Life Cycles and Mortality*

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May 2, 2012

Abstract

In this paper we build up on the classical life cycle models presented by Merton (1969) and (1971) as well as Bodie, Merton and Samuelson (1992) and include mortality risk into our analysis. As such we study the consumption, labor supply, and portfolio decisions of a representative agent facing age-dependent mortality risk, as presented in UK actuarial life tables. While working, the representative agent receives wage income as well as income from investment into one risky and one risk-free asset, depending on the current wage rate, the chosen labour supply and the chosen investment strategy. At any time prior to death, the agent can spend his wealth on consumption or further investment and is trying to maximize life time utility from consumption and leisure. Using Martingale techniques instead of the Hamilton-Jacobi-Bellman approach allows us to consider general time varying mortality risk. As in Yaari (1965) and Blanchard (1985) we assume the existence of life insurance markets. We derive closed-form solutions for optimal consumption, labor supply and investment strategy and show that the inclusion of mortality risk, and in fact the shape of the mortality risk curve, significantly affects the level of consumption as well as the decomposition of the investment portfolio.

*The authors acknowledge funding from the National Natural Science Foundation of China (Grant No. 11171372) and funding from the Zhejiang Natural Science Foundation in China (Grant No. Y6110721). Christian-Oliver Ewald also acknowledges funding from the Australian Research Council (Grant No. DP1095969) and the Fields Institute. This paper was presented at the Royal Economic Society Conference 2012 in Cambridge. We would like to acknowledge helpful comments from Roger Farmer.

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Keywords: Lifetime consumption and investment, mortality risk, Martingale method

JEL Subject Classification: C61; C73; G1; J22
1 Introduction

Lifetime consumption and investment models have been considered by various authors, including Merton (1969) and (1971), Bodie, Merton and Samuelson (1992) as well as Bodie (2004). The setup in all of these contributions is very similar, they all study the problem of maximizing expected discounted utility under consideration of a utility function which includes consumption and in some cases leisure, over the lifetime of a representative agent. Bodie, Merton and Samuelson (1992) consider an exogenously given retirement age and leave it as an open question, to determine the optimal retirement age within an optimal stopping context. This problem has now been considered by Dybvig and Liu (2010). Zhang (2010) considers retirement age as exogenously given, but allows for fully flexible labour supply, which in essence includes retirement as an option for the agent.

All of the above have in common that they include a stochastic investment asset and possibly stochastic wage income, but do not account for the affects of mortality risk.\(^1\) The contribution of this article is the inclusion of time varying, general mortality risk into a continuous time stochastic lifetime consumption model, where a representative agent chooses consumption, labor supply and portfolio investment into riskless and risky assets, assuming a CRRA type of utility function measuring utility from consumption against disutility from labor. Mortality risk has been considered by Yaari (1965) and in continuous time overlapping generations models, such as Blanchard (1985), but in both cases no risky investment assets had been included\(^2\). Furthermore, in Blanchard (1985) the mortality rate has been chosen as constant in time, which is a rather unrealistic assumption, if actual statistical life tables are taken into account. Our paper, as well as the related paper by Ewald, Zhang and Nolan (2012), bridges the gap between Blanchard and Yaari (1965) on one hand, and the literature following along the lines of Merton and Bodie on the other hand.

Admittingly, by using classical techniques such as the Hamilton-Jacobi-Bellman

\(^1\)Merton (1971) briefly discusses the case of constant mortality risk, pointing out that an "individual who faces an exponentially-distributed uncertain age of death acts as if he will live forever, but with a subjective rate of time preference equal to his "force of mortality", i.e., to the reciprocal of his life expectancy". Merton does not consider flexible labour.

\(^2\)In fact, once Blanchard (1985) has averaged over the mortality risk, his model is completely deterministic.
framework or the Pontryagin maximum principle, it is very difficult, if not impossible to allow for time varying mortality rates, in particular when real mortality curves as obtained from statistical life tables are supposed to be used. In this article we use a combination of Martingale techniques to circumvent these problems. In particular, we will avoid any form of partial or ordinary differential equations, and will in effect be able to deal with completely arbitrary mortality curves. Nevertheless, even in this far more complex setup, we are able to derive analytic forms for the optimal consumption, labour supply and portfolio investment process in the presence of mortality risk. We are further able to derive a compact form for the Euler equation of consumption growth. We find that the effect of mortality risk on consumption and labour supply is through the Lagrange multiplier of the associated constrained optimization problem only, and as such it shifts consumption and labor supply, but has no effect on the Euler equation. This is essentially a consequence of the availability of full life insurance, as proposed by Yaari (1965) and also used by Blanchard (1985). Mortality risk however affects optimal portfolio investment in a more subtle way.

Under the assumption of constant mortality rate, we are able to derive a closed form expression for the elasticity of consumption with respect to the mortality rate. Using realistic parameters we find that this elasticity is negative, within the range of 0 (at zero mortality rate) to $-0.53$, at mortality rate 0.002 which corresponds roughly to a 39 year old UK male. In the empirical part of the paper we use actual mortality curves as obtained from statistical life tables supplied by the UK’s Government Actuary’s Department covering the years from 1982 until 2006. Substituting these curves into our model we obtain that keeping all other parameters constant, a change in the mortality curves from 1982 to 2006 leads to a shift in consumption upwards of roughly 5%, contributing to a total of approximately 100% in real GDP growth in the UK over the same time period.\footnote{Historical data for real GDP have been obtained via \url{https://docs.google.com/spreadsheet/ccc?key=0AonYZs4MzlZbcGhOdG0zTG1EWkVPX1k1VVR6LTdU3c#gid=1}.}

We also observe from our model that optimal labour supply in effect of the same change of the mortality curve is reduced by 6%, from about 40.5 hours to 38 hours per week. Finally, portfolio investment into the risky asset increases by a factor of roughly 6%, mainly financing the reduction in labour and increase in consumption.
As such the message of this article is that historical changes in mortality risk do have a significant impact on consumption spending, labour supply and portfolio investment.

The remainder of the paper is organized as follows. In section 2 we set up our model and derive some basic equations, while in section 3 we consequently proceed by using martingale methods in order to transform the dynamic problem into a static problem, which we will solve. Section 4 contains both theoretical and empirically founded examples, while the main conclusions are summarized in section 5.

2 The Model

Let us consider a representative agent trying to maximize the following functional:

$$\max_{\pi_t, C_t, L_t} \mathbb{E} \left( \int_0^\tau e^{-\int_0^s \rho_s \, ds} u(C_t, L_t) \, dt \right).$$

(1)

Here \( \tau \) denotes the time of death, \( C_t \) denotes instantaneous consumption, \( L_t \) denotes instantaneous labour supply and \( \pi_t \) denotes the investment choice.\(^4\) We assume that \( C_t \geq 0, L_t \geq 0 \) and \( \pi_t \) are chosen by the agent depending on information contained in the sigma algebra \( \mathcal{F}_t \) which will be introduced below. The investment assets available to the agent will also be introduced below. The time preference rate \( \rho_s \) of the agent is assumed to be a deterministic and positive function, while the time of death will be considered as random, with

$$\mathbb{P} (\tau \in [t, t+dt] | \tau \geq t) = \nu_t dt,$$

(2)

where \( \nu_t \) is the time dependent instantaneous mortality rate. Intuitively, the mortality rate \( \nu_t \) describes the likelihood of the agent aged \( t \) dying in the interval \([t, t+dt)\). This rate can be easily obtained from actuarial life tables and in general differs regionally and historically. We assume that \( \nu_t \) is a deterministic function, but note that most of the following analysis could be carried out, if \( \nu_t \) were a doubly stochastic process, compare Duffie (2011) page 11. Under this assumption,

\(^4\)The subscript \( t \) denotes ‘at time \( t \)’ throughout, unless otherwise stated.
the agents likelihood of surviving until age $t$ is given by

$$P(\tau > t) = e^{-\int_0^t \nu_s ds}. \quad (3)$$

We assume that the random time $\tau$ is independent of any of the economic state variables, and hence using that $E(1_{\{t<\tau\}}) = P(\tau > t)$ we conclude that

$$E\left(\int_0^\tau e^{-\int_0^\cdot \rho_s ds} u(C_t, L_t) dt\right) = E\left(\int_0^\cdot e^{-\int_0^\cdot \rho_s ds} u(C_t, L_t) \cdot 1_{\{t<\tau\}} dt\right) \quad (4)$$

Defining the mortality adjusted discount rate

$$\hat{\rho}_t = \rho_t + \nu_t$$

we may write (1) as

$$\max_{\pi, C, L} E\left(\int_0^\infty e^{-\int_0^\cdot \hat{\rho}_s ds} u(C_t, L_t) dt\right) \quad (5)$$

Let us now introduce the investment assets in our model. We assume that the economy features one risk-less asset modeled as

$$dB_t = B_t r_t dt \quad (6)$$

and one risky asset

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t) \quad (7)$$

Here $W_t$ denotes a standard Brownian motion and we denote with $\mathcal{F}_t$ the filtration it generates. The parameters $r_t$, $\mu_t$ and $\sigma_t$ are allowed to vary deterministically in time. As in Blanchard (1985) we assume the existence of fairly priced life insurance, which replaces a bequest motive: "In the absence of a bequest motive, and if negative bequests are prohibited, agents will contract to have their wealth (positive or negative) returned to the life insurance company contingent on their death." The modeling framework in this article assumes that the representative agent represents "a large number of identical agents", and as such life insurance
contracts can be offered risk-less by life insurance companies. We assume that the market for life insurance contracts is competitive, and hence free entry and exit will result in a zero profit condition, which in turn implies that the fair pricing of the insurance contract obliges/entitles the holder to payments

\[ X_t \nu_t dt \]  

per infinitesimal time interval \( dt \), where \( X_t \) denotes the current wealth of the agent. Note that (8) represents a payment to be made by the agent to the insurance company, in case the agent has debt, i.e. \( X_t < 0 \) and otherwise presents an income, i.e. payment from the insurance company to the agent, in exchange for the agent giving up his wealth to the insurance company at the time of his death.

Denoting with \( \pi_t \) the fraction of wealth invested into the risky asset and with \( w_t \) the wage rate, the dynamics of the wealth process is described by

\[
dX_t = X_t \left\{ (r_t + \nu_t) dt + \pi_t \left[ (\mu_t - r_t) dt + \sigma_t dW_t \right] \right\} - C_t dt + w_t L_t dt, \tag{9}\]

with \( X_0 = x \geq 0 \).

As the analysis above has shown, the problem of the finitely lived agent, problem (1) subject to constraint (9), is equivalent to the problem of the infinitely lived agent, problem (5) subject to constraint (9), where the discount rate as well as the drift of the wealth process have been adjusted to accommodate the mortality risk. We further define

\[
\hat{r}_t = r_t + \nu_t \tag{10} \\
\hat{\mu}_t = \mu_t + \nu_t \tag{11}
\]

and note that the market price of financial risk

\[
\theta_t = \frac{\mu_t - r_t}{\sigma_t} = \frac{\hat{\mu}_t - \hat{r}_t}{\sigma_t} \tag{12}
\]

is unaffected by mortality risk.
3 Martingale Approach

In order to apply martingale methods to solve the problem discussed in the previous section, we define the stochastic discount factor $\hat{H}_t$ via

$$d\hat{H}_t = -\hat{H}_t (\hat{r}_t dt + \theta_t dW),$$
$$\hat{H}_0 = 1.$$  \hfill (13)

Note that the stochastic discount factor features the mortality adjusted rate $\hat{r}_t$ and the classical market price of risk $\theta_t$ in it. We can write $\hat{H}_t$ as

$$\hat{H}_t = e^{-\int_0^t \nu_s ds} H_t,$$ \hfill (14)

where $H_t$ is the classical stochastic discount factor and is defined by \(^5\)

$$H_t = e^{-\int_0^t (r_s + \frac{1}{2} \theta_s^2) ds - \int_0^t \theta_s dW_s}.$$ \hfill (15)

Hence the stochastic discount factor $\hat{H}_t$ splits up into two components, $e^{-\int_0^t \nu_s ds}$ is adjusting for mortality risk and $H_t$ is adjusting for financial risk.

Applying the Itô product rule, it is easy to verify that

$$d(\hat{H}_t X_t) = \hat{H}_t X_t (\pi_t \sigma_t - \theta_t) dW_t - \hat{H}_t C_t dt + \hat{H}_t w_t L_t dt.$$ \hfill (16)

Integrating (16) from $t$ to $\infty$ and imposing the following transversality condition\(^6\)

$$\lim_{u \to \infty} \mathbb{E}(\hat{H}_u X_u) = 0$$ \hfill (17)

we obtain

$$-\hat{H}_t X_t = \int_t^\infty \hat{H}_s X_s (\pi_s \sigma_s - \theta_s) dW_s - \int_t^\infty \hat{H}_s C_s ds + \int_t^\infty \hat{H}_s w_s L_s ds.$$ \hfill (18)

\(^5\)See for example Korn (2000).

\(^6\)The corresponding deterministic version of the this transversality condition appears in Blanchard (1985) on page 227, and prevents the case where an agent takes up more and more debt, while being covered by life insurance.
Denoting the conditional expectation with respect to $\mathcal{F}_t$ as $E_t$ we obtain

$$X_t = E_t \left[ \int_t^\infty \frac{\hat{H}_s}{H_t} C_s ds \right] - E_t \left[ \int_t^\infty \frac{\hat{H}_s}{H_t} w_s L_s ds \right]. \tag{19}$$

At time $t = 0$ we obtain the static budget constraint

$$E \left( \int_0^\infty \hat{H}_s C_s ds \right) = x + E \left( \int_0^\infty \hat{H}_s w_s L_s ds \right). \tag{20}$$

The intuition behind equation (20) is that expected stochastically discounted consumption need to be equal to initial wealth plus expected stochastically discounted wage income, where the discount factor takes both market risk and mortality risk into account.

We now obtain that problem (5) subject to the dynamic constraint (9) and transversality condition (17) is equivalent to problem (5) with the static budget constraint (20). In order to solve the latter problem we introduce the Lagrange function

$$L(\lambda, C_t, L_t) = E \left( \int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t) dt \right)$$

$$+ \lambda \left\{ x + E \left( \int_0^\infty \hat{H}_s w_s L_s ds \right) - E \left( \int_0^\infty \hat{H}_s C_s ds \right) \right\}. \tag{21}$$

In order to proceed to a closed form solution, we need to specify the utility function $u(C_t, L_t)$ at this point. We define

$$u(C_t, L_t) := \frac{C_t^{1-\gamma}}{1-\gamma} - b_t \frac{L_t^{1+\eta}}{1+\eta}. \tag{22}$$

The intuition behind (22) is to weigh up benefits from consumptions against costs from labour in constant relative risk aversion manner. The deterministic function $b_t > 0$ measures the relative cost of labour, which may vary between age classes. We assume $\gamma > 0$, $\gamma \neq 1$ and $\eta > 0$, consistent with decreasing marginal benefits from consumption and increasing marginal costs of labour.$^7$

$^7$The case $\gamma = 1$ corresponds to the case where utility from consumption is logarithmic and can in fact be obtained from the following results by considering the limit for $\gamma \to 1$. The
Differentiating the Lagrange function (21), we obtain the following first order conditions

\[ C_t^{-\gamma} = \frac{\partial u}{\partial C_t} = \lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t \]
\[ -b_t L_t^{\eta} = \frac{\partial u}{\partial L_t} = -\lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t \hat{w}_t. \]

(23)

Using (4) and (14), we obtain from (23) and (24) that

\[ C_t^{-\gamma} = \lambda e^{\int_0^t \hat{\rho}_s ds} H_t \]
\[ -b_t L_t^{\eta} = -\lambda e^{\int_0^t \hat{\rho}_s ds} H_t \hat{w}_t. \]

(24)

(25)

(26)

The mortality component hence cancels out of the time dependent component of consumption and labour supply represented by \( H_t \) above. It can therefore be concluded that mortality risk as such will have no effect on the growth rate of consumption \( \frac{d}{dt} E_t \left( \frac{dC_t}{C_t} \right) \). However, as we will see below, it will affect the value of the Lagrange multiplier \( \lambda \) and hence shift consumption to a different level. These results are in line with the results in Yaari (1965) and Blanchard (1985) and essentially a consequence of the availability of full life insurance. The optimal consumption and labour supply can be easily derived from (25) and (26) as

\[ C_t^* = \lambda^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma} \int_0^t \hat{\rho}_s ds} H_t^{-\frac{1}{\gamma}} \]
\[ L_t^* = \lambda^{\frac{1}{\eta}} e^{\frac{1}{\eta} \int_0^t \hat{\rho}_s ds} (H_t \hat{w}_t)^{\frac{1}{\eta}} \hat{b}_t^{-\frac{1}{\eta}}. \]

(27)

(28)

We will now derive an analytic expression for the Lagrange multiplier \( \lambda \) and by doing this identify the mortality dependence in (27) and (28). Substitution into (20) and using (14) once more we obtain

\[ \lambda^{-\frac{1}{\gamma}} \mathbb{E} \left( \int_0^\infty \left( e^{-\int_0^t \nu_s ds + \frac{1}{\eta} \rho_s} \right) H_t^{\frac{\gamma-1}{\gamma}} dt \right) \]
\[ = x + \lambda^{\frac{1}{\gamma}} \mathbb{E} \left( \int_0^\infty \left( e^{-\int_0^t \nu_s ds - \frac{1}{\eta} \rho_s} \right) \hat{b}_t^{-\frac{1}{\eta}} (H_t \hat{w}_t)^{\frac{\eta+1}{\eta}} dt \right). \]

(29)

same holds for the case \( \eta = 0 \), which corresponds to linear costs of labour, which has interesting implications on the elasticity of consumption, as discussed in section 4.
Using that everything, except \(H_t\) and and \(w_t\), is deterministic, we obtain
\[
\lambda^{-\frac{1}{\gamma}} \left( \int_0^\infty e^{-\int_0^t (\nu_s + \frac{1}{\gamma} \rho_s) ds} \mathbb{E} \left( \frac{H_t}{\gamma} \right)^{\gamma + 1} dt \right) = x + \lambda^\frac{1}{\gamma} \left( \int_0^\infty e^{-\int_0^t (\nu_s - \frac{1}{\gamma} \rho_s) ds} b_t^{-\frac{1}{\gamma}} \mathbb{E} \left( \left( H_t w_t \right)^{\eta + \frac{1}{\eta}} \right) dt \right).
\]

(30)

In order to proceed, we need to make assumptions about the dynamics of the wage rate \(w_t\). We assume that
\[
dw_t = w_t a_t dt,
\]
with \(a_t \geq 0\) a deterministic function. Using (15) and (31) we can compute
\[
\mathbb{E} \left( \frac{H_t^{\gamma + 1}}{\gamma} \right) = e^{-\int_0^t \frac{1}{\gamma} \left( \rho_s + \frac{r^2}{\gamma} \right) ds},
\]
\[
\mathbb{E} \left( \left( H_t w_t \right)^{\eta + \frac{1}{\eta}} \right) = \frac{w_0^{\eta + 1}}{\eta} e^{-\int_0^t \frac{1}{\eta} \left( r_s - a_s - \frac{r^2}{\eta} \right) ds}
\]
and substitution into (30) gives
\[
\lambda^{-\frac{1}{\gamma}} \int_0^\infty e^{-\frac{1}{\gamma} \int_0^t \left( \rho_s + (\gamma - 1) \left( r_s + \frac{r^2}{\gamma} \right) \right) ds} \cdot e^{-\int_0^t \frac{1}{\gamma} \nu_s ds} dt
\]
\[
= x + \lambda^\frac{1}{\gamma} \frac{w_0^{\eta + 1}}{\eta} \int_0^\infty e^{-\frac{1}{\gamma} \int_0^t \left( \rho_s - (\eta + 1) \left( r_s - a_s - \frac{r^2}{\eta} \right) \right) ds} \cdot e^{-\int_0^t \frac{1}{\gamma} \nu_s ds} \cdot b_t^{-\frac{1}{\gamma}} dt.
\]

(34)

Note that the computability of the integrals above, depends on the deterministic functions \(\rho_s, r_s, \theta_s, a_s, b_s\) and \(\nu_s\). If for example these are all constant, then it is straightforward to compute all the integrals in (34) explicitly. However, it will still not be possible to solve analytically (34) for \(\lambda\), as the equation \(\lambda^a = x + \lambda^b\) can not be solved explicitly for \(\lambda\) in the generic case. On the other side, in the most general case, it is a simple exercise to compute the integrals and \(\lambda\) from (34) numerically.

The non-existence of a bequest motive on the other hand can also be interpreted as that the representative agent is born at time \(t = 0\) without any means, i.e. \(x = 0\).
In this case we obtain an explicit solution for the Lagrange multiplier

\[
\lambda = \left( \begin{array}{c}
\frac{\eta+1}{\eta} \\
\int_0^\infty e^{\frac{1}{\eta} \int_0^u \left( \rho_s - (\eta+1) \left( r_s - a_s - \frac{\theta^2}{2} \right) \right) ds} \cdot e^{-\int_0^u \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} du \cdot \frac{u}{\eta+1}
\end{array} \right)
\]

(35)

We can see clearly in (35) the mortality dependence of the Lagrange multiplier. The classical model without mortality is obtained by setting \( \nu_s \equiv 0 \). Noticing once more that \( \mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds} \) we can see that compared to the classical case without mortality the integrands in the nominator and denominator in (34) are weighted by the probability of survival. We will later show that under realistic choices of parameters, the inclusion of mortality risk will have significant quantitative effects.

Let us summarize the results so far in the following theorem:

**Theorem 3.1.** The optimal consumption and labour supply strategy of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (17) are given by

\[
C_t^* = e^{-\frac{1}{\eta} \int_0^t \rho_s ds} H_t \left( \left( \frac{\eta+1}{\eta} \right) \int_0^\infty e^{\frac{1}{\eta} \int_0^u \left( \rho_s - (\eta+1) \left( r_s - a_s - \frac{\theta^2}{2} \right) \right) ds} \cdot e^{-\int_0^u \nu_s ds} \cdot b_u^{-\frac{1}{\eta}} du \cdot \frac{u}{\eta+1} \right)
\]

(36)

\[
L_t^* = e^{\frac{1}{\eta} \int_0^t \rho_s ds} (H_t w_t)^{-\frac{1}{\eta}} b_t^{-\frac{1}{\eta}} \left( \left( \frac{\eta+1}{\eta} \right) \int_0^\infty e^{\frac{1}{\eta} \int_0^u \left( \rho_s - (\eta+1) \left( r_s - a_s - \frac{\theta^2}{2} \right) \right) ds} \cdot e^{-\int_0^u \nu_s ds} \cdot b_u^{-\frac{1}{\eta}} du \cdot \frac{u}{\eta+1} \right)
\]

We will now turn our attention to the optimal investment strategy \( \pi_t^* \) which is part of the solution. From (19) we obtain that the wealth process \( X_t^* \) under the optimal plan \((\pi_t^*, C_t^*, L_t^*)\) is given by

\[
X_t^* = A_t - B_t,
\]

(36)
with

$$A_t = C_t^* \mathbb{E}_t \left( \int_t^\infty \frac{\hat{H}_t C^*_s}{H_t C_t^*} ds \right) \quad (37)$$

$$B_t = (w_t L^*_t) \mathbb{E}_t \left( \int_t^\infty \frac{\hat{H}_s (w_s L^*_s)}{H_t (w_t L^*_t)} ds \right). \quad (38)$$

In the following we will compute $A_t$ and $B_t$. Substituting the expressions for $C_t^*$ and $L_t^*$ from Theorem 3.1 gives

$$A_t = C_t^* \mathbb{E}_t \left( \int_t^\infty \frac{H_s C^*_s}{H_t C_t^*} \cdot e^{-\int_s^t \nu_u du} ds \right)$$

$$= C_t^* \mathbb{E}_t \left( \int_t^\infty \left( \frac{H_s}{H_t} \right)^{\frac{\gamma - 1}{\gamma}} \cdot e^{-f_t^* (\nu_u + \frac{1}{\gamma} \rho_u) du} ds \right)$$

$$= C_t^* \int_t^\infty \mathbb{E}_t \left( \left( \frac{H_s}{H_t} \right)^{\frac{\gamma - 1}{\gamma}} \cdot e^{-f_t^* (\nu_u + \frac{1}{\gamma} \rho_u) du} ds \right).$$

Now, using that $\frac{H_s}{H_t}$ is independent of $\mathcal{F}_t$ and distributed like a geometric Brownian motion with time varying drift term, we obtain

$$A_t = C_t^* f_t,$$

with

$$f_t = \int_t^\infty e^{-f_t^* \left( \left( \frac{\nu_u + \frac{1}{\gamma} \rho_u}{b_s/b_t} \right)^{\frac{\gamma - 1}{\gamma}} + \frac{1}{\gamma} \rho_u \right) du} \cdot e^{-f_t^* \nu_u du} ds. \quad (39)$$

Similarly we can compute

$$B_t = w_t L^*_t g_t,$$

with

$$g_t = \int_t^\infty e^{-f_t^* \left( \left( \frac{\nu_u - \frac{1}{\gamma} \rho_u}{b_s/b_t} \right)^{\frac{\gamma - 1}{\gamma}} - \frac{1}{\gamma} \rho_u \right) du} \cdot e^{-f_t^* \nu_u du} ds. \quad (40)$$

Using (39) and (40) we can write

$$X^*_t = f_t C_t^* - g_t w_t L^*_t,$$

with $f_t$ and $g_t$ are deterministic functions. Let us also note at this point that the
expression $e^{-\int_s^\tau \nu_u du} ds$ is equal to $\mathbb{P}(\tau > s|\tau > t)$, the probability of survival until $s$ given that the agent is still alive at time $t < s$.

Using the representation (41) together with (13) we compute

$$d \left( \hat{H}_t X_t^\pi \right) = \hat{H}_t f_t dC_t^\pi - \hat{H}_t g_t w_t dL_t^\pi + X_t^\pi d\hat{H}_t + \ldots dt.$$  \hspace{1cm} (42)

The terms indicated by $(\ldots)$ in front of $dt$ will be irrelevant for the following, which is why we omit them. In fact we will only be interested in the diffusion term, i.e. the expression in front of $dW_t$, within the expression (42). To identify this term, we compute

$$dC_t^\pi = -\frac{1}{\gamma} C_t^\pi H_t^{-1} dH_t + \ldots dt \hspace{1cm} (43)$$

$$dL_t^\pi = \frac{1}{\eta} L_t^\pi H_t^{-1} dH_t + \ldots dt \hspace{1cm} (44)$$

Furthermore, using (14) and

$$dH_t = -H_t (r_t dt + \theta_t dW_t), \hspace{1cm} (45)$$

we eventually obtain

$$d \left( \hat{H}_t X_t^\pi \right) = \frac{1}{\gamma} \hat{H}_t f_t C_t^\pi \theta_t dW_t + \frac{1}{\eta} \hat{H}_t g_t w_t L_t^\pi \theta_t dW_t - \hat{H}_t X_t^\pi \theta_t dW_t + \ldots dt, \hspace{1cm} (46)$$

where for the second equality we used (41). Since the diffusion term in the representation (46) must coincide with the diffusion term in the representation (16) for $X_t = X_t^\pi$, we obtain by noticing (41) once more and solving for $\pi_t$:

**Theorem 3.2.** The optimal investment strategy of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (16) is given by

$$\pi_t^\pi = \frac{1}{\gamma} \mu_t - \frac{r_t}{\sigma_t^2} + g_t \cdot \left( \frac{1}{\gamma} + \frac{1}{\eta} \right) \frac{\mu_t - r_t}{\sigma_t^2} \cdot \frac{w_t L_t^\pi}{X_t^\pi}. \hspace{1cm} (47)$$

Note that the function $g_t$ in (47) depends on mortality risk, and as such the
proportion between riskless and risky investment of the agent does so as well. In
fact expression (47) represents a modification of the classical Merton (1969) rule
\[ \pi_t = \frac{1}{\gamma} \frac{\mu - r_t}{\sigma^2_t}, \]
where the adjustment for flexible labour supply, wages and mortality risk is given by the term \( g_t \cdot \left( \frac{1}{\gamma} + \frac{1}{\eta} \right) \frac{\mu - r_t}{\sigma^2_t} \cdot \frac{w_t L_t}{X_t}. \) We observe from (40) that \( g_t \) is
always positive and hence that under the derived strategy (47) the agent invests a
higher proportion of her/his wealth into the risky asset. Further it can be observed
that when the mortality curve shifts down, i.e. mortality rates decrease uniformly,
the proportion of wealth invested into the risky asset increases. These features can
obviously not be observed in Yaari (1965) and Blanchard (1985), as these authors
only allow for investment in a risk-less asset. An empirical comparative analysis
of this expression will follow in the next section.

We have already indicated above, that consumption growth is not directly af-
affected by the inclusion of mortality risk. Nevertheless we believe that it is interesting
to derive the Euler equation for consumption growth at this point. Computing
the term in front of \( dt \) in (43) explicitly, we obtain
\[ dC^*_t = \frac{1}{\gamma} \theta_t C^*_t dW_t + \frac{1}{\gamma} \left( r_t - \rho_t + \frac{\gamma + 1}{2\gamma} \theta^2_t \right) C^*_t dt. \]
Dividing (48) by \( C^*_t \) and taking expectations, we obtain
\[ \frac{d}{dt} \mathbb{E} \left( \frac{dC^*_t}{C^*_t} \right) = \frac{1}{\gamma} \left( r_t - \rho_t + \frac{\gamma + 1}{2\gamma} \theta^2_t \right). \]

As expected, the consumption Euler equation does not depend on the mor-
tality risk parameter \( \nu_t \). This is because of the full insurance against loss of life.
But the uncertainty attached to the financial market does affect the individual’s
consumption decision (see the third term in the bracket of (49)).

4 Examples and Empirical Analysis

To begin with, we start with a toy example, in which all parameters, including
the mortality rate \( \nu_t \), are assumed to be constant. Furthermore assuming that the

\[ \text{We refer to Zhang (2010) for details on how the individual adjusts consumption according to financial risk. Note that Zhang (2010) did not take account of the mortality risk.} \]
representative agent is born without any initial wealth, i.e. \( x = 0 \), we can compute the Lagrange multiplier \( \lambda(\nu) \) in (35) as a function of the mortality rate \( \nu \) (\( \nu_t \equiv \nu \), for all \( t \)) explicitly:

\[
\lambda(\nu) = w_0^{\left(\frac{a+1}{\eta}\right)} b_{\left(\frac{1}{\gamma+\eta}\right)} \left( \frac{\nu + \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma} \left( r + \frac{\theta^2}{2\gamma} \right)}{\nu - \frac{a}{\eta} + \frac{\nu+1}{\eta} \left( r - a - \frac{\theta^2}{2\gamma} \right)} \right)^{-\frac{\gamma \eta}{\gamma + \eta}}
\]

(50)

In the following we will compute the elasticity of consumption with respect to mortality. This elasticity represents the percentage change in consumption for each percentage change in the mortality rate. It is rather simple to verify by using (27) and (50) that

\[
\frac{dC^*_t(\nu)}{C^*_t(\nu)} = -\frac{1}{\gamma} \frac{d\lambda(\nu)}{\lambda(\nu)}
\]

(51)

i.e. the elasticity of consumption is a constant fraction of the elasticity of the Lagrange multiplier.\(^9\) The constant factors \( w_0 \) and \( b \) do not affect the elasticities with respect to mortality.

Using (51) it is then a tedious but straightforward exercise to verify that

\[
\frac{dC^*_t(\nu)}{C^*_t(\nu)} = \frac{\nu + \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma} \left( r + \frac{\theta^2}{2\gamma} \right)}{\nu - \frac{a}{\eta} + \frac{\nu+1}{\eta} \left( r - a - \frac{\theta^2}{2\gamma} \right)} \eta \nu
\]

\[
\left( \frac{1}{\gamma + \eta} \right) \left( \nu + \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma} \left( r + \frac{\theta^2}{2\gamma} \right) \right)
\]

(52)

It can be concluded from (52) that for general parameters

\[
\left. \frac{dC^*_t(\nu)}{C^*_t(\nu)} \right|_{\nu=0} = 0.
\]

(53)

This means that at mortality rate \( \nu = 0 \) there is no first order effect on consumption.

\(^9\)Note that \( dC^*_t \) above is the change in consumption in effect of a change in mortality, and that equation (51) is a priori unrelated to the consumption Euler equation (49), where the change \( dC^*_t \) is in effect of a change in time \( t \).
by increasing the mortality rate. The two effects of increasing current consumption because of fear of death in the future and decreasing consumption because of a decrease in human wealth exactly offset each other. The same neutrality holds for linear costs of labour. It can be easily verified that in the limit for $\eta \rightarrow 0$, expression (52) converges to 0 as well, i.e.

$$\lim_{\eta \rightarrow 0} \frac{dC^*_t(\nu)}{C^*_t(\nu)} \bigg|_{\eta \rightarrow 0} = 0,$$

(54)

independent of $\gamma$. Neutrality ceases to hold however, when the mortality rate is positive and $\eta > 0$, as the following numerical example shows. For the analysis below we assume the following parameter values: $\rho = 0.06$; $\gamma = 2$; $r = 0.03$, $\mu = 0.09$, $\sigma = 0.35$; $a = 0.01$, $b = 0.5$ and $\eta = 3$. Figure 1 shows the elasticity of consumption depending on the level of the mortality rate, for mortality rates ranging from 0 to 0.025. The mortality rate 0.025 corresponds to a 70 year old male living in the UK in 2006, according to current UK historical interim life tables, see UK-GAD (2011).

We observe that the elasticities are all negative, meaning that with increasing mortality consumption declines. Furthermore, the effect of a change in the mortality rate is strongest at about $\nu = 0.001503$, which corresponds to the mortality rate of a 39 year old male living in the UK in 2006. At that age, the elasticity of consumption is approximately at $-0.53$, which can be loosely interpreted as saying that if the mortality rate of a 39 year old declines by 10%, then consumption will increase by about 5%. The mortality rate of a 39 year old male living in the UK in 1981 was 0.001682, and hence declined over the period of 25 years between 1981 and 2006 by 12% hence inducing a growth in consumption of about 6%. If we look further down, at around pension age of 66 the mortality rate in 1982 was 0.032541

\footnote{For the case $\gamma = 1$, which corresponds to logarithmic utility from consumption, we obtain in the limit

$$\lim_{\gamma \rightarrow 1} \frac{dC^*_t(\nu)}{C^*_t(\nu)} \bigg|_{\gamma \rightarrow 1} = \left(1 - \frac{\nu + \rho}{\nu - \frac{\eta}{\eta + 1}(\gamma - \frac{\rho}{\eta})}\right) \frac{\eta \nu}{(\eta + 1)(\nu + \rho)},$$

and observe, that even in this case, for $\nu \neq 0$, the elasticity of consumption with respect to mortality is non-zero, except in the case where $\eta = 0$, i.e. linear costs of labour.}
while in 2006 the mortality rate for the same age group was 0.017108, which means that the mortality rate has been reduced over the period by roughly 50%. The elasticity in consumption at that mortality rate is $-0.44$, so that the reduction in mortality of this age group effects in a growth of consumption by approximately 20%. Real GDP over the period from 1982 to 2006 in the UK grew by about 100%, which means that the simple analysis above, provides an indication that a reduction in mortality rates had a significant impact on real GDP growth, possibly explaining between $10\% - 25\%$ of it.\(^{11}\)

We are now considering the case of time dependent mortality rates. Figure 2 shows age dependent mortality rates for various years between 1982 and 2006 in the UK.

The figure clearly shows that mortality rates are on very similar levels until about age 40, but then diverge. The following Figure 3 represents the mortality

\[^{11}\text{In similar manner as the elasticity of consumption with respect to mortality, one can discuss the elasticity of labour supply with respect to mortality. In the simplified model with constant parameters, it is in fact true that} \]

\[
\frac{dL^*_t(\nu)}{L^*_t(\nu)} = \frac{1}{\gamma} \frac{d\lambda(\nu)}{\lambda(\nu)},
\]

and hence, it can be concluded from (51), that the elasticity of labour supply with regards to mortality is proportional to the elasticity of consumption with respect to mortality.
Figure 2: Age dependent mortality rates for selected years between 1982 and 2006.

It can be seen that the mortality rates in the more senior age groups has decreased very significantly over the years, while in the more junior age groups up to age 30, the effect is far less significant.

In the following we use the following set of parameters, which have been obtained from calibrating the model to realistic data: $\rho = 0.06$, $\gamma = 0.95$, $r = 0.03$, $mu = 0.045$, $\sigma = 0.5$, $a = 0$, $w = 288000^{12}$, $eta = 0.72$ and $b = 3$. Figure 4 has been obtained by computing $C^*_0$ in Theorem 1 with time dependent mortality rates obtained from UK life expectancy tables for UK males from the years 1982 to 2006.

The figure shows an upward trend, as expected. The overall growth in consumption caused by the changing mortality curves over the 25 year period in this case is about 5%, compared with the aforementioned 100% in real GDP growth over the same period. Changes in the mortality curves in this setting still seem to have a significant impact on GDP.\(^{13}\)

\(^{12}\)This is the annual wage for working non-stop every day, every minute, for one year. Under the assumption of a 40 hour working week, this value corresponds to 60,000 GBP annually.

\(^{13}\)The labour supply from Figure 5, together with the chosen salary level $w$ corresponds to
Figure 3: Mortality rates for selected age groups between 1982 and 2006.

Figure 4: Consumption under historical mortality.
Let us now have a look at the labour supply. With the same parameters as before, we compute labour supply from Theorem 1 for the above historical mortality curves and obtain the following Figure 5. Labour supply is expressed in terms of hours per week.

![Labour supply](image)

Figure 5: Labour supply under historical mortality in hours per week.

We observe a noticeable downward trend which can also be observed in reality. Specifically labour supply decreases by about 6% over the 25 year period due to changing mortality curves from about 40.5 hours per week in 1982 to 38 hours in 2006.

Finally, let us look at portfolio investment. We have already indicated that the optimal portfolio investment consists of a Merton type rule, which is adjusted by a markup, which is always positive, see (47) and the discussion following. For the same mortality data as used for consumption and labour supply, by fixing the ratio of wage income and wealth to 0.03 in order to make the results comparable, annual wage income of approx. 69.000 in 1982 and 65.000 in 2006. Note that $a = 0$, and that therefore wage does not grow over time, with the chosen parameter set. The increase in consumption is mainly financed by taking out more risky investment. Over all, the saving consumption ratio is in the range of 16% to 24%, which is slightly higher than observed in real UK data for the same period.
and otherwise using the same set of parameters we obtain the following figure 6

![Figure 6: Portfolio investment under historical mortality.](image)

As expected, we observe that over time, agents invest more into the risky asset and less into the risk-less asset, primarily as a consequence of the reduced mortality and the increased life expectancy, but note that life insurance plays a role in this too.

The following figure 7 displays investment into the risky asset under strategy (47) in excess of the classical Merton rule $\pi_t = \frac{1}{\gamma} \mu_s \frac{\gamma - r}{\sigma^2}$, depending on the age of the investor and year in history.

We observe that among all ages of investors between 1982 to 2006 the proportion of wealth invested into the risky asset increases. Additional, fixing any year in history between 1982 and 2006, the proportion of wealth invested into the risky asset declines with the age of the investor. This is intuitive of course, as with the age of the investor, her/his mortality risk increases, and safer short term investments are sought. While this effect can also be observed in reality, it is not present in Yaari’s (1965), because of the non-existence of a risky asset, and Merton’s (1971) or Blanchard’s (1985) model, where constant mortality rates are assumed, and a
Figure 7: Excess investment into risky asset as function of age and year in history.
25 year old investor uses the same portfolio strategy as a 95 year old investor.\textsuperscript{14}

5 Conclusions

We have extended the classical life cycle models presented by Merton (1969) and (1971) as well as Bodie, Merton and Samuelson (1992) and included mortality risk of the type considered in Yaari (1965) into our analysis. The difference to Blanchard (1985) is that mortality risk is not assumed to be constant, but in fact obtained from current actuarial life tables. Another difference from both, Yaari (1965) and Blanchard (1985), is that we allow for a stochastic investment asset. Because of the increased complexity, however, aggregation is not carried out. This is done for constant mortality risk in the related paper by Ewald, Zhang and Nolan (2012). To the best of our knowledge, our framework is the first in continuous time, where real actuarial life expectancy data can be fed into a stochastic investment and consumption model. We derived closed-form solutions for optimal consumption, labor supply and investment strategy and showed that the inclusion of mortality risk, and in fact the shape of the mortality risk curve, significantly affects the level of consumption as well as the decomposition of the investment portfolio. An empirical analysis with UK actuarial data from 1982 to 2006 underlines the results.

\textsuperscript{14}Note that Blanchard’s (1985) model in addition does not allow for investment into a risky asset.
References


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