_{s_2|s_1}[\omega(s_L, s_2)|0 \leq s_2 \leq \bar{s}_2(t_H)] \geq t_H > t_L,
\]

a contradiction.

Proof of (ii). Choose some \( s_L < s_M \) such that \( \bar{t}(s_L) > t_L \) and \( s_H \in (s_M, \bar{s}) \). Define \( t_H := \bar{t}(s_L) \). Again by Lemma 4, we have \( \bar{s}_2(t_H) < \bar{s}_2(t_L) \). Now, IC for type \( s_H \) requires

\[
\int_0^{\bar{s}_2(t_H)} [\omega(s_H, s_2) - t_H]dF(s_2|s_H) \leq \int_0^{\bar{s}_2(t_L)} [\omega(s_H, s_2) - t_L]dF(s_2|s_H).
\]

This inequality implies that:

\[
\frac{\int_{s_2(t_H)}^{\bar{s}_2(t_L)} \omega(s_H, s_2)dF(s_2|s_H)}{F(\bar{s}_2(t_L)|s_H) - F(s_2(t_H)|s_H)} > t_L.
\]
We can write this as:

\[
E[\omega(s_H, s_2) | s_2(t_H) \leq s_2 \leq s_2(t_L)] < t_L.
\]

Since \( \omega \) is nondecreasing in its first argument, we have:

\[
E[\omega(s_L, s_2) | 0 \leq s_2 \leq s_2(t_L)] \leq E[\omega(s_H, s_2) | s_2(t_H) \leq s_2 \leq s_2(t_L)] < t_L < t_H.
\]

The first and last expressions give the desired contradiction. \( \square \)

Proof of Theorem 3

Existence. Recall the differential equation (3):

\[
P'(s_1) = \begin{cases} 
\frac{\omega(s_1, h(s_1, f(s_1))| - f(s_1)) [\omega_1(s_1, s_1) + \omega_2(s_1, s_1) - \omega_2(h(s_1, f(s_1))| s_1)]}{\omega_1(h(s_1, f(s_1))| s_1) F(h(s_1, f(s_1))| s_1) - |\omega(s_1, h(s_1, f(s_1))| s_1)|} , & h(s_1, i) < 1 \\
0, & \text{else}
\end{cases}
\]

It is possible to show that \( \omega_1(0, 0) > 0 \) and and \( f(0|s_1 = 0) > 0 \) imply that \( 0 < P'(0) < \infty \).\(^{14}\) Now, the (local) inverse of \( l(s_1) \), denoted \( S(l) \), is defined by

\[
S(0) = 0
\]

\[
S'(l) = \frac{1 - \omega(S(l), h(S(l), l))| f(h(S(l), l)| S(l)) + F(h(S(l), l)| S(l)) \omega_1(h(S(l), l)| S(l))}{\omega_1(h(S(l), l)| S(l)) - l|f(h(S(l))| S(l))|\omega_1(S(l), S(l)) + \omega_2(S(l), S(l)) - \omega_2(h(S(l)), S(l))}.
\]

By the Inverse Function Theorem, \( S'(0) = \frac{1}{P'(0)} \in (0, \infty) \). Next, let \( l \) be such that \( S'(l) = 0 \). Then, \( \omega(S(l), h(S(l))) > l \) and:

\[
S''(l) = \frac{d}{dl} \left( \frac{F(h(S(l), l)| S(l))}{\omega_1(h(S(l), l), S(l))} \right) = \frac{\omega_1(h(S(l), l), S(l)) - l|\omega_1(S(l), S(l)) + \omega_2(S(l), S(l)) - \omega_2(h(S(l), l), S(l))|}{\omega_1(h(S(l), l), S(l)) - l^{2}\omega_1(S(l), S(l)) + \omega_2(S(l), S(l)) - \omega_2(h(S(l), l), S(l))}
\]

\(^{14}\)The algebra was carried out with Mathematica. The formulas are cumbersome, and so are not printed here.
Now, \( \frac{d}{dt} \left( \frac{f(h(S(t), \tilde{I})|S(t))}{f(h(S(t), \tilde{I})|S(t))} \right) > 0 \) by log-concavity and

\[
\omega_1(S(\tilde{I}), S(\tilde{I})) + \omega_2(S(\tilde{I}), S(\tilde{I})) - \omega_2(h(S(\tilde{I}), \tilde{I}), S(\tilde{I})) > 0
\]

by assumption A2. Thus, \( S''(\tilde{I}) \) is positive. This implies that if \( \tilde{I}'(s_1) \) is ever infinite, that \( s_1 \) is only an inflection point of \( \tilde{I}(\cdot) \). Thus, a equation (3) has a unique, continuous solution.

To show that this is an equilibrium, first consider bidder 2. If bidder 2 observes some \( t < t^* \), she believes bidder 1’s type is \( \tilde{I}^{-1}(t) \), and the obvious best response is \( \tilde{s}_2(t) = h(\tilde{I}^{-1}(t), t) \). If \( t = t^* \), bidder 2 updates her beliefs and places all weight on \( [s_1^*, 1] \), in which case it is again easy to show that the best response is \( \tilde{s}_2(t) = 1 \). If \( t > t^* \), her beliefs can be anything, and the best response will depend on the beliefs. However, no matter what bidder 2 believes and what her response to an action \( t > t^* \) is, bidder 1 will never offer such a \( t \), because she can always induce the same action from bidder 2 by offering a lower transfer (because \( \tilde{I}([0, t^*]) = [0, 1] \)).

So, consider bidder 1. As argued above, bidder 1 will never offer \( t > t^* \). Thus, we only need check that no type \( s_1 \) wants to offer some \( \tilde{I}(\hat{s}_1) \) for some \( \hat{s}_1 \neq s_1 \). First, define the function \( \chi(s_1, s_2, \hat{s}_1) = \frac{d}{ds_1} \left[ \left( \omega(s_1, s_2) - \tilde{I}(\hat{s}_1) \right) f(s_2|s_1) \right] = \frac{\partial \omega(s_1, s_2)}{\partial s_1} f(s_2|s_1) + \left( \omega(s_1, s_2) - \tilde{I}(\hat{s}_1) \right) \frac{\partial f(s_2|s_1)}{\partial s_1} \right] \). Then, note that the envelope theorem gives

\[
\frac{d\pi_1(s_1)}{ds_1} = \int_0^{\tilde{I}(s_1)} \chi(s_1, s_2, \hat{s}_1) ds_2
\]

Now, define a function \( \phi_1(s_1, \hat{s}_1) = U(s_1, \tilde{s}_2(\tilde{I}(\hat{s}_1)), \tilde{s}_2(\tilde{I}(\hat{s}_1))) - \pi(s_1) \), and consider \( \phi_2(s_1, \hat{s}_1) \):

\[
\phi_2(s_1, \hat{s}_1) = \begin{cases} 
\int_0^{\tilde{I}(s_1)} \chi(s_1, s_2, \hat{s}_1) ds_2 - \int_0^{\tilde{I}(s_1)} \chi(s_1, s_2, \hat{s}_1) ds_2, & s_1 \leq \tilde{s}_2(\tilde{I}(\hat{s}_1)) \\
\int_0^{\tilde{I}(s_1)} \chi(s_1, s_2, \hat{s}_1) ds_2 + \int_0^{s_1} \rho(s_1, s_2) ds_2 - \int_0^{\tilde{I}(s_1)} \chi(s_1, s_2, \hat{s}_1) ds_2, & s_1 > \tilde{s}_2(\tilde{I}(\hat{s}_1)) 
\end{cases}
\]

where \( \rho(s_1, s_2) = \frac{\partial \omega(s_1, s_2)}{\partial s_1} f(s_2|s_1) + \left[ \omega(s_1, s_2) - \omega(s_2, s_2) \right] \frac{\partial f(s_2|s_1)}{\partial s_1} \). Now, since \( \tilde{I}(\cdot) \) and
\( \tilde{s}_2(\cdot) \) are increasing, if \( s_1 \leq \tilde{s}_1 \), then \( \phi_{s_1}(s_1, \tilde{s}_1) \geq 0. \)

\[
U_1(s_1, \tilde{s}_2(\tilde{I}(s_1)), \tilde{I}(s_1)) = U_1(0, \tilde{s}_2(\tilde{I}(0)), \tilde{I}(0)) + \int_0^{s_1} \left[ \int_0^{\tilde{s}_2(\tilde{I}(s_1'))} \chi(s_1', s_2) ds_2 \right] ds_1'
\]

where \( \chi(s_1, s_2) = \frac{d}{ds_1} [\omega(s_1, s_2)f(s_2|s_1)] = \frac{\partial \omega(s_1, s_2)}{\partial s_1} f(s_2|s_1) + \omega(s_1, s_2) \frac{\partial f(s_2|s_1)}{\partial s_1}. \) Using this, we have

\[
U_1(s_1, \tilde{s}_2(\tilde{I}(s_1)), \tilde{I}(s_1)) - U_1(s_1, \tilde{s}_2(\tilde{I}(\tilde{s}_1)), \tilde{I}(\tilde{s}_1)) = \int_{\tilde{s}_1}^{s_1} \int_0^{\tilde{s}_2(\tilde{I}(s_1'))} \chi(s_1', s_2) ds_2 ds_1'
\]

Switching the order of integration, the RHS becomes

\[
\int_0^{\tilde{s}_2(\tilde{I}(s_1))} \int_{\tilde{s}_1}^{s_1} \chi(s_1', s_2) ds_1' ds_2 + \int_{\tilde{s}_2(\tilde{I}(\tilde{s}_1))}^{s_1} \int_{s_1}^{s_1^{-1}(s_2)} \chi(s_1', s_2) ds_1' ds_2
\]

where \( s_1^{-1}(s_2) \) is the \( s_1 \) such that \( \tilde{s}_2(\tilde{I}(s_1)) = s_2. \) Since \( \chi(s_1', s_2) \) is a derivative, the inner integral is easy to evaluate, and we get

\[
\int_0^{\tilde{s}_2(\tilde{I}(s_1))} [\omega(s_1, s_2)f(s_2|s_1) - \omega(s_1, s_2)f(s_2|\tilde{s}_1)] ds_2 \quad (5)
\]

\[
+ \int_{\tilde{s}_2(\tilde{I}(\tilde{s}_1))}^{\tilde{s}_2(\tilde{I}(s_1))} [\omega(s_1, s_2)f(s_2|s_1) - \omega(s_1, s_2^{-1}(s_2)) f(s_2|s_1^{-1}(s_2))] ds_2
\]

Consider also the equation \( U_1(s_1, \tilde{s}_2(\tilde{I}(s_1)), \tilde{I}(s_1)) - U_1(s_1, \tilde{s}_2(\tilde{I}(\tilde{s}_1)), \tilde{s}_1). \) This is:

\[
\begin{cases}
\int_0^{\tilde{s}_2(\tilde{I}(s_1))} [\omega(s_1, s_2) - \omega(s_1, s_2^{-1}(s_2))] f(s_2|s_1) ds_2 & \text{if } s_1 < \tilde{s}_2(\tilde{I}(s_1)) \\
\int_0^{\tilde{s}_2(\tilde{I}(s_1))} [\omega(s_1, s_2) - \omega(s_1, s_2^{-1}(s_2))] f(s_2|s_1) ds_2 + \int_0^{\tilde{s}_2(\tilde{I}(s_1))} \omega(s_1, s_2) ds_2 & \text{if } s_1 \geq \tilde{s}_2(\tilde{I}(s_1))
\end{cases}
\]

**Uniqueness.** We first show that in any continuous equilibrium, if there is pooling on some (non-degenerate) interval \([s_1', s_1'']\), then \( s_1'' = 1. \) Assume not, so that \( s_1' < s_1'' < 1 \) and let \( \tilde{t}'' = \tilde{I}(s_1''). \) Since the equilibrium is continuous, we have \( \tilde{I}(\cdot) \) strictly increasing
on \((s_1'', s_1' + \epsilon)\) for some small \(\epsilon > 0\), with \(\lim_{s_1 \to s_1''} \tilde{I}(s_1) = t''\). Bidder 2’s optimal cutoff, \(\bar{s}_2(t'')\), is implicitly defined as the solution to

\[
t'' = \int_{s_1''}^{\bar{s}_2(t'')} \left[ \omega(\bar{s}_2(t''), s_1) - \omega(s_1, s_1) \right] dF(s_1 | \bar{s}_2(t''))
\]

and is such that \(\bar{s}_2(t'') \in (h(s_1', t''), h(s_1'', t''))\). Also, note that \(s_1'' < h(s_1'', t'') < 1\). Now, consider a transfer \(t'' + \delta\) for some small \(\delta\). Since \(t'' + \delta\) is in a separating region, we have \(\bar{s}_2(t'' + \delta) > h(s_1'', t'') \geq s_1''\). Now, we calculate the change in utility from offering \(t'' + \delta\) minus that from offering \(t''\). There are two cases. When \(s_1'' > \bar{s}_2(t'')\), this expression is

\[
\int_{s_1''}^{\bar{s}_2(t'' + \delta)} \omega(s_1'', s_2) dF(s_2 | s_1'') + \int_{\bar{s}_2(t'')}^{s_1''} \omega(s_2, s_2) dF(s_2 | s_1'') \]

\[
- \int_{\bar{s}_2(t'')}^{s_1''} t'' dF(s_2 | s_1'') - \int_{0}^{\bar{s}_2(t'' + \delta)} \delta dF(s_2 | s_1'')
\]

We also have \(\lim_{\delta \to 0} \bar{s}_2(t'' + \delta) = h(s_1'', t'') > s_1'' > \bar{s}_2(t'')\), and, by Lemma A, \(t'' < \omega(\bar{s}_2(t''), \bar{s}_2(t''))\). So, as \(\delta \to 0\), the last term becomes arbitrarily small, while the sum of the first three terms approaches a number strictly greater than 0, meaning \(t'' + \delta\) is a profitable deviation for some \(\delta\) sufficiently small. A similar argument holds when \(s_1'' \leq \bar{s}_2(t'')\).

Thus, there can be at most one pooling interval. Since we assume that the equilibrium is continuous but not constant, this means that there exists an \(s_1'\) such that \(\tilde{I}(\cdot)\) is strictly increasing and on \([0, \hat{s}_1]\) and is constant on \((\hat{s}_1, 1]\) for some \(\hat{s}_1 \leq 1\). We next show that we must have \(\tilde{I}(0) = 0\). Assume that \(\tilde{I}(0) = t' > 0\). Since we know \(\tilde{I}(\cdot)\) is strictly increasing at 0, bidder 2 will use cutoff \(h(0, t')\), where \(\omega(h(0, t'), 0) = t'\). Bidder 1’s payoff will be \(\int_{0}^{h(0, t')} [\omega(0, s_2) - \omega(h(0, t'), 0)] dF(s_2 | \hat{s}_1)\). However, the integrand is strictly negative for all \(s_2 < h(0, t')\), and so bidder 1 can deviate to offering 0 and be better off. Thus, \(\tilde{I}(0) = 0; I'(\cdot)\) satisfies equation (3).

Last, we must show that \(\hat{s}_1 = s_1^*\), where \(s_1^*\) is defined implicitly by \(h(s_1^*, \tilde{I}(s_1^*)) = 1\).
Assume that \( \hat{s}_1 < s_1^* \). Then, \( h(\hat{s}_1, \hat{I}(\hat{s}_1)) < 1 \). Consider a type \( \hat{s}_1 - \epsilon \) for some small \( \epsilon \). If type \( \hat{s}_1 - \epsilon \) deviates to offering \( \hat{I}(\hat{s}_1) \), his change in utility is

\[
\int_{\tilde{s}_2(\hat{I}(\hat{s}_1))}^{\tilde{s}_2(\hat{I}(\hat{s}_1) - \epsilon)} [\omega(\hat{s}_1 - \epsilon, s_2) - \hat{I}(\hat{s}_1)]d F(s_2|\hat{s}_1 - \epsilon) - \int_{0}^{\tilde{s}_2(\hat{I}(\hat{s}_1) - \epsilon)} [\hat{I}(\hat{s}_1) - \hat{I}(\hat{s}_1 - \epsilon)]d F(s_2|\hat{s}_1 - \epsilon)
\]

Note that \( \lim_{\epsilon \to 0} \hat{I}(\hat{s}_1 - \epsilon) = \hat{I}(\hat{s}_1) \), but \( \lim_{\epsilon \to 0} \tilde{s}_2(\hat{I}(\hat{s}_1) - \epsilon) = h(\hat{s}_1, \hat{I}(\hat{s}_1)) < \tilde{s}_2(\hat{I}(\hat{s}_1)) < \hat{s}_1 \).

Thus, as \( \epsilon \) approaches 0, the second integral vanishes, while the first converges to a strictly positive number, and so this deviation is profitable for all \( \epsilon \) small enough.

If, on the other hand, \( \hat{s}_1 > s_1^* \), then some type \( s_1^* + \epsilon \) can deviate to offering \( t(s_1^*) \).

Since \( \tilde{s}_2(\hat{I}(s_1^* + \epsilon)) = \tilde{s}_2(\hat{I}(s_1^*)) = 1 \), but \( t(s_1^*) < t(s_1^* + \epsilon) \), this is a profitable deviation. Fix an equilibrium \( (\hat{I}, \hat{s}_2) \) and an out of equilibrium transfer \( \hat{t} > \hat{I}(s_1) \). Define \( \pi_1(s_1) = U_1(s_1, \hat{s}_2(\hat{I}(s_1)), \hat{I}(s_1)) \) to be bidder 1’s equilibrium payoff. Assume that there exists a type \( s_1 \) such that \( \pi_1(s_1) < U_1(s_1, 1, \hat{t}) \). Define \( \tilde{\sigma}(s_1) \) to be the solution to

\[
U_1(s_1, \tilde{\sigma}(s_1), \hat{t}) = \pi_1(s_1) \tag{6}
\]

whenever such a solution exists. Then, for all \( s_1 \) such that \( \tilde{\sigma}(s_1) \) exists and \( s_1 < \tilde{\sigma}(s_1) \), we have \( \tilde{\sigma}'(s_1) \leq 0 \).

**Proof.** That such a \( \tilde{\sigma}(s_1) \) exists and is continuous is trivial. To shorten equations, define the following functions: \( \chi(s_1, s_2) = \frac{\partial \omega(s_1, s_2)}{\partial s_1} f(s_2|s_1) + \omega(s_1, s_2) \frac{\partial f(s_2|s_1)}{\partial s_1} \) and \( \tilde{\omega}(s_1, s_2) = \omega(s_1, s_2) - \omega(s_2, s_2) \). Note that \( \omega(\cdot, \cdot) \) positive and increasing and \( \frac{\partial f(s_2|s_1)}{\partial s_1} = \frac{\partial^2 F(s_2|s_1)}{\partial s_1 \partial s_2} \geq 0 \) (by increasing differences) implies that \( \chi(s_1, s_2) \geq 0 \). The envelope theorem gives

\[
\pi'_1(s_1) = \int_{0}^{\tilde{s}_2(\hat{I}(s_1))} \left[ \chi(s_1, s_2) - \hat{I}(s_1) \frac{\partial f(s_2|s_1)}{\partial s_1} \right] ds_2 + \\
\int_{\max\{s_1, \tilde{s}_2(\hat{I}(s_1))\}}^{\tilde{s}_2(\hat{I}(s_1))} \left[ \omega_1(s_1, s_2) f(s_2|s_1) + \tilde{\omega}(s_1, s_2) \frac{\partial f(s_2|s_1)}{\partial s_1} \right] ds_2
\]

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If $\hat{t} > \bar{t}(s_1)$, then it must be that $\hat{\sigma}(s_1) > \tilde{s}_2(\bar{t}(s_1))$ (see footnote 15). Then, using the implicit function theorem on (6), we find that

$$\tilde{\sigma}'(s_1) = \begin{cases} 
-\int_{\tilde{s}_2(\bar{t}(s_1))}^{\hat{\sigma}(s_1)} \chi(s_1, s_2) ds_2 + \tau(s_1), & s_1 \leq \tilde{s}_2(\bar{t}(s_1)) \\
\frac{\omega(s_1, \hat{\sigma}(s_1)) - f(\hat{\sigma}(s_1))}{\partial s_1} \omega(s_2, s_2) \frac{\partial F(s_2)}{\partial s_1}, & s_1 > \hat{\sigma}(s_1)
\end{cases},$$

where $\tau(s_1) = \hat{t} \frac{\partial F(\hat{\sigma}(s_1))}{\partial s_1} - \bar{t}(s_1) \frac{\partial F(\tilde{s}_2(\bar{t}(s_1)))}{\partial s_1}$. Next, we want to show that this function is everywhere negative. The fact that $\hat{t} > \bar{t}(s_1)$ and $F(s_2|s_1)$ having increasing differences (so that $\partial F(s_2|s_1)/\partial s_1$ is increasing in $s_2$) implies that $\tau(s_1)$ is negative.

Lemma A shows that all of the denominators are positive. As for the numerators, recall that $\chi(s_1, s_2) \geq 0$. For the other terms, integration by parts gives

$$\int_{\tilde{s}_2(\bar{t}(s_1))}^{\hat{\sigma}(s_1)} \omega(s_2, s_2) g(s_2) ds_2 = \omega(\tilde{\sigma}(s_1), \hat{\sigma}(s_1)) \frac{\partial F(\tilde{\sigma}(s_1)|s_1)}{\partial s_1}$$

$$- \omega(\tilde{s}_2(\bar{t}(s_1)), \tilde{s}_2(\bar{t}(s_1))) \frac{\partial F(\tilde{s}_2(\bar{t}(s_1)))}{\partial s_1} - \int \frac{\partial F(s_2)}{\partial s_1} d\omega(s_2, s_2).$$

Note that the last integral is negative, and so we can say

$$- \int_{\tilde{s}_2(\bar{t}(s_1))}^{\hat{\sigma}(s_1)} \omega(s_2, s_2) g(s_2) ds_2 + \omega(\tilde{\sigma}(s_1), \hat{\sigma}(s_1)) \frac{\partial F(\tilde{\sigma}(s_1)|s_1)}{\partial s_1}$$

$$- \omega(\tilde{s}_2(\bar{t}(s_1)), \tilde{s}_2(\bar{t}(s_1))) \frac{\partial F(\tilde{s}_2(\bar{t}(s_1)))}{\partial s_1} \leq 0.$$
Consider an equilibrium \((\bar{t}, \bar{s}_2)\) and an out of equilibrium transfer \(\hat{t}\). Then, for all \(s_1\):

(i) \(\omega(\bar{s}_2(\bar{t}(s_1)), \bar{s}_2(\bar{t}(s_1))) > \bar{t}(s_1)\)

(ii) \(\omega(s_1, \bar{o}(s_1)) \geq \hat{t}\), if \(\bar{o}(s_1)\) exists.

Proof. Assume the opposite inequality holds. Note that bidder 1 can always bid 0 and guarantee himself a payoff of \(U_1(s_1, 0, 0) = \int_0^{s_1} [\omega(s_1, s_2) - \omega(s_1, s_1)]dF(s_2|s_1)\). So, individual rationality requires that \(U_1(s_1, \bar{s}_2(\bar{t}(s_1)), \bar{t}(s_1)) \geq U_1(s_1, 0, 0)\), or

\[
\int_0^{\bar{s}_2(\bar{t}(s_1))}[\omega(s_1, s_2) - \bar{t}(s_1)]dF(s_2|s_1) + I(s_1 > \bar{s}_2(\bar{t}(s_1))) \int_{\bar{s}_2(\bar{t}(s_1))}^{s_1}[\omega(s_1, s_2) - \omega(s_1, s_2)]dF(s_2|s_1) \geq \int_0^{s_1}[\omega(s_1, s_2) - \omega(s_1, s_2)]dF(s_2|s_1)
\]

First, consider the case of \(s_1 > \bar{s}_2(\bar{t}(s_1))\). The above equation can be rewritten as

\[
\int_0^{\bar{s}_2(\bar{t}(s_1))} \bar{t}(s_1)dF(s_2|s_1) \leq \int_0^{\bar{s}_2(\bar{t}(s_1))} \omega(s_2, s_2)dF(s_2|s_1)
\]

If \(\omega(\bar{s}_2(\bar{t}(s_1)), \bar{s}_2(\bar{t}(s_1))) \leq \bar{t}(s_1)\), then, since \(\omega\) is an increasing function, \(\omega(s_2, s_2) < \bar{t}(s_1)\) for all \(s_2 \in [0, \bar{s}_2(\bar{t}(s_1))]\), and hence \(\int_0^{\bar{s}_2(\bar{t}(s_1))} \omega(s_2, s_2)dF(s_2|s_1) < \int_0^{\bar{s}_2(\bar{t}(s_1))} \bar{t}(s_1)dF(s_2|s_1)\), which contradicts the equation above.

When \(s_1 < \bar{s}_2(\bar{t}(s_1))\), bidder 1 only wins the object when the transfer is accepted. The highest possible value for the object when he wins is:

\[
\omega(s_1, \bar{s}_2(\bar{t}(s_1))) < \omega(\bar{s}_2(\bar{t}(s_1)), \bar{s}_2(\bar{t}(s_1))) < \bar{t}(s_1)
\]
or \(\bar{t}(s_1) > \omega(s_1, \bar{s}_2(\bar{t}(s_1)))\), which means that bidder 1 is paying more than the object is worth and is hence guaranteeing himself a negative payoff, which is clearly not individually rational.

For the second statement, consider some out-of-equilibrium \(\hat{t}\) and corresponding
\(\tilde{\sigma}(s_1)\), and assume that \(\omega(s_1, \tilde{\sigma}(s_1)) < \hat{t}\) for some \(s_1\). If \(s_1 \leq \tilde{\sigma}(s_1)\), then we have \(U_1(s_1, \tilde{\sigma}(s_1), \hat{t}) < 0\), while \(\pi_1(s_1) \geq 0\), which is a contradiction to the definition of \(\tilde{\sigma}(s_1)\). Next, assume that \(s_1 > \tilde{\sigma}(s_1)\). There are two subcases. First, if \(\hat{t} > \bar{t}(s_1)\), then \(\tilde{\sigma}(s_1) > \tilde{s}_2(\bar{t}(s_1))\). Then, we can simplify equation (6) to:

\[
-\int_{\bar{s}_2(\bar{t}(s_1))}^{\tilde{\sigma}(s_1)} \omega(s_2, s_1) dF(s_2|s_1) = \hat{t}F(\tilde{\sigma}(s_1)|s_1) - \bar{t}(s_1)F(\tilde{s}_2(\bar{t}(s_1))|s_1)
\]

(7)

notice that the LHS is (strictly) negative, while the RHS is positive, which is a contradiction. When \(\hat{t} < \bar{t}(s_1)\), a similar argument to footnote 15 shows that \(\tilde{s}_2(\bar{t}(s_1)) > \tilde{\sigma}(s_1)\), and then we can derive an equation analogous to equation (7) to reach the desired contradiction.

\[\Box\]

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15To see this, note that \(U_1\) is (strictly) increasing in its second argument at \(\tilde{s}_2(\bar{t}(s_1))\) (otherwise, since \(U_1\) is quasiconvex in its second argument for any fixed \(s_1, t\), deviating to a transfer of 0 and inducing a cutoff of \(\tilde{s}_2 = 0\) would be a profitable deviation for bidder 1, contradicting the fact that \((\bar{t}, \tilde{s}_2)\) was an equilibrium). This implies that \(U_1(s_1, \tilde{\sigma}, \hat{t}) < U_1(s_1, \tilde{\sigma}, \bar{t}(s_1)) \leq U_1(s_1, \tilde{s}_2(\bar{t}(s_1)), \bar{t}(s_1))\) for all \(0 \leq \tilde{\sigma} \leq \tilde{s}_2(\bar{t}(s_1))\), where the first inequality follows from \(\hat{t} > \bar{t}(s_1)\) and the second from the quasiconvexity of \(U_1\) and the fact that \(U_1(s_1, \tilde{s}_2(\bar{t}(s_1)), \bar{t}(s_1)) \geq U_1(s_1, 0, 0) \geq U_1(s_1, 0, \bar{t}(s_1))\) by individual rationality. Thus, \(U_1(s_1, \tilde{\sigma}, \hat{t}) < \pi_1(s_1)\) for all \(\tilde{\sigma} \leq \tilde{s}_2(\bar{t}(s_1))\), and so no such \(\tilde{\sigma}\)'s can satisfy equation (6).