Learning of Risk Preference in Auctions: Nonparametric Identification and Estimation

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Abstract

This paper provides a nonparametric analysis of bidders’ risk preference in first-price sealed-bid auctions within the independent value paradigm. Under the assumption of heterogeneous risk preference among bidders, we show that the “learning rules” of bidders’ risk preference, i.e., how bidders update their risk preference based on the previous bidding results, is nonparametrically identified. Our nonparametric identification procedure does not depend on the specification of risk preference, and the procedure is constructive in that it implies a consistent estimator of bidders’ “learning rule" of risk preference. The identification and estimation are complete and we plan to conduct a Monte Carlo study to illustrate the performance of the proposed estimator. We will also apply the methodology to U.S. Forest Service timber auction data, and provide some evidence on bidders’ updating their risk preference by estimating the “learning rules".

JEL Classification: C14, D03, D44

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1 Introduction

The existing literature has addressed the importance of risk aversion in individuals’ strategic behavior in auctions(e.g., see Cox, Smith, and Walker (1983, 1988), Baldwin, Marshall, and
Richard (1997), Bajari and Hortaçsu (2005), Athey and Levin (2001), Campo, Guerre, Perrigne, and Vuong (2011)). However, the homogeneity of risk preference across individuals is mostly imposed with only few exceptions (e.g., Cox, Smith, and Walker (1988), and Campo (2006)). It is generally unknown how heterogeneity of risk preference can be incorporated in auction models and affects individuals' behavior. This paper allows heterogeneity of risk preference among bidders in the framework of first-price sealed-bid auctions, and every bidder may change her risk preference based on the previous bidding results. We show that the “learning rules” of bidders’ risk preference, i.e., the conditional probability of the bidder being one risk type conditional on her previous type and bidding result is nonparametrically identified. The procedure of non-parametric identification is constructive and independent of specification of risk preference. As a consequence, bidders’ learning rule of risk preference can be consistently estimated following the identification procedure. We plan to apply the methodology to U.S. Forest Service timber auction data, and estimate how bidders update their risk preference in field.

It has been shown by the existing literature that Bayesian-Nash equilibrium (BNE) with risk neutral bidders may be not appropriate to explain bidders’ behavior in both laboratory and field. Specifically, the phenomenon that bidders bid higher than BNE, i.e., overbidding is empirically observed from both experimental and field data,(e.g., see Cox, Smith, and Walker (1983, 1988), and Malmendier and Lee (2011) for experimental and field evidence, respectively) Therefore, researchers turn to models deviating risk-neutral Bayesian-Nash equilibrium and they show that observed bidders’ behavior may be better explained. For instance, Bajari and Hortaçsu (2005) estimate four alternative structural models using experimental data of first-price auctions, and shows that the risk averse BNE fits the data best. However, homogeneous risk preference is assumed by most of the existing work, even though the existence of heterogeneity in risk preference across individuals is addressed empirically in other uncertainty environment such as insurance ( see Cohen and Einav (2007)).

Focusing on first-price auctions, this paper allows not only heterogeneous risk preference across bidders but also the possibility that a bidder updates her risk preference based on the previous bidding results. The identification objective is the conditional probability of bidders being a risk type conditional on their risk type in the proceeding auction, and their bidding
results, i.e., winning or losing. We call the objective “learning rule” of bidders’ risk preference. To identify the learning rule, we assume bidders have discrete “risk type”, and we employ the idea of identification in Hu (2008) to develop a novel nonparametric approach. The procedure of identification requires bids from independent auctions contested by bidders with their identity being observed, and each bidder has three bids from three independent auctions. The essential assumption on the data is that bids are correlated only through risk type but independent conditional on risk type. In addition to the “learning rules” of risk preference, the identification procedure also permits us to recover the distribution of risk type for each period.

The advantage of the identification methodology is twofold. First, the procedure is fully nonparametric in the sense that it does not depend on the specification of bidders’ risk preference, i.e., CRRA or CARA. We only impose the assumption that more risk averse bidders bid more aggressively, which could be rationalized by some experimental evidence presented in Cox, Smith, and Walker (1988). Second, the procedure is constructive and it yields a nonparametric estimator for the “learning rules” of risk preference, and also an estimator of (discrete) type distribution for each period. Furthermore, the identification methodology may also be used to identify other unobserved heterogeneities across bidders.

This paper makes several contributions to the literature on auctions and individuals’ risk averse behavior. First, we nonparametrically identify the probability of bidders updating their risk preference under the assumption of heterogeneous risk preference. To our best knowledge, this is the first paper deals with such a problem. More generally, the identification procedure in this paper may be applied to other models with heterogeneity. Second, this paper will provide empirical results on bidders’ “learning rules” and distribution of bidders’ risk type in field. The results can not only help us understand risk averse bidders’ strategic behavior but also have some policy implications for policymakers to design optimal mechanism of auctions.

We organize the rest of the paper as follows. In section 2, we present a theoretical model of first-price sealed-bid auction with heterogeneous risk averse bidders. In section 3, we propose the nonparametric identification of the model. In section 4, we discuss the nonparametric estimation of the “learning rule” for risk preference. In section 5, we will provide Monte Carlo evidence on the performance of proposed estimator. In section 6, we will present an empirical illustration,
using the United States Forest Service (USFS) timber auction data. Section 7 concludes.

2 The Theoretical Model

We consider first-price sealed-bid auctions with bidders of heterogeneous risk preference.

2.1 Risk-averse BNE

Risk-averse Bayesian-Nash equilibrium, where bidders are assumed to be risk-averse, is an important approach to realized observed auction data. For example, Baldwin (1995) and Athey and Levin (2001) empirically argue that bidders possess risk-aversion in US Forest Timber auctions. The model is also employed to explain observed overbidding in first-price sealed-bid auctions, e.g., see Cox, Smith, and Walker (1983, 1988), Bajari and Hortaçsu (2005). The existing literature mainly focus on the symmetric risk-averse BNE, where all the bidders have the same utility function. As a matter of fact, the existence of heterogeneity in risk attitude across individuals is proved empirically in other environment with uncertainty (see Cohen and Einav (2007)). In this paper, we consider a heterogeneous risk-averse BNE in the sense that bidders’ von Neuman Morgenstern (vNM) utility functions are not identical, while the valuations of the bidders are drawn from the same CDF, $F(\cdot)$. Bidder $i$’s von Neuman Morgenstern (vNM) utility function is $U_i(\cdot)$, with $U'_i(\cdot) > 0$ and $U''_i(\cdot) < 0$. The $N$ bidders’ have $K$ different utility functions, $U_1(\cdot), U_2(\cdot), \ldots, U_K(\cdot), K < N$. Campo (2006) employs a similar model to discuss procurement auctions. The model is also discussed in Athey and Haile (2007).

The existence of an equilibrium for the first-price sealed-bid auction is guaranteed by the general existence results of Theorem 7(1) in Athey (2001).\footnote{We don’t have results on the uniqueness of the equilibrium.} At equilibrium, all the bidders with utility function $U_k(\cdot), k = 1, 2, \ldots, K$ according to an increasing bidding function $s_k(\cdot)$. Let $G_k(b)$ denote the bid distribution for these bidders, we have $G_k(b) = F(s_k^{-1}(b))$. Consequently,
the observed bid distribution $G(b)$ can be expressed as

$$G(b) = \sum_{k=1}^{K} \lambda_k G_k(b) = \sum_{k=1}^{K} \lambda_k F(s_k^{-1}(b)),$$

where $\lambda_k$ ($k = 1, 2, ..., K$) is the probability of the bidders being risk type $k$ and with utility function $U_k(\cdot)$.

For the bidding strategies $s_k(\cdot)$, we impose the following assumption:

**Assumption 1.** More risk averse bidders bid more aggressively, i.e., $s_k(v) > s_l(v)$ for all $v \in (v_l, v]$ if type $k$ is more risk averse than type $l$.

The theoretical properties of bidding strategies $s_k^{-1}(\cdot)$ are unclear in existing literature, but we have the following results in symmetric auctions.

**Lemma 1.** In a symmetric equilibrium of first-price sealed-bid auction, more risk-averse bidders bid more aggressively.

**Proof** The proof is based on the result in Krishna (2010) (p.39) which states that in a symmetric equilibrium of first-price sealed-bid auction, the bidding functions $s_1(\cdot)$ and $s_2(\cdot)$ satisfy

$$s_i'(v) = \frac{U_i(v - s_i(v))}{U'_i(v - s_i(v))} \times \frac{h(v)}{H(v)}, i = 1, 2,$$

where $H(\cdot) = F^{N-1}(\cdot)$ and $h(\cdot) = H'(\cdot)$. Without loss of generality, we assume bidders with utility function $U_1(\cdot)$ is more risk-averse. Then there exists a strictly concave function $T(\cdot)$ with $T(0) = 0$ such that

$$U_1(\cdot) = T(U_2(\cdot)).$$

We further prove the lemma by seeking contradictions. Suppose that the more risk-averse bidders bid less aggressively, i.e. $s_2(v) > s_1(v)$ for all $v$. Then $U_1(v - s_1(v)) > U_1(v - s_2(v))$ and $U'_1(v - s_1(v)) < U'_1(v - s_2(v))$ since $U_i$ is strictly concave. Consequently,

$$\frac{U_1(v - s_1(v))}{U'_1(v - s_1(v))} > \frac{U_1(v - s_2(v))}{U'_1(v - s_2(v))} = \frac{T(U_2(v - s_2(v)))}{T'(U_2(v - s_2(v)))U'_2(v - s_2(v))}.$$
The fact that $T(\cdot)$ is concave, and $T(0) = 0$ implies that

$$\frac{T(U_2(v - s_2(v)))}{T'(U_2(v - s_2(v)))} > U_2(v - s_2(v)).$$

Therefore,

$$\frac{U_1(v - s_1(v))}{U_1'(v - s_1(v))} > \frac{U_1(v - s_2(v))}{U_1'(v - s_2(v))} > \frac{U_2(v - s_2(v))}{U_2'(v - s_2(v))}.$$

Combining this result with Eq.(1), we obtain

$$s_1'(\cdot) > s_2'(\cdot).$$

We prove that $s_2(\cdot) > s_1(\cdot)$ implies $s_1'(\cdot) > s_2'(\cdot)$. Furthermore, $s_2(0) = s_1(0) = 0$. Then we conclude that $s_1(v) > s_2(v)$ for all $v$.

Even though Assumption 1 cannot be derived directly from Lemma 1, it helps rationalize the assumption. Furthermore, in an experimental setting, Cox, Smith, and Walker (1988) show that more risk averse bidders bid more aggressively.

## 3 Nonparametric Identification

In this section, we present the nonparametric procedure to identify “learning rules” of bidders’ risk aversion.

### 3.1 The Econometric Model

A single and indivisible object is sold through a first-price sealed-bid auction among $N(\geq 2)$ risk-neutral bidders with private values $v_1, v_2, ..., v_N$, which are random draws from $F(\cdot)$ which is a CDF with support $[\underline{v}, \overline{v}]$. The distribution is differentiable over $(\underline{v}, \overline{v})$ and the density functions $f(\cdot)$ is continuous and strictly positive over $(\underline{v}, \overline{v})$. In addition to valuation, each bidder is one of the $K (K \leq N)$ risk-aversion types, and we denote the type of $i$-th bidder $\tau_i$, for all $i \in \{1, 2, ..., N\}$. Bidder $i$’s bidding strategy $s_i$ is a mapping from value and type to her bid, i.e.,
$s_i(\cdot, \cdot) : [v, \overline{v}] \times \{1, 2, \ldots, K\} \rightarrow [v, \overline{v}]$. \(^2\) In such a setting, the distributions of values and types are common knowledge among bidders while unobserved by the econometrician a bidder may or may not know her opponents’ type: the information is contained in her bidding strategy $s_i(\cdot)$.

Further assumptions will be imposed when it is necessary for some specific structural auction models.

We consider the case where each of the $N$ bidders participates in three independent first-price sealed-bid auctions sequentially\(^3\). In auction $t$ ($t = 1, 2, 3$), we observe $(b_{ti}, w_{ti})$, where $b_{ti}$ and $w_{ti} \in \{0, 1\}$ are bidder $i$’s bid and indicator of winning with 1 indicating winning. In a short panel data, we observe a sample $\{w_{3i}, b_{3i}, w_{2i}, b_{2i}, w_{1i}, b_{1i}\}$ for $i = 1, 2, \ldots, N$, and each bidder’s identity. Bidder $i$’s type in auction $t$ is denoted as $\tau_{ti} \in \{1, 2, \ldots, K\}$, and the type may evolve across auctions. The identification objective is how bidders adjust their type across auctions based on their previous type and the indicator of winning, i.e., $f(\tau_{ti} | w_{t-1,i}, \tau_{t-1,i})$, which we call “learning rules” of risk preference. In the remaining part of this section, we show that this learning rules can be nonparametrically identified and estimated using the observed bids, indicator of winning, and bidders’ identity.

### 3.2 Nonparametric Identification

We assume that $\{w_{3i}, b_{3i}, \tau_{3i}, w_{2i}, b_{2i}, \tau_{2i}, w_{1i}, b_{1i}, \tau_{1i}\}$ is i.i.d for $i \in \{1, 2, \ldots, N\}$. For simplicity, we omit the subscript $i$.

We first consider the joint distribution of three bids $b_1, b_2, b_3$ and the indicator of winning $w_2$,

$$
\begin{align*}
\hat{f}_{b_3, w_2, b_2, b_1} &= \sum_{\tau_3} \sum_{\tau_2} \hat{f}_{b_3, \tau_3, w_2, \tau_2, b_1} \\
&= \sum_{\tau_3} \sum_{\tau_2} \hat{f}_{b_3 | \tau_3, w_2, \tau_2, b_1} \hat{f}_{\tau_3 | w_2, \tau_2, b_1} \hat{f}_{w_2 | b_2, \tau_2, b_1} \hat{f}_{b_2 | \tau_2, b_1} \hat{f}_{\tau_2, b_1}.
\end{align*}
$$

\(^2\) Notice that if a bidder’s value is $v$, her winning probability is zero and we assume she bids $v$. Also it is not necessary true that the bidding strategy $s_i(\cdot)$ is at equilibrium, hence our methodology also applies to some non-equilibrium auction models.

\(^3\) But the three auctions are not “sequential auctions” as analyzed in the literature.
Let the information set $\Omega_{<t}$ contains all the information before period $t$, i.e.,

$$\Omega_{<t} = \{(w_s, b_s, \tau_s) \text{ for } s = 1, 2, ..., t - 1\}.$$ 

For each auction a bidder participates, the bidding strategy is dependent on her own value, the value distribution, her own risk type and the distribution of risk types. Hence we impose the following assumption to our analysis.

**Assumption 2.** *(static auctions)* Conditional on type $\tau_t$, the bid $b_t$ is independent of the history $\Omega_{<t}$, i.e.,

$$f(b_t | \tau_t, \Omega_{<t}) = f(b_t | \tau_t).$$

By this assumption the dependence of the bid $b_t$ on the history $\Omega_{<t}$ is absorbed to the evolution of type across auctions. This assumption implies that we have both $f_{b_3 | \tau_3, w_2, b_2, \tau_2, b_1} = f_{b_3 | \tau_3}$ and $f_{b_2 | \tau_2, b_1} = f_{b_2 | \tau_2}$. Consequently, Eq.(2) can be simplified as

$$f_{b_3, w_2, b_2, b_1} = \sum_{\tau_3} \sum_{\tau_2} f_{b_3 | \tau_3} f_{\tau_3 | w_2, b_2, \tau_2, b_1} f_{w_2 | b_2, \tau_2, b_1} f_{b_2 | \tau_2} f_{\tau_2, b_1}.$$ 

In the equation above, $f_{\tau_3 | w_2, b_2, \tau_2, b_1} f_{w_2 | b_2, \tau_2, b_1}$ describes how bidders’ risk type evolves across auctions. To further analyze the learning process, we impose the following assumption on the learning rules,

**Assumption 3.** *(learning rule)* Conditional on the current winning indicator $w_t$ and the current type $\tau_t$, the next-period type $\tau_{t+1}$ is independent of the current bid $b_t$ and the history $\Omega_{<t}$, i.e.,

$$f(\tau_{t+1} | w_t, b_t, \tau_t, \Omega_{<t}) = f(\tau_{t+1} | w_t, \tau_t).$$

This assumption can be rationalized as the result of the independence of values for a bidder across auctions: since values are independent, a bidder can only learn from whether she wins and what her type in the last period. The independence of type $\tau_{t+1}$ and the information set $\Omega_{<t}$ is a simplification and it can be relaxed when more periods of data are available for each bidder. As a consequence of the assumption above, we have $f_{\tau_3 | w_2, b_2, \tau_2, b_1} = f_{\tau_3 | w_2, \tau_2}$. Hence Eq.(??) can be changed to

$$f_{b_3, w_2, b_2, b_1} = \sum_{\tau_3} \sum_{\tau_2} f_{b_3 | \tau_3} f_{\tau_3 | w_2, \tau_2} f_{w_2 | b_2, \tau_2, b_1} f_{b_2 | \tau_2} f_{\tau_2, b_1}.$$ 

(3)
To further analyze the R.H.S. of Eq.(3), we consider the winning probability for a bidder $i$ in the period $t = 2$, $\Pr(w_2 = 1) = \Pr(b_{2i} > b_{2j}), i \neq j$. Therefore, it is natural to impose the following condition on the model.

**Assumption 4. (probability of winning)** Given the current bid $b_t$, the probability of winning is independent of the current type $\tau_t$ and the previous bid $b_{t-1}$, i.e., $f_{w_2|b_2,\tau_2,b_1} = f_{w_2|b_2}$.

Based on the condition above, we have

$$f_{b_3,w_2,b_2,b_1} = \sum_{\tau_3} \sum_{\tau_2} f_{b_3|\tau_3} f_{\tau_3|w_2,\tau_2} f_{w_2|b_2} f_{b_2|\tau_2} f_{\tau_2,b_1}$$

$$= \sum_{\tau_2} \left( \sum_{\tau_3} f_{b_3|\tau_3} f_{\tau_3|w_2,\tau_2} \right) f_{w_2|b_2} f_{b_2|\tau_2} f_{\tau_2,b_1}$$

$$= \sum_{\tau_2} f_{b_3|w_2,\tau_2} f_{w_2|b_2} f_{b_2|\tau_2} f_{\tau_2,b_1}$$

$$= f_{w_2|b_2} \sum_{\tau_2} f_{b_3|w_2,\tau_2} f_{b_2|\tau_2} f_{\tau_2,b_1}.$$ 

Rewriting the L.H.S. of the equation above as $f_{b_3,b_1|w_2,b_2} f_{w_2|b_2} f_{b_2}$, we can then rearrange the equation as

$$f_{b_3,b_1|w_2,b_2} f_{b_2} = \sum_{\tau_2} f_{b_3|w_2,\tau_2} f_{b_2|\tau_2} f_{\tau_2,b_1} \quad (4)$$

Integrating out $b_2$ on both sides of the equation above, we obtain

$$\int f_{b_3,b_1|w_2,b_2} f_{b_2} db_2 = \sum_{\tau_2} f_{b_3|w_2,\tau_2} f_{\tau_2,b_1} \quad (5)$$

Since bidders’ risk type is discrete, it is convenient to express the equation above in matrix form. For given $w_2$ and $b_2$ we have

$$A \equiv B_{b_3|w_2,\tau_2} D_{b_2|\tau_2} C_{\tau_2,b_1} \quad (6)$$

$$E \equiv B_{b_3|w_2,\tau_2} C_{\tau_2,b_1} \quad (7)$$
where

\[
A_{ij} \equiv f(b_3 = i, b_1 = j|w_2, b_2)f_{b_2},
\]

\[
E_{ij} \equiv \int f(b_3 = i, b_1 = j|w_2, b_2)f_{b_2}db_2,
\]

\[
B_{b_3|w_2, \tau_2} \equiv [f(b_3 = i|w_2, \tau_2 = k)]_{ik},
\]

\[
C_{\tau_2, b_1} \equiv [f(\tau_2 = k, b_1 = j)]_{kj},
\]

\[
D_{b_2|\tau_2} \equiv \text{diag}[f(b_2|\tau_2 = 1) f(b_2|\tau_2 = 2) \cdots f(b_2|\tau_2 = m)].
\]

For a given value of \(w_2\), the matrix \(E = B_{b_3|w_2, \tau_2}C_{\tau_2, b_1}\) describes the joint distribution of two bids \(b_1\) and \(b_3\). As argued in An (2010), the rank of this matrix can be used to identify the number of types under two conditions: first, the support of \(\tau_t\) does not change along with \(t\); second, bid distribution of any type is not a linear combination of those for other types. We employ this insight here and make the following assumption.

**Assumption 5. (invertibility)** The matrix \(B_{b_3|w_2, \tau_2}C_{\tau_2, b_1}\) is invertible for any given \(w_2\).\(^4\)

The assumption of invertibility implies \(E^{-1} = C_{\tau_2, b_1}^{-1}B_{b_3|w_2, \tau_2}^{-1}\). Combining Eq.(6) with the relationship above, we obtain

\[
A \times E^{-1} = B_{b_3|w_2, \tau_2}D_{b_2|\tau_2}B_{b_3|w_2, \tau_2}^{-1}
\]

(8)

where \(D_{b_2|\tau_2}\) and \(B_{b_3|w_2, \tau_2}\) are matrices of eigenvalues and eigenvectors, respectively. Especially, the diagonal elements of \(D_{b_2|\tau_2}=k\), \(k \in \{1, 2, \ldots, K\}\) is the distribution of bids for bidders of type \(k\). Employing the strategies of identification proposed in Hu (2008), if the matrix decomposition in Eq.(8) is unique, then both \(f_{b_3|w_2, \tau_2}\) and \(f_{b_2|\tau_2}\) are identified since the L.H.S of the equation can be recovered from data. For the purpose of uniqueness of the decomposition, we need a normalization of eigenvector matrix \(B_{b_3|w_2, \tau_2}\) and ordering of the diagonal elements of the eigenvalue matrix \(D_{b_2|\tau_2}\). Considering that for a given value of \(w_2\), each element in the eigenvector matrix \(B_{b_3|w_2, \tau_2}\) is a conditional probability, hence each column of the matrix sums up to one. Then a plausible method of normalization is to divide each column by the corresponding column sum. As argued in Hu (2008), the method to correctly order the \(K\) eigenvalues \(f(b_2|\tau_2)\) is model specific and not unique. Therefore we make the following assumption on the ordering.

\(^4\)We will leave the relationship between the rank of \(E\) and the number of types in next section when we discuss specific structural auction models.
**Condition 1. (uniqueness)** There exists a function $\varpi(\cdot)$ such that $\varpi(b_2|\tau_2)$ is strictly increasing or decreasing in $\tau_2$.

This condition is guaranteed by Assumption 1: we have $F(s_k^{-1}(b)) \leq F(s_l^{-1}(b))$ for any given bid $b$ if type $k$ is more risk averse than type $l$. Moreover, we consider that the bidding function $s_k(\cdot)$, and $s_l(\cdot)$ are monotonically increasing, then $G(b|\tau = i) = Pr(B \leq b|\tau = i) = Pr(s_i(V) \leq b) = Pr(V \leq s_i^{-1}(b)) = F(s_i^{-1}(b)), i = k, l$. As a consequence, we obtain $G_k(b) = G(b|\tau = k) \leq G_l(b) = G(b|\tau = l)$, i.e., the function $\varpi(\cdot)$ can be chosen as any quantile of bid distribution.

Up till now, both $f_{b_3|w_2,\tau_2}$ and $f(b_2|\tau_2)$ are identified. The learning rule $f_{\tau_3|w_2,\tau_2}$ can be identified using $f_{b_3|w_2,\tau_2}$ and $f(b_2|\tau_2)$ from the relationship

$$f_{b_3|w_2,\tau_2} = \sum_{\tau_3} f_{b_3|\tau_3} f_{\tau_3|w_2,\tau_2},$$

where $f_{b_t|\tau_t} = f_{b_2|\tau_2}$ is implied by stationarity, which implies that bid distributions in different periods are the same for each type. This condition imposes no restriction on the model because it is assumed that auctioned objects are homogeneous and bidders' values are i.i.d. across auctions, the distribution of bids must be the same for each type in different periods.

Similarly, we can also identify another learning rule $f_{\tau_2|w_1,\tau_1}$. For this purpose, we consider the equality

$$f_{b_2, w_1, b_1} = \sum_{\tau_2} \sum_{\tau_1} f_{b_2, \tau_2, w_1, b_1, \tau_1}$$

$$= \sum_{\tau_2} \sum_{\tau_1} f_{b_2|\tau_2, w_1, b_1, \tau_1} f_{\tau_2|w_1, b_1, \tau_1} f_{w_1|b_1, \tau_1} f_{\tau_1}$$

$$= \sum_{\tau_2} \sum_{\tau_1} f_{b_2|\tau_2} f_{\tau_2|w_1, b_1, \tau_1} f_{w_1|b_1, \tau_1} f_{\tau_1}.$$  

Moving $f_{w_1|b_1}$ to L.H.S. of the equation above leads to

$$f_{b_2|w_1, b_1} f_{b_1} = \sum_{\tau_2} \sum_{\tau_1} f_{b_2|\tau_2} \left(f_{\tau_2|w_1, \tau_1} f_{\tau_1}\right) f_{b_1|\tau_1}. \quad (10)$$

Since we have identified $f_{b_2|\tau_2} = f_{b_1|\tau_1}$, we may solve for $f_{\tau_2|w_1, \tau_1} f_{\tau_1}$ from the equation above. Moreover, $f_{\tau_t}, t = 1, 2, 3$ can also be solved from

$$f_{b_t} = \sum_{\tau_t} f_{b_t|\tau_t} f_{\tau_t}.$$
Consequently, the learning rule \( f_{\tau_2|w_1,\tau_1} \) is identified.

In summary, we have identified \( f_{\tau_1} \), and the learning rules \( f_{\tau_2|w_1,\tau_1} \) and \( f_{\tau_3|w_2,\tau_2} \). The results are presented as follows.

**Theorem:** Under Assumptions 1-5, the learning rules \( f_{\tau_2|w_1,\tau_1} \) and \( f_{\tau_3|w_2,\tau_2} \), together with \( f_{\tau_1} \), are uniquely determined by \( f_{b_3|w_2,\tau_2,\tau_1} \).

### 3.3 Nonparametric Estimation

The identification result is constructive, and it implies a multiple step procedure of estimation. In the first step, we estimate \( f_{b_2|\tau_2} \) and \( f_{b_3|w_2,\tau_2} \) from the eigenvalue-eigenvector decomposition in Eq.(8). Then we proceed to estimate \( f_{\tau_3|w_2,\tau_2} \) based on the relationship Eq.(9). and \( f_{\tau_1} \) and \( f_{\tau_2|w_1,\tau_1} \) from Eq.(10) in the second step. Let \( N \) be the number of bidders or sample size, and \( n \) be the index of \( n \)th bidder.

**First step** To conduct the first step of estimation, we first estimate the two matrices on the R.H.S. of Eq.(8) for given \( b_2 \) and \( w_2 \).

\[
\hat{A} = \left( \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}(d_{n3} = i, d_{n1} = j|w_2, b_2) \right)_{i,j} f_{b_2} \\
\hat{E} = \int \left( \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}(d_{n3} = i, d_{n1} = j|w_2, b_2) \right)_{i,j} f_{b_2} db_2
\]

According to Hu (2008), \( B_{b_3|w_2,\tau_2} \) can be expressed as a non-stochastic analytical function (denote the function \( \phi(\cdot) \)) of \( A \times E^{-1} \). Consequently, an estimator of \( B_{b_3|w_2,\tau_2} \) for a given \( w_2 \) is

\[
\hat{B}_{b_3|w_2,\tau_2} = \phi \left( \hat{A} \times \hat{E}^{-1} \right).
\]

Similarly, the eigenvalue matrix \( D_{b_2|\tau_2} \), which consists of \( m \) possible conditional densities \( f(b_2|\tau_2 = 1), f(b_2|\tau_2 = 2), \ldots, f(b_2|\tau_2 = m) \), can be estimated from the decomposition in Eq.(8).
Second step  To estimate the learning rule based on the first-step estimation, we first rewrite Eq.(9) in matrix form,

$$B_{b_3|w_2,\tau_2} = D_{b_3|\tau_3} B_{\tau_3|w_2,\tau_2}.$$  

Then for a given $w_2$, the learning rule included in the matrix $B_{\tau_3|w_2,\tau_2}$ can be estimated as

$$\hat{B}_{\tau_3|w_2,\tau_2} = \hat{D}^{-1}_{b_3|\tau_3} \hat{B}_{b_3|w_2,\tau_2}.$$  

Employing a similar procedure, we can estimate another learning rule $f_{\tau_2|w_1,\tau_1}$ as follows.

$$\hat{B}_{\tau_2|w_1,\tau_1} = \hat{D}^{-1}_{b_2|\tau_2} \hat{B}_{b_2|w_1,b_1} \hat{B}_{b_1} \hat{D}^{-1}_{b_2|\tau_1} \hat{B}_{\tau_1}.$$  

where the matrices $B_{\tau_2|w_1,\tau_1}$, $B_{b_2|w_1,\tau_1}$, and $D_{b_2|\tau_2}$ are defined as before. $B_{b_1}$ and $B_{\tau_1}$ are both column matrices, and they consist of probability of $b_1$ and $\tau_1$, respectively. The estimator of $B_{\tau_1}$, $\hat{B}_{\tau_1}$ can be expressed explicitly as

$$\hat{B}_{\tau_1} = \hat{D}^{-1}_{b_1|\tau_1} \hat{B}_{b_1} = \hat{D}^{-1}_{b_2|\tau_2} \hat{B}_{b_1}.$$  

4 Monte Carlo Evidence

This section provides some Monte Carlo evidence on the performance of our estimator. We consider first-price auctions with bidders value $v_i \sim U[0, 1]$

5 Empirical Application: USFS Timber Data

In this section, we apply our methodology to U.S. Forest Service (U.S.F.S) timber auction data and provide some empirical evidence on how bidders update their risk preference based on the outcomes of their previous bidding behavior.
5.1 The Data

The USFS timber auction data has been extensively used for analysis of various auction models and some of the analyses provide evidence on bidders’ risk aversion. For example, the empirical results in Baldwin (1995) suggest that bidders might be of decreasing absolute risk aversion. Athey and Levin (2001) argue that bidders’ behavior is consistent with some amount of risk aversion. Under the specification of the constant relative risk aversion (CRRA) utility function, Campo, Guerre, Perrigne, and Vuong (2011) measure that the risk aversion parameter under different specification of risk aversion. These existing evidence of risk aversion implies that it is natural to analyze bidders’ risk preference using timber auction data.

We focus on those first-price sealed-bid ones in all of the Forest Service auctions. To adopt the private value paradigm, we only use those “scaled sale” auctions. In a “scaled sale” auction, bidders bid on a per unit basis (thousand board-feet or mbf) and the payments are based on the winning bidder’s unit prices and the actual volumes, which are measured by a third party at the time of harvest. Haile, Hong, and Shum (2003) empirically demonstrate that there is little evidence of common values for scaled sale auctions. Many other studies also assume private values for timber auctions, e.g., see Baldwin, Marshall, and Richard (1997), Haile (2001), and Haile and Tamer (2003) among others. We further assume that the reserve price is nonbinding for the auctions based on the empirical evidence proposed in Haile (1996). The property is also widely acknowledged by other studies using USFS timber auctions, e.g., Baldwin, Marshall, and Richard (1997), Haile (2001), and Campo, Guerre, Perrigne, and Vuong (2007). Our sample contains scaled sale sealed-bid first-price auctions conducted from 1982 to 1993 in all regions with both salvage and small-business set-aside sales being eliminated. We observe bidder’s identity and their bids (dollar per mbf), the volume and the appraisal value of each tract.

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6In USFS timber auction data, one can observe the identities of the highest and second highest bidders only and for the other bids, one cannot put a name on them. However, this is not a concern in this application since I only use those auctions with three bidders.
6 Conclusions

References


